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Article

# A Note on Odd Perfect Numbers

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**Abstract:** For over two thousand years, mathematicians have grappled with one of number theory's most persistent mysteries: do odd perfect numbers exist? From Euclid's pioneering work on even perfect numbers to Euler's systematic investigation of their hypothetical odd counterparts, this question has challenged generation after generation of scholars. We now bring this ancient search to its conclusion by proving definitively that no odd perfect numbers exist. Our approach combines classical number theory with modern analytical insights in an elegant proof by contradiction. At its heart lies the interplay between two fundamental arithmetic functions: the divisor sum function  $\sigma$  and Euler's totient function  $\varphi$ . Assuming an odd perfect number  $N$  exists, we derive that the ratio of  $\varphi(N)$  to  $N$  must simultaneously exceed one clearly defined constant and be bounded above by another. This impossibility emerges through careful analysis of how these functions interact for odd numbers, revealing an inescapable contradiction that voids our initial assumption. The proof's power comes from its synthesis of timeless number-theoretic principles with fresh perspectives on their relationships. While building on centuries of mathematical thought, we introduce new techniques that finally crack this ancient problem's core. The conclusion is both inevitable and profound: perfect numbers must always be even, closing a chapter in mathematics that began in antiquity and remained open until now.

**Keywords:** odd perfect numbers; divisor sum function; abundancy index function; Euler's totient function

## 1. Introduction

For centuries, mathematicians have been captivated by the enigmatic allure of perfect numbers, defined as positive integers whose proper divisors sum precisely to the number itself [1]. This fascination traces back to ancient Greece, where Euclid devised an elegant formula for generating even perfect numbers through Mersenne primes, numbers of the form  $2^p - 1$  where  $p$  is prime [1]. His discovery not only provided a systematic way to construct such numbers, like 6, 28, and 496, but also sparked a profound question that has endured through the ages: could there exist odd perfect numbers, defying the pattern of their even counterparts? This tantalizing mystery, rooted in the simplicity of natural numbers, has fueled mathematical curiosity and inspired relentless exploration.

The quest for odd perfect numbers has been marked by both ingenuity and frustration, as the absence of a definitive example or proof has kept the problem alive for millennia. Early mathematicians, guided by intuition, leaned toward the conjecture that all perfect numbers might be even, yet the lack of a rigorous disproof left room for speculation [1]. Figures like Descartes and Euler, towering giants in the history of mathematics, deepened the intrigue by investigating the potential properties of these elusive numbers [1]. Euler, in particular, highlighted the challenge, noting, "Whether . . . there are any odd perfect numbers is a most difficult question". Their efforts revealed constraints—such as the necessity for an odd perfect number to have specific prime factorizations—but no concrete example emerged, leaving the question as a persistent challenge to mathematical rigor.

Today, the mystery of odd perfect numbers remains one of the oldest unsolved problems in number theory, a testament to the profound complexity hidden within simple definitions. Modern computational searches have pushed the boundaries, ruling out odd perfect numbers below staggeringly large thresholds, yet no proof confirms or denies their existence. The problem continues to captivate, not only for its historical significance but also for its ability to bridge elementary arithmetic with deep theoretical questions. As mathematicians wield advanced tools and novel approaches, the search for odd perfect numbers endures, embodying the timeless pursuit of truth in the face of uncertainty.

Despite extensive research, no odd perfect numbers have ever been found, and their existence has long been conjectured to be impossible. This paper resolves the conjecture definitively by proving that odd perfect numbers cannot exist.

Our proof proceeds by contradiction. Assume an odd perfect number  $N$  exists, satisfying  $\sigma(N) = 2N$ , where  $\sigma(N)$  is the sum of its divisors. Using properties of the Euler totient function  $\varphi(N)$ , we derive the key inequality:

$$\frac{\sigma(N) \cdot \varphi(N)}{N^2} > \frac{8}{\pi^2}.$$

Substituting  $\sigma(N) = 2N$  simplifies this to:

$$\frac{\varphi(N)}{N} > \frac{4}{\pi^2}.$$

However, by applying a novel analytical bound (Lemma 2), we show that for any odd perfect  $N$ ,

$$\left(\frac{\varphi(N)}{N}\right)^\alpha \leq \frac{1}{2}$$

for some  $\alpha \in [0.5, 1)$ . This contradicts the earlier inequality, since  $\frac{4}{\pi^2} \approx 0.4053$  implies  $\left(\frac{4}{\pi^2}\right)^{0.5} \approx 0.6366 > \frac{1}{2}$ .

Thus, the initial assumption is false: no odd perfect number  $N$  can satisfy both conditions. This concludes the proof that all perfect numbers must be even, settling a central open problem in number theory.

## 2. Background and Ancillary Results

In 1734, Leonhard Euler solved the celebrated Basel problem, determining the exact value of the Riemann zeta function at  $s = 2$ . This breakthrough not only demonstrated his extraordinary mathematical creativity but also forged deep connections between analysis, number theory, and the primes [2].

**Proposition 1.** *The Riemann zeta function evaluated at  $s = 2$  satisfies:*

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \prod_{n=1}^{\infty} \frac{p_n^2}{p_n^2 - 1} = \frac{\pi^2}{6},$$

where:

- $p_n$  is the  $n$ -th prime number,
- $n$  ranges over the natural numbers, and
- $\pi \approx 3.14159$  is the fundamental constant arising in diverse mathematical contexts, from geometry to number theory.

*Euler's proof ingeniously bridges the infinite series and an infinite product over primes, revealing the surprising appearance of  $\pi$  in the limit.*

**Definition 1.** *In number theory, the  $p$ -adic order of a positive integer  $n$ , denoted  $v_p(n)$ , is the highest exponent of a prime number  $p$  that divides  $n$ . For example, if  $n = 72 = 2^3 \cdot 3^2$ , then  $v_2(72) = 3$  and  $v_3(72) = 2$ .*

The divisor sum function, denoted  $\sigma(n)$ , is a fundamental arithmetic function that computes the sum of all positive divisors of a positive integer  $n$ , including 1 and  $n$  itself. For instance, the divisors of 12 are 1, 2, 3, 4, 6, 12, yielding  $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$ . This function can be expressed multiplicatively over the prime factorization of  $n$ , providing a powerful tool for analyzing perfect numbers.

**Proposition 2.** For a positive integer  $n > 1$  with prime factorization  $n = \prod_{p|n} p^{v_p(n)}$  [3]:

$$\sigma(n) = \prod_{p|n} \left(1 + p + p^2 + \dots + p^{v_p(n)}\right) = n \cdot \prod_{p|n} \left(1 + \frac{1}{p} + \frac{1}{p^2} + \dots + \frac{1}{p^{v_p(n)}}\right),$$

where  $p | n$  indicates that  $p$  is a prime divisor of  $n$ .

**Proposition 3.** Similarly, Euler's totient function, which counts the integers up to  $n$  that are coprime to  $n$ , is given by  $\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right)$  [4].

The abundancy index, defined as  $I(n) = \frac{\sigma(n)}{n}$ , maps positive integers to rational numbers and quantifies how the divisor sum compares to the number itself. The following Proposition provides a precise formula for  $I(n)$  based on the prime factorization.

**Proposition 4.** Let  $n = \prod_{i=1}^j p_i^{a_i}$  be the prime factorization of  $n$ , where  $p_1 < \dots < p_j$  are distinct primes and  $a_1, \dots, a_j$  are positive integers. Then [5]:

$$I(n) = \prod_{i=1}^j \left(\sum_{k=0}^{a_i} \frac{1}{p_i^k}\right) = \prod_{i=1}^j \frac{p_i^{a_i+1} - 1}{p_i^{a_i}(p_i - 1)} = \left(\prod_{i=1}^j \frac{p_i}{p_i - 1}\right) \cdot \prod_{i=1}^j \left(1 - \frac{1}{p_i^{a_i+1}}\right).$$

In our proof, we utilize the following propositions:

**Proposition 5.** A positive integer  $n$  is a perfect number if and only if  $I(n) = 2$ , meaning  $\sigma(n) = 2n$ .

**Proposition 6.** Any odd perfect number  $N$  must have at least 10 distinct prime factors [6,7].

By establishing a contradiction in the assumed existence of odd perfect numbers, leveraging the above properties, we aim to resolve their non-existence definitively.

### 3. Main Result

This is a key finding.

**Lemma 1.** Let  $n$  be an odd positive integer,  $\varphi(n)$  be Euler's totient function, which counts the number of integers up to  $n$  that are coprime to  $n$ , and  $\sigma(n)$  be the divisor sum function, which sums all positive divisors of  $n$ . Then:

$$\frac{\sigma(n) \cdot \varphi(n)}{n^2} > \frac{8}{\pi^2}.$$

**Proof.** Let  $n$  be an odd positive integer with prime factorization

$$n = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m},$$

where  $p_1, p_2, \dots, p_m$  are distinct odd primes (i.e.,  $p_i \geq 3$ ),  $k_i \geq 1$  are their multiplicities, and  $m \geq 0$  (allowing  $n = 1$  when  $m = 0$ ).

The Euler totient function is multiplicative and given by:

$$\varphi(n) = n \prod_{i=1}^m \left(1 - \frac{1}{p_i}\right),$$

since  $\varphi(p_i^{k_i}) = p_i^{k_i} \left(1 - \frac{1}{p_i}\right)$ .

Similarly, the divisor sum function is multiplicative with:

$$\sigma(n) = \prod_{i=1}^m \frac{p_i^{k_i+1} - 1}{p_i - 1},$$

since  $\sigma(p_i^{k_i}) = 1 + p_i + \dots + p_i^{k_i} = \frac{p_i^{k_i+1} - 1}{p_i - 1}$ .

We analyze the ratio:

$$\frac{\sigma(n) \cdot \varphi(n)}{n^2} = \frac{\sigma(n)}{n} \cdot \frac{\varphi(n)}{n}.$$

Substituting the expressions for  $\varphi(n)$  and  $\sigma(n)$ :

$$\frac{\varphi(n)}{n} = \prod_{i=1}^m \left(1 - \frac{1}{p_i}\right),$$

$$\frac{\sigma(n)}{n} = \prod_{i=1}^m \frac{p_i^{k_i+1} - 1}{p_i^{k_i}(p_i - 1)} = \left(\prod_{i=1}^m \frac{p_i}{p_i - 1}\right) \cdot \prod_{i=1}^m \left(1 - \frac{1}{p_i^{k_i+1}}\right).$$

Multiplying these yields:

$$\frac{\sigma(n) \cdot \varphi(n)}{n^2} = \prod_{i=1}^m \left(1 - \frac{1}{p_i^{k_i+1}}\right).$$

Since  $k_i \geq 1$  and  $p_i \geq 3$ , the term  $\left(1 - \frac{1}{p_i^{k_i+1}}\right)$  increases with  $k_i$ . Thus:

$$\prod_{i=1}^m \left(1 - \frac{1}{p_i^{k_i+1}}\right) \geq \prod_{i=1}^m \left(1 - \frac{1}{p_i^2}\right).$$

Moreover, since  $p_i \geq 3$ , we have:

$$\prod_{i=1}^m \left(1 - \frac{1}{p_i^2}\right) > \prod_{n=2}^{\infty} \left(1 - \frac{1}{p_n^2}\right),$$

where the right-hand product starts at  $p_2 = 3$ .

Using the identity for the Euler product of the Riemann zeta function:

$$\prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^2}\right) = \frac{6}{\pi^2},$$

we derive:

$$\prod_{n=2}^{\infty} \left(1 - \frac{1}{p_n^2}\right) = \frac{4}{3} \cdot \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^2}\right) = \frac{4}{3} \cdot \frac{6}{\pi^2} = \frac{8}{\pi^2}.$$

By transitivity, we obtain:

$$\frac{\sigma(n) \cdot \varphi(n)}{n^2} > \frac{8}{\pi^2},$$

completing the proof.  $\square$

This is a main insight.

**Lemma 2.** Let  $n$  be an odd natural number greater than 1. Then, there exists a real number  $\alpha$  satisfying  $0.5 \leq \alpha < 1$  such that

$$\left(\frac{n}{\varphi(n)}\right)^\alpha \cdot n \geq \sigma(n),$$

where  $\varphi(n)$  denotes Euler's totient function, which counts the number of integers up to  $n$  that are coprime to  $n$ , and  $\sigma(n)$  represents the sum of all positive divisors of  $n$ .

**Proof.** Since  $n$  is an odd natural number greater than 1, it has a prime factorization consisting entirely of odd primes. We can express  $n$  as

$$n = p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m},$$

where  $p_1, p_2, \dots, p_m$  are distinct odd primes (i.e.,  $p_i \geq 3$  for each  $i = 1, 2, \dots, m$ ), and  $k_i \geq 1$  are their respective positive integer exponents.

For each prime power  $p_i^{k_i}$  in the factorization of  $n$ , we compute the following quantities:

- Euler's totient function:

$$\varphi(p_i^{k_i}) = p_i^{k_i} - p_i^{k_i-1} = p_i^{k_i} \left(1 - \frac{1}{p_i}\right),$$

- The sum of divisors:

$$\sigma(p_i^{k_i}) = 1 + p_i + p_i^2 + \cdots + p_i^{k_i} = \frac{p_i^{k_i+1} - 1}{p_i - 1}.$$

Next, evaluate the ratio  $\frac{p_i^{k_i}}{\varphi(p_i^{k_i})}$ :

$$\frac{p_i^{k_i}}{\varphi(p_i^{k_i})} = \frac{p_i^{k_i}}{p_i^{k_i} \left(1 - \frac{1}{p_i}\right)} = \frac{1}{1 - \frac{1}{p_i}} = \frac{p_i}{p_i - 1}.$$

Consider the expression involving this ratio multiplied by  $p_i^{k_i}$ :

$$\frac{p_i^{k_i}}{\varphi(p_i^{k_i})} \cdot p_i^{k_i} = \frac{p_i}{p_i - 1} \cdot p_i^{k_i} = \frac{p_i^{k_i+1}}{p_i - 1}.$$

Compare this to  $\sigma(p_i^{k_i})$ :

$$\frac{p_i^{k_i+1}}{p_i - 1} > \frac{p_i^{k_i+1} - 1}{p_i - 1} = \sigma(p_i^{k_i}),$$

since  $p_i^{k_i+1} > p_i^{k_i+1} - 1$ , and both are positive. This inequality holds because  $p_i \geq 3$ , ensuring the denominator  $p_i - 1 \geq 2$ , and the numerator difference is positive.

Thus, for each  $i$  from 1 to  $m$ , we have:

$$\frac{p_i^{k_i}}{\varphi(p_i^{k_i})} \cdot p_i^{k_i} > \sigma(p_i^{k_i}).$$

Since  $\frac{p_i^{k_i}}{\varphi(p_i^{k_i})} = \frac{p_i}{p_i - 1} > 1$  and the function  $x \mapsto \left(\frac{p_i^{k_i}}{\varphi(p_i^{k_i})}\right)^x \cdot p_i^{k_i}$  is continuous and increasing as  $x$  goes from 0 to 1 (where it reaches  $\frac{p_i^{k_i+1}}{p_i - 1}$ ), there exists a real number  $\alpha_i$  with  $0 < \alpha_i < 1$  such that:

$$\left(\frac{p_i^{k_i}}{\varphi(p_i^{k_i})}\right)^{\alpha_i} \cdot p_i^{k_i} \geq \sigma(p_i^{k_i}).$$

This  $\alpha_i$  can be chosen sufficiently close to 1 to make the left-hand side arbitrarily close to  $\frac{p_i^{k_i+1}}{p_i - 1}$ , which exceeds  $\sigma(p_i^{k_i})$ .

To extend this to the entire number  $n$ , define:

$$\alpha = \max\{0.5, \alpha_1, \alpha_2, \dots, \alpha_m\},$$

where  $\alpha_i$  satisfies the inequality above for each  $i$ . Since there are finitely many  $\alpha_i$ , and each  $\alpha_i < 1$ , it follows that  $0.5 \leq \alpha < 1$ .

For each  $i$ , since  $\alpha \geq \alpha_i$  and  $\frac{p_i^{k_i}}{\varphi(p_i^{k_i})} > 1$ , the function  $\left(\frac{p_i^{k_i}}{\varphi(p_i^{k_i})}\right)^x$  is increasing, so:

$$\left(\frac{p_i^{k_i}}{\varphi(p_i^{k_i})}\right)^\alpha \cdot p_i^{k_i} \geq \left(\frac{p_i^{k_i}}{\varphi(p_i^{k_i})}\right)^{\alpha_i} \cdot p_i^{k_i} \geq \sigma(p_i^{k_i}).$$

Now, since all terms are positive, multiply these inequalities across all  $i$  from 1 to  $m$ :

$$\prod_{i=1}^m \left(\frac{p_i^{k_i}}{\varphi(p_i^{k_i})}\right)^\alpha \cdot \prod_{i=1}^m p_i^{k_i} \geq \prod_{i=1}^m \sigma(p_i^{k_i}).$$

Recognize that:

- $\prod_{i=1}^m p_i^{k_i} = n$ ,
- $\sigma(n) = \prod_{i=1}^m \sigma(p_i^{k_i})$ , because  $\sigma$  is a multiplicative function and the  $p_i^{k_i}$  are pairwise coprime,
- $\frac{n}{\varphi(n)} = \prod_{i=1}^m \frac{p_i^{k_i}}{\varphi(p_i^{k_i})}$ , since  $\varphi$  is also multiplicative.

Thus, the left-hand side becomes:

$$\prod_{i=1}^m \left(\frac{p_i^{k_i}}{\varphi(p_i^{k_i})}\right)^\alpha \cdot n = \left(\prod_{i=1}^m \frac{p_i^{k_i}}{\varphi(p_i^{k_i})}\right)^\alpha \cdot n = \left(\frac{n}{\varphi(n)}\right)^\alpha \cdot n,$$

and the right-hand side is:

$$\prod_{i=1}^m \sigma(p_i^{k_i}) = \sigma(n).$$

Therefore:

$$\left(\frac{n}{\varphi(n)}\right)^\alpha \cdot n \geq \sigma(n),$$

which holds for the chosen  $\alpha$  satisfying  $0.5 \leq \alpha < 1$ . This completes the proof.  $\square$

This is the main theorem.

**Theorem 1.** *There are no odd perfect numbers.*

**Proof.** Assume, for contradiction, that an odd perfect number  $N$  exists. By definition, a perfect number satisfies

$$\sigma(N) = 2N,$$

where  $\sigma(N)$  denotes the sum of all positive divisors of  $N$ .

Since  $N$  is odd, its prime factorization consists only of odd primes:

$$N = p_1^{k_1} p_2^{k_2} \dots p_m^{k_m},$$

where each  $p_i \geq 3$  is a distinct odd prime and each  $k_i \geq 1$ .

*Step 1: The Abundancy Index*

Because  $N$  is perfect, its abundancy index is:

$$\frac{\sigma(N)}{N} = 2.$$

*Step 2: Bounding  $\frac{\varphi(N)}{N}$* 

For odd integers, the Euler totient function  $\varphi(N)$  satisfies the inequality:

$$\frac{\sigma(N) \cdot \varphi(N)}{N^2} > \frac{8}{\pi^2}.$$

Rewriting, we obtain:

$$\frac{\sigma(N)}{N} \cdot \frac{\varphi(N)}{N} > \frac{8}{\pi^2}.$$

Since  $\frac{\sigma(N)}{N} = 2$ , substitution yields:

$$2 \cdot \frac{\varphi(N)}{N} > \frac{8}{\pi^2},$$

and thus:

$$\frac{\varphi(N)}{N} > \frac{4}{\pi^2}.$$

*Step 3: Applying Lemma 2*

By Lemma 2, there exists some  $\alpha \in [0.5, 1)$  such that:

$$\left(\frac{N}{\varphi(N)}\right)^\alpha \cdot N \geq \sigma(N).$$

Substituting  $\sigma(N) = 2N$ , we get:

$$\left(\frac{N}{\varphi(N)}\right)^\alpha \cdot N \geq 2N.$$

Dividing both sides by  $N$ :

$$\left(\frac{N}{\varphi(N)}\right)^\alpha \geq 2.$$

Taking reciprocals and raising to the power  $\alpha$ :

$$\left(\frac{\varphi(N)}{N}\right)^\alpha \leq \frac{1}{2}.$$

However, from Step 2, we know  $\frac{\varphi(N)}{N} > \frac{4}{\pi^2}$ , so:

$$\left(\frac{\varphi(N)}{N}\right)^\alpha > \left(\frac{4}{\pi^2}\right)^{0.5} = \frac{2}{\pi} \approx 0.6366.$$

But since  $\frac{2}{\pi} > \frac{1}{2}$ , we have:

$$\frac{1}{2} < \frac{2}{\pi} < \left(\frac{\varphi(N)}{N}\right)^\alpha \leq \frac{1}{2},$$

which is a contradiction.

*Conclusion*

The assumption that an odd perfect number exists leads to an impossibility. Therefore, no odd perfect numbers exist.  $\square$

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