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Not peer-reviewed version

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Posted Date: 8 October 2024

doi: 10.20944/preprints202410.0547.v1

Keywords: Odd Perfect Numbers; Divisor Sum Function; Prime Numbers; Natural Logarithms doi:10.20944/preprints202410.0547.v1

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A Note on Odd Perfect Numbers

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Abstract: This paper definitively settles the longstanding conjecture regarding odd perfect numbers. A perfect number is one whose sum of divisors equals itself doubled. While Euclid's method for constructing even perfect numbers is well-known, the existence of odd ones has remained elusive. By employing elementary techniques and analyzing the properties of the divisor sum function, we conclusively prove that no odd perfect numbers exist.

Keywords: odd perfect numbers; divisor sum function; prime numbers; natural logarithms

MSC: Primary 11A41; Secondary 11A25

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1. Introduction

For centuries, mathematicians have been fascinated by perfect numbers − positive integers whose divisor sum is twice the number itself. Euclid's elegant method for constructing even perfect numbers using Mersenne primes ignited a quest for odd counterparts. While intuition suggested that all perfect numbers might be even, the lack of proof kept the question alive since ancient times. Descartes and Euler, two mathematical luminaries, added to the intrigue by exploring the potential properties of odd perfect numbers. Yet, the mystery of their existence persisted.

Proposition 1.1. An odd perfect number N must have the form:

- $N = q^{\alpha} \cdot p_1^{2 \cdot e_1} \cdot p_2^{2 \cdot e_2} \cdot \ldots \cdot p_k^{2 \cdot e_k}$ (where $q, p_1, p_2, \ldots, p_k$ are distinct odd primes),
- At least 10 distinct prime factors ($k \geq 9$),
- $\frac{1}{q} + \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} > \frac{\log k}{2 \cdot \log 2}$.

These conditions, established in [1,2] and [3,4], respectively, provide valuable insights into the potential structure of odd perfect numbers.

The divisor sum function, denoted by $\sigma(n)$, is an arithmetic function in number theory that calculates the sum of all positive divisors of a positive integer n. For example, the positive divisors of 12 are 1, 2, 3, 4, 6, and 12, and their sum is $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. Define $f(n)$ as $\frac{\sigma(n)}{n}$. A precise formula for $f(n)$ can be derived from its multiplicity:

Proposition 1.2. Let $\prod_{i=1}^{j} q_i^{a_i}$ be the representation of n as a product of prime numbers $q_1 < \ldots < q_j$ with natural numbers $a_1, \ldots, a_j$ as exponents. Then [5]:

$$f(n) = \left(\prod_{i=1}^{j} \frac{q_i}{q_i - 1} \right) \cdot \prod_{i=1}^{j} \left(1 - \frac{1}{q_i^{a_i+1}} \right).$$

The prime zeta function, denoted by $P(s)$, is a mathematical function analogous to the Riemann zeta function. It is defined as the infinite series $\sum_{p \in \text{primes}} \frac{1}{p^s}$, which converges for $\Re(s) > 1$. The value of $P(2)$ has been calculated to be approximately $0.4522474200410654985\ldots$ (source: OEIS A085548 [6]).

The following inequalities use the natural logarithm:

Proposition 1.3. For $t > 0$ [7]:

$$\frac{1}{t+1} < \log\left(1 + \frac{1}{t}\right).$$

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Proposition 1.4. For $u \geq 0$ [7]:

$$-\frac{u}{1-u} \leq \log(1-u).$$

By demonstrating an inherent contradiction in the assumption of odd perfect numbers, we can definitively prove their non-existence.

2. Main Result

This is a key finding.

Theorem 2.1. The inequality

$$\log f(N) > \frac{1}{q} + \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} - \left(P(2) - \frac{1}{4} + \frac{1}{8}\right)$$

holds for any odd perfect number N, where $q, p_1, p_2, \ldots, p_k$ are its distinct prime factors and $P(2)$ is the prime zeta function at 2.

Proof. The function $f(N)$ can be calculated using the formula

$$f(N) = \left(\frac{q}{q-1} \cdot \left(1 - \frac{1}{q^{\alpha+1}}\right)\right) \cdot \left(\prod_{i=1}^{k} \frac{p_i}{p_i - 1}\right) \cdot \prod_{i=1}^{k}\left(1 - \frac{1}{p_i^{2\cdot e_i+1}}\right),$$

where q, p_i, α, and e_i are as defined in Propositions 1.1 and 1.2. Using Proposition 1.3, we can show that $\log\left(\frac{r}{r-1}\right) = \log\left(1 + \frac{1}{r-1}\right) > \frac{1}{r}$ for any prime number r. Similarly, Proposition 1.4 implies that $\log\left(1 - \frac{1}{r^\beta}\right) \geq -\frac{1}{r^\beta - 1}$ for any prime power r^β. Therefore, we have:

$$\log f(N) > \frac{1}{q} + \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} - \left(\frac{1}{q^{\alpha+1} - 1} + \sum_{i=1}^{k} \frac{1}{p_i^{2\cdot e_i+1} - 1}\right).$$

We can further deduce that

$$\begin{aligned}\left(\frac{1}{q^{\alpha+1} - 1} + \sum_{i=1}^{k} \frac{1}{p_i^{2\cdot e_i+1} - 1}\right) &\leq \left(\frac{1}{q^{\alpha+1} - 1} + \sum_{p \,\in\, \text{odd primes}} \frac{1}{p^3 - 1}\right) \\ &\leq \left(\frac{1}{3^2 - 1} + \sum_{p \,\in\, \text{odd primes}} \frac{1}{p^2}\right) \\ &= \left(P(2) - \frac{1}{4} + \frac{1}{8}\right).\end{aligned}$$

We can finally conclude that

$$\log f(N) > \frac{1}{q} + \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} - \left(P(2) - \frac{1}{4} + \frac{1}{8}\right).$$

□

This is the main theorem.

Theorem 2.2. The existence of odd perfect numbers is impossible.

Proof. We will employ a proof by contradiction. Assuming the existence of a smallest odd perfect number N (i.e., $f(N) = 2$), we know from Proposition 1.1 that N has the form $N = q^\alpha \cdot p_1^{2\cdot e_1} \cdot p_2^{2\cdot e_2} \cdot \ldots \cdot p_k^{2\cdot e_k}$, where $q, p_1, p_2, \ldots, p_k$ are distinct odd primes, N has at least 10 distinct prime factors ($k \geq 9$), and $\frac{1}{q} + \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} > \frac{\log k}{2\cdot\log 2}$.

Therefore, it suffices to show that

$$\log f(N) > \frac{1}{q} + \frac{1}{p_1} + \frac{1}{p_2} + \ldots + \frac{1}{p_k} - \left(P(2) - \frac{1}{4} + \frac{1}{8}\right)$$

is false. This leads to

$$\log 2 > \frac{\log 9}{2 \cdot \log 2} - \left(P(2) - \frac{1}{4} + \frac{1}{8}\right),$$

which contradicts Theorem 2.1. Equivalently, we have

$$0.481 > (\log 2)^2 > \frac{\log 9}{2} - (\log 2) \cdot \left(P(2) - \frac{1}{4} + \frac{1}{8}\right) > 0.871,$$

a clear contradiction. This implies the impossibility of N and, by contradiction, the non-existence of odd perfect numbers. □

3. Conclusion

The pursuit of odd perfect numbers is more than an academic curiosity. It challenges the core principles of number theory, expanding our knowledge of integers and their relationships. This journey has not only revealed new aspects of perfect numbers but also inspired innovative techniques in the field. The non-existence of odd perfect numbers, as proven here, opens up avenues for further exploration and potential discoveries in the captivating world of number theory.

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