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[Artur Ishkhanyan](#) *

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Article

A new Quadratic Transformation Identity for the Confluent Heun Function

Artur Ishkhanyan

Institute for Physical Research, Ashtarak, 0204 Armenia; aishkhanyan@gmail.com

Abstract: We present a new quadratic transformation identity for the confluent Heun function. This transformation is independent of the previously discovered quadratic transformation (Heliyon 10, e36535, 2024) and offers additional insights into the properties of this special function. Like the first transformation, this new identity involves two arbitrary parameters and has potential applications in both mathematical and physical contexts, particularly in quantum mechanics.

Keywords: confluent Heun function; quadratic transformation; special functions

MSC: 33E30; 33E10; 34B30

1. Introduction

The confluent Heun function (HeunC) arises as a solution to the confluent Heun equation, a second-order linear differential equation that generalizes both the confluent and ordinary hypergeometric equations. The confluent Heun equation has three singularities: two regular singularities located at $z = 0$ and $z = 1$, and one irregular singularity of rank 1 at $z = \infty$ [1,2]:

$$\frac{d^2 u}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1} + \varepsilon \right) \frac{du}{dz} + \frac{\alpha z - q}{z(z-1)} u = 0, \quad (1)$$

The equation appears in a wide range of physical and mathematical problems, including classical and quantum mechanics, laser physics, general relativity, cosmology, and wave propagation in curved spacetimes [3-18]. Its applications have grown considerably in recent years, but despite its relevance, the confluent Heun function has not been studied as extensively as the hypergeometric functions. This limited exploration is particularly evident in the absence of well-established functional identities, such as quadratic or cubic transformations, which are common for ordinary hypergeometric functions [3].

Recently, a quadratic transformation for the confluent Heun function was reported ([19] Heliyon 10, e36535, 2024), marking the first step toward addressing this gap. In this paper, we extend this development by introducing a second, independent quadratic transformation identity for the confluent Heun function. Like the first transformation, the new identity involves two arbitrary parameters and offers additional insights into the function's properties. These transformations hold potential for both mathematical and physical applications, particularly in quantum mechanics and related areas.

2. Results: New Quadratic Transformation

The result we report is that for $0 < z < 1$ or any z with $\text{Im}(z) \neq 0$, the following relationship holds:

$$\begin{aligned}
e^{\frac{\varepsilon z}{2}} \text{HeunC}\left(\frac{\gamma(\varepsilon-\gamma)}{2}, \gamma\varepsilon, \gamma, \gamma, \varepsilon, z\right) = \\
C_1 \text{HeunC}\left(-\frac{\gamma^2}{8} - \frac{\varepsilon^2}{64}, -\frac{\varepsilon^2}{64}, \frac{1}{2}, \gamma, 0, (1-2z)^2\right) + \\
C_2 (1-2z) \text{HeunC}\left(-\frac{\gamma}{2} - \frac{\gamma^2}{8} - \frac{\varepsilon^2}{64}, -\frac{\varepsilon^2}{64}, \frac{3}{2}, \gamma, 0, (1-2z)^2\right),
\end{aligned} \quad (2)$$

where C_1 and C_2 are given by

$$C_1 = e^{\varepsilon/4} \text{HeunC}\left(\frac{\gamma(\varepsilon-\gamma)}{2}, \gamma\varepsilon, \gamma, \gamma, \varepsilon, \frac{1}{2}\right), \quad (3)$$

$$C_2 = -\frac{\varepsilon}{4} C_1 - \frac{e^{\varepsilon/4}}{2} \text{HeunCPrime}\left(\frac{\gamma(\varepsilon-\gamma)}{2}, \gamma\varepsilon, \gamma, \gamma, \varepsilon, \frac{1}{2}\right). \quad (4)$$

In this expression, HeunCPrime denotes the derivative of the confluent Heun function with respect to its argument. The validity of this relation can be confirmed by substituting $u(z) = e^{-\varepsilon z/2} v(y)/z$ and $y = (1-2z)^2$ into equation (1). The presented relation has been thoroughly tested through extensive simulations, utilizing Mathematica's implementation of Heun functions [20].

3. Applications in Physics and Mathematics

We conclude by noting that the confluent Heun function, with the parameters involved in equation (2), arises in various physical problems, which suggests a wide range of potential applications for the presented relation. For instance, consider the stationary one-dimensional Schrödinger equation for a particle of mass m and energy E :

$$\frac{d^2 \psi(x)}{dx^2} + \frac{2m}{\hbar^2} (E - V(x)) \psi(x) = 0 \quad (5)$$

for the potential

$$V(x) = V_0 \left(\frac{1}{x^2} + \frac{1}{(x-1)^2} \right). \quad (6)$$

It can then be verified, by direct substitution, that a fundamental solution to the Schrödinger equation (5) can be expressed as:

$$\psi(x) = x^{\gamma/2} (x-1)^{\gamma/2} e^{\frac{\varepsilon x}{2}} \text{HeunC}\left(\frac{\gamma(\varepsilon-\gamma)}{2}, \gamma\varepsilon, \gamma, \gamma, \varepsilon, x\right), \quad (7)$$

where the following notations are introduced:

$$\varepsilon = \pm \sqrt{\frac{-8mE}{\hbar^2}}, \quad \gamma = 1 \pm \sqrt{1 + \frac{8mV_0}{\hbar^2}}. \quad (8)$$

In this solution, any combination of plus and minus signs is applicable in equations (8). This observation can be used to construct the general solution of the Schrödinger equation by taking a linear combination, with arbitrary constant coefficients, of two independent solutions of the form (7) but with different parameters (8).

A concluding remark is that after shifting the variable $x \rightarrow x' + 1/2$, the Schrödinger equation (5) for the potential (6) exhibits symmetry under the reflection $x' \rightarrow -x'$, meaning the solution remains unchanged under reflection. Consequently, the method outlined in Ref. [21], Section 2.4, can

be applied to derive a solution based on the quadratic form of the argument. This ultimately shows that the solution satisfies a confluent Heun equation, from which relation (2) is obtained.

5. Discussion

In this paper, we introduced a second quadratic transformation identity for the confluent Heun function, broadening the range of known functional transformations for this important special function. This new identity, which is independent of the previously reported transformation, involves two arbitrary parameters and provides additional insights into the structure of the confluent Heun function.

The presented relation has been verified through extensive simulations using Mathematica's implementation of Heun functions. The transformation offers potential applications in various fields, particularly in solving problems in quantum mechanics, general relativity, and other areas of mathematical physics.

As the confluent Heun function becomes increasingly relevant in modern theoretical studies, the discovery of new functional identities like this one is crucial for simplifying complex mathematical systems. Future research may explore additional transformations and further applications of the confluent Heun function in both mathematical and physical contexts.

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