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Article

Robin's Criterion on Divisibility (II)

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Abstract: The Riemann hypothesis, renowned for its deep connection to the distribution of prime numbers, remains a central problem in mathematics. Understanding the distribution of primes is crucial for developing efficient algorithms and advancing our knowledge of number theory. The Riemann hypothesis is the assertion that all non-trivial zeros are complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. Several equivalent formulations of the Riemann hypothesis exist. Robin's criterion for the Riemann hypothesis is based on an inequality that divisor sum function σ must satisfy at natural numbers greater than 5040. We require the properties of superabundant numbers, that is to say left to right maxima of $n \mapsto \frac{\sigma(n)}{n}$. By using Robin's criterion on superabundant numbers, we present a novel approach that culminates in a complete proof of the Riemann hypothesis. This work is an expansion and refinement of the article "Robin's criterion on divisibility", published in The Ramanujan Journal.

Keywords: Riemann hypothesis; Robin's criterion; superabundant numbers; abundancy index function; prime numbers

MSC: Primary 11M26; Secondary 11A41; 11A25

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1. Introduction

The Riemann hypothesis (RH), concerning the nontrivial zeros of the Riemann zeta function, has been dubbed the "Holy Grail of Mathematics" by many [1,2]. Numerous equivalent formulations exist [3]. One of interest here is Robin's criterion [4], which states that if RH is true, then the inequality

$$\sigma(n) < e^{\gamma} \cdot n \cdot \log \log n$$

holds for all $n > 5040$. Here, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant, σ is the divisor sum function, log is the natural logarithm, and n is a natural number. The superabundant and colossally abundant numbers were discovered by Ramanujan [5] and studied later by Alaoglu and Erdős [6]. The falsity of the Riemann hypothesis would imply the existence of infinitely many counterexamples to Robin's inequality, including superabundant numbers exceeding 5040. Therefore, demonstrating the non-existence of these counterexamples would be enough to verify the Riemann hypothesis. Indeed, this is precisely what we demonstrate in this manuscript. Without a doubt, we provide a solution to this problem grounded in the properties of superabundant numbers and Robin's criterion.

2. Background and Ancillary Results

In mathematics, the Euler-Mascheroni constant, denoted by $\gamma \approx 0.57721$, is defined as

$$\gamma = \lim_{n \to \infty} (H_n - \log n),$$

where log represents the natural logarithm and $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the n^{th} harmonic number. As usual $\sigma(n)$ is the divisor sum function of n:

$$\sum_{d \mid n} d$$

where $d \mid n$ means the integer d divides n. We define the abundancy index $I : \mathbb{N} \to \mathbb{Q}$ with $I(n) = \frac{\sigma(n)}{n}$. A precise formula for $I(n)$ can be derived from its multiplicity:

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Proposition 2.1. *Let $n = \prod_{i=1}^{r} p_i^{a_i}$ be the prime factorization of n, where $p_1 < \ldots < p_r$ are distinct prime numbers and $a_1, \ldots, a_r$ are positive integers. Then [7, Lemma 1 (2) pp. 2]:*

$$I(n) = \left(\prod_{i=1}^{r} \frac{p_i}{p_i - 1}\right) \cdot \prod_{i=1}^{r} \left(1 - \frac{1}{p_i^{a_i+1}}\right)$$
$$= \prod_{i=1}^{r} I(p_i^{a_i}).$$

Proposition 2.2. *For $n > 1$ [8, (2.7) pp. 362]:*

$$I(n) < \prod_{p|n} \frac{p}{p-1}.$$

Definition 2.3. We say that $\mathsf{Robin}(n)$ holds provided that

$$I(n) < e^{\gamma} \cdot \log\log n.$$

The Chebyshev function $\theta(x)$ is defined as

$$\theta(x) = \sum_{p \leq x} \log p$$

where the sum is taken over all prime numbers p less than or equal to x. The following inequality holds for the Chebyshev function:

Proposition 2.4. *For $x \geq 7232121212$ [9, Lemma 2.7 (4) pp. 19]:*

$$\theta(x) \geq \left(1 - \frac{0.01}{\log^3(x)}\right) \cdot x.$$

Proposition 2.5. *For $x \geq 2278382$ [9, Lemma 2.7 (5) pp. 19]:*

$$\prod_{p \leq x} \frac{p}{p-1} \leq e^{\gamma} \cdot (\log x) \cdot \left(1 + \frac{0.2}{\log^3(x)}\right).$$

Ramanujan's Theorem asserts that the Riemann hypothesis implies $\mathsf{Robin}(n)$ for sufficiently large n [5]. Moreover, Robin's Theorem establishes a stronger result:

Proposition 2.6. $\mathsf{Robin}(n)$ *holds for all natural numbers $n > 5040$ if and only if the Riemann hypothesis is true [4, Theorem 1 pp. 188].*

Ramanujan's unpublished notes from 1997 presented generalized highly composite numbers, which encompass both superabundant and colossally abundant numbers [5]. These numbers were also explored by Alaoglu and Erdős in 1944 [6]. Given the first k consecutive primes $p_1 = 2, p_2 = 3, \ldots, p_k$, an integer of the form $\prod_{i=1}^{k} p_i^{a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ is termed an Hardy-Ramanujan integer [8, pp. 367]. A natural number n is classified as superabundant if, for all natural numbers $m < n$, the inequality $I(m) < I(n)$ holds.

Proposition 2.7. *If n is superabundant, then n is an Hardy-Ramanujan integer [6, Theorem 1 pp. 450].*

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Proposition 2.8. *Let n be a superabundant number. Denoting its largest prime factor by p, we have [6, Theorem 7 pp. 454]:*

$$p \sim \log n, \quad (n \to \infty).$$

A number n is said to be colossally abundant if, for some $\epsilon > 0$,

$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \text{ for } (m > 1).$$

Superabundant and colossally abundant numbers are closely related.

Proposition 2.9. *Every colossally abundant number is superabundant [6, pp. 455].*

Several analogues of the Riemann hypothesis have been proven. Many researchers anticipate or hope that the original hypothesis is true. However, there are potential consequences if the Riemann hypothesis were false.

Proposition 2.10. *If $n > 5040$ is the smallest integer such that $\mathsf{Robin}(n)$ does not hold, then $p > e^{31.018189471}$ where p is the largest prime factor of n [10, Theorem 4.2 pp. 748].*

Proposition 2.11. *If the Riemann hypothesis is false, then there exist infinitely many colossally abundant numbers $n > 5040$ such that $\mathsf{Robin}(n)$ does not hold [4, Proposition 1 pp. 204].*

By combining these results, we present a proof of the Riemann hypothesis.

3. Main Result

Definition 3.1. The *p-adic* order of an integer n is the exponent of the highest power of the prime number p that divides n. It is denoted $\nu_p(n)$.

These are significant findings.

Lemma 3.2. *For any superabundant number n with largest prime factor p, we have $p \leq q^{\nu_q(n)+1}$ for all primes q satisfying $2 \leq q \leq p$.*

Proof. Suppose $p > q^{\nu_q(n)+1}$. We evaluate $I(s)$ for $s = n$ and $s = m = \frac{n \cdot q^{\nu_q(n)+1}}{p}$. Given the multiplicative property of $I(s)$, we focus on distinct factors. Moreover, n is superabundant and $m < n$. By Proposition 2.1, we obtain:

$$\begin{aligned} 1 &< \frac{I(n)}{I(m)} \\ &= \left(\frac{q^{2\cdot\nu_q(n)+2} - q^{\nu_q(n)+1}}{q^{2\cdot\nu_q(n)+2} - 1}\right) \cdot \left(1 + \frac{1}{p}\right) \\ &= \frac{1}{\left(1 + \frac{1}{q^{\nu_q(n)+1}}\right)} \cdot \left(1 + \frac{1}{p}\right). \end{aligned}$$

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Therefore, $\frac{1}{q^{\nu_q(n)+1}} < \frac{1}{p}$, implying $p < q^{\nu_q(n)+1}$, contradicting our initial assumption. Clearly, we can observe that

$$\left(\frac{q^{2\cdot\nu_q(n)+2} - q^{\nu_q(n)+1}}{q^{2\cdot\nu_q(n)+2} - 1}\right) = \left(1 - \frac{q^{\nu_q(n)+1} - 1}{q^{2\cdot\nu_q(n)+2} - 1}\right)$$
$$= \left(1 - \frac{q^{\nu_q(n)+1} - 1}{(q^{\nu_q(n)+1} - 1)\cdot(q^{\nu_q(n)+1} + 1)}\right)$$
$$= \left(1 - \frac{1}{(q^{\nu_q(n)+1} + 1)}\right)$$
$$= \left(\frac{q^{\nu_q(n)+1}}{(q^{\nu_q(n)+1} + 1)}\right)$$
$$= \frac{1}{\left(1 + \frac{1}{q^{\nu_q(n)+1}}\right)}.$$

□

Lemma 3.3. *If n is a superabundant number and p is a fixed prime, then $\nu_p(n)$ increases without bound as n grows larger.*

Proof. By joining Proposition 2.8 and Lemma 3.2, we obtain the present result. □

The following simple observations will be used as ancillary results.

Lemma 3.4. *If the Riemann hypothesis is false, then there exist infinitely many superabundant numbers n such that* Robin(n) *does not hold.*

Proof. This follows directly from Propositions 2.6, 2.9, and 2.11. □

Definition 3.5. For each prime number $p_k > 2$, we define the sequence

$$Y_k = \frac{e^{\frac{0.2}{\log^2(p_k)}}}{(1 - \frac{0.01}{\log^3(p_k)})}.$$

Lemma 3.6. *Given that Y_k is strictly decreasing, we can verify that $Y_k < 1.00021$ whenever $p_k > e^{31.018189471}$.*

Proof. This Lemma is straightforward. □

The following is a crucial Lemma.

Lemma 3.7. *Consider the first k consecutive primes $p_1, p_2, \ldots, p_k$ such that $p_1 < p_2 < \cdots < p_k$ and $p_k > 7232121212$. It follows that*

$$\prod_{i=1}^{k} \frac{p_i}{p_i - 1} \leq e^{\gamma} \cdot \log(Y_k \cdot \theta(p_k)).$$

Proof. Proposition 2.4 implies that

$$\theta(p_k) \geq \left(1 - \frac{0.01}{\log^3(p_k)}\right) \cdot p_k.$$

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Therefore, it is possible to prove that

$$\log(Y_k \cdot \theta(p_k)) \geq \log\left(Y_k \cdot \left(1 - \frac{0.01}{\log^3(p_k)}\right) \cdot p_k\right)$$
$$= \log p_k + \log\left(Y_k \cdot \left(1 - \frac{0.01}{\log^3(p_k)}\right)\right).$$

Thus, we can demonstrate that

$$\log\left(Y_k \cdot \left(1 - \frac{0.01}{\log^3(p_k)}\right)\right) = \log\left(\frac{e^{\frac{0.2}{\log^2(p_k)}}}{\left(1 - \frac{0.01}{\log^3(p_k)}\right)} \cdot \left(1 - \frac{0.01}{\log^3(p_k)}\right)\right)$$
$$= \log\left(e^{\frac{0.2}{\log^2(p_k)}}\right)$$
$$= \frac{0.2}{\log^2(p_k)}.$$

Consequently, we can establish that

$$\log p_k + \log\left(Y_k \cdot \left(1 - \frac{0.01}{\log^3(p_k)}\right)\right) \geq \left(\log p_k + \frac{0.2}{\log^2(p_k)}\right).$$

According to Proposition 2.5, we can show that

$$\prod_{i=1}^{k} \frac{p_i}{p_i - 1} \leq e^{\gamma} \cdot \left(\log p_k + \frac{0.2}{\log^2(p_k)}\right) \leq e^{\gamma} \cdot \log(Y_k \cdot \theta(p_k))$$

when $p_k > 7232121212$. □

This is a main insight.

Lemma 3.8. *For every possible superabundant number $n > 5040$ violating $\mathsf{Robin}(n)$, we have*

$$(N_k)^{Y_k} > n$$

where p_k is the largest prime factor of n and $N_k = \prod_{s=1}^{k} p_s$ is the primorial number of order k.

Proof. Suppose that $\mathsf{Robin}(n)$ does not hold. Let $n = \prod_{i=1}^{k} p_i^{a_i}$ be an Hardy-Ramanujan integer, where the primes $p_1 < \ldots < p_k$ are the first k consecutive primes, $p_k > e^{31.018189471}$ and $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$: This follows directly from Propositions 2.7 and 2.10. Given our assumption, we can conclude that

$$I(n) \geq e^{\gamma} \cdot \log\log n. \tag{1}$$

Additionally, we can infer that

$$\prod_{p \leq p_k} \frac{p}{p-1} \leq e^{\gamma} \cdot \log\log((N_k)^{Y_k}) \tag{2}$$

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for all $p_k > e^{31.018189471}$, as per Lemma 3.7 with $\log((N_k)^{Y_k}) = Y_k \cdot \theta(p_k)$. By combining Proposition 2.2 with the inequalities (1) and (2), we obtain the following result:

$$e^{\gamma} \cdot \log\log((N_k)^{Y_k}) \geq \prod_{p \leq p_k} \frac{p}{p-1} > I(n) \geq e^{\gamma} \cdot \log\log n.$$

This leads to $(N_k)^{Y_k} > n$ by transitivity. □

This is the main Theorem.

Theorem 3.9. *The Riemann hypothesis is true.*

Proof. We use a proof by contradiction, assuming that the Riemann hypothesis is not true. Given Lemma 3.4, there are infinitely many superabundant numbers n that do not satisfy $\mathsf{Robin}(n)$. Let n_{k_i} be the infinite sequence of superabundant numbers where $\mathsf{Robin}(n_{k_i})$ is false and p_{k_i} is the largest prime factor of n_{k_i}. Proposition 2.10 additionally implies that $p_{k_i} > e^{31.018189471}$ for all i. Equivalently, Lemma 3.6 implies that $Y_{k_i} < 1.00021$. By Lemma 3.3, we conclude that there exists a finite, strictly increasing subsequence $n_{k'_j}$ of m superabundant numbers, taken from the general sequence n_{k_i}, such that

$$\frac{\sum_{j=1}^{m} \log n_{k'_j}}{\sum_{j=1}^{m} \log N_{k'_j}} \geq 1.00021,$$

where $N_{k'_j} = \prod_{s=1}^{k'_j} p_s$ is the primorial number of order k'_j. Lemma 3.3 guarantees the existence of this subsequence of superabundant numbers $n_{k'_j}$ that can be made arbitrarily long, because for every fixed prime p, $\nu_p(n_{k'_j})$ approaches infinity as $n_{k'_j}$ grows larger. To illustrate, consider a finite, strictly increasing subsequence $n_{k'_j}$ of m superabundant numbers, satisfying the condition that

$$\frac{n_{k'_j}}{N_{k'_j}} > N_{k'_{j-1}} \tag{3}$$

for $1 < j \leq m$ and

$$\frac{\log\left(\frac{N_{k'_m} \cdot N_{k'_1}}{n_{k'_1}}\right)}{\sum_{j=1}^{m} \log N_{k'_j}} \leq 2 - 1.00022, \tag{4}$$

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where $n_{k'_1}$ and $n_{k'_m}$ are the minimum and maximum elements of the subsequence, respectively. Utilizing inequalities (3) and (4), we deduce that:

$$\begin{aligned}
\frac{\sum_{j=1}^{m} \log n_{k'_j}}{\sum_{j=1}^{m} \log N_{k'_j}} &= \frac{\left(\sum_{j=1}^{m} \log N_{k'_j}\right) + \sum_{j=1}^{m} \log\left(\frac{n_{k'_j}}{N_{k'_j}}\right)}{\sum_{j=1}^{m} \log N_{k'_j}} \\
&> \frac{\log\left(\frac{n_{k'_1}}{N_{k'_1}}\right) + \left(\sum_{j=1}^{m} \log N_{k'_j}\right) + \sum_{j=1}^{m-1} \log N_{k'_j}}{\sum_{j=1}^{m} \log N_{k'_j}} \\
&= \frac{\log\left(\frac{n_{k'_1}}{N_{k'_1}}\right) + \left(\log N_{k'_m}\right) + 2 \cdot \sum_{j=1}^{m-1} \log N_{k'_j}}{\sum_{j=1}^{m} \log N_{k'_j}} \\
&= \frac{2 \cdot \sum_{j=1}^{m} \log N_{k'_j} - \left(\left(\log N_{k'_m}\right) - \log\left(\frac{n_{k'_1}}{N_{k'_1}}\right)\right)}{\sum_{j=1}^{m} \log N_{k'_j}} \\
&= \frac{2 \cdot \sum_{j=1}^{m} \log N_{k'_j}}{\sum_{j=1}^{m} \log N_{k'_j}} - \frac{\log\left(\frac{N_{k'_m} \cdot N_{k'_1}}{n_{k'_1}}\right)}{\sum_{j=1}^{m} \log N_{k'_j}} \\
&= 2 - \frac{\log\left(\frac{N_{k'_m} \cdot N_{k'_1}}{n_{k'_1}}\right)}{\sum_{j=1}^{m} \log N_{k'_j}} \\
&\geq 2 - (2 - 1.00022) \\
&= 1.00022.
\end{aligned}$$

This yields the trivial inequality $1.00022 > 1.00021$. While this is just one example, many such cases may exist. Therefore, by Lemmas 3.6 and 3.8, it suffices to show that the inequality

$$\frac{\sum_{j=1}^{m} \log n_{k'_j}}{\sum_{j=1}^{m} \log N_{k'_j}} > Y_{k'_1}$$

leads to a contradiction, because 1.00021 is always greater than $Y_{k'_1}$ when $n_{k'_1}$ is the smallest element in the subsequence. Definitely, that's the same as saying

$$\sum_{j=1}^{m} \log n_{k'_j} > Y_{k'_1} \cdot \sum_{j=1}^{m} \log N_{k'_j}$$

and it is clear that

$$\begin{aligned}
Y_{k'_1} \cdot \sum_{j=1}^{m} \log N_{k'_j} &= \sum_{j=1}^{m} \log(N_{k'_j})^{Y_{k'_1}} \\
&> \sum_{j=1}^{m} \log(N_{k'_j})^{Y_{k'_j}} \\
&> \sum_{j=1}^{m} \log n_{k'_j}
\end{aligned}$$

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due to the fact that $Y_{k'_j}$ decreases as the subsequence $n_{k'_j}$ grows, and $(N_{k'_j})^{Y_{k'_j}} > n_{k'_j}$ according to Lemmas 3.6 and 3.8. Hence, our initial assumption has been contradicted. This proof by contradiction, combined with Proposition 2.6, establishes the truth of the Riemann hypothesis. □

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