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Robin's Criterion on Divisibility (II)

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Abstract: Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \cdot n \cdot \log\log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The possible smallest counterexample $n > 5040$ of the Robin inequality implies that $(N_m)^{Y_m} > n$, $(\log q_m) \cdot \left(1 + \frac{1}{\log^2(q_m)}\right) > \log\log n$ and $\left(1 + \frac{0.2}{\log^3(q_m)}\right) > \frac{\log\log N_m}{\log q_m}$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m, q_m is the largest prime divisor of n and $Y_m = \frac{e^{\frac{0.2}{\log^2(q_m)}}}{\left(1 - \frac{1}{\log^3(q_m)}\right)}$. By combining these results, we present a proof of the Riemann hypothesis. This work is an expansion and refinement of the article "Robin's criterion on divisibility", published in The Ramanujan Journal.

Keywords: riemann hypothesis; robin inequality; sum-of-divisors function; prime numbers; riemann zeta function

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1. Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n:

$$\sum_{d|n} d$$

where $d \mid n$ means the integer d divides n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. We say that $\mathsf{Robin}(n)$ holds provided that

$$f(n) < e^{\gamma} \cdot \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. An upper bound for $f(n)$ can be derived from its multiplicity:

Proposition 1.1. *For $n > 1$ ([4] (2.7) pp. 362):*

$$f(n) < \prod_{q|n} \frac{q}{q-1}.$$

The following inequality is based on natural logarithms:

Proposition 1.2. *For $t > 0$ [5]:*

$$\log\left(1 + \frac{1}{t}\right) < \frac{1}{t}.$$

The Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

where $p \leq x$ means all the prime numbers p that are less than or equal to x. It is known that

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Proposition 1.3. *For $x \geq 7232121212$ ([3] Lemma 2.7 (4) pp. 19):*

$$\theta(x) \geq \left(1 - \frac{0.01}{\log^3(x)}\right) \cdot x.$$

Proposition 1.4. *For $x \geq 2278382$ ([3] [Lemma 2.7 (5) pp. 19]):*

$$\prod_{q \leq x} \frac{q}{q-1} \leq e^{\gamma} \cdot (\log x) \cdot \left(1 + \frac{0.2}{\log^3(x)}\right).$$

Proposition 1.5. *For $x > 1$ ([8] [Corollary 1 (3.30) pp. 70]):*

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \cdot (\log x) \cdot \left(1 + \frac{1}{\log^2(x)}\right).$$

The Ramanujan's Theorem states that if the Riemann hypothesis is true, then $\mathsf{Robin}(n)$ holds for large enough n [6]. Next, we have the Robin's Theorem:

Proposition 1.6. $\mathsf{Robin}(n)$ *holds for all natural numbers $n > 5040$ if and only if the Riemann hypothesis is true [7] [Theorem 1 pp. 188].*

In 1997, Ramanujan's old notes were published where it was defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers [6]. These numbers were also studied by Leonidas Alaoglu and Paul Erdős (1944) [2]. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [4] [pp. 367]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Proposition 1.7. *If n is superabundant, then n is a Hardy-Ramanujan integer [2] [Theorem 1 pp. 450].*

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann hypothesis could be false.

Proposition 1.8. *If $n > 5040$ is the smallest integer such that* $\mathsf{Robin}(n)$ *does not hold, then n must be a superabundant number [1] [Theorem 3 pp. 273].*

Proposition 1.9. *If $n > 5040$ is the smallest integer such that* $\mathsf{Robin}(n)$ *does not hold, then $q < \log n$ where q is the largest prime factor of n [4] [Lemma 6.1 pp. 369].*

Proposition 1.10. *If $n > 5040$ is the smallest integer such that* $\mathsf{Robin}(n)$ *does not hold, then $q > e^{31.018189471}$ where q is the largest prime factor of n [9] [Theorem 4.2 pp. 748].*

By combining these results, we present a proof of the Riemann hypothesis.

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2. Main Result

Definition 2.1. For every prime number $p_n > 2$, we define the sequence

$$Y_n = \frac{e^{\frac{0.2}{\log^2(p_n)}}}{(1 - \frac{0.01}{\log^3(p_n)})}.$$

The following is a key Lemma.

Lemma 2.2. *Let $q_1, q_2, \ldots, q_m$ denote the first m consecutive primes such that $q_1 < q_2 < \ldots < q_m$ and $q_m > 7232121212$. Then*

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \cdot \log(Y_m \cdot \theta(q_m)).$$

Proof. By Proposition 1.3, we know that

$$\theta(q_m) \geq \left(1 - \frac{0.01}{\log^3(q_m)}\right) \cdot q_m.$$

In this way, we can show that

$$\log(Y_m \cdot \theta(q_m)) \geq \log\left(Y_m \cdot \left(1 - \frac{0.01}{\log^3(q_m)}\right) \cdot q_m\right)$$
$$= \log q_m + \log\left(Y_m \cdot \left(1 - \frac{0.01}{\log^3(q_m)}\right)\right).$$

We notice that

$$\log\left(Y_m \cdot \left(1 - \frac{0.01}{\log^3(q_m)}\right)\right) = \log\left(\frac{e^{\frac{0.2}{\log^2(q_m)}}}{\left(1 - \frac{0.01}{\log^3(q_m)}\right)} \cdot \left(1 - \frac{0.01}{\log^3(q_m)}\right)\right)$$
$$= \log\left(e^{\frac{0.2}{\log^2(q_m)}}\right)$$
$$= \frac{0.2}{\log^2(q_m)}.$$

Consequently, we obtain that

$$\log q_m + \log\left(Y_m \cdot \left(1 - \frac{0.01}{\log^3(q_m)}\right)\right) \geq \left(\log q_m + \frac{0.2}{\log^2(q_m)}\right).$$

By Proposition 1.4, we can prove that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \cdot \left(\log q_m + \frac{0.2}{\log^2(q_m)}\right) \leq e^{\gamma} \cdot \log(Y_m \cdot \theta(q_m))$$

when $q_m > 7232121212$. □

This is the main insight.

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Lemma 2.3. *If $n > 5040$ is the smallest integer such that* $\mathsf{Robin}(n)$ *does not hold, then* $(N_m)^{Y_m} > n$, $(\log q_m) \cdot \left(1 + \frac{1}{\log^2(q_m)}\right) > \log\log n$ *and* $\left(1 + \frac{0.2}{\log^3(q_m)}\right) > \frac{\log\log N_m}{\log q_m}$*, where* $N_m = \prod_{i=1}^{m} q_i$ *is the primorial number of order m and* $n = \prod_{i=1}^{m} q_i^{a_i}$.

Proof. By Propositions 1.7 and 1.8, the primes $q_1 < \ldots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \ldots \geq a_m \geq 0$. In addition, we know that $q_m > e^{31.018189471}$ by Proposition 1.10. If $n > 5040$ is the smallest integer such that $\mathsf{Robin}(n)$ does not hold, then we deduce that

$$f(n) \geq e^{\gamma} \cdot \log\log n$$

and

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} > f(n)$$

by Proposition 1.1. In addition, we know that

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \cdot \log\log((N_m)^{Y_m})$$

for all $q_m > e^{31.018189471}$ by Lemma 2.2 since $\log((N_m)^{Y_m}) = Y_m \cdot \theta(q_m)$. As result, we obtain that $(N_m)^{Y_m} > n$ since

$$e^{\gamma} \cdot \log\log((N_m)^{Y_m}) > e^{\gamma} \cdot \log\log n$$

by transitivity. By Proposition 1.1 and 1.5, we can see that

$$\begin{aligned} e^{\gamma} \cdot (\log q_m) \cdot \left(1 + \frac{1}{\log^2(q_m)}\right) &> \prod_{q \leq q_m} \frac{q}{q-1} \\ &> f(n) \\ &\geq e^{\gamma} \cdot \log\log n \end{aligned}$$

under the assumption that $\mathsf{Robin}(n)$ does not hold. This implies that

$$(\log q_m) \cdot \left(1 + \frac{1}{\log^2(q_m)}\right) > \log\log n.$$

We claim that

$$\prod_{q \leq q_m} \frac{q}{q-1} > e^{\gamma} \cdot \log\log N_m \tag{1}$$

under the assumption that $\mathsf{Robin}(n)$ does not hold. Certainly, if we assume that

$$\prod_{q \leq q_m} \frac{q}{q-1} \leq e^{\gamma} \cdot \log\log N_m,$$

then we would have

$$\begin{aligned} e^{\gamma} \cdot \log\log N_m &\geq \prod_{q \leq q_m} \frac{q}{q-1} \\ &> f(n) \\ &\geq e^{\gamma} \cdot \log\log n \end{aligned}$$

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where this implies that $N_m > n$ which is a trivial contradiction according to the Proposition 1.7. By Proposition 1.4, we can infer from (1) the following result:

$$\left(1 + \frac{0.2}{\log^3(q_m)}\right) \geq \frac{\prod_{q \leq q_m} \frac{q}{q-1}}{e^{\gamma} \cdot \log q_m} > \frac{\log \log N_m}{\log q_m}$$

which directly implies that

$$\left(1 + \frac{0.2}{\log^3(q_m)}\right) > \frac{\log \log N_m}{\log q_m}.$$

Therefore, the proof is done. □

This is the main Theorem.

Theorem 2.4. *The Riemann hypothesis is true.*

Proof. We will proceed by contradiction. Assume that $n > 5040$ is the smallest integer such that $\mathsf{Robin}(n)$ does not hold. By Propositions 1.7 and 1.8, the primes $q_1 < \ldots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \ldots \geq a_m \geq 0$. By Proposition 1.10, this also implies that $q_m > e^{31.018189471}$. By Lemma 2.3, we deduce that $(N_m)^{Y_m} > n$ which is the same as

$$\log Y_m > \log \log n - \log \log N_m$$

after of applying the logarithm and distributing the terms. Certainly, we get this inequaltiy following the next steps:

1. First, we obtain $Y_m \cdot \log(N_m) > \log n$ after of applying the logarithm to the both sides.
2. Next, we get $Y_m > \frac{\log n}{\log(N_m)}$ when we distribute the terms.
3. Finally, we arrive at $\log Y_m > \log \log n - \log \log N_m$ if we apply the logarithm to the both sides once again.

That is equivalent to

$$\frac{\log Y_m}{\log q_m} > 1 - \frac{\log \log N_m}{\log \log n}$$

after dividing both sides by $\log \log n$ and under the assumption that $\frac{1}{\log q_m} > \frac{1}{\log \log n}$ since $q_m < \log n$ by Proposition 1.9. By Proposition 1.2, we obtain that

$$\begin{aligned}
\log Y_m &= \frac{0.2}{\log^2(q_m)} + \log\left(\frac{\log^3(q_m)}{\log^3(q_m) - 0.01}\right) \\
&= \frac{0.2}{\log^2(q_m)} + \log\left(1 + \frac{0.01}{\log^3(q_m) - 0.01}\right) \\
&< \frac{0.2}{\log^2(q_m)} + \frac{0.01}{\log^3(q_m) - 0.01}
\end{aligned}$$

for all $q_m > e^{31.018189471}$. So, we would have

$$\frac{\log Y_m}{\log q_m} < \frac{0.2}{\log^3(q_m)} + \frac{0.01}{(\log(q_m)) \cdot (\log^3(q_m) - 0.01)} < \frac{1}{\log^3(q_m)}$$

for all $q_m > e^{31.018189471}$. We arrive at:

$$\log\left(\frac{1}{\log^3(q_m)}\right) > \log\left(1 - \frac{\log \log N_m}{\log \log n}\right)$$

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after of applying the logarithm. That would be

$$-\log\left(\frac{1}{\log^3(q_m)}\right) < -\log\left(1 - \frac{\log\log N_m}{\log\log n}\right)$$

which is

$$\log\log^3(q_m) < \log\left(\frac{\log\log n}{\log\log n - \log\log N_m}\right)$$

and

$$\log^3(q_m) < \frac{\log\log n}{\log\log n - \log\log N_m}$$

after of multiplying both sides by -1 and applying the exponentiation. By Lemma 2.3, we can further deduce that

$$\frac{\log\log n}{\log\log n - \log\log N_m} < \frac{(\log q_m)\cdot\left(1+\frac{1}{\log^2(q_m)}\right)}{(\log q_m)\cdot\left(1+\frac{1}{\log^2(q_m)}\right) - \log\log N_m}$$

where

$$\frac{(\log q_m)\cdot\left(1+\frac{1}{\log^2(q_m)}\right)}{(\log q_m)\cdot\left(1+\frac{1}{\log^2(q_m)}\right) - \log\log N_m} = \frac{\left(1+\frac{1}{\log^2(q_m)}\right)}{\left(1+\frac{1}{\log^2(q_m)}\right) - \frac{\log\log N_m}{\log q_m}}.$$

Furthermore, we can infer that

$$\frac{\left(1+\frac{1}{\log^2(q_m)}\right)}{\left(1+\frac{1}{\log^2(q_m)}\right) - \frac{\log\log N_m}{\log q_m}} < \frac{\left(1+\frac{1}{\log^2(q_m)}\right)}{\left(1+\frac{1}{\log^2(q_m)}\right) - \left(1+\frac{0.2}{\log^3(q_m)}\right)}$$

where

$$\frac{\left(1+\frac{1}{\log^2(q_m)}\right)}{\left(1+\frac{1}{\log^2(q_m)}\right) - \left(1+\frac{0.2}{\log^3(q_m)}\right)} = \frac{\left(1+\frac{1}{\log^2(q_m)}\right)}{\left(\frac{1}{\log^2(q_m)} - \frac{0.2}{\log^3(q_m)}\right)}.$$

Putting all together yields the following inequality:

$$\log^3(q_m) < \frac{\left(1+\frac{1}{\log^2(q_m)}\right)}{\left(\frac{1}{\log^2(q_m)} - \frac{0.2}{\log^3(q_m)}\right)}$$

which is

$$(\log^3(q_m))\cdot\left(\frac{1}{\log^2(q_m)} - \frac{0.2}{\log^3(q_m)}\right) < \left(1+\frac{1}{\log^2(q_m)}\right).$$

Hence, it is enough to show that

$$\log(q_m) - 0.2 < \left(1+\frac{1}{\log^2(q_m)}\right)$$

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does not hold for all $q_m > e^{31.018189471}$ since

$$(\log^3(q_m)) \cdot \left(\frac{1}{\log^2(q_m)} - \frac{0.2}{\log^3(q_m)} \right) = \log(q_m) - 0.2.$$

Thus our original assumption that $\textsf{Robin}(n)$ does not hold has led to a final contradiction. By reductio ad absurdum, we prove that the Riemann hypothesis is true by Proposition 1.6. □

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