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Proof of the Riemann Hypothesis

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Abstract: By the strong relation between ξ function and the Riemann ζ function, we will prove the Riemann hypothesis which states that "All non-trivial zeros $z = x + y\,i$ of ζ must lie on the critical line $x = \frac{1}{2}$".

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1. Introduction

This work is only concerned with proving the Riemann Hypothesis ([1]), so it is very short and focuses on the proof only. The Riemann Hypothesis concerns about the Riemann zeta function ([2,3]) which is defined for $\Re(z) > 1$ by the following infinite sum:

$$\zeta(z) := \sum_{n=1}^{\infty} \frac{1}{n^z} = \sum_{n=1}^{\infty} \frac{1}{e^{z\,\ln n}} = \sum_{n=1}^{\infty} \frac{e^{-i\,(y\,\ln n)}}{n^x} \tag{1}$$

Riemann does not refer to analytic continuation of the function ζ beyond the half plane $\Re(z) > 1$. Instead, he focuses on finding a formula that applies to all z. First, he derives his formula for $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ which is valid for every $z : \Re(z) > 0$, he achieved this by relating the function ζ to the function $\Gamma(z) = \int_0^{\infty} t^{z-1}\,e^{-t}\,dt$ with the following relation

$$\zeta(z) = \frac{1}{\Gamma(z)} \int_0^{\infty} \frac{t^{z-1}}{e^t - 1}\,dt \tag{2}$$

Next, Riemann derives his formula to remains valid for all z by the relation

$$\pi^{-\frac{z}{2}}\,\Gamma\left(\left(\frac{z}{2}\right)\zeta(z) = \int_1^{\infty} \left[t^{\frac{z}{2}} + t^{\frac{1-z}{2}}\right] \frac{\psi(t)}{t}\,dt - \frac{1}{z(1-z)} \tag{3}$$

where $\psi(t) := \sum_{n=1}^{\infty} e^{-\pi\,n^2\,t}$.

The functional equation of the zeta function is demonstrated by the fact that the right side of (3) remains unchanged when $z = 1 - z$:

$$\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z) = \pi^{\frac{-(1-z)}{2}}\Gamma\left(\frac{1-z}{2}\right)\zeta(z-1) \tag{4}$$

In the symmetric form of the functional equation, the function $\pi^{\frac{-z}{2}}\Gamma\left(\frac{z}{2}\right)\zeta(z)$, has removable poles at $z = 0$ and $z = 1$, Riemann multiplies it by $\frac{1}{2}\,z(z-1)$ and define

$$\xi(z) := \frac{1}{2}\,z(z-1)\,\pi^{-\frac{z}{2}}\,\Gamma\left(\frac{z}{2}\right)\zeta(z) \tag{5}$$

2. Main Results

Lemma 1. *If $z = x + y\,i$ and $y \neq 0$, then*

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I. $\overline{\zeta(z)} = \zeta(\overline{z})$ *and* $\overline{\Gamma(z)} = \Gamma(\overline{z})$
II. $\overline{\xi(z)} = \xi(\overline{z})$
III. $\xi(z) = \xi(1-z)$
IV. $\overline{\xi(z)} = \xi(1-\overline{z})$
V. *If* $\xi((1-x)+y\,i) = \xi(x+y\,i)$, *then* $x = \frac{1}{2}$
VI. *The non-trivial zeros of* ζ *is the same the non-trivial zeros of* ξ

Proof. I. First we prove that $\overline{\zeta(z)} = \zeta(\overline{z})$

$$\begin{aligned}
\overline{\zeta(z)} &= \overline{\sum_{n=1}^{\infty} n^{-z}} = \overline{\sum_{n=1}^{\infty} n^{-x} e^{-i\,y \ln n}} \\
&= \overline{\sum_{n=1}^{\infty} n^{-x} cos(y \ln n) - i \sum_{n=1}^{\infty} n^{-x} \sin(y \ln n)} \\
&= \sum_{n=1}^{\infty} n^{-x} cos(y \ln n) + i \sum_{n=1}^{\infty} n^{-x} \sin(y \ln n) \\
&= \sum_{n=1}^{\infty} n^{-x} e^{i\,y \ln n} = \sum_{n=1}^{\infty} n^{-(x-y\,i)} = \zeta(\overline{z})
\end{aligned}$$

Similarly, we can prove that $\overline{\Gamma(z)} = \Gamma(\overline{z})$

II. By the definition of ξ and using I., we have

$$\begin{aligned}
\overline{\xi(z)} &= \overline{\frac{1}{2} z(z-1)\, \pi^{-\frac{z}{2}}\, \Gamma\left(\frac{z}{2}\right) \zeta(z)} \\
&= \frac{1}{2} \overline{z(z-1)}\, \overline{\pi^{-\frac{z}{2}}}\, \overline{\Gamma\left(\frac{z}{2}\right)}\, \overline{\zeta(z)} \\
&= \frac{1}{2} \overline{z}(\overline{z}-1)\, \pi^{-\frac{\overline{z}}{2}}\, \Gamma\left(\frac{\overline{z}}{2}\right) \zeta(\overline{z}) = \xi(\overline{z})
\end{aligned}$$

III. It is obvious from (4) that $\xi(z) = \xi(1-z)$

IV. From II. and III., we obtain IV.

VI. Let $\xi((1-x)+y\,i) = \xi(x+y\,i)$. By substitute in the definition of $\xi(z)$ and using (3), we have

$$(a-b\,i)\int_1^{\infty}\left[t^{\frac{(1-x)+y\,i}{2}} + t^{\frac{x-y\,i}{2}}\right]\frac{\psi(t)}{t}\,dt = (a+bi)\int_1^{\infty}\left[t^{\frac{x+y\,i}{2}} + t^{\frac{(1-x)-y\,i}{2}}\right]\frac{\psi(t)}{t}\,dt$$

where $a = x^2 - y^2 - x$ and $b = 2xy - y$.

After simplification, we obtain

$$\begin{aligned}
&y(2x-1)\int_1^{\infty}\left[t^{\frac{1}{2}x} - t^{\frac{1}{2}(1-x)}\right]\cos(y\,\ln t)\,\frac{\psi(t)}{t}\,dt \\
&+ \left(x^2 - x - y^2\right)\int_1^{\infty}\left[t^{\frac{1}{2}x} - t^{\frac{1}{2}(1-x)}\right]\sin(y\,\ln t)\,\frac{\psi(t)}{t}\,dt = 0
\end{aligned}$$

this is true only if $t^{\frac{1}{2}x} = t^{\frac{1}{2}(1-x)}$ for all $t \geq 1$, which implies to $x = \frac{1}{2}$.

VI. Since $\pi^{-\frac{z}{2}}$ and $\Gamma\left(\frac{z}{2}\right)$ have no zeros, then,VI. is verified.

□

Theorem 1. *If* $z = x + y\,i$ *and* $y \neq 0$, *then* $\xi(z)$ *is real if and only if* $x = \frac{1}{2}$.

Proof. First, let $z = x + i\,y$, $y \neq 0$, and $\overline{\xi(z)} = \xi(z)$. From IV. in Lemma1, $\overline{\xi(z)} = \xi(1-\overline{z})$, then $\xi((1-x)+y\,i) = \xi(x+y\,i)$ From V. in Lemma1, $x = \frac{1}{2}$.

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Second, $\overline{\xi\left(\frac{1}{2}+yi\right)} = \xi\left(\overline{\frac{1}{2}+yi}\right) = \xi\left(\frac{1}{2}-yi\right) = \xi\left(1-\left(\frac{1}{2}-yi\right)\right) = \xi\left(\frac{1}{2}+yi\right)$ this complete the proof. $\square$

Since zero is a real number, then from Theorem 1, we have the following corollary:

Corollary 1. *All non-trivial zeros $z = x + y\,i$ of ξ must lie on the critical line $x = \frac{1}{2}$.*

From Lemma 1,VI., we obtain:

Corollary 2. (**Riemann Hypothesis**) *All non-trivial zeros $z = x + y\,i$ of ζ must lie on the critical line $x = \frac{1}{2}$.*

In [4] a real function

$$F(y) := \frac{\pi^{\frac{-iy}{2}}\,\Gamma\left(\frac{1}{4}+\frac{iy}{2}\right)\zeta\left(\frac{1}{2}+\frac{iy}{2}\right)}{\left|\Gamma\left(\frac{1}{4}+\frac{iy}{2}\right)\right|} \tag{6}$$

has been constructed. A zero of this function corresponds precisely to the imaginary part of a zero of the zeta function. In this way zeta's zeros can be plotted (Figures 1 and 2) and calculated (Tables 1 and 2).

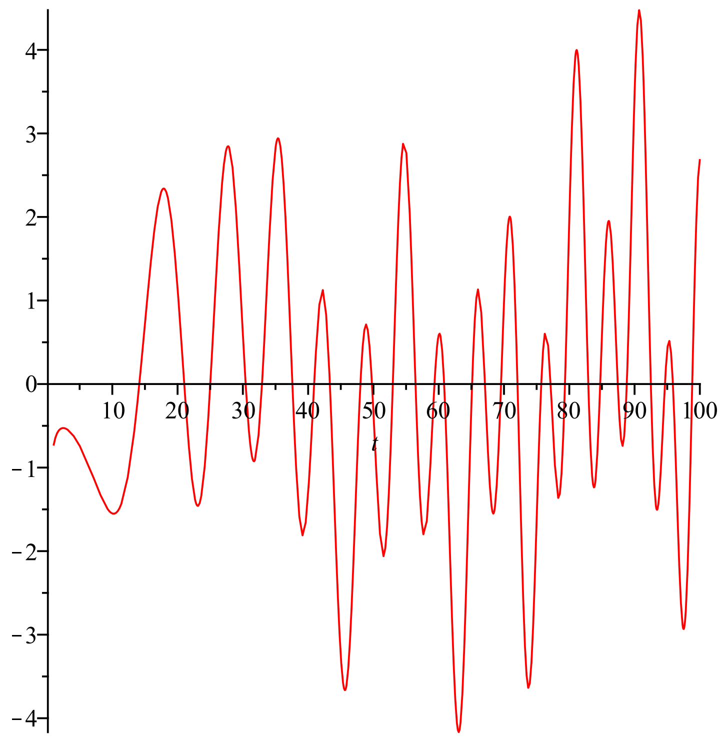

Figure 1. $F(y), \quad 0 < y < 100$

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Table 1. The zeros $z = \frac{1}{2} + y\,i, \quad 0 < y < 100$

14.13472514	21.02203964	25.01085758	30.42487613
32.93506159	37.58617816	40.91871901	43.32707328
48.00515088	49.77383248	52.97032148	56.44624770
59.34704400	60.83177852	65.11254405	67.07981053
69.54640171	72.06715767	75.70469070	77.14484007
79.33737502	82.91038085	84.73549298	87.4252746
88.80911121	92.49189927	94.65134404	95.87063423
98.83119422			

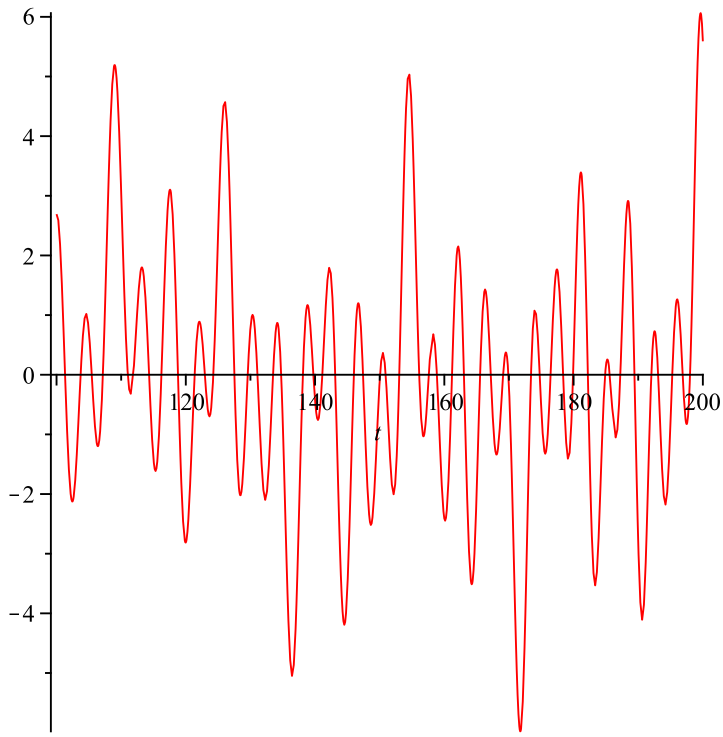

Figure 2. $F(y), \quad 100 < y < 200$

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Table 2. The zeros $z = \frac{1}{2} + y\,i, \quad 0 < y < 100$

101.33	103.72	25.01	105.21
107.15	110.88	114.02	115.96
118.65	121.19	122.83	124.02
127.16	129.25	130.89	133.28
134.62	137.91	139.70	140.74
142.83	145.82	147.16	149.95
150.76	152.88	155.92	157.43
158.75	161.07	162.89	165.42
167.14	168.96	169.77	173.31
174.72	176.36	178.32	179.86
182.12	184.69	185.41	186.95
189.21	191.88	192.91	195.17
196.71	197.95		

3. Conclusion

We have proven a very important principle for the function ξ that is "if z is a complex number with nonzero imaginary part, then $\xi(z)$ is real if and only if z have a real part equal $\frac{1}{2}$". From this principle and the relation between the zeros of ζ and ξ functions, the Riemann hypothesis have been verified.

Conflict of interest: The author declare that he has no conflict of interest.

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