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Article

Influence of Riemann Space-time Curvature on the Laws of Electromagnetism

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Abstract: In this paper, we focus on the influence of Riemann space-time curvature on the laws of electromagnetism, such as Gauss's law, Maxwell-Ampére's law, and Faraday's law. Furthermore, we also introduce the work done on the unification of the gravitation and electromagnetic fields.

Keywords: riemann space-time curvature; Gauss's law; Maxwell-Ampére's law; Faraday's law

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1. Electromagnetism in Curved Space-time

The background space-time curvature of the geometry influences the electromagnetic fields [35–37]. Here, we consider the electromagnetic fields in a curved space-time geometry; however, we will ignore the influence of the electromagnetic field on the space-time geometry, which is discussed in the next section. For that, we describe the electromagnetic field equations, derived from the action principle.

The covariant derivative of the electromagnetic field covariant vector potential A_μ (or contravariant vector A^μ) in the torsionless space-time manifold is

$$\begin{aligned} \mathbb{D}^\nu A_\mu &= \nabla^\nu A_\mu \\ \mathbb{D}_\nu A^\mu &= \nabla_\nu A^\mu \end{aligned} \tag{1}$$

and hence, the anti-symmetric part of the connection, $\Gamma^\mu_{\sigma\nu} A^\sigma = 0$. Thus, the field strength tensor in covariant form is:

$$F_{\mu\nu} = \mathbb{D}_\mu A_\nu - \mathbb{D}_\nu A_\mu = \nabla_\mu A_\nu - \nabla_\nu A_\mu \tag{2}$$

and its contravariant form is

$$F^{\mu\nu} = \mathbb{D}^\mu A^\nu - \mathbb{D}^\nu A^\mu = \nabla^\mu A^\nu - \nabla^\nu A^\mu \tag{3}$$

Furthermore, the electric and magnetic fields of the electromagnetic field components are given as follows:

$$\begin{aligned} F_{0i} &= \nabla_0 A_i - \nabla_i A_0 = \frac{E_i}{c} = -\frac{E^i}{c} \\ F_{ij} &= \nabla_i A_j - \nabla_j A_i = -B_{ij} = -\varepsilon_{kij} B^k \end{aligned} \tag{4}$$

Thus, $F_{\mu\nu}$ are the components of the electromagnetic field strength 2-form **F** and A_μ are the components of the electromagnetic field potential 1-form **A**. On the other hand [21], the electric field vector **E** is related to the line integrand, and hence it is represented by a covector or a 1-form in three-dimensional space; the magnetic field vector **B** is related to surface and it is represented by a 2-form with components B_{ij} that are expressed in terms of the contravariant components of the density vector B^k being surface integrands:

$$B_{ij} = \varepsilon_{kij} B^k \tag{5}$$

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The Maxwell's equations are written as

$$\begin{aligned} \boldsymbol{\nabla} \cdot \mathbf{D} &= \rho \\ \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \boldsymbol{\nabla} \cdot \mathbf{B} &= 0 \\ \boldsymbol{\nabla} \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \end{aligned} \tag{6}$$

which are equivalent to conservation laws of charge and magnetic flux. These equations, under the influence of the curved space-time geometry, are written in terms of covariant derivative as

$$\begin{aligned} \mathbb{D}_\mu F^{\mu\nu} &= \mu_0 J^\nu \\ \mathbb{D}_\mu F_{\nu\alpha} + \mathbb{D}_\nu F_{\alpha\mu} + \mathbb{D}_\alpha F_{\mu\nu} &= 0 \end{aligned} \tag{7}$$

Besides, it is required that

$$\mathbb{D}_\sigma g_{\mu\nu} = 0 \tag{8}$$

Note that Eq. (8) indicates that during the covariant differentiation the metric tensor components $g_{\mu\nu}$ can be considered a constant.

The actions of the covariant and contravariant derivatives on a vector can be written as

$$\begin{aligned} \mathbb{D}^\nu A_\mu &= \nabla^\nu A_\mu + A_\sigma \Gamma^{\sigma\nu}{}_\mu \\ \mathbb{D}_\nu A^\mu &= \nabla_\nu A^\mu + A^\sigma \Gamma^\mu{}_{\sigma\nu} \\ \mathbb{D}^\mu &= g^{\mu\nu}(\mathbf{x}) \mathbb{D}_\nu \end{aligned} \tag{9}$$

where $\Gamma^\mu{}_{\sigma\nu}$ is the space-time connection, the so-called Christoffel's symbols, and $\mathbf{x}$ is a vector characterising x^μ point in the space-time. Besides, the actions of the covariant derivative on the second-rank tensors are determined as

$$\begin{aligned} \mathbb{D}_\mu F^{\sigma\nu} &= \nabla_\mu F^{\sigma\nu} + \Gamma^\sigma{}_{\lambda\mu} F^{\lambda\nu} + \Gamma^\nu{}_{\lambda\mu} F^{\sigma\lambda} \\ \mathbb{D}_\mu F_{\sigma\nu} &= \nabla_\mu F_{\sigma\nu} + \Gamma^\lambda{}_{\sigma\mu} F_{\lambda\nu} + \Gamma^\lambda{}_{\nu\mu} F_{\sigma\lambda} \end{aligned} \tag{10}$$

For that, the general form of the covariant derivative of the anti-symmetric electromagnetic field strength tensor $F^{\mu\nu}$ in Riemann curved space-time is used, given as

$$\begin{aligned} \mathbb{D}_\mu F^{\mu\nu} &= \frac{1}{\sqrt{-g}} \nabla_\mu \left(\sqrt{-g} F^{\mu\nu} \right) \\ \Gamma^\sigma{}_{\mu\nu} &= \nabla_\mu \left(\ln\left(\sqrt{-g} \right) \right) \end{aligned} \tag{11}$$

Combining Eq. (7) and Eq. (11), we obtain

$$\nabla_\mu F^{\mu\nu} + \frac{1}{\sqrt{-g}} \nabla_\mu \left(\sqrt{-g} \right) F^{\mu\nu} = \mu_0 J^\nu \tag{12}$$

Therefore,

$$\begin{aligned} & g^{\sigma\mu} g^{\lambda\nu} \nabla_\mu F_{\sigma\lambda} \\ & + F_{\sigma\lambda} \left(\nabla_\mu \left(g^{\lambda\nu} g^{\sigma\mu} \right) + g^{\sigma\mu} g^{\lambda\nu} \frac{1}{\sqrt{-g}} \nabla_\mu \left(\sqrt{-g} \right) \right) = \mu_0 J^\nu \end{aligned} \tag{13}$$

where the following relations were used:

$$
\begin{aligned}
F^{\mu\nu} &= g^{\sigma\mu} g^{\lambda\nu} F_{\sigma\lambda} \\
\nabla_\mu F^{\mu\nu} &= \left(\nabla_\mu \left(g^{\sigma\mu} g^{\lambda\nu}\right)\right) F_{\sigma\lambda} + g^{\lambda\nu} g^{\sigma\mu} (\nabla_\mu F_{\sigma\lambda})
\end{aligned} \tag{14}
$$

In terms of the components of the electric and magnetic fields, Eq. (13) can be written as

$$
\begin{aligned}
&\frac{1}{c}(\nabla_\mu E_i)\left(g^{0\mu} g^{i\nu} - g^{i\mu} g^{0\nu}\right) + \frac{1}{c} E_i \\
&\times \left(\nabla_\mu \left(g^{0\mu} g^{i\nu} - g^{i\mu} g^{0\nu}\right) + \frac{1}{\sqrt{-g}} \nabla_\mu(\sqrt{-g}) \left(g^{0\mu} g^{i\nu} - g^{i\mu} g^{0\nu}\right)\right) \\
&- \varepsilon_{klm} \left(\nabla_\mu B^k\right) g^{l\mu} g^{m\nu} - \varepsilon_{klm} B^k \\
&\times \left(\nabla_\mu \left(g^{l\mu} g^{m\nu}\right) + \frac{1}{\sqrt{-g}} \nabla_\mu(\sqrt{-g}) \left(g^{l\mu} g^{m\nu}\right)\right) = \mu_0 J^\nu
\end{aligned} \tag{15}
$$

where $J^\nu = (\rho c, \mathbf{J})$ (with ρ being the charge density and $\mathbf{J} = \rho \mathbf{v}$ the current density, where c is the speed of light in vacuum and $\mathbf{v}$ velocity vector in three-dimensional space.)

For $\nu = 0$, Eq. (15) can be written as

$$
\begin{aligned}
&(\nabla_\mu E_i)\left(g^{0\mu} g^{i0} - g^{i\mu} g^{00}\right) \\
&+ E_i \left(\nabla_\mu \left(g^{0\mu} g^{i0} - g^{i\mu} g^{00}\right) + \frac{1}{\sqrt{-g}} \nabla_\mu(\sqrt{-g}) \left(g^{0\mu} g^{i0} - g^{i\mu} g^{00}\right)\right) \\
&- c\varepsilon_{klm} \left(\nabla_\mu B^k\right) g^{l\mu} g^{m0} - c\varepsilon_{klm} B^k \\
&\times \left(\nabla_\mu \left(g^{l\mu} g^{m0}\right) + \frac{1}{\sqrt{-g}} \nabla_\mu(\sqrt{-g}) \left(g^{l\mu} g^{m0}\right)\right) = \mu_0 \rho c^2 = \frac{\rho}{\epsilon_0}
\end{aligned} \tag{16}
$$

where the relation $c^2 = 1/(\mu_0 \epsilon_0)$ is used. Note that Eq. (16) is the generalisation of Gauss's law of Maxwell's equations in Minkowski (flat) space-time for the case of Riemann (curved) space-time. Interestingly, this equation is showing some new phenomena of the electromagnetism in the presence of the gravitational fields modifying the curvature of the space-time. For instance, in the absence of charges (that is, $\rho = 0$), a magnetic field (including a static magnetic field) creates an electric field. Furthermore, in the case of the weak gravitational fields, when the off-diagonal elements of metric tensor are zero, the magnetic field cancels out of Eq. (16). In particular, for $g^{0i} = g^{i0} = g^{ij} = 0$ (for $i \neq j$), we get

$$
- g^{jj} g^{00} \nabla_j E_j - E_j \left(g^{jj} g^{00} \frac{1}{\sqrt{-g}} \nabla_j(\sqrt{-g}) + \nabla_j \left(g^{jj} g^{00}\right)\right) = \frac{\rho}{\epsilon_0} \tag{17}
$$

From Eq. (17), for the Riemann space-time geometries for which the second term is non-zero, then the electric field in vacuum (that is, $\rho = 0$) is non-vanishing and it is necessary non-uniform. Therefore, the curvature of the space-time introduces a spatial change on the electric field in vacuum, even for weak gravitational fields. Furthermore, in the case of Minkowski space-time (that is, $g^{00} = 1$ and $g^{jj} = -1$), Eq. (17) reduces to the first Maxwell's equation, $\boldsymbol{\nabla} \cdot \mathbf{E} = \rho/\epsilon_0$.

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For $\nu = 1,2,3$, we obtain the generalisation of Maxwell-Ampére's law in the Riemann's curved space-time. In particular, from Eq. (15), for $\nu \equiv j = 1,2,3$, we obtain

$$\begin{aligned}
&\frac{1}{c}(\nabla_\mu E_i)\left(g^{0\mu}g^{ij} - g^{i\mu}g^{0j}\right) + \frac{1}{c}E_i \\
&\times \left(\nabla_\mu\left(g^{0\mu}g^{ij} - g^{i\mu}g^{0j}\right) + \frac{1}{\sqrt{-g}}\nabla_\mu(\sqrt{-g})\left(g^{0\mu}g^{ij} - g^{i\mu}g^{0j}\right)\right) \\
&- \varepsilon_{klm}\left(\nabla_\mu B^k\right)g^{l\mu}g^{mj} - \varepsilon_{klm}B^k \\
&\times \left(\nabla_\mu\left(g^{l\mu}g^{mj}\right) + \frac{1}{\sqrt{-g}}\nabla_\mu(\sqrt{-g})\left(g^{l\mu}g^{mj}\right)\right) = \mu_0 J^j
\end{aligned} \tag{18}$$

For the case of the weak gravitational fields (i.e., $g^{0i} = g^{i0} = 0$), we obtain

$$\begin{aligned}
&\frac{1}{c}(\nabla_0 E_i)\left(g^{00}g^{ij}\right) + \frac{1}{c}E_i\left(\nabla_0\left(g^{00}g^{ij}\right) + \frac{1}{\sqrt{-g}}\nabla_0(\sqrt{-g})\left(g^{00}g^{ij}\right)\right) \\
&- \varepsilon_{klm}\left(\nabla_n B^k\right)g^{ln}g^{mj} - \varepsilon_{klm}B^k \\
&\times \left(\nabla_n\left(g^{ln}g^{mj}\right) + \frac{1}{\sqrt{-g}}\nabla_n(\sqrt{-g})\left(g^{ln}g^{mj}\right)\right) = \mu_0 J^j
\end{aligned} \tag{19}$$

Or,

$$\begin{aligned}
&\frac{1}{c}(\nabla_0 E_i)\left(g^{00}g^{ij}\right) + \frac{1}{c}E_i\left(\nabla_0\left(g^{00}g^{ij}\right) + \frac{1}{\sqrt{-g}}\nabla_0(\sqrt{-g})\left(g^{00}g^{ij}\right)\right) \\
&- (\nabla_n B_{lm})g^{ln}g^{mj} - B_{lm} \\
&\times \left(\nabla_n\left(g^{ln}g^{mj}\right) + \frac{1}{\sqrt{-g}}\nabla_n(\sqrt{-g})\left(g^{ln}g^{mj}\right)\right) = \mu_0 J^j
\end{aligned} \tag{20}$$

where the following relation is used

$$B_{lm} = \varepsilon_{lmk}B^k \tag{21}$$

which transforms the electromagnetic vector field to the corresponding 2-form. Thus, Eq. (20) becomes

$$\begin{aligned}
&\frac{1}{c^2}\left(g^{00}g^{ij}\right)(\nabla_t E_i) \\
&+ \frac{1}{c^2}E_i\left(\nabla_t\left(g^{00}g^{ij}\right) + \frac{1}{\sqrt{-g}}\nabla_t(\sqrt{-g})\left(g^{00}g^{ij}\right)\right) \\
&- g^{ln}g^{mj}(\nabla_n B_{lm}) \\
&- B_{lm}\left(\nabla_n\left(g^{ln}g^{mj}\right) + \frac{1}{\sqrt{-g}}\nabla_n(\sqrt{-g})\left(g^{ln}g^{mj}\right)\right) = \mu_0 J^j
\end{aligned} \tag{22}$$

where the following relation is used

$$\nabla_0 = \frac{\partial}{\partial x^0} = \frac{\partial}{\partial(ct)} = \frac{1}{c}\frac{\partial}{\partial t} = \frac{1}{c}\nabla_t \tag{23}$$

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Next, we consider a metric with vanishing the off-diagonal elements (i.e., $g^{ij} = 0$, for $i \neq j$), and Eq. (22) reduces to

$$\begin{aligned} &\frac{1}{c^2}\left(g^{00}g^{jj}\right)(\nabla_t E_j) \\ &+ \frac{1}{c^2}E_j\left(\nabla_t\left(g^{00}g^{jj}\right) + \frac{1}{\sqrt{-g}}\nabla_t(\sqrt{-g})\left(g^{00}g^{jj}\right)\right) \\ &- g^{ll}g^{jj}\left(\nabla_l B_{lj}\right) \\ &- B_{lj}\left(\nabla_l\left(g^{ll}g^{jj}\right) + \frac{1}{\sqrt{-g}}\nabla_l(\sqrt{-g})\left(g^{ll}g^{jj}\right)\right) = \mu_0 J^j \end{aligned} \tag{24}$$

Or,

$$\begin{aligned} &- \varepsilon_{ljk}g^{ll}g^{jj}\left(\nabla_l B^k\right) \\ &- \varepsilon_{ljk}B^k\left(\nabla_l\left(g^{ll}g^{jj}\right) + \frac{1}{\sqrt{-g}}\nabla_l(\sqrt{-g})\left(g^{ll}g^{jj}\right)\right) \\ &= \mu_0 J^j + \mu_0\epsilon_0\left(-g^{00}g^{jj}\right)\nabla_t E_j \\ &+ \mu_0\epsilon_0 E_j\left(\nabla_t\left(-g^{00}g^{jj}\right) - \frac{1}{\sqrt{-g}}\nabla_t(\sqrt{-g})\left(g^{00}g^{jj}\right)\right) \end{aligned} \tag{25}$$

Interestingly, Eq. (25) indicates that the coupling between the magnetic and electric fields does not vanish from Maxwell-Ampére's law, even for weak gravitational fields and diagonal Riemann metric, in contrast to Gauss's law. Furthermore, if the geometry is time-independent, then Eq. (25) reduces to

$$\begin{aligned} &- \varepsilon_{ljk}g^{ll}g^{jj}\left(\nabla_l B^k\right) \\ &- \varepsilon_{ljk}B^k\left(\nabla_l\left(g^{ll}g^{jj}\right) + \frac{1}{\sqrt{-g}}\nabla_l(\sqrt{-g})\left(g^{ll}g^{jj}\right)\right) \\ &= \mu_0 J^j + \mu_0\epsilon_0\left(-g^{00}g^{jj}\right)\nabla_t E_j \end{aligned} \tag{26}$$

because $\nabla_t(-g^{00}g^{jj}) - \frac{1}{\sqrt{-g}}\nabla_t(\sqrt{-g})(g^{00}g^{jj}) = 0$. However, the time derivative term of electric field does not cancel out (to cancel this term out, a time-varying charge density is required based on Gauss's law). For time-varying space-time geometry, both terms on the right-hand side of Eq. (26), because a non-stationary space-time geometry, will certainly induce a time-varying electric field (based on Gauss's law). Furthermore, the gravitational waves will directly influence the magnetic fields. If we introduce a generalised charge current density as

$$\mathbb{J}^j = \sqrt{-g}J^j \tag{27}$$

a generalised displacement current as

$$\begin{aligned} \mathbb{J}_D^j = \epsilon_0\sqrt{-g}\Big(&-g^{00}g^{jj}\nabla_t E_j \\ &- E_j\left(\nabla_t\left(g^{00}g^{jj}\right) - \frac{1}{\sqrt{-g}}\nabla_t(\sqrt{-g})\left(g^{00}g^{jj}\right)\right)\Big) \end{aligned} \tag{28}$$

and by introducing the following five-dimensional object

$$\mathbb{B}^{lljjk} = \sqrt{-g}g^{ll}g^{jj}B^k \tag{29}$$

then Eq. (26) takes the form of the generalised Maxwell-Ampére's law as

$$\varepsilon_{ljk}\nabla_l \mathbb{B}^{lljjk} = \mu_0(\mathbb{J}^j + \mathbb{J}^j_D) \tag{30}$$

Therefore, in the absence of the currents (that is, $\mathbb{J} = 0$), the electric field can be the source of the magnetic field, where an extra contribution is added from the displacement current that is induced by the time-varying space-time geometry. For stationary space-time geometry that contribution is zero; however, for high frequency gravitational waves (strongly varying waves), that contribution might be significant.

The homogeneous equations of the electromagnetic fields are expressed as follows:

$$\begin{aligned} 0 &= \mathbb{D}_\mu F_{\nu\alpha} + \mathbb{D}_\nu F_{\alpha\mu} + \mathbb{D}_\alpha F_{\mu\nu} \\ &= \nabla_\mu F_{\nu\alpha} + \Gamma^\sigma_{\ \nu\mu} F_{\sigma\alpha} + \Gamma^\sigma_{\ \alpha\mu} F_{\nu\sigma} \\ &+ \nabla_\nu F_{\alpha\mu} + \Gamma^\sigma_{\ \alpha\nu} F_{\sigma\mu} + \Gamma^\sigma_{\ \mu\nu} F_{\alpha\sigma} \\ &+ \nabla_\alpha F_{\mu\nu} + \Gamma^\sigma_{\ \mu\alpha} F_{\sigma\nu} + \Gamma^\sigma_{\ \nu\alpha} F_{\mu\sigma} \end{aligned} \tag{31}$$

Thus, for a torsionless manifold, as it is the case of Riemann space-time geometry, we get

$$\nabla_\mu F_{\nu\alpha} + \nabla_\nu F_{\alpha\mu} + \nabla_\alpha F_{\mu\nu} = 0 \tag{32}$$

Eq. (31) indicates that the homogeneous equations of the electromagnetic fields are not influenced by the curvature of Riemann's space-time, and hence they are equivalent to Faraday's law (for $\mu = 1,2,3$) and Gauss's law for magnetic field (for $\mu = 0$):

$$\begin{aligned} \nabla_t B^i &= -\varepsilon^{ijk}\nabla_j E_k \\ \nabla_j B^j &= 0 \end{aligned} \tag{33}$$

where the summation of repeating indices is assumed.

1.1. Equation of Electromagnetic Field Potential Wave

In the following, the inhomogeneous equations will be expressed in terms of the electromagnetic field 4-potential, namely A^μ. For that, the covariant derivative of field strength tensor is

$$\mathbb{D}_\mu F^{\mu\nu} = \mathbb{D}_\mu \mathbb{D}^\mu A^\nu - g^{\sigma\nu}\left(\left[\mathbb{D}_\mu, \mathbb{D}_\sigma\right] A^\mu + \mathbb{D}_\sigma \mathbb{D}_\mu A^\mu\right) \tag{34}$$

where

$$\left[\mathbb{D}_\mu, \mathbb{D}_\sigma\right] = \mathbb{D}_\mu \mathbb{D}_\sigma - \mathbb{D}_\sigma \mathbb{D}_\mu \tag{35}$$

Furthermore,

$$\left[\mathbb{D}_\mu, \mathbb{D}_\sigma\right] A^\nu = R^\nu_{\ \lambda\mu\sigma} A^\lambda \tag{36}$$

where $R^\nu_{\ \lambda\mu\sigma}$ are the components of Riemann tensor. Therefore, we obtain Maxwell's equation in the following form:

$$\mathbb{D}_\mu \mathbb{D}^\mu A^\nu - g^{\sigma\nu} R_{\lambda\sigma} A^\lambda - \mathbb{D}^\nu(\mathbb{D}_\mu A^\mu) = \mu_0 J^\nu \tag{37}$$

where $R_{\lambda\sigma}$ is Ricci tensor

$$R_{\lambda\sigma} = R^\mu_{\ \lambda\mu\sigma} \tag{38}$$

Using Lorentz condition for the electromagnetic field potential in the curved space-time:

$$\mathbb{D}_\mu A^\mu = \frac{1}{\sqrt{-g}} \nabla_\mu \left(\sqrt{-g} A^\mu \right) = 0 \tag{39}$$

we obtain the following

$$\mathbb{D}_\mu \mathbb{D}^\mu A^\nu - g^{\sigma\nu} R_{\lambda\sigma} A^\lambda = \mu_0 J^\nu \tag{40}$$

The formula of generalised Laplacian in the curved space-time is

$$\mathbb{D}_\mu \mathbb{D}^\mu \Phi = \frac{1}{\sqrt{-g}} \nabla_\mu \left(\sqrt{-g} g^{\mu\sigma} \nabla_\sigma \Phi \right) \tag{41}$$

Combining Eq. (40) and Eq. (41), we get

$$\nabla_\mu \nabla^\mu A^\nu + \frac{1}{\sqrt{-g}} \nabla_\mu \left(\sqrt{-g} g^{\mu\sigma} \right) \nabla_\sigma A^\nu - g^{\sigma\nu} R_{\lambda\sigma} A^\lambda = \mu_0 J^\nu \tag{42}$$

Using the relationship with Christoffel's symbol for Riemann's space-time geometry:

$$\frac{1}{\sqrt{-g}} \nabla_\mu \left(\sqrt{-g} g^{\mu\sigma} \right) = -g^{\mu\nu} \Gamma^\sigma_{\mu\nu} \tag{43}$$

Eq. (42) can also be written as follows:

$$\nabla_\mu \nabla^\mu A^\nu - g^{\mu\nu} \Gamma^\sigma_{\mu\nu} \nabla_\sigma A^\nu - g^{\sigma\nu} R_{\lambda\sigma} A^\lambda = \mu_0 J^\nu \tag{44}$$

Consider a diagonal metric in vacuum (that is, $J^\nu = 0$), Eq. (42) reduces to

$$\nabla_\mu \nabla^\mu A^\nu + \frac{1}{\sqrt{-g}} \nabla_\mu \left(\sqrt{-g} g^{\mu\mu} \right) \nabla_\mu A^\nu - g^{\nu\nu} R_{\lambda\nu} A^\lambda = 0 \tag{45}$$

Thus, the 4-potential components of the electromagnetic field are coupled in either Eq. (42) or Eq. (45), which is not the case of Minkowski flat space-time geometry. Furthermore, Eq. (45) indicates that even when Ricci's tensor components are zero, there will be a coupling between the components of the 4-potential of the electromagnetic field. Therefore, these results indicate that presence of the geometry dependent terms for the gravitational fields suggests new phenomena may arise for electromagnetism under the influence of space-time curvature.

1.2. Equations of Electromagnetic Waves

Using the electromagnetic field equations in vacuum ($J^\mu = 0$), we write

$$\begin{aligned} &\mathbb{D}_\mu F^{\mu\nu} = \nabla_\mu F^{\mu\nu} + \frac{1}{\sqrt{-g}} \nabla_\mu \left(\sqrt{-g} \right) F^{\mu\nu} = 0 \\ &\mathbb{D}_\mu F_{\alpha\beta} + \mathbb{D}_\alpha F_{\beta\mu} + \mathbb{D}_\beta F_{\mu\alpha} = 0 \end{aligned} \tag{46}$$

These two equations are gauge invariant.

First, consider the first expression in Eq. (46) for $\nu \equiv j = 1, 2, 3$ (which is Maxwell-Ampére's law). Then using Eq. (19) for $J^j = 0$ (that is, vacuum), we obtain:

$$\begin{aligned} &\frac{1}{c}(\nabla_\mu E_i)\left(g^{0\mu}g^{ij} - g^{i\mu}g^{0j}\right) + \frac{1}{c}E_i \\ &\times\left(\nabla_\mu\left(g^{0\mu}g^{ij} - g^{i\mu}g^{0j}\right) + \frac{1}{\sqrt{-g}}\nabla_\mu(\sqrt{-g})\left(g^{0\mu}g^{ij} - g^{i\mu}g^{0j}\right)\right) \\ &- \varepsilon_{klm}\left(\nabla_\mu B^k\right)g^{l\mu}g^{mj} - \varepsilon_{klm}B^k \\ &\times\left(\nabla_\mu\left(g^{l\mu}g^{mj}\right) + \frac{1}{\sqrt{-g}}\nabla_\mu(\sqrt{-g})\left(g^{l\mu}g^{mj}\right)\right) = 0 \end{aligned} \tag{47}$$

For weak gravitational fields (i.e., $g^{0i} = g^{i0} = 0$) with a diagonal metric (i.e., $g^{ij} = 0$ for $i \neq j$), from Eq. (47), we obtain

$$\begin{aligned} &\frac{1}{c^2}g^{00}g^{jj}\nabla_t E_j + \frac{1}{c^2}E_j\left(\nabla_t\left(g^{00}g^{jj}\right) + \nabla_t(\ln\sqrt{-g})\left(g^{00}g^{jj}\right)\right) \\ &= \varepsilon_{ljk}g^{ll}g^{jj}\nabla_l B^k + \varepsilon_{ljk}B^k\left(\nabla_l\left(g^{ll}g^{jj}\right) + \nabla_l(\ln\sqrt{-g})\left(g^{ll}g^{jj}\right)\right) \end{aligned} \tag{48}$$

where the following identity is used

$$\frac{1}{\sqrt{-g}}\nabla_t(\sqrt{-g}) = \nabla_t(\ln\sqrt{-g}) \tag{49}$$

Taking the time derivative of both sides of Eq. (48), we get

$$\begin{aligned} &\frac{1}{c^2}g^{00}g^{jj}\nabla^2_{tt}E_j \\ &\quad + \frac{1}{c^2}(\nabla_t E_j)\left(2\nabla_t\left(g^{00}g^{jj}\right) + \nabla_t(\ln\sqrt{-g})\left(g^{00}g^{jj}\right)\right) \\ &\quad + \frac{1}{c^2}E_j\left(\nabla^2_{tt}\left(g^{00}g^{jj}\right) + \nabla^2_{tt}(\ln\sqrt{-g})\left(g^{00}g^{jj}\right)\right. \\ &\quad \left. + \nabla_t(\ln\sqrt{-g})\nabla_t\left(g^{00}g^{jj}\right)\right) \\ &= \varepsilon_{ljk}\nabla_t\left(g^{ll}g^{jj}\right)\left(\nabla_l B^k\right) \\ &\quad + \varepsilon_{ljk}g^{ll}g^{jj}\left(\nabla_l\nabla_t B^k\right) \\ &\quad + \varepsilon_{ljk}(\nabla_t B^k)\left(\nabla_l\left(g^{ll}g^{jj}\right) + \nabla_l(\ln\sqrt{-g})\left(g^{ll}g^{jj}\right)\right) \\ &\quad + \varepsilon_{ljk}B^k\left(\nabla^2_{lt}\left(g^{ll}g^{jj}\right) + \nabla^2_{lt}(\ln\sqrt{-g})\left(g^{ll}g^{jj}\right)\right. \\ &\quad \left. + \nabla_l(\ln\sqrt{-g})\nabla_t\left(g^{ll}g^{jj}\right)\right) \end{aligned} \tag{50}$$

Using Faraday's law in the following form

$$-\varepsilon^k{}_{mn}\nabla^m E^n = \nabla_t B^k \tag{51}$$

and the mathematical identity

$$\varepsilon_{ljk}\varepsilon^k{}_{mn} = \delta_{lm}\delta_{jn} - \delta_{ln}\delta_{jm} \tag{52}$$

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then, we have

$$\begin{aligned}\varepsilon_{ljk} g^{ll} g^{jj} \left(\nabla_l \nabla_t B^k \right) &= -\varepsilon_{ljk} g^{ll} g^{jj} \left(\varepsilon^k{}_{mn} \nabla_l \nabla^m E^n \right) \\ &= -\delta_{lm}\delta_{jn} g^{ll} g^{jj} \nabla_l \nabla^m E^n \\ &\quad + \delta_{ln}\delta_{jm} g^{ll} g^{jj} \nabla_l \nabla^m E^n \\ &= -g^{mm} g^{nn} \nabla_m \nabla^m E^n + g^{nn} g^{mj} \nabla_n \nabla^m E^n \\ &= -g^{nn} \nabla^m \nabla_m E_n \\ &= -g^{jj} \nabla^k \nabla_k E_j \end{aligned} \tag{53}$$

where $g^{mj} = 0$, $g^{mm}\nabla_m = \nabla^m$, and $E_j = g_{kj} E^k$ are used.

Furthermore,

$$\begin{aligned} &\varepsilon_{ljk} (\nabla_t B^k) \left(\nabla_l \left(g^{ll} g^{jj} \right) + \nabla_l (\ln \sqrt{-g}) \left(g^{ll} g^{jj} \right) \right) \\ &= \varepsilon_{ljk} (-\varepsilon^k{}_{mn} \nabla^m E^n) \left(\nabla_l \left(g^{ll} g^{jj} \right) + \nabla_l (\ln \sqrt{-g}) \left(g^{ll} g^{jj} \right) \right) \\ &= (-\delta_{lm}\delta_{jn} + \delta_{ln}\delta_{jm}) \nabla^m E^n \left(\nabla_l \left(g^{ll} g^{jj} \right) + \nabla_l (\ln \sqrt{-g}) \left(g^{ll} g^{jj} \right) \right) \\ &= -\nabla^m E^n \left(\nabla_l \left(g^{ml} g^{nj} \right) + \nabla_l (\ln \sqrt{-g}) \left(g^{ml} g^{nj} \right) \right) \\ &+ \nabla^m E^n \left(\nabla_l \left(g^{nl} g^{mj} \right) + \nabla_l (\ln \sqrt{-g}) \left(g^{nl} g^{mj} \right) \right) \\ &= 0 \end{aligned} \tag{54}$$

where $g^{ml} = g^{nj} = g^{nl} = g^{mj} = 0$ is used.

Using Gauss's law in vacuum ($\rho = 0$), from Eq. (18), we write

$$g^{jj} g^{kk} \nabla_k E_k - g^{jj} g_{00} E_k a^k(\mathbf{x}) = 0 \tag{55}$$

where

$$a^k(\mathbf{x}) = -\left(g^{kk} g^{00} \nabla_k (\ln \sqrt{-g}) + \nabla_k \left(g^{kk} g^{00} \right) \right) \tag{56}$$

which is a function of the space-time point.

Taking the derivative ∇_j of both sides and summing for $j = 1, 2, 3$, we obtain

$$\begin{aligned} &\nabla_j (g^{jj} g^{kk}) \nabla_k E_k + g^{jj} g^{kk} \nabla_j \nabla_k E_k \\ &+ (\nabla_j g^{jj} g_{00}) a^k E_k + g^{jj} g_{00} a^k \nabla_j E_k + g^{jj} g_{00} (\nabla_j a^k) E_k = 0 \end{aligned} \tag{57}$$

Or,

$$\begin{aligned} &\left(\nabla_j (g^{jj} g^{kk}) + g^{jj} g^{kk} \nabla_j \right) \nabla_k E_k \\ &+ \left((\nabla_j g^{jj} g_{00}) a^k + g^{jj} g_{00} (\nabla_j a^k) \right) E_k + g^{jj} g_{00} a^k \nabla_j E_k = 0 \end{aligned} \tag{58}$$

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Combining Eq. (50), Eq. (53), Eq. (54), and Eq. (58) we obtain

$$
\begin{aligned}
g^{jj}\left(\frac{g^{00}}{c^2}\nabla^2_{tt}E_j + \nabla^k\nabla_k E_j\right) = {} & \alpha^{jj}\nabla_t E_j \\
& + \eta^{jkk}\nabla_k E_k + \theta^{jjk}\nabla_j E_k \\
& + \beta^{jj}E_j + \psi^{jk}E_k \\
& + \gamma^{jl}{}_k\left(\nabla_l B^k\right) + \sigma^j{}_k B^k
\end{aligned} \tag{59}
$$

where

$$
\begin{aligned}
\alpha^{jj} &= -\frac{1}{c^2}(\nabla_t E_j)\left(2\nabla_t\left(g^{00}g^{jj}\right) + \nabla_t(\ln\sqrt{-g})\left(g^{00}g^{jj}\right)\right) \\
\beta^{jj} &= -\frac{1}{c^2}\left(\nabla^2_{tt}\left(g^{00}g^{jj}\right) + \nabla^2_{tt}(\ln\sqrt{-g})\left(g^{00}g^{jj}\right)\right. \\
&\quad \left. + \nabla_t(\ln\sqrt{-g})\nabla_t\left(g^{00}g^{jj}\right)\right) \\
\gamma^{jl}{}_k &= -\varepsilon_{ljk}\nabla_t\left(g^{ll}g^{jj}\right) \\
\sigma^j{}_k &= -\varepsilon_{ljk}\left(\nabla^2_{lt}\left(g^{ll}g^{jj}\right) + \nabla^2_{lt}(\ln\sqrt{-g})\left(g^{ll}g^{jj}\right)\right. \\
&\quad \left. + \nabla_l(\ln\sqrt{-g})\nabla_t\left(g^{ll}g^{jj}\right)\right) \\
\eta^{jkk} &= \nabla_j(g^{jj}g^{kk}) + g^{jj}g^{kk}\nabla_j \\
\theta^{jjk} &= g^{jj}g_{00}a^k \\
\psi^{jk} &= (\nabla_j g^{jj}g_{00})a^k + g^{jj}g_{00}\nabla_j a^k
\end{aligned} \tag{60}
$$

Eq. (53) represents the wave equation of the electric field in a curved space-time geometry. It can be seen that is not a homogeneous second order differential equation, as derived for the flat space-time geometry. The presence of non-vanishing terms on the right-hand side indicates the influence of the space-time curvature on the electric field polarisation. Furthermore, the wave equation of the electric field is not independent on the magnetic field, even in the case of stationary space-time curvature indicating that the electromagnetic wave is a mixture of transverse and longitudinal waves, which is in contrast to Lorentz flat space-time geometry.

The following steps can be taken to derive the wave equation for magnetic field. From homogeneous electromagnetic equation in Eq. (46), for $\mu = 0$, we obtain

$$\varepsilon^{klj}\nabla_l E_j = -\nabla_t B^k \tag{61}$$

Taking the derivative for time t of both sides in Eq. (61), we will get

$$\varepsilon^{klj}\nabla_t\nabla_l E_j = -\nabla^2_{tt}B^k \tag{62}$$

Furthermore,

$$\nabla_t\nabla_l E_j = \varepsilon_{ljk}\nabla^2_{tt}B^k \tag{63}$$

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Taking the derivative ∇_l in both sides of Eq. (48) and summing for $l = 1,2,3$, we obtain

$$\begin{aligned}
&\frac{1}{c^2}\nabla_l(g^{00}g^{jj})(\nabla_t E_j) + \frac{1}{c^2}g^{00}g^{jj}\nabla_t\nabla_l E_j \\
&\quad + \frac{1}{c^2}(\nabla_l E_j)\left(\nabla_t\left(g^{00}g^{jj}\right) + \nabla_t(\ln\sqrt{-g})\left(g^{00}g^{jj}\right)\right) \\
&\quad + \frac{1}{c^2}E_j\left(\nabla_l\nabla_t\left(g^{00}g^{jj}\right) + \nabla_l\nabla_t(\ln\sqrt{-g})\left(g^{00}g^{jj}\right)\right. \\
&\quad \left. + \nabla_t(\ln\sqrt{-g})\nabla_l\left(g^{00}g^{jj}\right)\right) \\
&= \varepsilon_{ljk}(\nabla_l g^{ll}g^{jj})\left(\nabla_l B^k\right) + \varepsilon_{ljk}g^{ll}g^{jj}\left(\nabla_l\nabla_l B^k\right) \\
&\quad + \varepsilon_{ljk}\nabla_l B^k\left(\nabla_l\left(g^{ll}g^{jj}\right) + \nabla_l(\ln\sqrt{-g})\left(g^{ll}g^{jj}\right)\right) \\
&\quad + \varepsilon_{ljk}B^k\left(\nabla_l\nabla_l\left(g^{ll}g^{jj}\right) + \nabla_l\nabla_l(\ln\sqrt{-g})\left(g^{ll}g^{jj}\right)\right. \\
&\quad \left. + \nabla_l(\ln\sqrt{-g})\nabla_l\left(g^{ll}g^{jj}\right)\right)
\end{aligned} \tag{64}$$

Substituting Eq. (63) into Eq. (64), we get

$$\begin{aligned}
&\frac{1}{c^2}\nabla_l(g^{00}g^{jj})(\nabla_t E_j) + \varepsilon_{ljk}\frac{g^{00}}{c^2}g^{jj}\nabla^2_{tt}B^k \\
&\quad + \frac{1}{c^2}(\nabla_l E_j)\left(\nabla_t\left(g^{00}g^{jj}\right) + \nabla_t(\ln\sqrt{-g})\left(g^{00}g^{jj}\right)\right) \\
&\quad + \frac{1}{c^2}E_j\left(\nabla_l\nabla_t\left(g^{00}g^{jj}\right) + \nabla_l\nabla_t(\ln\sqrt{-g})\left(g^{00}g^{jj}\right)\right. \\
&\quad \left. + \nabla_t(\ln\sqrt{-g})\nabla_l\left(g^{00}g^{jj}\right)\right) \\
&= \varepsilon_{ljk}(\nabla_l g^{ll}g^{jj})\left(\nabla_l B^k\right) + \varepsilon_{ljk}g^{jj}\left(\nabla^l\nabla_l B^k\right) \\
&\quad + \varepsilon_{ljk}\nabla_l B^k\left(\nabla_l\left(g^{ll}g^{jj}\right) + \nabla_l(\ln\sqrt{-g})\left(g^{ll}g^{jj}\right)\right) \\
&\quad + \varepsilon_{ljk}B^k\left(\nabla_l\nabla_l\left(g^{ll}g^{jj}\right) + \nabla_l\nabla_l(\ln\sqrt{-g})\left(g^{ll}g^{jj}\right)\right. \\
&\quad \left. + \nabla_l(\ln\sqrt{-g})\nabla_l\left(g^{ll}g^{jj}\right)\right)
\end{aligned} \tag{65}$$

Or,

$$\begin{aligned}
&g^{jj}\left(\frac{g^{00}}{c^2}\nabla^2_{tt}B^k + \nabla^l\nabla_l B^k\right) = A^{jjl}\nabla_l B^k + C^{jj}B^k \\
&\quad + D^{jjjk}\nabla_t E_j + F^{jjjk}{}_l\nabla^l E_j + M^{jjjk}E_j
\end{aligned} \tag{66}$$

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where

$$
\begin{aligned}
A^{jjl} &= -\left(2\nabla_l\left(g^{ll}g^{jj}\right) + \nabla_l(\ln\sqrt{-g})\left(g^{ll}g^{jj}\right)\right) \\
C^{jj} &= -\left(\nabla_l\nabla_l\left(g^{ll}g^{jj}\right) + \nabla_l\nabla_l(\ln\sqrt{-g})\left(g^{ll}g^{jj}\right)\right. \\
&\quad \left. + \nabla_l(\ln\sqrt{-g})\nabla_l\left(g^{ll}g^{jj}\right)\right) \\
D^{jjjk} &= -\varepsilon_{ljk}\frac{1}{c^2}\nabla_l(g^{00}g^{jj}) \\
F^{jjjk}{}_l &= -\varepsilon_{ljk}\frac{1}{c^2}\left(\nabla_t\left(g^{00}g^{jj}\right) + \nabla_t(\ln\sqrt{-g})\left(g^{00}g^{jj}\right)\right) \\
M^{jjjk} &= -\varepsilon_{ljk}\frac{1}{c^2}\left(\nabla_l\nabla_t\left(g^{00}g^{jj}\right) + \nabla_l\nabla_t(\ln\sqrt{-g})\left(g^{00}g^{jj}\right)\right. \\
&\quad \left. + \nabla_t(\ln\sqrt{-g})\nabla_l\left(g^{00}g^{jj}\right)\right)
\end{aligned} \tag{67}
$$

Eq. (66) represents the wave equation of the magnetic field in a curved space-time geometry. Again, it is not a homogeneous second order differential equation, in contrast to the wave equation derived for the flat space-time geometry. The presence of non-vanishing terms on the right-hand side indicates the influence of the space-time curvature on the magnetic field. Furthermore, the wave equation of the magnetic field is coupled to the electric field, even in the case of stationary space-time curvature indicating that the electromagnetic wave is a mixture of transverse and longitudinal waves, which is in contrast to Lorentz flat space-time geometry.

2. On Unification of Gravitation and Electromagnetism in Framework of General Relativity

2.1. System of Discrete Point-like Particles

Consider a system of discrete relativistic non-interacting point-like particles with a rest mass of m_i. The mass density at the position $\mathbf{r}$ can be given as

$$\rho(\mathbf{r}) = \sum_i m_i\delta(\mathbf{r}-\mathbf{r}_i) \tag{68}$$

where $\mathbf{r}_i$ denotes the position of the i-th particle at some reference frame and $\delta(\cdots)$ is the delta-function. Furthermore, the 4-momentum density can be defined as

$$p^\mu = \rho u^\mu, \quad \mu = 0,1,2,3 \tag{69}$$

where u^μ is the 4-velocity, $u^\mu = (\gamma c, \gamma\mathbf{v})$; therefore,

$$p^\mu = (\gamma\rho c, \gamma\rho\mathbf{v}) = \left(\frac{\varepsilon}{c}, \mathbf{p}\right) \tag{70}$$

where ε is the relativistic particle energy density and $\mathbf{p}$ is the relativistic three-dimensional momentum density vector.

We propose the following scalar form of Lagrangian density function:

$$\sqrt{-g}\mathcal{L} = -\frac{1}{\gamma}\rho g_{\alpha\beta}u^\alpha u^\beta = -\frac{1}{\gamma}g_{\alpha\beta}u^\alpha p^\beta \tag{71}$$

where the summation of repeating indices is assumed.

The energy-momentum density tensor of the system is

$$\mathcal{G}^{\mu\nu} = \frac{1}{\gamma}u^\mu p^\nu \tag{72}$$

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The spatial components of the energy-momentum density tensor are given as

$$\mathcal{G}^{0i} = \frac{1}{\gamma}u^0 p^i = \frac{1}{\gamma}\rho u^0 u^i = c p^i = c p_i \tag{73}$$
$$\mathcal{G}^{ij} = \frac{1}{\gamma}u^i p^j = \frac{1}{\gamma}\rho u^i u^j$$

where p_i is the Cartesian component of the relativistic three-dimensional momentum density vector and p^i is the contravariant component.

From Eq. 73, it can be seen that the energy-momentum density tensor is symmetric; that is, $\mathcal{G}^{0i} = \mathcal{G}^{i0}$ and $\mathcal{G}^{ij} = \mathcal{G}^{ji}$. Moreover, the zeroth component (which equals the energy density) is given as

$$\mathcal{G}^{00} = \frac{1}{\gamma}u^0 p^0 = \varepsilon \tag{74}$$

where ε is the relativistic energy density given as

$$\varepsilon = \gamma \rho c^2 = \frac{\rho c^2}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{75}$$

Here,

$$\varepsilon_0 = \rho c^2 \tag{76}$$

is the rest energy density.

Now, consider a system of discrete relativistic non-interacting point-like particles with charges q_i, then the charge density at the position $\mathbf{r}$ will be as

$$\rho_e(\mathbf{r}) = \sum_i q_i \delta(\mathbf{r} - \mathbf{r}_i) \tag{77}$$

The 4-current charge density is defined as

$$J^\mu = (\rho_e c, \rho_e \mathbf{v}), \quad \mu = 0,1,2,3 \tag{78}$$

and its covariant form is

$$J_\mu = (\rho_e c, -\rho_e \mathbf{v}), \quad \mu = 0,1,2,3 \tag{79}$$

We propose the following scalar form of Lagrangian density function for this system:

$$\sqrt{-g}\mathcal{L} = -\kappa_e \rho_e g_{\alpha\beta} u^\alpha u^\beta = -\kappa_e g_{\alpha\beta} J^\alpha u^\beta \tag{80}$$

We obtain the energy-momentum density tensor as

$$\mathcal{T}^{\mu\nu} = \kappa_e J^\mu u^\nu \tag{81}$$

The spatial components of the energy-momentum density tensor due to the current charges are written as

$$\mathcal{T}_q^{0i} = \kappa_e J^0 u^i = \gamma \kappa_e \rho_e c v^i \tag{82}$$
$$\mathcal{T}_q^{ij} = \kappa_e J^i u^j = \gamma \kappa_e \rho_e v^i v^j$$

where κ_e is a scaling factor [11]

$$\kappa_e = \frac{4\pi\sqrt{G}/s^2}{\kappa} = \frac{c^4}{2\sqrt{G}s^2} \tag{83}$$

where $s = 1$ and its SI unit is $\mathrm{m}^2/\mathrm{s}^2$.

Similarly, see also Eq. 82, the energy-momentum density tensor is symmetric; that is, $\mathcal{T}_q^{0i} = \mathcal{T}_q^{i0}$ and $\mathcal{T}_q^{ij} = \mathcal{T}_q^{ji}$. The zeroth component is given as

$$\mathcal{T}_q^{00} = \gamma\kappa_e\rho_e c^2 \tag{84}$$

Note that in Eq. 84 the term $\mathcal{T}_q^{00}$ equals some relativistic charge energy density, similar to the relativistic energy density of the mass (see also Eq. 74). Therefore, the presence of the charge and current density of the matter in the Riemann space-time deforms the space-time [11], similar to the presence of mass. Here, $\mathcal{T}_q^{\mu\nu}$ are the energy-momentum density tensor components of the deformable charged medium, which depends on the velocity $\mathbf{v}$ and charge density ρ_e.

2.2. Macroscopic Masses

For the ideal fluid of non-charged masses, the energy-momentum density tensor is also defined by $\mathcal{G}_{\mu\nu}$ (or $\mathcal{G}^{\mu\nu}$) [4–7]. The macroscopic mass is considered continuous bodies. The flux of momentum through an element $d\mathbf{A}$ of the surface of the mass equals the force on that surface element. Therefore, the ith component of the force vector acting on the surface element is

$$F_i = \sigma_{ij} dA_i \tag{85}$$

In a reference frame in which a volume element is at rest, using the Pascal's law, the pressure P is equal in all directions and it is perpendicular to the surface element, and thus

$$\sigma_{ij} dA_i = P dA_i \tag{86}$$

Therefore, the stress tensor is

$$\sigma_{ij} = P\delta_{ij} \tag{87}$$

Thus, in the reference frame in which the macroscopic body is at rest, Lagrangian density function is suggested as

$$\sqrt{-g}\mathcal{L} = -P \tag{88}$$

and in any arbitrary reference frame as

$$\sqrt{-g}\mathcal{L} = -\left(\rho + \frac{P}{c^2}\right) g_{\alpha\beta} u^\alpha u^\beta - P \tag{89}$$

We find that the energy-momentum density tensor of the macroscopic mass in an arbitrary reference frame is

$$\mathcal{G}^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right) u^\mu u^\nu + g^{\mu\nu} P \tag{90}$$

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where ρ is the mass density of macroscopic mass distribution and P is the pressure. In Eq. 90, $g^{\mu\nu}$ is the metric tensor of Riemann space-time. The zeroth component is

$$\begin{aligned}\mathcal{G}^{00} &= \left(\rho + \frac{P}{c^2}\right)\gamma^2 c^2 + g^{00}P \\ &= \gamma^2 \rho c^2 + \gamma^2 P + g^{00}P \\ &= \gamma\varepsilon + P\left(\gamma^2 + g^{00}\right)\end{aligned} \tag{91}$$

where ε is the relativistic energy density. Eq. 91 indicates that $\mathcal{G}^{00} > 0$, as expected.

The other components are given as

$$\mathcal{G}^{0i} = \left(\rho + \frac{P}{c^2}\right)\gamma^2 c v^i + g^{0i}P = cp^i = \frac{S^i}{c}, \quad i = 1, 2, 3 \tag{92}$$

where p^i is the ith 4-momentum density component

$$p^i = \gamma^2(\varepsilon_0 + P)\frac{v^i}{c^2} + g^{0i}\frac{P}{c} \tag{93}$$

and $\mathbf{S}$ is energy density flow vector:

$$S^i = \gamma^2(\varepsilon_0 + P)v^i + g^{0i}cP \tag{94}$$

where ε_0 is the rest energy density, $\varepsilon_0 = \rho c^2$.

Introducing, the three-dimensional components of the stress density tensor as follows:

$$\sigma^{ij} = \gamma^2(\varepsilon_0 + P)\frac{v^i v^j}{c^2} + Pg^{ij} \tag{95}$$

then,

$$\mathcal{G}^{ij} = \gamma^2(\varepsilon_0 + P)\frac{v^i v^j}{c^2} + Pg^{ij} = \sigma^{ij} \tag{96}$$

In general, the covariant form of the energy-momentum density tensor of the macroscopic body is

$$\mathcal{G}_{\mu\nu} = \frac{1}{c^2}(\varepsilon_0 + P)u_\mu u_\nu + Pg_{\mu\nu} \tag{97}$$

and its contravariant form is

$$\mathcal{G}^{\mu\nu} = \frac{1}{c^2}(\varepsilon_0 + P)u^\mu u^\nu + Pg^{\mu\nu} \tag{98}$$

where the metric of the space-time is Riemann metric tensor of the curved space-time geometry. In a mixed-tensor form, we can write

$$\mathcal{G}^\mu_\nu = \frac{1}{c^2}(\varepsilon_0 + P)u^\mu u_\nu + P\delta^\mu_\nu \tag{99}$$

Besides, if there is an electromagnetic field present in the matter, then we will discuss two different approaches. In the first approach, the energy-momentum density tensor of the electromagnetic field contributes to the total energy-momentum density tensor of the matter, which is given as follows

$$T^{\mu\nu} = \mathcal{G}^{\mu\nu} + \mathcal{T}^{\mu\nu}_q + \mathcal{T}^{\mu\nu}_{\text{EM}} \tag{100}$$

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where the space-time curvature of Riemann geometry is determined by the gravitational field.

In Eq. 100, $\mathcal{T}^{\mu\nu}_{\mathrm{EM}}$ is energy-momentum density tensor of the electromagnetic field, which is calculated by taking Lagrangian density function as

$$\sqrt{-g}\mathcal{L} = -\frac{1}{4\mu_0} F_{\alpha\beta}F^{\alpha\beta} \tag{101}$$

We obtain the energy-momentum density tensor of the electromagnetic field:

$$\begin{aligned}
\mathcal{T}^{00} &= \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \\
\mathcal{T}^{0i} &= \frac{1}{c\mu_0}(\mathbf{E}\times\mathbf{B})_i = \frac{1}{c}(\mathbf{E}\times\mathbf{H})_i = \frac{S_i}{c} \\
\mathcal{T}^{ij} &= -\epsilon_0\left[E_iE_j + c^2B_iB_j - \frac{1}{2}\delta_{ij}\left(E^2 + c^2B^2\right)\right]
\end{aligned} \tag{102}$$

In Eq. 100, $\mathcal{T}^{\mu\nu}_{\mathrm{EM}}$ is the energy-momentum density tensor of electromagnetic field:

$$\mathcal{T}^{\mu\nu}_{\mathrm{EM}} = \frac{1}{\mu_0} g^{\mu\alpha}F_{\alpha\beta}F^{\beta\nu} + \frac{1}{4\mu_0} g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{103}$$

In this case, however, the Maxwell's laws (such as Gauss's laws for electric and magnetic fields, Faraday's law, and Maxwell-Ampére's law) must be revised due the space-time curvature caused by gravitational field, which will be described in a future work.

In the second approach (see also Refs. [13–15]), one can introduce a *joint* space-time, which is a result of both gravitational and electromagnetic fields, characterised by the joint metric tensor of space-time curvature $\tilde{g}_{\mu\nu}$ and joint manifold connection symbols $\Gamma^{\nu}{}_{\mu\nu}$. In this case, the total energy-momentum density tensor of the matter is

$$T^{\mu\nu} = \mathcal{G}^{\mu\nu} + \mathcal{T}^{\mu\nu}_{q} \tag{104}$$

The covariant form is:

$$\kappa^{-1}\Lambda g_{\mu\nu} = T_{\mu\nu} - \kappa^{-1}G_{\mu\nu} \tag{105}$$

$T^{\mu\nu}$ gives the energy-momentum density tensor of the matter, which may includes the mass distribution and they current of charges in matter. $\kappa^{-1}\Lambda g_{\mu\nu}$ is the total energy-momentum density tensor; and $-\kappa^{-1}G_{\mu\nu}$ gives the energy-momentum density tensor of the gravitation field only. That is the viewpoint of the un-unified theory of the gravitation and electromagnetic fields.

Furthermore, there is an essential difference between the $-\kappa^{-1}G_{\mu\nu}$, which represents the storage of gravitation field energy and momentum density surrounding the masses, and the gravitational potential energy, which expresses the interaction energy between the masses, and hence it represents the interaction forces between the masses and it depends on the mass distribution. Therefore, their physical origin is completely different. The gravitational field energy and momentum distribution (which is the reality) is represented by the curvature of the space-time described by the Riemann metric tensor $g_{\mu\nu}(x^\sigma)$ a function of the space-time point x^σ (which is a picture of that reality). Besides, both can not be localised in space; thus, to obtain both, one has to integrate overall space. Moreover, the gravitational potential energy density is always negative and it vanishes in the empty space-time.

On the other hand, the gravitation field energy density is always negative inside the matter, and outside it depends on the metric used; for instance, for some metrics, it is positive and thus the cosmological constant $\Lambda > 0$. Λ is a constant parameter, which depends on the metric ds^2 [7,26], and it is proportional to the total energy density concerning the metric under consideration. Λ is positive concerning Friedmann-Lemaître-Robertson-Walker metric (de Sitter space), see also Ref. [18]; however, Λ is negative for the metric of the celestial body (anti-de Sitter space). Thus, the cosmological constant

Λ is connected to the cosmology in the universe under the Friedmann-Lemaître-Robertson-Walker metric. In the case of Friedmann-Lemaître-Robertson-Walker metric, the cosmological constant has a value of $\Lambda = 1.1056 \times 10^{-52}\ m^{-2}$. In contrast, κ is a universal constant, and hence it is the same constant in every reference frame.

One can also think about the electromagnetic field energy (which is a reality) and the electromagnetic interactions between the currents and the electromagnetic field; in this view, the electromagnetic field energy represents the energy stored around the space of the sources that created it, and the electromagnetic potential energy represents the interaction between the charges and the currents with the electromagnetic field, and so it depends on the charge and the current distribution of the matter. Therefore, in analogy, the distribution of the stored electromagnetic field energy and momentum (which is the reality) can introduce a new fabrication of the geometry, which can be described by an effective metric tensor, namely $\tilde{g}_{\mu\nu}$ [13–15]:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + \alpha F_{\mu\nu} \tag{106}$$

which is a projection of the reality influenced by both gravitational and electromagnetic fields. In Eq. (106), $F_{\mu\nu}$ denotes the electromagnetic field strength tensor, and α is a constant having the inverse units of $F_{\mu\nu}$ such that $\alpha F_{\mu\nu}$ is dimensionless.

Then, we have the following covariant form:

$$\kappa^{-1}\Lambda g_{\mu\nu} = \mathcal{G}_{\mu\nu} + \mathcal{T}^{(q)}_{\mu\nu} - \kappa^{-1}G_{\mu\nu} \tag{107}$$

In Eq. (107), $\kappa^{-1}\Lambda g_{\mu\nu}$ is the total energy-momentum density tensor; $\mathcal{G}_{\mu\nu} + \mathcal{T}^{(q)}_{\mu\nu}$ gives the energy-momentum density tensor of the matter (which includes the mass distribution and the current of charges in matter only); and $-\kappa^{-1}G_{\mu\nu}$ gives the energy-momentum density tensor of the gravitation field and electromagnetic field. Therefore, the rays of the electromagnetic field bend when they travel as they would bend in a space with only gravitation field with the effective metric tensor $\tilde{g}_{\mu\nu}$ [15].

3. Conclusions

We focused on the influence of Riemann space-time curvature on the laws of electromagnetism, such as Gauss's law, Maxwell-Ampére's law, and Faraday's law. Furthermore, we also introduced some work done on the unification of the gravitation and electromagnetic fields.

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