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An Averaged Halpern Type Algorithm for Solving Fixed Point Problems and Variational Inequality Problems

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Abstract: In this paper we propose and study in the setting of a Hilbert space an averaged Halpern type algorithm for approximating the solution of a common fixed point problem for a couple of nonexpansive and demicontractive mappings with a variational inequality constraint. The strong convergence of the sequence generated by the algorithm is established under feasible assumptions on the parameters involved. In particular, we also obtain the common solution of the fixed point problem for nonexpansive or demicontractive mappings and of a variational inequality problem. Our results extend and generalize various important related results in literature that were established for the pairs (nonexpansive, nonspreading) or (nonexpansive, strongly quasi-nonexpansive) mappings. Numerical tests to illustrate the superiority of our algorithm over the ones existing in literature are also reported.

Keywords: Hilbert space; nonexpansive mapping; strictly pseudocontractive mapping; quasi-nonexpansive mapping; strongly quasinonexpansive mapping; nonspreading mapping; demicontractive mapping; averaged Halpern algorithm; fixed point; common fixed point; strong convergence; variational inequality

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1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$. Let $D \subset \mathcal{H}$ be a closed convex set, and consider the self-mapping $G : D \to D$. Throughout this paper the set of all fixed points of G in D is denoted by

$$Fix(G) = \{u \in D \ : \ Gu = u\}.$$

The mapping G is said to be

(a) *nonexpansive* if

$$\|Gu - Gv\| \leq \|u - v\|, \quad \text{for all } u, v \in D; \tag{1}$$

(b) *quasi-nonexpansive* if $Fix(G) \neq \emptyset$ and

$$\|Gu - v\| \leq \|u - v\|, \quad \text{for all } u \in D \text{ and } v \in Fix(G); \tag{2}$$

(c) *β-demicontractive* if $Fix(G) \neq \emptyset$ and there exists a positive number $\beta < 1$ such that

$$\|Gu - v\|^2 \leq \|u - v\|^2 + \beta\|u - Gu\|^2, \tag{3}$$

for all $u \in D$ and $v \in Fix(G)$.

(d) *strongly quasi-nonexpansive* if $Fix(G) \neq \emptyset$, G is quasi-nonexpansive and $u_p - Gu_p \to 0$ whenever $\{u_p\}$ is a bounded sequence such that $\|u_p - u^*\| - \|Gu_p - u^*\| \to 0$ for some $u^* \in Fix(G)$.

By the previous definitions, it is obvious that any nonexpansive mapping G with $Fix(G) \neq \emptyset$ is quasi-nonexpansive, any strongly quasi-nonexpansive is quasi-nonexpansive, and that any quasi-

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nonexpansive mapping is demicontractive, too, but the reverses are no more true, as illustrated by the next example.

Example 1 ([4]). *Let $\mathcal{H}$ be the real line with the usual norm and $D = [0,1]$. Define F on D as*

$$F(u) = \begin{cases} 7/8, & \text{if } 0 \le u < 1 \\ 1/4, & \text{if } u = 1. \end{cases} \tag{4}$$

Then F is $\frac{2}{3}$-demicontractive but F is neither nonexpansive nor quasi-nonexpasive (and hence not strongly quasi-nonexpansive).

There are several papers in literature which are devoted to the approximation of common fixed points of nonexpansive type mappings. For example, in order to approximate the common fixed points of a pair of nonexpansive self mappings (T_1, T_2) with $Fix\,(T_1) \cap Fix\,(T_2) \neq \emptyset$, Takahashi and Tamura [21] considered the following iterative procedure:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T_1[(1-\beta_n)x_n + \beta_n T_2 x_n], n \ge 1, \end{cases} \tag{5}$$

for which they established a weak convergence theorem.

Moudafi [15] considered a slightly different Krasnoselsij-Mann iterative procedure for the same problem, that he called "hierarchical fixed-point problem":

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n[(1-\beta_n)T_2 x_n + \beta_n T_1 x_n], n \ge 1, \end{cases} \tag{6}$$

where $Fix\,(T_1)$ and $Fix\,(T_2)$ are assumed to be nonempty.

Further, Iemoto and Takahashi [21] considered the problem of approximating the common fixed points of a nonexpansive mapping T and of a nonspreading mappings S in a Hilbert space, and utilized the iterative scheme

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n[(1-\beta_n)T x_n + \beta_n S x_n], n \ge 1, \end{cases} \tag{7}$$

for which they formulated and proved some weak convergence theorems.

Starting from this background, our aim in this paper is to solve the common fixed point problem in the setting of Hilbert spaces for the case of the larger class of demicontractive mappings, thus extending and unifying the main results in Cianciaruso et al. [7], Falset et al. [8], Iemoto and Takahashi [9] and many others.

Our main result (Theorem 1) provides a convergence theorem for an averaged iterative Halpern type algorithm used to approximate a solution of the common fixed point problem for a pair consisting of a nonexpansive mapping and a demicontractive mapping, which also solves a certain variational inequality problem.

2. Preliminaries

We recall some important lemmas used in the proofs of our main results. The following two lemmas are taken from Berinde [3].

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Lemma 1. [3] *Let $\mathcal{H}$ be a real Hilbert space and $D \subset \mathcal{H}$ a closed and convex set. If $G : D \to D$ is β-demicontractive, then the Krasnoselskij perturbation $G_\nu = (1-\nu)I + \nu G$ of G is $(1+\beta/\nu - 1/\nu)$-demicontractive.*

Lemma 2. [3] *Let $\mathcal{H}$ be a real Hilbert space and $D \subset \mathcal{H}$ a closed and convex set. If $G : D \to D$ is β-demicontractive, then for any $\nu \in (0, 1-\beta)$*

$$G_\nu = (1-\nu)I + \nu G$$

is quasi-nonexpansive.

Lemma 3 (Zhou [30])**.** *Let C be a nonempty subset of a real Hilbert space and let $T : C \to C$ be a k-strictly pseudocontractive mapping. Then the averaged mapping $T_\lambda = (1-\lambda)I + \lambda T$ is nonexpansive for any $\lambda \in (0, 1-k)$.*

Lemma 4. *Let $\mathcal{H}$ be a real Hilbert space, $D \subset \mathcal{H}$ a closed and convex set and $F : D \to D$ a mapping. Then, for any $\nu \in (0,1)$, we have $Fix(F_\nu) = Fix(F)$.*

Lemma 5. [24] *Let $\{\alpha_n\}$ be a sequence of nonnegative numbers such that*

$$\alpha_n \leq (1-c_n)\alpha_n + c_n\mu_n + \delta_n, n \geq 0,$$

where $\{c_n\}$ is a sequence in $[0,1]$ and $\{\mu_n\}$ is a sequence in $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} c_n = \infty, \quad \limsup_{n\to\infty} \mu_n \leq 0, \quad \delta_n \geq 0, \quad \text{and} \sum_{n=1}^{\infty} \delta_n < \infty.$$

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 6. *Let $\{\gamma_p\}$ be a sequence of real numbers that has a subsequence $\{\gamma_{p_k}\}$ which satisfies $\gamma_{p_k} < \gamma_{p_k+1}$ for all $k \in \mathbb{N}$. There exists an increasing sequence of integers $\{\tau(p)\}_{p \geq p_0}$ satisfying:*

$$\lim_{p\to\infty} \tau(p) = \infty, \qquad \gamma_{\tau(p)} \leq \gamma_{\tau(p)+1}, \qquad \gamma_p \leq \gamma_{\tau(p)+1}, \quad \forall p \geq p_0.$$

3. Fixed Points and Variational Inequalities

In this section we state and prove our main results. To do this, we first consider the following property.

A mapping G satisfies **Condition A** if $u_p - Gu_p \to 0$ whenever $\{u_p\}$ is a bounded sequence such that

$$\|u_p - u^*\| - \|(1-\nu)u_p + \nu Gu_p - u^*\| \to 0$$

for some $u^* \in Fix(G)$ and $\nu \in [0,1]$.

Theorem 1. *Let $\mathcal{H}$ be a Hilbert space and C be a closed convex subset of $\mathcal{H}$. Let $F : C \to C$ be a nonexpansive mapping and $G : C \to C$ be a β-demicontractive mapping satisfying Condition A such that $I - G$ is demiclosed at 0. Assume that $Fix(F) \cap Fix(G) \neq \emptyset$. Let $\{a_p\}$ and $\{b_p\}$ be sequences in $[0,1]$ such that $a_p \to 0$ and $\sum_{p=1}^{\infty} a_p = \infty$. Let $\{u_p\}$ be the sequence generated in the following manner:*

$$\begin{cases} x, u_1 \in C, \\ u_{p+1} = a_p x + (1-a_p)[b_p F u_p + (1-b_p)((1-\alpha)u_p + \alpha G u_p)], \quad p \geq 1. \end{cases} \tag{8}$$

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Then, the following assertions hold.

(I) *If* $\sum_{p=1}^{\infty}(1-b_p) < \infty$ *and* $\sum_{p=1}^{\infty}|a_p = a_{p+1}| < \infty$, *then* $\{u_p\}$ *strongly converges to* $u^* \in Fix(F)$ *which is the unique point in* $Fix(F)$ *that solves the variational inequality*

$$\langle u^* - x, u - u^* \rangle \geq 0, \qquad \forall u \in Fix(F), \tag{9}$$

i.e. $u^* = P_{Fix(F)}x$.

(II) *If* $\sum_{p=1}^{\infty}(1-b_p) < \infty$ *and* $\frac{b_p}{a_p} \to 0$, *then* $\{u_p\}$ *converges strongly to* $v^* \in Fix(G)$ *which is the unique point in* $Fix(G)$ *that solves the variational inequality*

$$\langle u^* - x, u - v^* \rangle \geq 0, \qquad \forall u \in Fix(G), \tag{10}$$

i.e. $v^* = P_{Fix(G)}x$.

(III) *If* $\liminf_{p\to\infty} b_p(1-b_p) > 0$, *then* $\{u_p\}$ *strongly converges to* $\bar{u} \in Fix(F) \cap Fix(G)$ *that is the unique solution of the variational inequality*

$$\langle \bar{u} - x, u - \bar{u} \rangle \geq 0, \qquad \forall u \in Fix(F) \cap Fix(G), \tag{11}$$

i.e. $\bar{u} = P_{Fix(F)\cap Fix(G)}x$.

Proof. Since G is a β-demicontractive mapping, by Lemma 2 it follows that the averaged mapping $G_\alpha = (1-\alpha)I + \alpha G$ is quasi-nonexpansive, for $\alpha \in (0, 1-\beta)$. Clearly, $G_\alpha - I$ is demiclosed at zero. One can also see that G_α is strongly quasi-nonexpansive from the fact that G satisfies Condition A. Now we can write

$$u_{p+1} = a_p x + (1-a_p)[b_p F u_p + (1-b_p) G_\alpha u_p], \quad p \in \mathbb{N}.$$

Let w be a common fixed point of F and G_α. Define

$$T_p = b_p F + (1-b_p)G_\alpha, \quad \forall p \in \mathbb{N}.$$

For all $p \in \mathbb{N}$,

$$\begin{aligned} \|u_{p+1} - w\| &= \|a_p x + (1-a_p)T_p u_p - w\| \\ &\leq (1-a_p)\|u_p - w\| + a_p\|x - w\| \\ &\leq \max\{\|u_p - w\|, \|x - w\|\} \\ &\leq \max\{\|u_1 - w\|, \|x - w\|\}, \end{aligned}$$

that is, $\{u_p\}$ is a bounded sequence.

Furthermore, since $a_p \to 0$ as $p \to \infty$, we have

$$u_{p+1} - T_p u_p = a_p(x - T_p u_p) \to 0, \text{ as } p \to \infty. \tag{12}$$

To prove **(I)**, for all $p \in \mathbb{N}$ compute

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$$
\begin{aligned}
\|u_{p+1}-u_p\| &= \|(1-a_p)(T_pu_p - T_{p-1}u_{p-1}) - (a_p - a_{p-1})T_{p-1}u_{p-1} + (a_p - a_{p-1})x\| \\
&= \|(1-a_p)(T_pu_p - T_{p-1}u_{p-1}) + (a_{p-1} - a_p)[T_{p-1}u_{p-1} - x]\| \\
&= \|(a_{p-1} - a_p)[T_{p-1}u_{p-1} - x] \\
&\quad + (1-a_p)[b_pFu_p + (1-b_p)G_\alpha Fu_p - b_{p-1}Fu_{p-1} - (1-b_{p-1})G_\alpha u_{p-1}]\| \\
&= \|(a_{p-1} - a_p)[T_{p-1}u_{p-1} - x] + (1-a_p)[b_p(Fu_p - Fu_{p-1}) \\
&\quad + (1-b_p)(G_\alpha u_p - G_\alpha u_{p-1}) + (b_p - b_{p-1}(Fu_{p-1} - G_\alpha u_{p-1})]]\| \\
&\le |a_{p-1} - a_p|\|T_{p-1}u_{p-1} - x\| + (1-a_p)[b_p\|u_p - u_{p-1}\| \\
&\quad + (1-b_p)\|G_\alpha u_p - G_\alpha u_{p-1}\| + |b_p - b_{p-1}|\|Fu_{p-1} - G_\alpha u_{p-1}]\| \\
&= (1-c_p)\|u_p - u_{p-1}\| + \mu_p,
\end{aligned}
$$

where $c_p = 1 - b_p + a_pb_p$ and

$$\mu_p = |a_{p-1} - a_p|\|T_{p-1}u_{p-1} - x\| + (1-a_p)[(1-b_p) + |b_p - b_{p-1}|\|Fu_{p-1} - G_\alpha u_{p-1}]\|.$$

We see that $c_p \to 0$, $\sum_{p=1}^{\infty} c_p = \infty$ and $\sum_{p=1}^{\infty} \mu_p < \infty$. Hence, by Lemma 5 we conclude that $u_{p+1} - u_p \to 0$. This and (12) imply that

$$u_p - T_pu_p \to 0. \tag{13}$$

Moreover,

$$
\begin{aligned}
\|u_p - T_pu_p\| &= \|u_p - b_pFu_p - (1-b_p)\| \\
&\ge \|u_p - b_pFu_p\| - (1-b_p)\|G_\alpha u_p\|, \\
\|u_p - b_pFu_p\| &\le \|u_p - T_pu_p\| + (1-b_p)\|G_\alpha u_p\|,
\end{aligned}
$$

and thus, from the hypothesis that $\sum_{p=1}^{\infty}(1-b_p) < \infty$, we also have

$$u_p - b_pFu_p \to 0.$$

We can conclude that $u_p - Fu_p \to 0$. Hence, any weak limit of $\{u_p\}$ is in $Fix(F)$. Let $\{u_{p_k}\}$ be a subsequence of $\{u_p\}$ such that

$$\limsup_{p\to\infty}\langle u_p - u^*, x - u^*\rangle = \lim_{k\to\infty}\langle u_{p_k} - u^*, x - u^*\rangle \tag{14}$$

and $u_{p_k} \to y$. Thus, $y \in Fix(F)$ and

$$\limsup_{p\to\infty}\langle u_p - u^*, x - u^*\rangle = \langle y - u^*, x - u^*\rangle,$$

which is nonpositive by the definition of u^*. We obtain

$$
\begin{aligned}
\limsup_{p\to\infty}\langle T_pu_p - u^*, x - u^*\rangle &= \limsup_{p\to\infty}[\langle u_p - u^*, x - u^*\rangle + \langle T_pu_p - u_p, x - u^*\rangle] \\
&= \limsup_{p\to\infty}\langle u_p - u^*, x - u^*\rangle \le 0.
\end{aligned}
$$

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Finally,

$$
\begin{aligned}
\|u_{p+1} - u^*\|^2 &= \|(1-a_p)(T_p u_p - u^*) + a_p(x - u^*)\|^2 \\
&= (1-a_p)^2\|T_p u_p - u^*\|^2 + a_p^2\|x - u^*\|^2 \\
&\quad + 2a_p\langle(1-a_p)(T_p u_p - u^*), x - u^*\rangle \\
&= (1-a_p)^2\|b_p(Fu_p - u^*) + (1-b_n)(G_\alpha u_p - u^*)\|^2 + a_p^2\|x - u^*\|^2 \\
&\quad + 2a_p\langle T_p u_p - u^*, x - u^*\rangle - 2a_p^2\langle T_p u_p - u^*, x - u^*\rangle \\
&\leq (1-a_p)^2\Big[b_p\|u_p - u^*\| + (1-b_p)\|G_\alpha u_p - u^*\|^2\Big] + a_p^2\|x - u^*\|^2 \\
&\quad + 2a_p\langle T_p u_p - u^*, x - u^*\rangle \\
&\leq (1-a_p)^2\|u_p - u^*\|^2 + (1-b_p)\|G_\alpha u_p - u^*)\|^2 + a_p\|x - u^*\|^2 \\
&\quad + 2a_p\langle T_p u_p - u^*, x - u^*\rangle \\
&= (1-t_p)\|u_p - u^*\|^2 + t_p r_p + s_p,
\end{aligned}
$$

where

$$
\begin{aligned}
&t_p = 2a_p - a_p^2, \quad r_p = a_p\Big[\|x - u^*\|^2 + 2\langle T_p u_p - u^*, x - u^*\rangle\Big], \\
&s_p = (1-b_p)\|G_\alpha u_p - u^*\|^2.
\end{aligned}
$$

Now, Lemma 5 implies that $u_p \to u^*$.

To prove **(II)**, let v^* be the unique solution of the variational inequality (10) and compute

$$
\begin{aligned}
\|u_{p+1} - v^*\|^2 &= \|a_n x + (1-a_n)T_p u_p - v^* + a_p v^* - a_p v^*\|^2 \\
&= \|a_p(x - v^*) + (1-a_n)(T_p u_p - v^*)\|^2 \\
&\leq (1-a_n)^2\|(T_p u_p - v^*)\|^2 + 2a_p\langle x - v^*, u_{p+1} - v^*\rangle \\
&= (1-a_n)^2\|b_p(Fu_p - v^*) + (1-b_p)(G_\alpha u_p - v^*)\|^2 \\
&\quad + 2a_p\langle x - v^*, u_{p+1} - v^*\rangle \\
&\leq (1-a_p)^2\|u_p - v^*\|^2 + b_p\|Fu_p - v^*\|^2 \\
&\qquad + 2a_p\langle x - v^*, u_{p+1} - v^*\rangle
\end{aligned} \tag{15}
$$

We have two cases, namely, the sequence $\{\|u_p - v^*\|\}$ is eventually not increasing or not eventually not increasing.

Case (II) 1. There exists $p_0 \in \mathbb{N}$ such that $\|u_{p+1} - v^*\| \leq \|u_p - v^*\|$ for all $p \geq p_0$. Put

$$
\begin{aligned}
&\varphi_p = 2\langle u_{p+1} - v^*, x - v^*\rangle, \text{ and} \\
&\mu_p = (1-b_p)\|(G_\alpha u_p - v^*)\|^2.
\end{aligned}
$$

Since $(1-a_p)^2 \leq (1-a_p)$, we have

$$
\|u_{p+1} - v^*\|^2 \leq (1-a_p)\|u_p - v^*\|^2 + a_p\varphi_p + \mu_p.
$$

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Since $\{\|u_p - v^*\|\}$ is not eventually increasing, $\lim_{p\to\infty} \|u_p - v^*\|$ exists. Thus,

$$\begin{aligned}
0 &= \lim_{p\to\infty} (\|u_{p+1} - v^*\| - \|u_p - v^*\|) \\
&\le \liminf_{p\to\infty} (a_p\|x - v^*\| + (1-a_n)\|T_p u_p - v^*\| - \|u_p - v^*\|) \\
&= \liminf_{p\to\infty} (\|T_p u_p - v^*\| - \|u_p - v^*\|) \\
&= \liminf_{p\to\infty} (\|b_p(Fu_p - v^*) + (1-b_p)(G_\alpha u_p - v^*)\| - \|u_p - v^*\|) \\
&= \liminf_{p\to\infty} (\|G_\alpha u_p - v^*\| - \|u_p - v^*\|) \\
&= \limsup_{p\to\infty} (\|u_p - v^*\| - \|u_p - v^*\|) = 0
\end{aligned}$$

Hence,

$$\lim_{p\to\infty} (\|G_\alpha u_p - v^*\| - \|u_p - v^*\|) = 0.$$

From the strong quasi-nonexpansiveness of G_α, we conclude that

$$G_\alpha u_p - u_p \to 0. \tag{16}$$

The rest of the proof is similar to the proof of (I).

Case (II) 2. The sequence $\{\|u_p - v^*\|\}$ is not eventually not increasing. There exists a subsequence $\{\|u_{p_k} - v^*\|\}$ such that $\|u_{p_k} - v^*\| < \|u_{p_k+1} - v^*\|$ for all $k \in \mathbb{N}$. By Lemma 6, there exists an increasing sequence of integers $\{\tau(p)\}$ satisfying

$$\begin{aligned}
&\lim_{p\to\infty} \tau(p) = \infty, \qquad \|u_{\tau(p)} - v^*\| \le \|u_{\tau(p)+1} - v^*\| \\
&\|u_p - v^*\| \le \|u_{\tau(p)+1} - v^*\|, \quad \forall p \ge p_0.
\end{aligned} \tag{17}$$

Thus,

$$0 \le \liminf_{p\to\infty} (\|u_{\tau(p)+1} - v^*\| - \|u_{\tau(p)} - v^*\|).$$

Using (16) with $\tau(p)$ instead of p, we obtain

$$\lim_{p\to\infty} \|G_\alpha u_{\tau(p)} - v^*\| - \|u_{\tau(p)} - v^*\| = 0.$$

The strong quasi-nonexpasiveness of G_α implies

$$G_\alpha u_{\tau(p)} - u_{\tau(p)} \to 0, \tag{18}$$

Since $I - G_\alpha$ is demiclosed at 0, we conclude that

$$\limsup_{p\to\infty} \langle u_{\tau(p)+1} - v^*, x - v^* \rangle \le 0. \tag{19}$$

One may observe that

$$\begin{aligned}
\|u_{\tau(p)+1} - u_{\tau(p)}\| &= a_{\tau(p)}\|x - u_{\tau(p)}\| + (1 - a_{\tau(p)})b_{\tau(p)}O(1) \\
&\quad + (1 - a_{\tau(p)})\|G_\alpha u_{\tau(p)} -_{\tau(p)} \|,
\end{aligned}$$

where $O(1)$ represents a bounded sequence. Thus, from (18) it follows that

$$u_{\tau(p)+1} - u_{\tau(p)} \to 0.$$

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Replacing p by $\tau(p)$ in (15) yields

$$\begin{aligned}\|u_{\tau(p)+1} - v^*\| &\leq (1-a_{\tau(p)})^2\|u_{\tau(p)} - v^*\|^2 \\ &\quad + 2a_{\tau(p)}\langle u_{\tau(p)+1} - v^*, x - v^*\rangle + b_{\tau(p)}\|Fu_{\tau(p)} - v^*\|^2 \\ &\leq (1-a_{\tau(p)})^2\|u_{\tau(p)+1} - v^*\|^2 \\ &\quad + 2a_{\tau(p)}\langle u_{\tau(p)+1} - v^*, x - v^*\rangle + b_{\tau(p)}\|Fu_{\tau(p)} - v^*\|^2.\end{aligned}$$

As a consequent, we have

$$\begin{aligned}2a_{\tau(p)}\|u_{\tau(p)+1} - v^*\|^2 &\leq (a_{\tau(p)})^2\|u_{\tau(p)+1} - v^*\|^2 \\ &\quad + 2a_{\tau(p)}\langle u_{\tau(p)+1} - v^*, x - v^*\rangle + b_{\tau(p)}\|Fu_{\tau(p)} - v^*\|^2.\end{aligned}$$

Dividing by $a_{\tau(p)}$ gives

$$\begin{aligned}0 \leq 2\|u_{\tau(p)+1} - v^*\|^2 &\leq a_{\tau(p)}\|u_{\tau(p)+1} - v^*\|^2 \\ &\quad + 2a_{\tau(p)}\langle u_{\tau(p)+1} - v^*, x - v^*\rangle + \frac{b_{\tau(p)}}{a_{\tau(p)}}\|Fu_{\tau(p)} - v^*\|^2.\end{aligned}$$

Since $b_p/a_p \to 0$, by (19) we obtain

$$\lim_{p\to\infty}\|u_{\tau(p)} - v^*\| \leq \lim_{p\to\infty}\|u_{\tau(p)+1} - v^*\| = 0.$$

From (17), we obtain that $u_p \to v^*$.

To prove (III), let $\bar{u}$ be the unique point in $Fix(F)\cap Fix(G)$ that satisfies the variational inequality (11). We have

$$\begin{aligned}\|T_pu_p - \bar{u}\|^2 &= \|b_p(Fu_p - \bar{u}) + (1-b_p)(Gu_p - \bar{u}\|^2 \\ &= b_p\|Fu_p - \bar{u}\|^2 + (1-b_p)\|G_\alpha u_p - \bar{u}\|^2 - b_p(1-b_p)\|Fu_p - G_\alpha u_p\|^2 \\ &\leq \|u_p - \bar{u}\|^2 - b_p(1-b_p)\|Fu_p - G_\alpha u_p\|^2,\end{aligned}$$

$$\begin{aligned}\|u_{p+1} - \bar{u}\|^2 &= \|a_p(x-\bar{u}) + (1-a_p)(T_pu_p - \bar{u}\|^2 \\ &\leq (1-a_p)^2\|T_pu_p - \bar{u}\|^2 + a_p^2\|x - \bar{u}\|^2 + a_p\|(x-\bar{u})(T_pu_p - \bar{u})\| \\ &\leq \|u_p - \bar{u}\|^2 - b_p(1-b_p)\|Fu_p - G_\alpha u_p\|^2 \\ &\quad + a_p^2\|x-\bar{u}\|^2 + a_p\|(x-\bar{u})(T_pu_p - \bar{u})\|. \end{aligned} \tag{20}$$

Similar to (II), we have two cases.

Case (III) 1. $\{\|u_p - \bar{u}\|\}$ is eventually not increasing. There exists $p_0 \in \mathbb{N}$ such that $\|u_{p+1} - \bar{u}\| \leq \|u_p - \bar{u}\|$, for all $n \geq n_0$. Thus, $\lim_{p\to\infty}\|u_p - \bar{u}\|$ exists. We have

$$\begin{aligned}b_p(1-b_p)\|Fu_p - Gu_p\| \leq &\|u_p - \bar{u}\|^2 - \|u_{p+1} - \bar{u}\|^2 + a_p^2\|x - \bar{u}\|^2 \\ &+ a_p\|(x-\bar{u})(T_pu_p - \bar{u})\|.\end{aligned}$$

Since $\liminf_{p\to\infty} b_p(1-b_p) > 0$, we have

$$Fu_p - G_\alpha u_p \to 0. \tag{21}$$

Moreover,

$$
\begin{aligned}
0 &= \lim_{p\to\infty}(\|u_{p+1}-\bar{u}\| - \|u_p-\bar{u}\|) \\
&\leq \liminf_{p\to\infty}(a_p\|x-\bar{u}\| + (1-a_p)\|T_pu_p-\bar{u}\| - \|u_p-\bar{u}\|) \\
&= \liminf_{p\to\infty}(\|T_pu_p-\bar{u}\| - \|u_p-\bar{u}\|) \\
&= \liminf_{p\to\infty}(\|b_p(Fu_p-\bar{u}+(1-b_p)(G_\alpha u_p-\bar{u})\| - \|u_p-\bar{u}\|) \\
&\leq \liminf_{p\to\infty}(b_p(\|Fu_p-\bar{u}\| - \|u_p-\bar{u}\|) + (1-b_p)(\|G_\alpha u_p-\bar{u}\| - \|u_p-\bar{u}\|)) \\
&\leq \limsup_{p\to\infty}(b_p(\|Fu_p-\bar{u}\| - \|u_p-\bar{u}\|) + (1-b_p)(\|G_\alpha u_p-\bar{u}\| - \|u_p-\bar{u}\|)) \\
&\leq 0
\end{aligned}
$$

Therefore

$$\lim_{p\to\infty}(b_p(\|Fu_p-\bar{u}\| - \|u_p-\bar{u}\|) + (1-b_p)(\|G_\alpha u_p-\bar{u}\| - \|u_p-\bar{u}\|)) = 0.$$

It follows that

$$\lim_{p\to\infty}(b_p(\|Fu_p-\bar{u}\| - \|u_p-\bar{u}\|) = \lim_{p\to\infty}(1-b_p)(\|G_\alpha u_p-\bar{u}\| - \|u_p-\bar{u}\|)) = 0. \tag{22}$$

Thus,

$$\lim_{p\to\infty}((\|Fu_p-\bar{u}\| - \|u_p-\bar{u}\|) = \lim_{p\to\infty}(\|G_\alpha u_p-\bar{u}\| - \|u_p-\bar{u}\|)) = 0. \tag{23}$$

From strong quasi-nonexpansiveness of G_α it follows

$$Su_p - u_p \to 0. \tag{24}$$

Since $u_p - Fu_p = u_p - G_\alpha u_p + G_\alpha u_p - Fu_p$, we obtain

$$u_p - Fu_p \to 0. \tag{25}$$

Choose a subsequence $u_{p_k} \to z$ such that

$$\limsup_{p\to\infty}\langle u_p-\bar{u}, x-\bar{u}\rangle = \lim_{p\to\infty}\langle u_{p_k}-\bar{u}, x-\bar{u}\rangle = \langle z-\bar{u}, x-\bar{u}\rangle.$$

Since both F and G_α are demiclosed at 0 and by (24), (25), one can conclude that $z \in Fix(F) \cap Fix(G_\alpha)$. Hence, by the definition of $\bar{u}$, we obtain (26). Furthermore, from (24) and (25) we have

$$T_pu_p - u_p \to 0. \tag{26}$$

Finally

$$
\begin{aligned}
\|u_{p+1}-\bar{u}\|^2 &= \|(1-a_p)(T_pu_p-\bar{u}) + a_p(x-\bar{u})\| \\
&= (1-a_p)^2\|T_pu_p-\bar{u}\|^2 + a_p^2\|x-\bar{u}\|^2 + 2a_p\langle(1-a_p)(T_pu_p-\bar{u}), x-\bar{u}\rangle \\
&\leq (1-a_p)^2\|u_p-\bar{u}\|^2 + a_p^2[\|x-\bar{u}\|^2 - 2\langle T_pu_p-\bar{u}, x-\bar{u}\rangle] \\
&\quad + 2a_p\langle T_pu_p-u_p, x-\bar{u}\rangle + 2a_p\langle u_p-\bar{u}, x-\bar{u}\rangle \\
&\leq (1-a_p)\|u_p-\bar{u}\|^2 + a_p[\|x-\bar{u}\|^2 - 2\langle T_pu_p-\bar{u}, x-\bar{u}\rangle] \\
&\quad + 2a_p\langle T_pu_p-u_p, x-\bar{u}\rangle + 2a_p\langle u_p-\bar{u}, x-\bar{u}\rangle.
\end{aligned}
$$

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Putting

$$s_p = 1 - (1 - a_p)^2,$$
$$\sigma_p = 2\langle T_p u_p - u_p, x - p_0\rangle + 2\langle u_p - \bar{u}, x - \bar{u}\rangle + a_p[\|x - \bar{u}\|^2 - 2\langle T_p u_p - \bar{u}, x - \bar{u}\rangle],$$

the conclusion follows from Lemma 5.

Case (III) 2. The sequence $\{\|u_p - \bar{u}\|\}$ is not eventually not increasing, i.e., there exists a subsequence $\{\|u_{p_k} - \bar{u}\|\}$ such that $\|u_{p_k} - \bar{u}\| < \|u_{p_{k+1}} - \bar{u}\|, \forall j \in \mathbb{N}$.

Lemma 6 implies that there exists an increasing sequence of integers $\{\tau(p)\}_{p\in\mathbb{N}}$ satisfying (17). Therefore

$$\begin{aligned}
o &\le \liminf_{p\to\infty}(\|u_{\tau(p)+1} - \bar{u}\| - \|u_{\tau(p)} - \bar{u}\|) \\
&\le \limsup_{p\to\infty}(\|u_{\tau(p)+1} - \bar{u}\| - \|u_{\tau(p)} - \bar{u}\|) \\
&\le \limsup_{p\to\infty}(\|u_{p+1} - \bar{u}\| - \|u_p - \bar{u}\|) \\
&\le \limsup_{p\to\infty}(\|(1 - a_p)(T_p u_p - \bar{u}) + a_p(x - \bar{u})\| - \|u_p - \bar{u}\|) \\
&\le \limsup_{p\to\infty}((1 - a_p)\|u_p - \bar{u}\| + a_p\|x - \bar{u}\| - \|u_p - \bar{u}\|) \\
&= 0
\end{aligned}$$

We obtain

$$\lim_{p\to\infty}(\|u_{\tau(p)+1} - \bar{u}\| - \|u_{\tau(p)} - \bar{u}\|) = 0 \tag{27}$$

Now, using (24) with $\tau(p)$ instead of p, we have

$$G_\alpha u_{\tau(p)} - u_{\tau(p)} \to 0, \tag{28}$$

and (20) can be written as

$$\begin{aligned}
0 &\le b_{\tau(p)}(1 - b_{\tau(p)})\|F u_{\tau(p)} - G_\alpha u_{\tau(p)}\|^2 \\
&\le \|u_{\tau(p)} - \bar{u}\|^2 - \|u_{\tau(p)+1} - \bar{u}\|^2 + a_p\|x - \bar{u}\|(1 + \|(T_p u_p - \bar{u})\|).
\end{aligned}$$

From (27) and since $\liminf_{p\to\infty} b_p(1 - b_p) > 0$,

$$F u_{\tau(p)} - G_\alpha u_{\tau(p)} \to 0. \tag{29}$$

We also obtain that

$$u_{\tau(p)} - F u_{\tau(p)} = u_{\tau(p)} - G_\alpha u_{\tau(p)} + G_\alpha u_{\tau(p)} - F u_{\tau(p)}.$$

By (28) and (29), we have that

$$u_{\tau(p)} - F u_{\tau(p)} \to 0 \text{ and } u_{\tau(p)} - T_{\tau(p)} u_{\tau(p)} \to 0. \tag{30}$$

Similar to (26) changing $\tau(p)$ with p, we have

$$\limsup_{p\to\infty}\langle u_{\tau(p)} - \bar{u}, x - \bar{u}\rangle \le 0.$$

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Following the same proof of(26), replacing p with $\tau(p)$, we obtain

$$\limsup_{p\to\infty}\langle u_{\tau(p)} - \bar{u}, x - \bar{u}\rangle \leq 0. \tag{31}$$

Now we compute,

$$\begin{aligned}
\|u_{\tau(p)+1} - \bar{u}\|^2 &\leq (1 - a_{\tau(p)})^2\|u_{\tau(p)} - \bar{u}\|^2 \\
&\quad + a_{\tau(p)}^2[\|x - \bar{u}\|^2 - 2\langle T_{\tau(p)}u_{\tau(p)} - \bar{u}, x - \bar{u}\rangle] \\
&\quad + 2a_{\tau(p)}\langle T_{\tau(p)}u_{\tau(p)} - u_{\tau(p)}, x - \bar{u}\rangle + 2a_{\tau(p)}\langle u_{\tau(p)} - \bar{u}, x - \bar{u}\rangle \\
&\leq (1 - a_{\tau(p)})^2\|u_{\tau(p)+1} - \bar{u}\|^2 \\
&\quad + a_{\tau(p)}^2[\|x - \bar{u}\|^2 - 2\langle T_{\tau(p)}u_{\tau(p)} - \bar{u}, x - \bar{u}\rangle] \\
&\quad + 2a_{\tau(p)}\langle T_{\tau(p)}u_{\tau(p)} - u_{\tau(p)}, x - \bar{u}\rangle + 2a_{\tau(p)}\langle u_{\tau(p)} - \bar{u}, x - \bar{u}\rangle
\end{aligned}$$

Consequently,

$$\begin{aligned}
2a_{\tau(p)}\|u_{\tau(p)+1} - \bar{u}\|^2 &\leq a_{\tau(p)}^2[\|x - \bar{u}\|^2 - 2\langle T_{\tau(p)}u_{\tau(p)} - \bar{u}, x - \bar{u}\rangle] \\
&\quad + 2a_{\tau(p)}\langle u_{\tau(p)} - \bar{u}, x - \bar{u}\rangle \\
&\quad + 2a_{\tau(p)}\langle T_{\tau(p)}u_{\tau(p)} - u_{\tau(p)}, x - \bar{u}\rangle
\end{aligned}$$

dividing by $a_{\tau(p)}$, we get

$$\begin{aligned}
0 &\leq 2\|u_{\tau(p)+1} - \bar{u}\|^2 \\
&\leq a_{\tau(p)}[\|x - \bar{u}\|^2 - 2\langle T_{\tau(p)}u_{\tau(p)} - \bar{u}, x - \bar{u}\rangle] + 2\langle u_{\tau(p)} - \bar{u}, x - \bar{u}\rangle \\
&\quad + 2\langle T_{\tau(p)}u_{\tau(p)} - u_{\tau(p)}, x - \bar{u}\rangle
\end{aligned}$$

Taking the limsup and recalling the hypothesis (30) and (31), we obtain

$$\lim_{p\to\infty}\|u_{\tau(p)} - \bar{u}\| \leq \lim_{p\to\infty}\|u_{\tau(p)+1} - \bar{u}\| = 0$$

Now by (17), we conclude that $u_p \to \bar{u}$. $\square$

A more general result could be proved similarly to the proof of Theorem 1.

Theorem 2. *Let $\mathcal{H}$ be a Hilbert space and C be a closed convex subset of $\mathcal{H}$. Let $F : C \to C$ be a k-strictly pseudocontractive mapping and $G : C \to C$ be a β-demicontractive mapping satisfying Condition A such that $I - G$ is demiclosed at 0. Assume that $Fix(F) \cap Fix(G) \neq \emptyset$. Let $\{a_p\}$ and $\{b_p\}$ be sequences in $[0, 1]$ such that $a_p \to 0$ and $\sum_{p=1}^{\infty} a_p = \infty$. Let $\{u_p\}$ be a sequence generated in the following manner:*

$$\begin{cases} x, u_1 \in C, \\ u_{p+1} = a_p x + (1 - a_p)[b_p F_\lambda u_p + (1 - b_p)((1 - \alpha)u_p + \alpha G u_p)], \quad p \geq 1. \end{cases} \tag{32}$$

where $F_\lambda u = (1 - \lambda)u + \lambda F u$, with $\lambda \in (0, 1 - k)$.

Then, the following assertions hold.

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(I) *If* $\sum_{p=1}^{\infty}(1-b_p) < \infty$, $\sum_{p=1}^{\infty}|a_p = a_{p+1}| < \infty$, *then* $\{u_p\}$ *strongly converges to* $u^* \in Fix(F)$ *that is the unique point in* $Fix(F)$ *that solves the variational inequality*

$$\langle u^* - x, u - u^* \rangle \geq 0, \qquad \forall u \in Fix(F),$$

i.e. $u^* = P_{Fix(F)}x$.

(II) *If* $\sum_{p=1}^{\infty}(1-b_p) < \infty$, $\frac{b_p}{a_p} \to 0$, *then* $\{u_p\}$ *converges strongly to* $v^* \in Fix(G)$ *that is the unique point in* $Fix(G)$ *that solves the variational inequality*

$$\langle v^* - x, u - v^* \rangle \geq 0, \qquad \forall u \in Fix(G), \tag{33}$$

i.e. $v^* = P_{Fix(G)}x$.

(III) *If* $\liminf_{p\to\infty} b_p(1-b_p) > 0$, *then* $\{u_p\}$ *strongly converges to* $\bar{u} \in Fix(F) \cap Fix(G)$ *that is the unique solution of the variational inequality*

$$\langle \bar{u} - x, u - \bar{u} \rangle \geq 0, \qquad \forall u \in Fix(F) \cap Fix(G),$$

i.e. $\bar{u} = P_{Fix(F)\cap Fix(G)}x$.

Proof. As F is k-strictly pseudocontractive, by Lemma 3 we have that the averaged mapping

$$F_\lambda u = (1-\lambda)u + \lambda F u$$

is nonexpansive, for any $\lambda \in (0, 1-k)$ and that $Fix\,(F) = Fix\,(F_\lambda)$. We apply Theorem 1 for F_λ and G and get the conclusion. □

Remark 1. *Most of the results obtained in Takahashi and Tamura [21], Moudafi [15], Cianciaruso et al. [7], Falset et al. [8], Iemoto and Takahashi [9] could be obtained as corollaries of our main results or could be slightly improved by considering our averaged Halpern type algorithm (8).*

We illustrate this fact in the following for four different instances.

- *If F is nonexpansive and G is nonspreading, then by Theorem 1 we obtain an improvement of Theorem 4.1 in Iemoto and Takahashi [9], in the sense that for our averaged Halpern type algorithm (8) we have strong convergence, while for the Krasnoselsij-Mann iterative procedure (5) only weak convergence was obtained by Iemoto and Takahashi [9];*
- *If F is nonexpansive and G is nonspreading, then by Theorem 1 we obtain the main result (i.e., Theorem 14) in Cianciaruso et al. [7];*
- *If F and G are both nonexpansive, then by Theorem 1 we obtain an improvement of the main result in Takahashi and Tamura [21], in the sense that for our averaged Halpern type algorithm (8) we get strong convergence, while for the Krasnoselsij-Mann iterative procedure (5) only weak convergence is obtained by Takahashi and Tamura [21];*
- *If F is nonexpansive and G is strongly quasi-nonexpansive, then by Theorem 1 we obtain the main result (i.e., Theorem 3) in Falset et al. [8];*
- *...*

4. Numerical Illustrations

In this section, we consider some numerical examples to illustrate the numerical behaviour of Algorithm (8), for approximating a common fixed point for a nonexpansive mapping and a β-demicontractive mapping.

Example 2. *Let* $\mathcal{H}$ *be the real line with the usual norm and* $D = [0, 1]$. *Define F and G on D as follows as*

$$F(u) = 5/3 - u, \quad u \in D \tag{34}$$

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and

$$G(u) = \begin{cases} 5/6, & \text{if } 0 \le u < 1 \\ 1/3, & \text{if } u = 1. \end{cases} \tag{35}$$

Note that F is nonexpansive, and G is $1/2$-demicontractive. It is easy to check that $Fix(F) \cap Fix(G) = \{5/6\}$. The mapping G is neither quasi-nonexpasive nor nonexpansive (and hence it is neither strongly quasi-nonexpansive nor nonspeading).

Therefore, we cannot apply any of the results in Takahashi and Tamura [21], Moudafi [15], Cianciaruso et al. [7], Falset et al. [8], Iemoto and Takahashi [9] etc. to solve the common fixed point problem for F and G.

If we put

$$a_p = \frac{1}{rp}, \quad b_p = \frac{2p}{1+3p}, \qquad p \in \mathbb{N}, \quad r \in \mathbb{R} \text{ and } r \ge 1,$$

then all assumptions in Theorem 1 part (iii) are satisfied. This implies that the sequence $\{u_p\}$ generated by the algorithm (8) converges to $5/6$, the unique common fixed point of F and G.

Several numerical experiments were conducted in MATLAB using the algorithm (8) with different values of the parameters.

The numerical results for three initial values with $r = 1000$ are presented in Table 1.

Table 2 shows numerical results for three initial values with $r = 5000$ and $x = 0.7$. One can see that for x near the common fixed point and r large, the iterations converge faster.

Table 1. Numerical results for $x = 0.2, r = 1000$ with initial values $u_0 = 1, u_0 = 0.7$ and $u_0 = 0.1$.

Iteration (p)	u_p	u_p	u_p
0	1.000000	0.700000	0.100000
1	0.786549	0.822542	0.774552
2	0.837948	0.834349	0.839148
3	0.832452	0.833106	0.832234
4	0.833503	0.833354	0.833553
5	0.833264	0.833303	0.833251
6	0.833332	0.833321	0.833335
7	0.833316	0.833319	0.833315
8	0.833323	0.833322	0.833323
9	0.833323	0.833323	0.833322
10	0.833324	0.833324	0.833324
...	...	...	...
15	0.833327	0.833327	0.833327
...	...	...	...
20	0.833329	0.833329	0.833329
...	...	...	...
50	0.833332	0.833332	0.833332
...	...	...	...
51	0.833332	0.833332	0.833332
...	...	...	...
111	**0.833333**	**0.833333**	**0.833333**
112	0.833333	0.833333	0.833333

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Table 2. Numerical results for $x = 0.7, r = 5000$ with initial values $u_0 = 1, u_0 = 0.7$ and $u_0 = 0.1$.

Iteration (p)	u_p	u_p	u_p
0	1.000000	0.700000	0.100000
1	0.786649	0.822642	0.774652
2	0.837988	0.834389	0.839188
3	0.832478	0.833133	0.832260
4	0.833522	0.833373	0.833572
5	0.833279	0.833318	0.833266
6	0.833344	0.833330	0.833348
7	0.833326	0.833330	0.833325
8	0.833332	0.833331	0.833332
9	0.833332	0.833331	0.833332
10	0.833332	0.833331	0.833332
...	...	...	...
15	0.833332	0.833332	0.833332
...	...	...	...
20	0.833332	0.833332	0.833332
...	...	...	...
26	**0.833333**	**0.833333**	**0.833333**
27	0.833333	0.833333	0.833333

5. Conclusions

- We have introduced an averaged iterative Halpern type algorithm intended to find a common fixed point for a pair consisting of a nonexpansive mapping and a demicontractive mapping which also solves a certain variational inequality problem;
- We established a strong convergence theorem (Theorem 1) for the sequence generated by our algorithm;
- We extended Theorem 1 to the more general case of a pair of mappings consisting of a k-strictly pseudocontractive mapping F and a β-demicontractive mapping G (Theorem 2), by considering the double averaged Halpern type algorithm (32).
- We validated the effectiveness of our general theoretical results by some appropriate numerical experiments, corresponding to part (iii) of Theorem 1, which are reported in Section 4. These results clearly illustrate the progress of our convergence results over existing literature.
- For other related results we refer the reader to Agwu et al. [1], Araveeporn et al. [2], Ceng and Yao [5], Ceng and Yuan [6], Jaipranop and Saejung [10], Kraikaew and Saejung [12], Mebawondu et al. [14], Nakajo et al. [17], Petruşel and Yao [18], Rizvi [19], Sahu et al. [20], Thuy [22], Uba et al. [23], Xu [25], Yao et al. [26,27], Yotkaew et al. [28],...

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