Article

Not peer-reviewed version

Spin(8, C)-Higgs Bundles and the Hitchin Integrable System

Álvaro Antón-Sancho *

Posted Date: 24 September 2024

doi: 10.20944/preprints202409.1678.v1

Keywords: outer automorphism; triality; Higgs bundle; fixed point; Hitchin integrable system

Article

Spin$(8, \mathbb{C})$-Higgs Bundles and the Hitchin Integrable System

Álvaro Antón-Sancho

Department of Mathematics and Experimental Science, Fray Luis de Leon University College, Catholic University of Ávila, C/Tirso de Molina, 44, 47010 Valladolid, Spain; alvaro.anton@frayluis.com or alvaro.anton@ucavila.es

Abstract: Let $\mathcal{M}(\mathrm{Spin}(8, \mathbb{C}))$ be the moduli space of Spin$(8, \mathbb{C})$-Higgs bundles over a compact Riemann surface X of genus $g \geq 2$. The triality automorphism of Spin$(8, \mathbb{C})$ acts on $\mathcal{M}(\mathrm{Spin}(8, \mathbb{C}))$ and those Higgs bundles that admit a reduction of structure group to G_2 are fixed points of this action. This defines a map of moduli spaces of Higgs bundles $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8, \mathbb{C}))$. In this work, the action of the triality automorphism is extended to an action on the Hitchin integrable system associated to $\mathcal{M}(\mathrm{Spin}(8, \mathbb{C}))$. In particular, it is checked that the map $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8, \mathbb{C}))$ restricts to a map at the level of Prym varieties. Necessary and sufficient conditions are also provided for the Prym varieties associated with the moduli spaces of G_2 and Spin$(8, \mathbb{C})$-Higgs bundles to be disconnected. Finally, some consequences are drawn from the above results in relation to the geometry of the Prym varieties involved.

Keywords: Outer automorphism; triality; Higgs bundle; fixed point; Hitchin integrable system

MSC: 14H60, 14H10, 57R57

---

1. Introduction

Let X be a compact Riemann surface of genus $g \geq 2$ and G be a semisimple complex Lie group with Lie algebra $\mathfrak{g}$. A G-Higgs bundle over X is defined to be a pair (E, φ) where E is a holomorphic principal G-bundle over X and φ is a holomorphic global section of the adjoint bundle of E, $E(\mathfrak{g})$, twisted by the canonical bundle, K. The section φ is called Higgs field of the Higgs bundle. Suitable notions of stability and polystability can be given for G-Higgs bundles that extend that given by Ramanathan [28,29] for principal G-bundles and in [27] for stable principal G-bundles, obtaining that the moduli space of polystable G-Higgs bundles, $\mathcal{M}(G)$, is a complex algebraic variety of dimension $2 \dim G(g-1)$.

Higgs bundles were introduced by Hitchin in his groundbreaking 1987 paper [23] and posses a remarkable wealth of geometric structures, so they are of interest in many different areas and have been intensively studied. Indeed, G-Higgs bundles provide the framework for the extension of the theorem of Narasimhan and Seshadri [26], whose analogue states that the moduli space of polystable G-Higgs bundles is isomorphic to the moduli space of reductive representations of the fundamental group $\pi_1(X)$ in G [13,14,31–33]. In other directions, G-Higgs bundles are of interest in different areas of mathematics and physics, including gauge theory, mirror symmetry, Langlands duality, or symplectic, Kähler and hyperkähler geometry [8,20,25].

Hitchin proved the existence of an integrable system in the moduli space of polystable G-Higgs bundles over an algebraic curve for any reductive complex Lie group G [22]. A relevant and classical theorem by Chevalley [12] states that, for any complex reductive Lie group G with adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, the algebra of all Ad-invariant polynomials is finitely generated and the degrees $d_1, \ldots, d_r$, where $r = \mathrm{rk}\, G$, of the elements of a basis of homogeneous polynomials are well-defined. Given any principal G-bundle E over X, an Ad-invariant (or G-invariant, or simply invariant) homogeneous polynomial p of degree d defines a map $p : H^0(X, E(\mathfrak{g}) \otimes K) \to H^0(X, K^d)$ by the evaluation of p on the corresponding Higgs field φ (and also a spectral curve S is induced by each G-Higgs bundle (E, φ) by taking the characteristic polynomial of the Higgs field through the adjoint

 doi:10.20944/preprints202409.1678.v1

representation, which is a cover of X whose fibers correspond to the eigenvalues of φ). This can be computed for a basis of invariant polynomials to finally obtain a map

$$\mathcal{M}(G) \to B(G) = H^0(X, K^{d_1}) \oplus \cdots \oplus H^0(X, K^{d_r}).$$

This map, called *Hitchin map*, is ideed proper. The space $B(G)$ is an affine space called the *base of the Hitchin map*. For each stable and simple principal G-bundle E (i.e. a stable principal G-bundle whose only automorphisms are those induced by the action of the center of the structure group G, which is a smooth point in the moduli space of principal G-bundles), the space $H^0(X, E(\mathfrak{g}) \otimes K)$ is isomorphic, by Serre duality, to $H^1(X, E(\mathfrak{g}))^*$, the cotangent bundle to the moduli space of stable and simple principal G-bundles at E. This cotangent bundle is naturally embedded in the moduli space $\mathcal{M}(G)$ of polystable G-Higgs bundles. Hitchin defined a global symplectic structure in $\mathcal{M}(G)$ that extends the natural symplectic structure of that cotangent space in a way that the Hitchin map defines an integrable system. The Hitchin fibration has proven to be an essential tool for understanding the Geometric Langlands program, as Kapustin and Witten [25] pointed out.

A fruitful way of studying the geometry of the moduli spaces $\mathcal{M}(G)$ of G-Higgs bundles is by describing the subvarieties and maps between these moduli spaces [2,17,19]. Specifically, given an automorphism of $\mathcal{M}(G)$, a subvariety is naturally defined by taking the of fixed points of that automorphism. It is then useful to study automorphisms of finite order of $\mathcal{M}(G)$. The case of involutions of the moduli space of $\mathrm{SL}(n,\mathbb{C})$-Higgs bundles was developed by García-Prada in [17], where he related Higgs bundles with representations of the fundamental group of the surface in the real forms of the group. A larger family of finite-order automorphisms is studied in [4], but in the context of orthogonal bundles over a curve. Also, the case of involutions of $\mathcal{G}$ induced by the action of outer automorphisms of order 2 of G have been studied in [19] and, with different techniques for simple classical complex Lie groups in [5].

This paper is interested in Higgs bundles whose structure group is $\mathrm{Spin}(8,\mathbb{C})$. This group is the only simple complex Lie group that admits an outer automorphism of order 3, called triality automorphism. This unique fact makes the geometric structures related to this group (including $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundles) have both interesting and very specific geometric features, which usually require specific studies [3,24,34]. In particular, in the previous literature it has been proved that the triality automorphism acts on the moduli space $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ and its fixed points can be described as reductions of structure group to the subgroups G_2 or $\mathrm{PSL}(3,\mathbb{C})$ of $\mathrm{Spin}(8,\mathbb{C})$. This leads the existence of two maps of algebraic varieties $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ and $\mathcal{M}(\mathrm{PSL}(3,\mathbb{C}) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$. In this work, the study of the first map $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ is deepened. The way to carry out this deepening is through the study of the Hitchin integrable system associated to the two moduli spaces involved. Specifically, if B is the basis of the Hitchin map of $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ and B' is the basis of the Hitchin map of $\mathcal{M}(G_2)$, it is proved that the triality automorphism acts on B, that there exists a homomorphism $j : B \to B'$ compatible with the map between moduli spaces, and that the image of j is formed by fixed points of the action of the triality automorphism on B (Lemma 4.2). This allows to state that the map between moduli spaces is restricted to maps between Prym varieties $\mathrm{Prym}(X_a) \to \mathrm{Prym}(X_{j(a)})$, where $a \in B'$ and X_a and $X_{j(a)}$ are the associated spectral curves. Following this, in this paper the geometry of the Prym varieties involved is studied to the extent of providing necessary and sufficient conditions for these Prym varieties to be disjoint (Proposition 5.4). As a consequence of the above results, sufficient conditions are provided for the two Prym varieties involved to be disjoint. Thus, the paper provides innovative results on the geometry of the map $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ and also provides novel techniques for the mentioned geometryc study (consisting specifically in studying the Hitchin integrable system). In particular, it is intended to provide tools to advance the knowledge about the map $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ and to be able in the future to prove properties such as, for example, whether it is injective, in the spirit of Serman [30].

In addition, the map $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ has been considered because the preceding literature provides sufficient results on the Hitchin integrable system of G_2 [24] and on the relationship

 doi:10.20944/preprints202409.1678.v1

between G_2-invariant polynomials and $\mathrm{Spin}(8,\mathbb{C})$-invariant polynomials [9] to be able to carry out the analysis intended here. Indeed, Hitchin [24] deepened the study of the integralble system of the moduli space of G_2-Higgs bundles, because the special characteristics of the group G_2 make it of great interest in differential equations [10], or geometry and physics [2,11]. Thus, Hitchin [24] described the spectral curves S associated to the Hitchin fibration for the group G_2 and the associated Prym varieties. In particular, he proved the existence of an intermediate curve C such that the covering $S \to X$ factors through C, and an involution σ of S so that $\mathrm{Prym}(S,X)$ is given by those $L \in \mathrm{Prym}(S,C)$ satisfying $L^* \cong \sigma(L)$. He also proved that the G_2-Higgs bundle can be reconstructed form the associated spectral curve. It has also been proved that every G_2-invariant polynomial is also a $\mathrm{Spin}(8,\mathbb{C})$-invariant polynomial fixed by the action of the triality automorphism [9].

The article is organized in the following way. In Section 2 some foundations on the geometry of the Lie group $\mathrm{Spin}(8,\mathbb{C})$ and the triality automorphism are recalled. The action of the group $\mathrm{Out}(G)$ of outer automorphisms of any semisimple complex Lie group G on the moduli space $\mathcal{M}(G)$ of G-Higgs bundles over X, introduced in [2], is described and studied in Section 3. It is also explained in a more detailed way the particular case of $G = \mathrm{Spin}(8,\mathbb{C})$ and the specific characteristics of the triality automorphism. In Section 4, the action of the triality automorphism on the base and on the fibers of the Hitchin integrable system is constructed. Section 5 is devoted to providing the main geometric features on the Prym varieties coming from the Hitchin integrable system associated to $\mathrm{Spin}(8,\mathbb{C})$. Finally, the main conclusions of the paper are drawn.

2. The Group $\mathrm{Spin}(8,\mathbb{C})$ and the Triality Automorphism

In this section, some basics on the Lie group $\mathrm{Spin}(8,\mathbb{C})$, its subgroups, and the triality automorphism are provided. Suitable references for this topic are [1–3,16]. The group $G = \mathrm{Spin}(8,\mathbb{C})$ is the only simple and simply connected complex Lie group of type D_4. Its Lie algebra is $\mathfrak{so}(8,\mathbb{C})$, its center is isomorphic to $Z = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. It is indeed the double cover of the special orthogonal Lie group $\mathrm{SO}(8,\mathbb{C})$ and then it can be described as an extension of $\mathrm{SO}(8,\mathbb{C})$ by $\mathbb{Z}_2$:

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(8,\mathbb{C}) \to \mathrm{SO}(8,\mathbb{C}) \to 1. \tag{1}$$

Of course, the action of the group $\mathrm{Aut}(\mathrm{Spin}(8,\mathbb{C}))$ of automorphisms of $\mathrm{Spin}(8,\mathbb{C})$ leaves the center $Z = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ invariant, hence there is a homomorphism of $\mathrm{Aut}(\mathrm{Spin}(8,\mathbb{C}))$ into the group $S(Z^2)$ of permutations of the set $Z^2 = Z \setminus \{1\}$ of central elements of order 2. The subgroup $\mathrm{Inn}(\mathrm{Spin}(8,\mathbb{C}))$ of inner automorphisms clearly acts trivially on Z, thus this induces a homomorphism of the group $\mathrm{Out}(\mathrm{Spin}(8,\mathbb{C}))$ of outer automorphisms of $\mathrm{Spin}(8,\mathbb{C})$ into $S(Z^2) \cong S_3$, which is actually an isomorphism. Recall that $\mathrm{Out}(\mathrm{Spin}(8,\mathbb{C}))$ is the quotient of the group $\mathrm{Aut}(\mathrm{Spin}(8,\mathbb{C}))$ of automorphisms of $\mathrm{Spin}(8,\mathbb{C})$ by the normal subgroup $\mathrm{Inn}(\mathrm{Spin}(8,\mathbb{C}))$ of inner automorphisms. The triality automorphism is then a choice τ of an order 3 outer automorphism (the other outer automorphism of order 3 is τ^{-1}).

Since $\mathrm{Spin}(8,\mathbb{C})$ is the simply connected complex Lie group with Lie algebra $\mathfrak{so}(8,\mathbb{C})$, there is an isomorphism of short exact sequences of the form

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Inn}(\mathrm{Spin}(8,\mathbb{C})) & \longrightarrow & \mathrm{Aut}(\mathrm{Spin}(8,\mathbb{C})) & \longrightarrow & \mathrm{Out}(\mathrm{Spin}(8,\mathbb{C})) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \mathrm{Inn}(\mathfrak{so}(8,\mathbb{C})) & \longrightarrow & \mathrm{Aut}(\mathfrak{so}(8,\mathbb{C})) & \longrightarrow & \mathrm{Out}(\mathfrak{so}(8,\mathbb{C})) & \longrightarrow & 1.
\end{array}$$

Given any complex Lie algebra $\mathfrak{g}$, if $\alpha, \beta \in \mathrm{Aut}(\mathfrak{g})$, it is stated that $\alpha \sim_i \beta$ if there exists $\theta \in \mathrm{Inn}(\mathfrak{g})$ such that $\alpha = \theta \circ \beta \circ \theta^{-1}$. Thus defined, $\sim_i$ is an equivalence relation such that the obvious map

$$\mathrm{Aut}(\mathfrak{g})/\sim_i \to \mathrm{Out}(\mathfrak{g}) \tag{2}$$

is well-defined [1,3].

Notice that the order of an automorphism of $\mathfrak{g}$ clearly coincides with the order of its class modulo $\sim_i$. Then, if $\mathrm{Aut}_3(\mathfrak{so}(8,\mathbb{C}))$ denotes the subset of automorphisms of order 3 of $\mathfrak{so}(8,\mathbb{C})$ and an analogous definition is given for $(\mathrm{Aut}(\mathfrak{so}(8,\mathbb{C}))/\sim_i)_3$ and for $\mathrm{Out}_3(\mathfrak{so}(8,\mathbb{C}))$, it is satisfied that

$$\mathrm{Aut}_3(\mathfrak{so}(8,\mathbb{C}))/\sim_i = (\mathrm{Aut}(\mathfrak{so}(8,\mathbb{C}))/\sim_i)_3.$$

It is also clear that the automorphisms of order 3 are sent to elements of $\mathrm{Out}(\mathfrak{so}(8,\mathbb{C}))$ of order 3 or to the identity through the map defined in (2). This implies that $\mathrm{Aut}_3(\mathfrak{g})/\sim_i$ is sent onto $\mathrm{Out}_3(\mathfrak{g}) \cup \{1\}$ through the natural map, that is,

$$\mathrm{Aut}_3(\mathfrak{g})/\sim_i \to \mathrm{Out}_3(\mathfrak{g}) \cup \{1\}. \tag{3}$$

There are exactly two pre-images of the triality automorphism τ by the map defined in (3) [1,3]. Then there are two possibilities for the subalgebra of fixed points of an automorphism of order 3 of $\mathfrak{so}(8,\mathbb{C})$ representing τ. Wolf and Gray [35, Theorem 5.5] proved that these two different representatives of τ by the map (3) have $\mathfrak{g}_2$ and $\mathfrak{sl}(3,\mathbb{C})$ as subalgebras of fixed points (with simply connected subgroups G_2 and $\mathrm{PSL}(3,\mathbb{C})$), respectively.

3. The Action of the Triality Automorphism on the Moduli Space of $\mathrm{Spin}(8,\mathbb{C})$-Higgs Bundles

Let X be a compact Riemann surface of genus $g \geq 2$. A principal $\mathrm{SO}(8,\mathbb{C})$-bundle over X is a complex rank-8 and trivial determinant vector bundle equipped with a globally-defined holomorphic non-degenerate symmetric bilinear form. The set of isomorphism classes of principal $\mathrm{SO}(8,\mathbb{C})$-bundles is parametrized by the cohomology set $H^1(X,\mathrm{SO}(8,\mathbb{C}))$. A map $\omega_2 : H^1(X,\mathrm{SO}(8,\mathbb{C})) \to H^2(X,\mathbb{Z}_2) \cong \mathbb{Z}_2$ is defined which assigns to each principal $\mathrm{SO}(8,\mathbb{C})$-bundle E its second Stiefel-Whitney class $\omega_2(E)$. The bundle E lifts to a principal $\mathrm{Spin}(8,\mathbb{C})$-bundle over X if and only if $\omega_2(E) = 0$, two of such lifts differing in a line bundle of order 2. However, every principal $\mathrm{Spin}(8,\mathbb{C})$-bundle admits an associated $\mathrm{SO}(8,\mathbb{C})$-bundle through the covering map $\mathrm{Spin}(8,\mathbb{C}) \to \mathrm{SO}(8,\mathbb{C})$ defined in (1).

In the next definitions, the notions of Higgs bundles are specified for the structure groups $\mathrm{SO}(8,\mathbb{C})$ and $\mathrm{Spin}(8,\mathbb{C})$ following the original notion of G-Higgs bundle introduced by Hitchin [23].

Definition 3.1. *An* $\mathrm{SO}(8,\mathbb{C})$*-Higgs bundle over X is a pair (E,φ) where E is a principal* $\mathrm{SO}(8,\mathbb{C})$*-bundle over X with associated bilinear form q and $\varphi : E \to E \otimes K$ is a complex vector bundle homomorphism preserving the bilinear form q, where K denotes the canonical bundle over X.*

Definition 3.2. *A* $\mathrm{Spin}(8,\mathbb{C})$*-Higgs bundle over X is a pair (E,φ) where E is a principal* $\mathrm{Spin}(8,\mathbb{C})$*-bundle over X and $\varphi \in H^0(X, E(\mathfrak{so}(8,\mathbb{C})) \otimes K)$. Here, K is the canonical bundle over X, $E(\mathfrak{so}(8,\mathbb{C}))$ is the adjoint vector bundle of E and $\varphi : E \to E \otimes K$ is a vector bundle morphism preserving the bilinear form with which the special orthogonal bundle associated to E is equipped.*

For both principal $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundles and $\mathrm{SO}(8,\mathbb{C})$-Higgs bundles, the element φ is called Higgs field and it can be understood on each fiber of the orthogonal bundle E as an 8-dimensional complex matrix of the Lie algebra $\mathfrak{so}(8,\mathbb{C})$.

There are suitable notions of stability and polystability that allows to construct the moduli space of G-Higgs bundles for any complex reductive Lie group G [18], which will be denoted by $\mathcal{M}(G)$. These notions extend to Higgs pairs the notions given by Ramanathan [28,29] for principal bundles. The moduli space of $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundles over X is then the algebraic variety which parameterizes isomorphism classes of polystable $\mathrm{Spin}(8,\mathbb{C})$-bundles over X.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 September 2024 doi:10.20944/preprints202409.1678.v1

Given any $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundle (E,φ), an automorphism of (E,φ) is an automorphism $f: E \to E$ of the principal $\mathrm{Spin}(8,\mathbb{C})$-bundle E such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[d, "f"'] & E \otimes K \arrow[d, "f\otimes 1_K"] \\ E \arrow[r, "\varphi"] & E \otimes K, \end{tikzcd}$$

that is, $(f \otimes 1_K) \circ \varphi = \varphi \circ f$.

Consider now any semisimple complex Lie group G with Lie algebra $\mathfrak{g}$. In [2] it is proved that the following defines an action of the group $\mathrm{Out}(G)$ on the moduli space $\mathcal{M}(G)$ of G-Higgs bundles: if $\rho \in \mathrm{Out}(\mathrm{Spin}(8,\mathbb{C}))$ and $(E,\varphi) \in \mathcal{M}(G)$, then $\rho \cdot (E,\varphi)$ is defined to be the G-Higgs bundle

$$\rho \cdot (E,\varphi) = (A(E), dA(\varphi)), \tag{4}$$

where $A \in \mathrm{Aut}(G)$ is an automorphism of G representing ρ and $A(E)$ is the principal G-bundle whose total space is that of E but it is equipped with the right action of G given by $e \diamond g = eA^{-1}(g)$, for $e \in E$ and $g \in G$. It can be proved that this action preserves the stability and polystability of the G-Higgs bundles and that it does not depend on the choice of the representative A of ρ, since inner automorphisms act trivially on G-Higgs bundles [2].

Let $\tau \in \mathrm{Out}(\mathrm{Spin}(8,\mathbb{C}))$ be the triality automorphism and let (E,φ) be a $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundle over X fixed by the action of τ. If $A \in \mathrm{Aut}(\mathrm{Spin}(8,\mathbb{C}))$ is an automorphism of $\mathrm{Spin}(8,\mathbb{C})$ representing τ, then $(E,\varphi) \cong (A(E), dA(\varphi))$. Recall that there are only two possibilities for the group $\mathrm{Fix}(A)$ of fixed points of A depending on the lifting of the triality automorphism by the relation $\sim_i$. These two possibilities are G_2 or $\mathrm{PSL}(3,\mathbb{C})$ [35, Theorem 5.5]. In [2] it is proved that the fixed points of the action of the triality automorphism on the moduli space of $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundles are those which admit a reduction of structure group to G_2 or $\mathrm{PSL}(3,\mathbb{C})$. Then there are maps $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ and $\mathcal{M}(\mathrm{PSL}(3,\mathbb{C})) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ such that their images complete the subvariety of fixed points of the action of the triality automorphism. In this work, the study of map $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ is deepened, taking advantage of the existence of results relating the invariant polynomials of G_2 and $\mathrm{Spin}(8,\mathbb{C})$, which will be key in the study.

4. The Triality Automorphism and the Hitchin Integrable System

Let G be a semisimple complex Lie group with Lie algebra $\mathfrak{g}$ and $p_1, \dots, p_r$ be a basis of the ring of invariant homogeneous polynomials of $\mathfrak{g}$, where r is the rank of $\mathfrak{g}$. The action of each polynomial on the Higgs field defines a map

$$(p_1, \dots, p_r) : H^0(X, E(\mathfrak{g}) \otimes K) \to \bigoplus_{i=1}^{r} H^0(X, K^{d_i}),$$

where $d_i = \deg p_i$ for $i = 1, \dots, r$. This map allows to define the so-called *Hitchin map*,

$$\mathcal{H} : \mathcal{M}(G) \to \bigoplus_{i=1}^{r} H^0(X, K^{d_i}), \tag{5}$$

where $\mathcal{M}(G)$ is the moduli space of G-Higgs bundles. The vector space

$$B = \bigoplus_{i=1}^{r} H^0(X, K^{d_i})$$

is called the *base of the Hitchin map*. In [22] it is proved that $\dim \mathcal{M}(G) = \dim B$. If n is the dimension of $\mathcal{M}(G)$, the Hitchin map induces n complex valued functions $f_1, \dots, f_n$ defined on $T^*M_*(G)$, where by

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 September 2024 doi:10.20944/preprints202409.1678.v1

$M_*(G)$ we denote the moduli space of stable and simple principal G-bundles, which is a dense open subset of the moduli space $M(G)$ of principal G-bundles. The tangent space to $M_*(G)$ at an element $E \in M_*(G)$ is isomorphic to $H^1(X, E(\mathfrak{g}))$, which coincides with $H^0(X, E(\mathfrak{g}) \otimes K)^*$ by Serre duality. Hitchin also proved that the n functions f_i Poisson-commute with the canonical symplectic structure of the cotangent bundle [22, Proposition 4.5]. This then defines a completely integrable system on $\mathcal{M}(G)$ [6].

As will be proved below, the group $\mathrm{Out}(G)$ of outer automorphisms of G acts on the base of the Hitchin map so that this action is compatible qith the action of $\mathrm{Out}(G)$ on $\mathcal{M}(G)$ when G is simply connected (which is the case of $\mathrm{Spin}(8, \mathbb{C})$).

Lemma 4.1. *Let G be a semisimple and simply connected complex Lie group. Let f be an automorphism of the Lie algebra $\mathfrak{g}$ of G and p be an invariant homogeneous polynomial of G. Then $p \circ f$ is also an invariant homogeneous polynomial of G.*

Proof. Since f is a linear map, it is clear that $p \circ f$ is a homogeneous polynomial. Therefore, it suffices to show that it is invariant.

Let $F \in \mathrm{Aut}(G)$ be an automorphism of G such that $dF = f$ (such automorphism exists because G is simply connected) and let $g \in G$. Then for any $x \in \mathfrak{g}$,

$$\begin{aligned} p \circ f(\mathrm{Ad}(g)(x)) &= p \circ dF \circ di_g(x) = p \circ d(F \circ i_g)(x) \\ &= p\Big(di_{F(g)}(f(x))\Big) = p \circ f(x) \end{aligned}$$

(where $i_g : G \to G$ is defined as the inner automorphism induced by g, $i_g(h) = ghg^{-1}$), so $p \circ f$ is invariant, as desired. $\square$

From this, the announced action of $\mathrm{Out}(G)$ on the base B of the Hitchin map can be defined for a semisimple and simply-connected complex Lie group G. Hitchin [22] proved that the Hitchin map $\mathcal{H} : \mathcal{M}(G) \to B$ is surjective. Then, for each $\alpha \in B$, there exists $(E, \varphi) \in \mathcal{M}(G)$ such that $\alpha = (p_1(\varphi), \ldots, p_r(\varphi))$. If $\rho \in \mathrm{Out}(G)$, it is defined

$$\rho \cdot (p_1(\varphi), \ldots, p_r(\varphi)) = (p_1 \circ f_\rho(\varphi), \ldots, p_r \circ f_\rho(\varphi)), \tag{6}$$

where f_ρ is an automorphism of G representing ρ.

Lemma 4.2. *If G is a semisimple and simply connected complex Lie group, then the action of* $\mathrm{Out}(G)$ *on the base of the Hitchin map defined in* (6) *is well-defined and it is compatible with the action of* $\mathrm{Out}(G)$ *on* $\mathcal{M}(G)$ *defined in* (4).

Proof. To check that the action given in (6) is well-defined, notice the following:

- It is clear from Lemma 4.1 that $p_1 \circ f_\rho, \ldots, p_r \circ f_\rho$ are invariant homogeneous polynomials of G.
- If f is an inner automorphism of G, then, since $p_1, \ldots, p_r$ are invariant under this kind of automorphisms, the action is trivial, so the above action descends to an action of $\mathrm{Out}(G)$.
- If (E, φ) and (E', ψ) are polystable G-Higgs bundles over X such that

$$(p_1(\psi), \ldots, p_r(\psi)) = (p_1(\varphi), \ldots, p_r(\varphi)),$$

  then, since $p_1, \ldots, p_r$ is a basis of invariant polynomials of G, $p(\psi) = p(\varphi)$ for every invariant polynomial p of G, so, in particular,

$$(p_1 \circ \tau(\psi), \ldots, p_r \circ \tau(\psi)) = (p_1 \circ \tau(\varphi), \ldots, p_r \circ \tau(\varphi)).$$

The above three points lead to the conclusion that the action under consideration is well-defined.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 September 2024 doi:10.20944/preprints202409.1678.v1

Let now $\rho \in \mathrm{Out}(G)$. To check that the action of $\mathrm{Out}(G)$ on the base of the Hitchin map is compatible with the action on $\mathcal{M}(G)$ it suffices to show that

$$\rho \cdot \mathcal{H}((E,\varphi)) = \mathcal{H}((\rho(E),\rho(\varphi)))$$

for all $(E,\varphi) \in \mathcal{M}(G)$. Notice that

$$\begin{aligned}
\rho \cdot \mathcal{H}((E,\varphi)) &= \rho \cdot (p_1(\varphi),\dots,p_r(\varphi)) \\
&= (p_1 \circ \rho(\varphi),\dots,p_r \circ \rho(\varphi)) \\
&= (p_1(\rho(\varphi)),\dots,p_r(\rho(\varphi))) \\
&= \mathcal{H}((\rho(E),\rho(\varphi))),
\end{aligned}$$

as it was intended to prove. □

Remark. Since, by Lemma 4.2 the action of an outer automorphism ρ on the base of the Hitchin map does not depend on the representative of ρ chosen in $\mathrm{Aut}(G)$, if $\alpha = (p_1(\varphi),\dots,p_r(\varphi))$ is an element if the base of the Hitchin map for some G-Higgs bundle (E,φ), it can be denoted

$$\rho \cdot (p_1(\varphi),\dots,p_r(\varphi)) = (p_1 \circ \rho(\varphi),\dots,p_r \circ \rho(\varphi)).$$

The above study will now be particularized to the case of $\mathrm{Spin}(8,\mathbb{C})$ and G_2-Higgs bundles. The algebra of invariant polynomials of $\mathrm{Spin}(8,\mathbb{C})$ is generated by four invariant homogeneous polynomials

$$A = \langle p_2, p_4, p_6, p_4' \rangle,$$

where $\deg p_i = i$ for each i and p_4 is the Pfaffian, so $\deg p_4' = 4$ [22]. Indeed,

$$\begin{aligned}
p_2(\varphi) &= \frac{1}{2}\,\mathrm{Tr}(\varphi^2), \\
p_4(\varphi) &= \frac{1}{4}\,\mathrm{Tr}(\varphi^2)^2 + \frac{1}{8}\,\mathrm{Tr}(\varphi^4), \\
p_6(\varphi) &= \frac{1}{48}\,\mathrm{Tr}(\varphi^2)^3 - 6\,\mathrm{Tr}(\varphi^2)\,\mathrm{Tr}(\varphi^4) + 8\,\mathrm{Tr}(\varphi^6).
\end{aligned}$$

Then the base of the Hitchin map is

$$B = H^0(X,K^2) \oplus H^0(X,K^4) \oplus H^0(X,K^6) \oplus H^0(X,K^4). \tag{7}$$

Given a quadruple $a = (a_2,a_4,a_6,b_4) \in B$, it induces the polynomial

$$b_4^2 + a_6 t^2 + a_4 t^4 + a_2 t^6 + t^8 \tag{8}$$

in the sense that the characteristic polynomial of the Higgs field of any (E,φ) in the fiber of the Hitchin map at a is $\det(tI-\varphi) = b_4^2(\varphi) + a_6(\varphi)t^2 + a_4(\varphi)t^4 + a_2(\varphi)t^6 + t^8$, where each a_i or b_i is a holomorphic global section of K^i over X and $a = (a_2,a_4,a_6,b_4)$ defines the Hitchin map $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C})) \to B$.

The mentioned characteristic polynomial defines the spectral curve in the total space of the canonical bundle $\pi : K \to X$. Then, given any $a \in B$, it defines the equation of the corresponding spectral curve, which will be called X_a as the divisor of a section of $\pi^* K^8$ defined by the induced polynomial where x is the tautological section of $\pi^* K$ and it is a single-valued eigenvalue of the Higgs field φ.

Consider now the Lie group G_2, seen as a subgroup of $\mathrm{Spin}(8,\mathbb{C})$. The algebra of invariant polynomials of G_2 is generated by two polynomials, q_2, q_6, of degrees 2 and 6, respectively

$$A' = \langle q_2, q_6 \rangle.$$

Then the base of the Hitchin map of the moduli space of G_2-Higgs bundles is

$$B' = H^0(X, K^2) \oplus H^0(X, K^6), \tag{9}$$

and, given a pair $(a_2, a_6) \in B'$, the induced invariant polynomial is $a_6 t^2 + \frac{a_2^2}{4} t^4 + a_2 t^6 + t^8$, so by taking a common factor t^2 it follows that the polynomial

$$a_6 + \frac{a_2^2}{4} t^2 + a_2 t^4 + t^6 \tag{10}$$

is also G_2-invariant.

Some facts on spectral curves and Prym varieties associated to the moduli spaces $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ and $\mathcal{M}(G_2)$ will be now recalled. Given a $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundle (E,φ) over X, the characteristic polynomial of the Higgs field φ defines an element $a \in B$, so a spectral cover $X_{(E,\varphi)} = X_a$ and an 8-sheeted covering map $\pi_{(E,\varphi)} : X_{(E,\varphi)} \to X$ are defined, whose fibers are identified with the eigenvalues of φ. A detailed construction of this curve can be found in [7,15,24]. There exists an exact sequence on the spectral curve $X_{(E,\varphi)}$

$$0 \to L \otimes \pi^* K^{-7} \to \pi^* E \overset{x-\varphi}{\to} \pi^*(E \otimes K) \to L \otimes \pi^* K \to 0, \tag{11}$$

where E denotes both the principal $\mathrm{Spin}(8,\mathbb{C})$-bundle and the orthogonal bundle that it induces ([7,24]). By dualizing the sequence (11) and tensoring with $\pi^* K$ the following is obtained:

$$0 \to L^* \to \pi^* E^* \to \pi^*(E^* \otimes K) \to L^* \otimes \pi^* K^8 \to 0. \tag{12}$$

Here, L is a line bundle which satisfies that $E = \pi_* L$ (as a vector bundle). It is constructed as the coker of $\pi^*\varphi - x$, where x denotes the tautological global section of $\pi^* K$, and $X_{(E,\varphi)}$ can be identified with the support of L. The covering map $\pi_{(E,\varphi)} : X_{(E,\varphi)} \to X$ gives a norm map Nm defined by $\mathrm{Nm}(\sum_i a_i \cdot x_i) = \sum_i a_i \pi_{(E,\varphi)}(x_i)$ on divisor classes. The norm defines a map at the level of Jacobian varieties $\mathrm{Nm}_J : J(X_{(E,\varphi)}) \to J(X)$. Then the Prym variety of the spectral curve $\pi_{(E,\varphi)} : X_{(E,\varphi)} \to X$ is defined to be the connected component of the kernel of Nm_J. This Prym variety is then an abelian subvariety of the jacobian of the sectral curve $X_{(E,\varphi)}$. Observe that, since φ takes values in $\mathfrak{so}(8,\mathbb{C})$, the opposite of an eigenvalue of φ is also an eigenvalue of φ, so an involution σ of $X_{(E,\varphi)}$ is defined by change of sign and the eigenspace V of eigenvalue λ of φ is moved to $\sigma^* V$ for eigenvalue $-\lambda$. Then $L^* \cong L \otimes \pi^* K^{-7}$ by (11) and (12), so $L^2 \cong \pi^* K^7$. This means that $M = L \otimes \pi^* K^{-3/2}$ satisfies that $M \otimes \sigma^* M$ is trivial (the choice of a square root of K is required here). Then the Prym variety is the collection of those elements M of the Jacobian of $X_{(E,\varphi)}$ such that $M \otimes \sigma^* M$ is trivial. This is equivalent to stating that the desired Prym variety coincides with $\mathrm{Prym}(X_{(E,\varphi)}, X_{(E,\varphi)}/\sigma)$, the last Prym variety being defined from the covering map $X_{(E,\varphi)} \to X_{(E,\varphi)}/\sigma$. Notice that, given any $M \in \mathrm{Prym}(X_{(E,\varphi)}, X)$, one has that $E \cong \pi_* L$, where $L = \pi^* K^{3/2} \otimes M$ and, since $L^2 \cong \pi^* K^7$, then $E \cong E^*$ as a consequence of (11) and (12), so the special orthogonal bundle is obtained from the element of the Prym variety. Then the fiber $\mathcal{H}^{-1}(\mathcal{H}(E,\varphi))$ is a 2^{2g}-sheeted covering of the Prym variety of $X_{(E,\varphi)} \to X$ (details on this construction can be found in [24]).

Consider now the Lie group G_2. The fundamental complex representation of G_2 has rank 7 and defines, in addition, an inclusion $G_2 \hookrightarrow \mathrm{SO}(7,\mathbb{C})$. A holomorphic antisymmetric 3-form can be defined in $\mathbb{C}^7$ in a way that G_2 is the group of elements of $\mathrm{SL}(7,\mathbb{C})$ which preserves this 3-form [11]. For any G_2-Higgs bundle (E,φ) over X, Hitchin [24] considered the spectral curve $X_{(E,\varphi)}$ assocated to it and constructed an intermediate curve C such that the covering map $\pi_{(E,\varphi)}$ admits a factorization

$$X_{(E,\varphi)} \overset{p}{\to} C \overset{p_C}{\to} X$$

and proved the existence of an involution σ of $X_{(E,\varphi)}$ such that $p \circ \sigma = p$ and $\mathrm{Prym}(X_{(E,\varphi)}, X)$ is the subspace of those $L \in \mathrm{Prym}(X_{(E,\varphi)}, C)$ for which $L \otimes \sigma^* L$ is trivial. From this description, Hitchin [24] proved that the globally-defined holomorphic antisymmetric 2-form defined in E by its G_2-structure can be reconstructed from the corresponding point of the Prym variety $\mathrm{Prym}(X_{(E,\varphi)}, X)$. In this way, it was proved that the fiber of the Hitchin map of $\mathcal{M}(G_2)$ is isomorphic to $\mathrm{Prym}(X_{(E,\varphi)}, X)$ [24].

Notice that if (E, φ) is a G_2-Higgs bundle, then, seen as a $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundle through the contention of groups $G_2 \hookrightarrow \mathrm{Spin}(8,\mathbb{C})$, it satisfies that the Pfaffian $b_4(\varphi)$ is 0. Then, if a is the G_2-invariant polynomial $a_6 + \frac{a_2^2}{4}t^2 + a_2 t^4 + t^6$ and X_a is the associated spectral curve, b is the $\mathrm{Spin}(8,\mathbb{C})$-invariant polynomial and X_b is the associated spectral curve, the map $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C})$ restricts to a map

$$\mathrm{Prym}(X_a) \to \mathrm{Prym}(X_b). \tag{13}$$

Proposition 4.1. *Let B and B' be the base of the Hitchin maps of* $\mathrm{Spin}(8,\mathbb{C})$ *and G_2 defined in* (7) *and* (9), *respectively. Then there exists a map $j : B' \to B$ such that the diagram*

$$\begin{array}{ccc} \mathcal{M}(G_2) & \xrightarrow{F} & \mathcal{M}(\mathrm{Spin}(8,\mathbb{C})) \\ {\scriptstyle \mathcal{H}^{G_2}}\downarrow & & \downarrow{\scriptstyle \mathcal{H}^{\mathrm{Spin}(8,\mathbb{C})}} \\ B' & \xrightarrow{j} & B \end{array}$$

is commutative and $j(B')$ is given by fixed points for the action of the triality automorphism of $\mathrm{Spin}(8,\mathbb{C})$ *on B. Moreover, the generating homogeneous G_2-invariant polynomials q_2 and q_6 can be described as*

$$q_2 = \frac{1}{2}p_2,$$
$$q_6 = \frac{1}{16}p_2^3 - 5p_2p_4 + 8p_6,$$

where p_2, p_4, p_6 are the generating homogeneous $\mathrm{Spin}(8,\mathbb{C})$*-invariant polynomials given in* (7).

Proof. The algebra of homogeneous G_2-invariant polynomials is naturally embedded in the algebra of homogeneous polynomials of $\mathrm{Spin}(8,\mathbb{C})$ which are invariant for the action of the subgroup G_2, and there is also a surjective map of this subalgebra on the algebra of homogeneous $\mathrm{Spin}(8,\mathbb{C})$-invariant polynomials [9, Corollary 2.2.3]. From this, the map j making the diagram above commutative is defined by the composition of the two maps described. It has been also proved that the image by j of the algebra of homogeneous G_2-invariant polynomials are exactly the fixed points of the action of the triality automorphism on the algebra of homogeneous $\mathrm{Spin}(8,\mathbb{C})$-invariant polynomials [9, Corollary 2.2.3]. Moreover, Hitchin proved that a matrix of the Lie algebra $\mathfrak{g}_2$, when considered as a matrix of $\mathfrak{so}(8,\mathbb{C})$, has eigenvalues $(0, 0, \lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \lambda_3, -\lambda_3)$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$ and that the homogeneous invariant polynomials q_2 and q_6 which generate the algebra of invariant polynomials of G_2 take values $q_2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$ and $q_6 = (\lambda_1\lambda_2\lambda_3)^2$, respectively [24]. This readily leads to the equations of the statement for q_2 and q_6. □

Remark. Notice that, if $a \in B'$, X_a is the spectral curve of the G_2-invariant polynomial $a_6 + \frac{a_2^2}{4}t^2 + a_2t^4 + t^6$ induced by a and $X_{j(a)}$ is the spectral curve associated to $j(a)$, where j is defined in Proposition 4.1, then the map $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ restricts to a map $\mathrm{Prym}(X_a) \to \mathrm{Prym}(X_{j(a)})$. The image of $\mathrm{Prym}(X_a)$ are fixed points of the action of the triality automorphism on $\mathrm{Prym}(X_{j(a)})$.

5. Connectedness Criteria for the Prym Varieties

In this section the geometry of the Prym varieties $\mathrm{Prym}(X_a)$ and $\mathrm{Prym}(X_{j(a)})$ of $\mathcal{M}(G_2)$ and $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$, respectively, will be studied, where $a \in B'$ defined in (9). In particular, necessary and sufficient conditions for the above Prym varieties to be disconnected will be established.

 doi:10.20944/preprints202409.1678.v1

Lemma 5.1. *Let G be the complex simple Lie group $\mathrm{Spin}(8,\mathbb{C})$ or G_2. Let $\mathcal{M}(G)$ be the moduli space of G-Higgs bundles over X and $\mathcal{H} : \mathcal{M}(G) \to B$ be the Hitchin map defined in (5), where B denotes the base of the Hitchin map. Let $a \in B$ and let X_a be the associated spectral curve. Then X_a admits a copy of X in it if and only if the polynomial defined in (8) or (10), respectively, induced by a admits a global linear factor.*

Proof. Consider the polynomial on t induced by a defined in (8) or (10), which will also be called a for simplicity. Let d be the degree of a. Suppose that a admits a linear factor, say $\beta_0 + \beta_1 t$, where $\beta_0 \in H^(X, K^d)$ and $\beta_1 \in H^0(X, K^{d-1})$. Then $\beta_1 \neq 0$ on a dense open subset U of X. The element β_1 defines an isomorphism

$$0 \to \mathcal{O}|_U \to K^{d-1}\Big|_U \to 0.$$

By taking the dual, it is induced an isomorphism

$$0 \to K^{1-d}\Big|_U \to \mathcal{O}|_U \to 0.$$

Then there exists a unique element $\beta_1^{-1} : H^0(U, K^{1-d}) \to H^0(U, \mathcal{O}_U)$ such that $\beta_1\beta_1^{-1} = 1_{\mathcal{O}_U}$. Since $\beta_0 + \beta_1 t | a$, this implies that $\beta_0(x)\beta_1^{-1}(x) \in X_a$ for all $x \in U$. This then defines a morphism $\beta_0\beta_1^{-1} : U \to X_a$ such that $\pi_a \circ \beta_0\beta_1^{-1} = \mathrm{id}_U$, where $\pi_a : X_a \to X$ is the spectral covering. Therefore, $\beta_0\beta_1^{-1}$ is injective. Since X and X_a are projective curves, $\beta_0\beta_1^{-1}$ extends to the desired morphism.

Reciprocally, suppose that $\beta : X \hookrightarrow X_a$ is an inclusion of curves. It is then clear that $\beta + t$ is a linear factor of a. $\square$

Proposition 5.1. *Let $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ be the moduli space of $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundles over X and B be the base of the Hitchin map defined in (7). Let $a \in B$ and let X_a be the associated spectral curve. Let $b_4^2 + a_6t^2 + a_4t^4 + a_2t^6 + t^8$ be the invariant polynomial defined by a, where $b_4 \in H^0(X, K^4)$ and $a_i \in H^0(X, K^i)$ for all i. This polynomial will also be denoted by a. Then*

1. *The polynomial induced by a admits a linear factor if and only if X_a admits a copy of X.*
2. *The polynomial induced by a admits an irreducible factor of order two if and only if there exist $\beta_0 \in H^0(X, \mathcal{O}_X)$, $\beta_1 \in H^0(X, K)$, and $\beta_2 \in H^0(X, K^2)$ such that the following identities in $H^0(X, K^{14})$ and $H^0(X, K^{15})$, respectively, hold:*

$$\begin{aligned}
&\beta_0\beta_2^6a_2 + (\beta_0\beta_1^2\beta_2^4 - \beta_0^2\beta_2^5)a_4 + (\beta_0\beta_1^4\beta_2^2 + \beta_0^3\beta_2^4 - 3\beta_0^2\beta_1^2\beta_2^3)a_6 + \\
&\quad + (\beta_0\beta_1^6 - \beta_0^4\beta_2^3 + 6\beta_0^3\beta_1^2\beta_2^2 - 5\beta_0^2\beta_1^4\beta_2)b_4^2 = \beta_2^7 \\
&\beta_1\beta_2^6a_2 + (\beta_1^3\beta_2^4 - 2\beta_0\beta_1\beta_2^5)a_4 + (\beta_1^5\beta_2^2 + 3\beta_0^2\beta_1\beta_2^4 - 4\beta_0\beta_1^3\beta_2^3)a_6 + \\
&\quad + (\beta_1^7 - 4\beta_0^3\beta_1\beta_2^3 - 6\beta_0\beta_1^5\beta_2 + 10\beta_0^2\beta_1^3\beta_2^2)b_4^2 = 0.
\end{aligned}$$

3. *The polynomial induced by a admits an irreducible factor of order three if and only if there exist $\beta_0 \in H^0(X, \mathcal{O}_X)$, $\beta_1 \in H^0(X, K)$, $\beta_2 \in H^0(X, K^2)$, and $\beta_3 \in H^0(X, K^3)$ such that the following identities in $H^0(X, K^{18})$, $H^0(X, K^{19})$, and $H^0(X, K^{20})$, respectively, hold:*

$$\begin{aligned}
&\beta_0\beta_2\beta_3^4a_4 + (2\beta_0\beta_1\beta_2\beta_3^3 - \beta_0\beta_2^3\beta_3^2 - \beta_0^2\beta_3^4)a_6 + \\
&\qquad + (3\beta_0\beta_1\beta_2^3\beta_3 - 3\beta_0^2\beta_2^2\beta_3^2 - 3\beta_0\beta_1^2\beta_2\beta_3^2 + 2\beta_0^2\beta_1\beta_3^3)b_4^2 = \beta_3^6 \\
&(\beta_1\beta_2\beta_3^4 - \beta_0\beta_3^5)a_4 + (2\beta_1^2\beta_2\beta_3^3 - \beta_1\beta_2^3\beta_3^2 - 2\beta_0\beta_1\beta_3^4 + \beta_0\beta_2^2\beta_3^3)a_6 \\
&\qquad + (3\beta_1^2\beta_2^3\beta_3 - 6\beta_0\beta_1\beta_2^2\beta_3^2 - 3\beta_1^3\beta_2\beta_3^2 + 3\beta_0\beta_1^2\beta_3^3 + 2\beta_0^2\beta_2\beta_3^3)b_4^2 = 0 \\
&\beta_3^6a_2 + (\beta_1\beta_3^5 - \beta_2^2\beta_3^4)a_4 + (\beta_1^2\beta_3^4 + 2\beta_0\beta_2\beta_3^4 + \beta_2^4\beta_3^2 - 3\beta_1\beta_2^2\beta_3^3)a_6 + \\
&\qquad + (\beta_0^2\beta_3^4 - \beta_1^3\beta_3^3 - 6\beta_0\beta_1\beta_2\beta_3^3 + 6\beta_1^2\beta_2^2\beta_3^2 + 3\beta_0\beta_2^3\beta_3^2 - 3\beta_1\beta_2^4\beta_3)b_4^2 = 0.
\end{aligned}$$

 doi:10.20944/preprints202409.1678.v

4. *The polynomial induced by a admits an irreducible factor of order four if and only if there exist* $\beta_0 \in H^0(X, \mathcal{O}_X)$, $\beta_1 \in H^0(X, K)$, $\beta_2 \in H^0(X, K^2)$, $\beta_3 \in H^0(X, K^3)$, *and* $\beta_4 \in H^0(X, K^4)$ *such that the following identities in* $H^0(X, K^{20})$, $H^0(X, K^{21})$, $H^0(X, K^{22})$, *and* $H^0(X, K^{23})$, *respectively, hold:*

$$\begin{aligned}
&\beta_0\beta_4^4 a_4 + ((\beta_0\beta_2\beta_4^3 - \beta_0\beta_3^2\beta_4^2)a_6 + \\
&\quad + (\beta_0^2\beta_4^3 - \beta_0\beta_2^2\beta_4^2 + 3\beta_0\beta_2\beta_3^2\beta_4 - 2\beta_0\beta_1\beta_3\beta_4^2 - \beta_0\beta_3^4)b_4^2 = -\beta_4^5 \\
&\beta_1\beta_4^4 a_4 + (\beta_0\beta_3\beta_4^3 + \beta_1\beta_2\beta_4^3 - \beta_1\beta_3^2\beta_4^2)a_6 + \\
&\quad + (\beta_0\beta_3^3\beta_4 - 2\beta_0\beta_2\beta_3\beta_4^2 + 2\beta_0\beta_1\beta_4^3 - \beta_1\beta_2^2\beta_4^2 + 3\beta_1\beta_2\beta_3^2\beta_4 \\
&\quad - 2\beta_1^2\beta_3\beta_4^2 - \beta_1\beta_3^4)b_4^2 = 0 \\
&\beta_4^5 a_2 + \beta_2\beta_4^4 a_4 + (\beta_2^2\beta_4^3 + \beta_1\beta_3\beta_4^3 - \beta_2\beta_3^2\beta_4^2 - \beta_0\beta_4^4)a_6 + \\
&\quad + (\beta_2\beta_4^3 - \beta_0\beta_3^2\beta_4^2 + \beta_1^2\beta_4^3 + \beta_1\beta_3^3\beta_4 + \beta_0\beta_2\beta_4^3 - \beta_2^3\beta_4^2 + 3\beta_2^2\beta_3^2\beta_4 \\
&\quad - 4\beta_1\beta_2\beta_3\beta_4^2 - \beta_2\beta_3^4)b_4^2 = 0 \\
&\beta_3\beta_4^4 a_4 + (2\beta_2\beta_3\beta_4^3 - \beta_3^3\beta_4^2 - \beta_1\beta_4^4)a_6 + \\
&\quad + (2\beta_1\beta_2\beta_4^3 + 2\beta_0\beta_3\beta_4^3 - 3\beta_2^2\beta_3\beta_4^2 + 4\beta_2\beta_3^3\beta_4 - 3\beta_1\beta_3^2\beta_4^2 - \beta_3^5)b_4^2 = 0.
\end{aligned}$$

Proof. The first part is a consequence of Lemma 5.1. For the second part notice that the polynomial induced by a admits a factor of degree 2 if and only if there exist elements $\alpha_i \in H^0(X, K^i)$ for $i = 0,1,2,3,4,5,6$ and $\beta_i \in H^0(X, K^i)$ for $i = 0,1,2$ such that

$$\left(\beta_2 + \beta_1 t + \beta_0 t^2\right)\left(\alpha_6 + \alpha_5 t + \alpha_4 t^2 + \alpha_3 t^3 + \alpha_2 t^4 + \alpha_1 t^5 + \alpha_0 t^6\right)$$
$$= b_4^2 + a_6 t^2 + a_4 t^4 + a_2 t^6 + t^8.$$

This gives the following system of equations:

$$\begin{aligned}
\beta_2\alpha_6 &= b_4^2 \\
\beta_2\alpha_5 + \beta_0\alpha_6 &= 0 \\
\beta_2\alpha_4 + \beta_1\alpha_5 + \beta_0\alpha_6 &= a_6 \\
\beta_2\alpha_3 + \beta_1\alpha_4 + \beta_0\alpha_5 &= 0 \\
\beta_2\alpha_2 + \beta_1\alpha_3 + \beta_0\alpha_4 &= a_4 \\
\beta_2\alpha_1 + \beta_1\alpha_2 + \beta_0\alpha_3 &= 0 \\
\beta_2\alpha_0 + \beta_1\alpha_1 + \beta_0\alpha_2 &= a_2 \\
\beta_1\alpha_0 + \beta_0\alpha_1 &= 0 \\
\beta_0\alpha_0 &= 1_{\mathcal{O}_X}
\end{aligned}$$

Since, by hypothesis, the factor $\beta_2 + \beta_1 t + \beta_0 t^2$ is irreducible, it follws that $\beta_2 \neq 0$. Therefore, from the first seven equations it is obtained that the α_i's are determined by the β_j's, so the other two equations give the necessary and sufficient conditions which the β_j's should satisfy for the existence of a solution. Straightforward computations show that these two conditions are those announced in the statement.

The third part is analogous to the case above by considering the system in $\alpha_i \in H^0(X, K^i)$ for $i = 0,1,2,3,4,5$ induced by the expression

$$\left(\beta_3 + \beta_2 t + \beta_1 t^2 + \beta_0 t^3\right)\left(\alpha_5 + \alpha_4 t + \alpha_3 t^2 + \alpha_2 t^3 + \alpha_1 t^4 + \alpha_0 t^5\right)$$
$$= b_4^2 + a_6 t^2 + a_4 t^4 + a_2 t^6 + t^8,$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 September 2024 doi:10.20944/preprints202409.1678.v1

where $\beta_i \in H^0(X, K^i)$ for $i = 0, 1, 2, 3$. That is,

$$\begin{aligned}
\beta_3\alpha_5 &= b_4^2 \\
\beta_3\alpha_4 + \beta_2\alpha_5 &= 0 \\
\beta_3\alpha_3 + \beta_2\alpha_4 + \beta_1\alpha_5 &= a_6 \\
\beta_3\alpha_2 + \beta_2\alpha_3 + \beta_1\alpha_4 + \beta_0\alpha_5 &= 0 \\
\beta_3\alpha_1 + \beta_2\alpha_2 + \beta_1\alpha_3 + \beta_0\alpha_4 &= a_4 \\
\beta_3\alpha_0 + \beta_2\alpha_1 + \beta_1\alpha_2 + \beta_0\alpha_3 &= 0 \\
\beta_2\alpha_0 + \beta_1\alpha_1 + \beta_0\alpha_2 &= a_2 \\
\beta_1\alpha_0 + \beta_0\alpha_1 &= 0 \\
\beta_0\alpha_0 &= 1_{\mathcal{O}_X}
\end{aligned}$$

The α_i's are determined by the β_i's from the first six equations since it must be $\beta_3 \neq 0$. Then the necessary and sufficient conditions on the β_j's for the existence of a solution are given by the last three equations. These conditions are those of the statement.

Finally, the fourth par is similar. It follows from the expression

$$\left(\beta_4 + \beta_3 t + \beta_2 t^2 + \beta_1 t^3 + \beta_0 t^4\right)\left(\alpha_4 + \alpha_3 t + \alpha_2 t^2 + \alpha_1 t^3 + \alpha_0 t^4\right) = b_4^2 + a_6 t^2 + a_4 t^4 + a_2 t^6 + t^8,$$

which induces a system on the α_i's with coefficients in β_i's of the form

$$\begin{aligned}
\beta_4\alpha_4 &= b_4^2 \\
\beta_4\alpha_3 + \beta_3\alpha_4 &= 0 \\
\beta_4\alpha_2 + \beta_3\alpha_3 + \beta_2\alpha_4 &= a_6 \\
\beta_4\alpha_1 + \beta_3\alpha_2 + \beta_2\alpha_3 + \beta_1\alpha_4 &= 0 \\
\beta_4\alpha_0 + \beta_3\alpha_1 + \beta_2\alpha_2 + \beta_1\alpha_3 + \beta_0\alpha_4 &= a_4 \\
\beta_3\alpha_0 + \beta_2\alpha_1 + \beta_1\alpha_2 + \beta_0\alpha_3 &= 0 \\
\beta_2\alpha_0 + \beta_1\alpha_1 + \beta_0\alpha_2 &= a_2 \\
\beta_1\alpha_0 + \beta_0\alpha_1 &= 0 \\
\beta_0\alpha_0 &= 1_{\mathcal{O}_X}
\end{aligned}$$

From the first five equations, the α_i's can be expressed in terms of the β_i's (notice that, in this case, it must be $\beta_4 \neq 0$), so the four conditions of the statement follow from the four last equations of the system. □

Consider now the moduli space of G_2-Higgs bundles over X. From Lemma 5.1, similar computations as made in Proposition 5.1 allow to prove the following analogous result.

Proposition 5.2. *Let $\mathcal{M}(G_2)$ the moduli space of G_2-Higgs bundles over X and B' be the base of the Hitchin map of $\mathcal{M}(G_2)$ defined in (9). Let $a \in B'$ and X_a be the induced spectral curve. Suppose that the invariant polynomial induced by a, which will be also called a, is $a_6 + \frac{a_2^2}{4}t^2 - a_2 t^4 + t^6$, where $a_i \in H^0(X, K^i)$ for all $i = 2, 6$. Then*

1. *The polynomial induced by a admits a linear factor if and only if X_a admits a copy of X.*

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 24 September 2024 doi:10.20944/preprints202409.1678.v1

2. *The polynomial induced by a admits an irreducible factor of order two if and only if there exist* $\beta_0 \in H^0(X, \mathcal{O}_X)$, $\beta_1 \in H^0(X, K)$, *and* $\beta_2 \in H^0(X, K^2)$ *such that the following identities on* $H^0(X, K^{10})$ *and* $H^0(X, K^{11})$ *respectively hold:*

$$
\begin{aligned}
&(\beta_0\beta_1^4 + \beta_0^3\beta_2^2 - 3\beta_0^2\beta_1^2\beta_2)a_6 + \frac{\beta_0\beta_1^2\beta_2^2 - \beta_0^2\beta_2^3}{4}a_2^2 = \beta_2^5 \\
&(\beta_0\beta_1^4 - 2\beta_0^2\beta_1^2\beta_2 + \beta_0^3\beta_2^2 - \beta_0\beta_1^2\beta_2)a_6 \\
&\qquad - \frac{\beta_0^2\beta_1\beta_2^3 + \beta_0\beta_1\beta_2^3 - \beta_0\beta_1^3\beta_2^2}{4}a_2^2 - \beta_1\beta_2^4 a_2 = 0.
\end{aligned}
$$

3. *The polynomial induced by a admits an irreducible factor of order three if and only if there exist* $\beta_0 \in H^0(X, \mathcal{O}_X)$, $\beta_1 \in H^0(X, K)$, $\beta_2 \in H^0(X, K^2)$, *and* $\beta_3 \in H^0(X, K^3)$ *such that the following identities on* $H^0(X, K^{12})$, $H^0(X, K^{13})$, *and* $H^0(X, K^{14})$ *respectively hold:*

$$
\begin{aligned}
&(2\beta_0\beta_1\beta_2\beta_3 - \beta_0\beta_2^3 - \beta_0^2\beta_3^2)a_6 - \frac{\beta_0\beta_2\beta_3^2}{4}a_2^2 = \beta_3^4 \\
&(2\beta_1^2\beta_2\beta_3 - \beta_1\beta_2^3 - 2\beta_0\beta_1\beta_3^2 + \beta_0\beta_2^2\beta_3)a_6 + \frac{\beta_0\beta_3^3 - \beta_1\beta_2\beta_3^2}{4}a_2^2 = 0 \\
&(3\beta_1\beta_2^2\beta_3 - \beta_2^4 - 2\beta_0\beta_2\beta_3^2 - \beta_1^2\beta_3^2)a_6 + \frac{\beta_1\beta_3^3 - \beta_2^2\beta_3^2}{4}a_2^2 + \beta_3^4 a_2 = 0.
\end{aligned}
$$

Proof. The first part is an immediate consequence of Lemma 5.1. For the second, note that the polynomial induced by a admits a factor of degree two if and only if there exist elements $\alpha_i \in H^0(X, K^i)$ for $i = 0, 1, 2, 3, 4$ and $\beta_i \in H^0(X, K^i)$ for $i = 0, 1, 2$ such that

$$
\left(\beta_2 + \beta_1 t + \beta_0 t^2\right)\left(\alpha_4 + \alpha_3 t + \alpha_2 t^2 + \alpha_1 t^3 + \alpha_0 t^4\right) = a_6 + \frac{a_2^2}{4}t^2 - a_2 t^4 + t^6.
$$

This gives the following system of equations.

$$
\begin{aligned}
\beta_2\alpha_4 &= a_6 \\
\beta_2\alpha_3 + \beta_1\alpha_4 &= 0 \\
\beta_2\alpha_2 + \beta_1\alpha_3 + \beta_0\alpha_4 &= \frac{a_2^2}{4} \\
\beta_2\alpha_1 + \beta_1\alpha_2 + \beta_0\alpha_3 &= 0 \\
\beta_2\alpha_0 + \beta_1\alpha_1 + \beta_0\alpha_2 &= -a_2 \\
\beta_1\alpha_0 + \beta_0\alpha_1 &= 0 \\
\beta_0\alpha_0 &= 1_{\mathcal{O}_X}
\end{aligned}
$$

It must be $\beta_2 \neq 0$, since, by hypothesis, the factor $\beta_2 + \beta_1 t + \beta_0 t^2$ is irreducible. It is then clear that the α_i's are determined by the β_j's from the first five equations of this system, so the other two equations give the necessary and sufficient conditions which the β_j's should verify for the existence of a solution. Straightforward computations then readily show that these two conditions are those of the statement.

The third part is analogous. Now, the expression

$$
\left(\beta_3 + \beta_2 t + \beta_1 t^2 + \beta_0 t^3\right)\left(\alpha_3 + \alpha_2 t + \alpha_1 t^2 + \alpha_0 t^3\right) = a_6 + \frac{a_2^2}{4}t^2 - a_2 t^4 + t^6
$$

induces a system of the form

$$\begin{aligned}
\beta_3\alpha_3 &= a_6 \\
\beta_3\alpha_2 + \beta_2\alpha_3 &= 0 \\
\beta_3\alpha_1 + \beta_2\alpha_2 + \beta_1\alpha_3 &= \frac{a_2^2}{4} \\
\beta_3\alpha_0 + \beta_2\alpha_1 + \beta_1\alpha_2 + \beta_0\alpha_3 &= 0 \\
\beta_2\alpha_0 + \beta_1\alpha_1 + \beta_0\alpha_2 &= -a_2 \\
\beta_1\alpha_0 + \beta_0\alpha_1 &= 0 \\
\beta_0\alpha_0 &= 1_{\mathcal{O}_X}
\end{aligned}$$

Here, it also must be $\beta_3 \neq 0$, so the first four equations allow to express the α_i's in terms of the β_i's. The rest three equations give the conditions announced. □

A result of Hausel and Pauly [21, Corollary 1.3] is recalled here, as it will be useful in the to study the geometry of Prym varieties associated to the Hitchin system in the moduli spaces $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ and $\mathcal{M}(G_2)$. Although [21] works with the $\mathrm{SL}(n,\mathbb{C})$-case, the results that will be used here are directly applicable to the cases of interest in the present work, since the representations that considered allow to interpret the curves as spectral curves associated to some $\mathrm{SL}(n,\mathbb{C})$-Higgs bundle which admits an additional structure.

Proposition 5.3. *Let $X' \to X$ be an n-sheeted spectral cover of the complex irreducible projective curve X. The Prym variety $Prym(X')$ is not connected if and only if there exists a prime number d with $d|n$ such that the spectral cover $X' \to X$ comes from a degree n/d spectral cover over the étale Galois cover of degree d over X.*

Consider the moduli space $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ of $\mathrm{Spin}(8,\mathbb{C})$-Higgs bundles over X. Let B be the base of the Hitchin map of $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ defined in (7). In Proposition 5.3, Hausel and Pauly prove that the Prym variety of a given spectral curve X_a with $a \in B$ is connected if and only if the polynomial $\beta_0 t^d + \beta_d$ does not divide the polynomial defined by a in (8) whatever $\beta_0 \in H^0(X, \mathcal{O}_X)$, $\beta_d \in H^0(X, K^d)$, and d is a prime number which divides the degree of the cover $X_a \to X$.

Similarly, if the moduli space $\mathcal{M}(G_2)$ of G_2-Higgs bundles over X is considered and B' deonotes the base of its Hitchin map defined in (9), Proposition 5.3 states that, for any $a \in B'$ with associated spectral curve X_a, the Prym variety $\mathrm{Prym}(X_a)$ is connected if and only if for every $\beta_0 \in H^0(X, \mathcal{O}_X)$, $\beta_d \in H^0(X, K^d)$ and every prime number d such that d divides the degree of the cover $X_a \to X$, the polynomial $\beta_0 t^d + \beta_d$ does not divide the polynomial defined by a in (10).

From Proposition 5.3 and the discussion above, one can prove the following result, which determines the condition of a Prym variety given by the groups $\mathrm{Spin}(8,\mathbb{C})$ or G_2 to be connected.

Proposition 5.4. *Let G be the complex simple Lie group $\mathrm{Spin}(8,\mathbb{C})$ or G_2 and let $\mathcal{M}(G)$ be the moduli space of G-Higgs bundles over X. Let a be an element of the base of the associated Hitchin map. Then the following is satisfied:*

1. *If $G = \mathrm{Spin}(8,\mathbb{C})$ and the invariant polynomial associated to a is $b_4^2 + a_6 t^2 + a_4 t^4 + a_2 t^6 + t^8$ with $a_i \in H^0(X, K^i)$ for $i = 2,4,6$ and $b_4 \in H^0(X, K^4)$, then the Prym variety $Prym(X_a)$ is not connected if and only if one of the following conditions holds:*
   (a) *X_a contains a copy of X.*
   (b) *There exists a global section $\beta \in H^0(X, K^2)$ such that β is a solution of the polynomial*
   $$b_4^2 + a_6 t + a_4 t^2 + a_2 t^3 + t^4,$$
   which takes values in $H^0(X, K^8)$.

2. *If $G = G_2$ and the invariant polynomial associated to a is $a_6 + \frac{a_2^2}{4}t^2 - a_2t^4 + t^6$ with $a_i \in H^0(X, K^i)$ for $i = 2, 6$, then the Prym variety $\mathrm{Prym}(X_a)$ is not connected if and only if one of the following conditions holds:*

   (a) X_a contains a copy of X.
   (b) There exists a global section $\beta \in H^0(X, K^2)$ such that β is a solution of the polynomial

   $$-a_6 + \frac{a_2^2}{4}t + t^3,$$

   which takes values in $H^0(X, K^6)$.
   (c) $a_2 = 0$ and there exists a global section $\beta \in H^0(X, K^3)$ such that β is a solution of the polynomial

   $$a_6 + t^2,$$

   which takes values in $H^0(X, K^6)$.

Proof. As a consequence of Proposition 5.3, in both cases $\mathrm{Prym}(X_a)$ is disconnected if and only if there exists a polynomial of the form $\beta_0 t^d + \beta_d$, with $\beta_0 \in H^0(X, \mathcal{O}_X)$ and $\beta_d \in H^0(X, K^d)$, which divides the polynomial associated to a and such that d is a prime number which divides the degree of the polynomial associated to a. Consequently:

1. If $G = \mathrm{Spin}(8, \mathbb{C})$, the only possibility for d is $d = 2$, since the degree of the polynomial associated to a is 8. There are two possibilities for the polynomial $b_4^2 + a_6t^2 + a_4t^4 + a_2t^6 + t^8$ to have a factor of the form $\beta_0t^2 + \beta_2$: first, the polynomial $\beta_0t^2 + \beta_2$ is reducible if and only if X_a contains a copy of X, by the first part of Proposition 5.1; secondly, the polynomial $\beta_0t^2 + \beta_2$ is irreducible if and only if $\beta_0, \beta_1 = 0, \beta_2$ satisfy the conditions of the second part of Proposition 5.1. Notice that, since $\beta_0t^2 + \beta_2$ is supposed to be irreducible, both sections, β_0 and β_2, should be non-zero. By taking $\beta_1 = 0$ in that expressions it follows that $\beta_2\beta_0^{-1}$ is a solution of the polynomial $b_4^2 + a_6t + a_4t^2 + a_2t^3 + t^4$, which takes values in $H^0(X, K^8)$.
2. If $G = G_2$, since the polynomial associated to a has degree 6, there are two possibilities for de degree d:

   (a) If $d = 2$, the polynomial $\beta_0t^2 + \beta_2$ may be reducible or irreducible. It is reducible if and only if X_a contains a copy of X, by the first part of Proposition 5.2; it is irreducible if and only if $\beta_0, \beta_1 = 0, \beta_2$ satisfy the two conditions given in the second part of Proposition 5.2. Since $\beta_0t^2 + \beta_2$ is supposed to be irreducible, β_0 and β_2 are nonzero sections and by taking $\beta_1 = 0$ in that expressions it is deduced that $-\beta_2\beta_0^{-1}$ is a solution of the polynomial $-a_6 + \frac{a_2^2}{4}t + t^3$, which takes values in $H^0(X, K^6)$.
   (b) If $d = 3$ then, as in the previous item, the polynomial $\beta_0t^3 + \beta_3$ may be reducible or irreducible. It is easily seen that the reducible case falls in the preceding case. It is irreducible if and only if $\beta_0, \beta_1 = 0, \beta_2 = 0, \beta_3$ satisfy the conditions given in the third part of Proposition 5.2. Since it should be $\beta_0 \neq 0$ and $\beta_3 \neq 0$, taking $\beta_1 = 0$ and $\beta_2 = 0$ to that expressions, it follows that $a_2 = 0$ and $\beta_0^{-1}\beta_3$ is a solution of $a_6 + t^2$ in $H^0(X, K^6)$.

□

Remark. Proposition 5.4 gives necessary and sufficient conditions on the elements $a \in B'$ of the base of the Hitchin map of $\mathcal{M}(G_2)$ for the Prym varieties $\mathrm{Prym}(X_a)$ and $\mathrm{Prym}(X_{j(a)})$ to be connected or disconnected, where j is the map defined in Proposition 4.1. Recall that these two Prym varieties are related to each other through the map $\mathrm{Prym}(X_a) \to \mathrm{Prym}(X_{j(a)})$ defined in the Remark after Proposition 4.1. At the present moment, it is not easy, in the light of what has been proved, to give a simple criterion for deriving, for example, the connection of $\mathrm{Prym}(X_{j(a)})$ from the connection of $\mathrm{Prym}(X_a)$. To illustrate this difficulty, consider criterion 2(c) of Proposition 5.4. In the case illustrated in that item, $a_2 = 0$, so by Proposition 4.1, the polynomial associated to $j(a)$ is $\frac{a_6}{8}t + t^4 = t(\frac{a_6}{8} + t^3)$.

Even if criterion 2(c) of disconnection of $\mathrm{Prym}(X_a)$ (existence of $\beta \in H^0(X, K^3)$ such that $a_6 + \beta^2 = 0$) is satisfied, it is not possible to ensure that $\mathrm{Prym}(X_{j(a)})$ is disjoint, because this would require the existence of $\beta' \in H^0(X, K^2)$ such that $a_6 + \beta'^3 = 0$. However, taking the intersection of both criteria, it is possible to give a sufficient condition for the two Prym varieties to be disjoint, as it is stated in the following result, which is an immediate consequence of Proposition 5.4.

Corolary 5.1. *Let $a \in B'$ be an element of the base of the Hitchin map of $\mathcal{M}(G_2)$ defined in (9) whose associated polynomial is $a_6 + t^6$ (that is $a_2 = 0$). Then, if there exists $\beta \in H^0(X, K^6)$ such that $a_6 + \beta^6 = 0$, then the Prym varieties $\mathrm{Prym}(X_a)$ and $\mathrm{Prym}(X_{j(a)})$ are both disconnected, where j is defined in Proposition 4.1.*

6. Conclusion

Let X be a compact Riemann surface of genus $g \geq 2$ and let $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ and $\mathcal{M}(G_2)$ be the moduli spaces of $\mathrm{Spin}(8,\mathbb{C})$ and G_2-Higgs bundles over X, respectively. It is known that there is a map of algebraic varieties $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ whose image is composed by fixed points of the action of the triality automorphism τ on $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$. In this paper it has been proved that τ acts on the base B of the Hitchin map given by the Hitchin integrable system associated to $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$, that there exists a homomorphism $j : B' \to B$, where B' is the base of the Hitchin map of $\mathcal{M}(G_2)$ and that the image of j is composed by fixed points of the action of τ. This led the definition of maps between Prym varieties $\mathrm{Prym}(X_a) \to \mathrm{Prym}(X_{j(a)})$, where $a \in B'$, given by restriction of the map $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$. As a way of studying the geometry of the maps $\mathrm{Prym}(X_a) \to \mathrm{Prym}(X_{j(a)})$ (which, in turn, deepens the study of the grometry of the map $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$), results on the geometry of the Prym varieties involved have been provided. In particular, it has been characterized when the Prym varieties are disjoint. As a consequence, a sufficient condition is provided for both Prym varieties, $\mathrm{Prym}(X_a)$ and $\mathrm{Prym}(X_{j(a)})$, to be disjoint. All this contributes to a better understanding and deepening of the map $\mathcal{M}(G_2) \to \mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$ and, therefore, of the geometry of the moduli spaces of Higgs bundles involved. From this, future lines of research are proposed to prove whether the map above is injective and to explicitly identify the image of the homomorphism $B' \to B$, where B' is the base of the Hitchin map of $\mathcal{M}(G_2)$ and B is the base of the Hitchin map of $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$. It would also be interesting to analyze the effect of the action of triality automorphism on certain subvarieties of $\mathcal{M}(\mathrm{Spin}(8,\mathbb{C}))$, such as the Lagrangian subspaces that are defined as reductions to real forms of the structure group coming from the action of involutions of $\mathrm{Spin}(8,\mathbb{C})$, which are identified in [8]. This would strengthen the applications of the study to mirror symmetry and Langlands duality.

References

1. A. Antón-Sancho, *Principal* Spin*-bundles and triality*. Rev. Colombiana Mat. 49 (2) (2015), 235–259.
2. A. Antón-Sancho, *Automorphisms of order three of the moduli space of* Spin*-Higgs bundles*. Hokkaido Math. J. 47(2) (2018), 387–426.
3. Á. Antón-Sancho, *Triality and automorphisms of principal bundles moduli spaces*. Adv. Geom. 24(3) (2024), 421–435. https://doi.org/10.1515/advgeom-2024-0013
4. Á. Antón-Sancho, *Principal* $\mathrm{SO}(2n,\mathbb{C})$*-bundle fixed points over a compact Riemann surface*. J. Math. Phys. Anal. Geom. 20 (2024), 139–152. https://doi.org/10.15407/mag20.02.139
5. Á. Antón-Sancho, *Fixed points of involutions of G-Higgs bundle moduli spaces over a compact Riemann surface with classical complex structure group*. Front. Math. 19(6) (2024), 1025–1039. https://doi.org/10.1007/s11464-023-0014-0
6. M. Adler and P. van Moerbeke, *Completely integrable systems, Euclidean Lie algebras and curves*. Adv. in Math. 38 (3) (1980), 267–317.
7. A. Beauville, M.S. Narasimhan and S. Ramanan, *Spectral curves and the generalised theta divisor*. J. Reine Angew. Math. 398 (1989), 169-179.
8. D. Baraglia, and L.P. Schaposnik, *Real structures on moduli spaces of Higgs bundles*. Adv. Theor. Math. Phys. 20(3) (2016), 525–551. https://doi.org/10.4310/ATMP.2016.v20.n3.a2

9. F. Beck, R. Donagi, and K. Wendland, *Folding of Hitchin systems and crepant resolutions*. Int. Math. Res. Not. 2022(11) (2022), 8370–8419. https://doi.org/10.1093/imrn/rnaa375
10. J. Beckers, V. Hussin, and P. Winternitz, *Complex parabolic subgroups of G_2 and nonlinear differential equations*. Lett. Math. Phys. 11, 81–86 (1986). https://doi.org/10.1007/BF00417468
11. R.L. Bryant, *Some remarks on G_2-structures*. Proceedings of Gökova Geometry-Topology Conference 2005, 75-109, Gökova Geometry/Topology Conference (GGT), Gökova, 2006.
12. C. Chevalley, *Invariants of finite groups generated by reflections*. Amer. Journal of Math. 77 (4) (1955), 778-782.
13. K. Corlette, *Flat G-bundles with canonical metrics*. J. Diff. Geom. 28 (1988), 361-382.
14. S.K. Donaldson, *A new proof of a theorem of Narasimhan and Seshadri*. J. Diff. Geom. 18 (1983), 269-277.
15. R. Donagi, *Decomposition of spectral covers*. Astérisque 218 (1993), 145-175.
16. E. Fulton and J. Harris, *Representation Theory. A first course*. GTM 129. Springer-Verlag, 1991.
17. O. García-Prada, *Involutions of the moduli space of* $\mathrm{SL}(n,\mathbb{C})$*-Higgs bundles and real forms*. In Vector Bundles and Low Codimensional Subvarieties: State of the Art and Recent Developments Quaderni di Matematica, Editors: G. Casnati, F. Catanese and R. Notari, 2007.
18. O. García-Prada, P. Gothen, and I. Mundet, *Higgs bundles and the surface group representations of surface groups in the real symplectic group*, J. Topol. 6 (2013), 64-118.
19. O. García-Prada, and S. Ramanan, *Involutions and higher order automorphisms of Higgs bundle moduli spaces*. Proc. London Math. Soc. 119(3) (2019), 681–732. https://doi.org/10.1112/plms.12242
20. T. Hausel and M. Thaddeus, *Mirror symmetry, Langlands duality, and the Hitchin system*. Invent. Math. 153 (1) (2003), 197-229.
21. T. Hausel and C. Pauly, *Prym varieties of spectral covers*. Geometry & Topology 16 (2012), 1609-1638.
22. N. Hitchin, *Stable bundles and integrable systems*. Duke Math. J. 54(1) (1987), 91-114.
23. N. Hitchin, *The self-duality equations on a Riemann surface*. Proc. London Math. Soc. 55(3) (1987), 59-126.
24. N. Hitchin, *Langlands duality and G_2-spectral curves*. Q. J. Math. 58(3) (2007), 319-344.
25. A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*. Commun. Number Theory Phys. 1(1) (2007), 1-236.
26. M.S. Narasimhan and C.S. Seshadri, *Stable and unitary bundles on a compact Riemann surface*. Ann. of Math. 82 (1965), 540-564.
27. A. Ramanathan, *Stable principal bundles on a compact Riemann surface*. Math. Ann. 213 (1975), 129-152.
28. A. Ramanathan, *Moduli for principal bundles over algebraic curves. I*, Proc. Indian Acad. Sci. Math. Sci. 106 (1966), no. 3, 301-328.
29. A. Ramanathan, *Moduli for principal bundles over algebraic curves. II*, Proc. Indian Acad. Sci. Math. Sci. 106 (1966), no. 4, 421-449.
30. O. Serman, *Moduli spaces of orthogonal and symplectic bundles over an algebraic curve*, Compos. Math. 144 (2008), 721-733.
31. C.T. Simpson, *Higgs Bundles and Local Systems*. Publications mathématiques de l'IHES 75 (1992), 5-95.
32. C.T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety I*. Inst. Hautes Études Sci. Publ. Math. 79 (1994), 47-129.
33. C.T. Simpson, *Moduli of representations of the fundamental group of a smooth projective variety II*. Inst. Hautes Études Sci. Publ. Math. 80 (1994), 5-79.
34. F. Witt, *Special metrics and triality*. Adv. Math. 219(6) (2008), 1972–2005. https://doi.org/10.1016/j.aim.2008.07.017
35. J.A. Wolf and A. Gray, *Homogeneous Spaces defined by Lie groups automorphisms. I*, J. Differential Geometry 2 (1968), 77-114.