

Article

Not peer-reviewed version

---

# Ulam Stability of Second Order Periodic Linear Differential Equations with Periodic Damping

---

[Jin Li](#)<sup>\*</sup>, Xiaofang Xia, Chuanfang Zhang, Renlian Chen

Posted Date: 20 September 2024

doi: 10.20944/preprints202409.1562.v1

Keywords: Ulam stability; periodic coefficient; linear differential equation; minimal Ulam constant



Preprints.org is a free multidiscipline platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This is an open access article distributed under the Creative Commons Attribution License which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Disclaimer/Publisher's Note: The statements, opinions, and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions, or products referred to in the content.

Article

# Ulam Stability of Second Order Periodic Linear Differential Equations with Periodic Damping

Jin Li<sup>1\*</sup>, Xiaofang Xia<sup>2</sup>, Chuanfang Zhang<sup>3</sup> and Renlian Chen<sup>4</sup>

<sup>1</sup> College of Science, Guangdong University of Petrochemical Technology, Maoming 525000, China

<sup>2</sup> Library, Guangdong University of Petrochemical Technology, Maoming 525000, China

<sup>3</sup> College of Science, Guangdong University of Petrochemical Technology, Maoming 525000, China

<sup>4</sup> College of Science, Guangdong University of Petrochemical Technology, Maoming 525000, China

\* Correspondence: lxjin25@163.com

**Abstract:** We deal with Ulam stability of second order periodic linear differential equations with periodic damping. Necessary and sufficient condition for the Ulam stability of second order periodic linear differential equations is obtained by giving an Ulam stability theorem and an Ulam instability theorem. These results extend the conclusions in relevant literature. In addition, for the situation of constant coefficients, the minimum Ulam constant is given.

**Keywords:** Ulam stability; periodic coefficient; linear differential equation; minimal Ulam constant

We are concerned with Ulam stability of the following second-order periodic linear homogeneous equation

$$y''(x) + f(x)y'(x) + g(x)y = 0, \quad (0.1)$$

where  $f(x)$  and  $g(x)$  are continuously differentiable  $\omega$ -periodic functions.

We say that (1) has Ulam stability (abbreviated as "US") on  $\mathbb{R}$  means that If there exists a constant  $k > 0$  such that, for every  $\varepsilon > 0$  and every  $\eta \in C^2(\mathbb{R}, \mathbb{R})$  satisfying

$$\eta''(x) + f(x)\eta'(x) + g(x)\eta = 0, \quad (0.2)$$

there exists a solutions  $y \in C^2(\mathbb{R}, \mathbb{R})$  of (1) such that

$$|y(x) - \eta(x)| \leq k\varepsilon, \quad \forall x \in \mathbb{R}.$$

The constant  $k$  is called US constant of (1) on  $\mathbb{R}$ .

The concept of Ulam stability, applied firstly to functional equations, was posed by Ulam [1,2] and has been well developed subsequently. In 1998, this concept was shifted in the field of differential equations [3]. From then on, Ulam stability on differential equations has attracted the attention of many researchers (see [4-19]). Recently, Fukutaka and Onitsuka [9,10] consider Ulam stability of the equation

$$y'(x) - \lambda(x)y(x) = f(x) \quad (0.3)$$

on  $\mathbb{R}$ , where  $\lambda(x)$  is continuous periodic function. Let  $I = (a, b)$ ,  $-\infty \leq a < b \leq \infty$ . Fukutaka and Onitsuka [10] establish the following theorem.

**Theorem 1.** Let  $A(x)$  and  $\Gamma(x)$  be antiderivatives of  $\lambda(x)$  and  $e^{-A(x)}$  on  $\mathbb{R}$ , respectively. Suppose that  $\lambda(x)$  is a periodic function with period  $\omega > 0$  on  $\mathbb{R}$ . Then the following hold:

(i) if  $b = \infty$  and  $\int_0^\omega \lambda(x)dx > 0$ , then (3) has US with minimum US constant

$$\max_{x \in (0, \omega]} \left\{ \left( \lim_{x \rightarrow \infty} \Gamma(x) \right) - \Gamma(x) \right\} e^{A(x)}$$

on  $I$ ;

(ii) if  $a = -\infty$  and  $\int_0^\omega \lambda(x)dx < 0$ , then (3) has US with minimum US constant

$$\max_{x \in (0, \omega]} \left\{ \Gamma(x) - \left( \lim_{x \rightarrow -\infty} \Gamma(x) \right) \right\} e^{A(x)}$$

on  $I$ .

In [10], they also obtain a necessary and sufficient condition as follows.

**Theorem 2.** Suppose that  $\lambda(x)$  is a periodic function with period  $\omega > 0$  on  $\mathbb{R}$ . Then (3) has US on  $\mathbb{R}$  if and only if  $\int_0^\omega \lambda(x)dx \neq 0$ .

Soon afterwards, Fukutaka and Onitsuka [11] study Hill's equation

$$y''(x) - (\lambda^2(x) - \lambda'(x))y(x) = 0 \quad (0.4)$$

and use Theorem 1.1 to prove the following theorem.

**Theorem 3.** Let  $\Lambda(x)$ ,  $\Gamma^+(x)$  and  $\Gamma^-(x)$  be antiderivatives of  $\lambda(x)$ ,  $e^{\Lambda(x)}$  and  $e^{-\Lambda(x)}$  on  $\mathbb{R}$ , respectively. Suppose that  $\lambda(x)$  is a periodic function with period  $\omega > 0$  on  $\mathbb{R}$ . Then the following hold:

(i) if  $\int_0^\omega \lambda(x)dx > 0$ , then (3) has US with US constant

$$\left[ \max_{x \in (0, \omega]} \left\{ \Gamma^+(x) - \left( \lim_{x \rightarrow -\infty} \Gamma^+(x) \right) \right\} e^{-\Lambda(x)} \right] \left[ \max_{x \in (0, \omega]} \left\{ \left( \lim_{x \rightarrow \infty} \Gamma^-(x) \right) - \Gamma^-(x) \right\} e^{\Lambda(x)} \right]$$

on  $\mathbb{R}$ ;

(ii) if  $\int_0^\omega \lambda(x)dx < 0$ , then (3) has US with US constant

$$\left[ \max_{x \in (0, \omega]} \left\{ \left( \lim_{x \rightarrow \infty} \Gamma^+(x) \right) - \Gamma^+(x) \right\} e^{-\Lambda(x)} \right] \left[ \max_{x \in (0, \omega]} \left\{ \Gamma^-(x) - \left( \lim_{x \rightarrow -\infty} \Gamma^-(x) \right) \right\} e^{\Lambda(x)} \right]$$

on  $\mathbb{R}$ .

Obviously, the differential equation (3) is a special case of the equation (1). Cădariu, Popa and Raşa [12] discuss (1) and get the following result.

**Theorem 4.** Suppose that there exists  $K > 0$  such that

$$\pm g(x) \geq \frac{1}{K}, \quad x \in I \quad (0.5)$$

and the Riccati equation  $u' = -u^2 + fu + f' \mp g$  has a solution  $u \in C^1(I, \mathbb{R})$ , with  $u(a) = f(a)$ . Then the equation (1) is Ulam stable with the Ulam constant  $K$ .

Motivated by Fukutaka and Onitsuka [9–11], this paper is devoted to study the equation (1) and establish US condition without the restriction (0.5). Since the equation we are studying has a damping coefficient, our purpose is also to extend Theorem 1.3.

In the next section, an Ulam stability theorem and two Ulam instability theorems are displayed. Furthermore, a necessary and sufficient condition is given. Section 3 focuses on minimum Ulam stability constant for second order constant coefficient differential equation.

## 1. Main Results

In this section, we first present a US result of (1).

**Theorem 5.** Suppose that the Riccati equation

$$p' = p^2 + fp - f' + g \quad (1.1)$$

has a  $\omega$ -periodic solution  $p(x)$  satisfying  $\int_0^\omega p(x)dx \neq 0$ . Let  $\Lambda_p(x)$ ,  $\Lambda_q(x)$ ,  $\Gamma_p(x)$  and  $\Gamma_q(x)$  be antiderivatives of  $p(x)$ ,  $-p(x) - f(x)$ ,  $e^{-\Lambda_p(x)}$  and  $e^{-\Lambda_q(x)}$  on  $\mathbb{R}$ , respectively. Then the following hold:

(i) if  $\int_0^\omega p(x)dx > 0$  and  $\int_0^\omega (p(x) + f(x))dx < 0$ , then (1) has US with US constant

$$\max_{x \in (0, \omega]} \left\{ \left( \lim_{x \rightarrow \infty} \Gamma_p(x) \right) - \Gamma_p(x) \right\} e^{\Lambda_p(x)} \max_{x \in (0, \omega]} \left\{ \left( \lim_{x \rightarrow \infty} \Gamma_q(x) \right) - \Gamma_q(x) \right\} e^{\Lambda_q(x)}$$

on  $\mathbb{R}$ ;

(ii) if  $\int_0^\omega p(x)dx > 0$  and  $\int_0^\omega (p(x) + f(x))dx > 0$ , then (1) has US with US constant

$$\max_{x \in (0, \omega]} \left\{ \left( \lim_{x \rightarrow \infty} \Gamma_p(x) \right) - \Gamma_p(x) \right\} e^{\Lambda_p(x)} \left[ \max_{x \in (0, \omega]} \left\{ \Gamma_q(x) - \left( \lim_{x \rightarrow -\infty} \Gamma_q(x) \right) \right\} e^{\Lambda_q(x)} \right]$$

on  $\mathbb{R}$ ;

(iii) if  $\int_0^\omega p(x)dx < 0$  and  $\int_0^\omega (p(x) + f(x))dx < 0$ , then (1) has US with US constant

$$\left[ \max_{x \in (0, \omega]} \left\{ \Gamma_p(x) - \left( \lim_{x \rightarrow -\infty} \Gamma_p(x) \right) \right\} e^{\Lambda_p(x)} \right] \max_{x \in (0, \omega]} \left\{ \left( \lim_{x \rightarrow \infty} \Gamma_q(x) \right) - \Gamma_q(x) \right\} e^{\Lambda_q(x)}$$

on  $\mathbb{R}$ ;

(iv) if  $\int_0^\omega p(x)dx < 0$  and  $\int_0^\omega (p(x) + f(x))dx > 0$ , then (1) has US with US constant

$$\left[ \max_{x \in (0, \omega]} \left\{ \Gamma_p(x) - \left( \lim_{x \rightarrow -\infty} \Gamma_p(x) \right) \right\} e^{\Lambda_p(x)} \right] \left[ \max_{x \in (0, \omega]} \left\{ \Gamma_q(x) - \left( \lim_{x \rightarrow -\infty} \Gamma_q(x) \right) \right\} e^{\Lambda_q(x)} \right]$$

on  $\mathbb{R}$ .

**Proof.** Let  $q(t) = -f(x) - p(x)$ . Then  $p, q$  are continuously differentiable on  $\mathbb{R}$  and  $f = -p - q$ . It follows from (2.1) that

$$g = -q' + pq.$$

Then the equation (1) becomes to the equation

$$y'' - (p + q)y' + (pq - q')y = 0,$$

which is also equivalent to the equation

$$(y' - qy)' - p(y' - qy) = 0. \quad (1.2)$$

For  $\forall \varepsilon > 0$ , we assume that  $\eta \in C^2(\mathbb{R})$  satisfies

$$|(\eta' - q\eta)' - p(\eta' - q\eta)| \leq \varepsilon, \quad \forall x \in \mathbb{R}.$$

Let  $\varphi(x) = \eta'(x) - q(x)\eta(x)$ ,  $x \in \mathbb{R}$ . Then  $\varphi \in C^1(\mathbb{R})$  and

$$|\varphi' - p\varphi| \leq \varepsilon, \quad \forall x \in \mathbb{R}.$$

Now, we prove (i). Let  $\phi$  be a solution of the equation

$$\phi' - p\phi = 0.$$

Using Theorem 1.1, we get from  $\int_0^\omega p(x)dx > 0$  that

$$|\eta' - q\eta - \phi| = |\varphi - \phi| \leq \max_{x \in (0, \omega]} \left\{ \left( \lim_{x \rightarrow \infty} \Gamma_p(x) \right) - \Gamma_p(x) \right\} e^{\Lambda_p(x)} \varepsilon.$$

Let  $z$  be a solution of the equation

$$z' - qz - \phi = 0.$$

Using Theorem 1.1 again, one has from  $\int_0^\omega q(x)dx > 0$  that

$$|\eta(x) - z(x)| \leq \max_{x \in (0, \omega]} \left\{ \left( \lim_{x \rightarrow \infty} \Gamma_p(x) \right) - \Gamma_p(x) \right\} e^{\Lambda_p(x)} \max_{x \in (0, \omega]} \left\{ \left( \lim_{x \rightarrow \infty} \Gamma_q(x) \right) - \Gamma_q(x) \right\} e^{\Lambda_q(x)} \varepsilon.$$

The proof for (ii), (iii), and (iv) is similar to (i), so we omit it. This completes the proof.  $\square$

Next, we discuss the instability of (1). Let us first consider the scenario of  $g \equiv 0$  and we have the following theorem.

**Theorem 6.** Suppose that  $\int_0^\omega f(x)dx \neq 0$  and  $g(x) \equiv 0$ , then (1) is not Ulam stable.

**Proof.** Set  $\bar{f} = \frac{1}{\omega} \int_0^\omega f(x)dx$  and  $\tilde{f} = f - \bar{f}$ . Then  $\int_0^\omega \tilde{f}(x)dx = 0$  and  $\bar{f} \neq 0$ . Let  $F(x)$  and  $\tilde{F}(x)$  be antiderivatives of  $f(x)$  and  $\tilde{f}(x)$  on  $\mathbb{R}$ , respectively. There exist two positive numbers  $m$  and  $M$  such that

$$m \leq e^{-\tilde{F}(x)} \leq M$$

for  $x \in \mathbb{R}$ . For any  $\varepsilon > 0$ , let  $k > 0$  be such that

$$\left| \frac{f(x)}{k} \right| \leq \varepsilon.$$

Then the function

$$\eta(x) = c_1 \int_0^x e^{-F(s)} ds + c_2 + \frac{x}{k}$$

solves the equation

$$\eta'' + f(x)\eta' = \frac{f(x)}{k}.$$

At the same time, the equation

$$y'' + f(x)y' = 0$$

has the general solution

$$y(x) = c_3 \int_0^x e^{-F(s)} ds + c_4.$$

Hence, we get

$$|\eta(x) - y(x)| = \left| (c_1 - c_3) \int_0^x e^{-F(s)} ds + (c_2 - c_4) + \frac{x}{k} \right|$$

If  $c_1 - c_3 \geq 0$ , it is easy to see that  $\lim_{x \rightarrow \infty} |\eta(x) - y(x)| = \infty$ . If  $c_1 - c_3 < 0$  and  $\bar{f} > 0$ , for  $x < 0$ ,

$$\begin{aligned} |\eta(x) - y(x)| &= \left| (c_1 - c_3) \int_0^x e^{-F(s)} ds + (c_2 - c_4) + \frac{x}{k} \right| \\ &= \left| (c_1 - c_3) \int_0^x e^{-\bar{f}s - \bar{F}(s)} ds + (c_2 - c_4) + \frac{x}{k} \right| \\ &\geq (c_1 - c_3) \int_0^x e^{-\bar{f}s - \bar{F}(s)} ds - |c_2 - c_4| - \left| \frac{x}{k} \right| \\ &\geq m(c_1 - c_3) \int_0^x e^{-\bar{f}s} ds - |c_2 - c_4| - \left| \frac{x}{k} \right| \\ &= \frac{m(c_1 - c_3)}{\bar{f}} (1 - e^{-\bar{f}x}) - |c_2 - c_4| - \left| \frac{x}{k} \right|, \end{aligned}$$

which implies  $\lim_{x \rightarrow -\infty} |\eta(x) - y(x)| = \infty$ . If  $c_1 - c_3 < 0$  and  $\bar{f} < 0$ , we can obtain  $\lim_{x \rightarrow \infty} |\eta(x) - y(x)| = \infty$  by using a similar discussion. This completes the proof.  $\square$

Immediately after, we will consider the instability of (1) for the general situation.

**Theorem 7.** Suppose that the Riccati equation

$$p' = p^2 + fp - f' + g$$

has a  $\omega$ -periodic solution  $p(x)$ . If  $\int_0^\omega p(x) dx = 0$  or  $\int_0^\omega (p(x) + f(x)) dx = 0$ , then (1) is not Ulam stable.

**Proof.** Set  $q(x) = -p(x) - f(x)$ , then the equation (1) becomes to (1.2). For any  $\varepsilon > 0$ , the function

$$\eta(x) = \left\{ c_1 \int_0^x e^{\Lambda_p(s) - \Lambda_q(s)} ds + c_2 + \varepsilon \int_0^x \left( e^{\Lambda_p(s) - \Lambda_q(s)} \int_0^s e^{-\Lambda_p(u)} du \right) ds \right\} e^{\Lambda_q(x)}$$

is the general solution of the equation

$$(\eta' - q\eta)' - p(\eta' - q\eta) = \varepsilon.$$

Let

$$y(x) = \left\{ c_3 \int_0^x e^{\Lambda_p(s) - \Lambda_q(s)} ds + c_4 \right\} e^{\Lambda_q(x)},$$

then it is the general solution of the equation (1.2). We obtain that

$$\begin{aligned} |\eta(x) - y(x)| &= \left| (c_1 - c_3) \int_0^x e^{\Lambda_p(s) - \Lambda_q(s)} ds + (c_2 - c_4) \right. \\ &\quad \left. + \varepsilon \int_0^x \left( e^{\Lambda_p(s) - \Lambda_q(s)} \int_0^s e^{-\Lambda_p(u)} du \right) ds \right| e^{\Lambda_q(x)} \\ &= \left| \frac{c_2 - c_4}{\varepsilon} + \int_0^x e^{\Lambda_p(s) - \Lambda_q(s)} \left( \frac{c_1 - c_3}{\varepsilon} + \int_0^s e^{-\Lambda_p(u)} du \right) ds \right| e^{\Lambda_q(x)} \varepsilon. \end{aligned}$$

If  $\int_0^\omega p(x) dx = 0$  and  $\int_0^\omega (p(x) + f(x)) dx = 0$  (i.e.  $\int_0^\omega q(x) dx = 0$ ), we have that there exist four numbers  $m_p, m_q, M_p$  and  $M_q$  such that

$$m_p \leq \Lambda_p(x) \leq M_p$$

and

$$m_q \leq \Lambda_q(x) \leq M_q$$

for  $x \in \mathbb{R}$ . For  $s > 0$ ,

$$e^{\Lambda_p(s)-\Lambda_q(s)} \left( \frac{c_1 - c_3}{\varepsilon} + \int_0^s e^{-\Lambda_p(u)} du \right) \geq e^{m_p - M_q} e^{-M_p s} - e^{M_p - m_q} \frac{|c_1 - c_3|}{\varepsilon}.$$

This implies that, for  $x > 0$ ,

$$\begin{aligned} \frac{c_2 - c_4}{\varepsilon} + \int_0^x e^{\Lambda_p(s)-\Lambda_q(s)} \left( \frac{c_1 - c_3}{\varepsilon} + \int_0^s e^{-\Lambda_p(u)} du \right) ds \\ \geq \frac{1}{2} e^{m_p - M_q} e^{-M_p x^2} - e^{M_p - m_q} \frac{|c_1 - c_3|}{\varepsilon} x + \frac{c_2 - c_4}{\varepsilon}. \end{aligned}$$

Then there exists a  $x_1 > 0$  such that, for  $x \geq x_1$ ,

$$\frac{1}{2} e^{m_p - M_q} e^{-M_p x^2} - e^{M_p - m_q} \frac{|c_1 - c_3|}{\varepsilon} x + \frac{c_2 - c_4}{\varepsilon} > x.$$

Hence, for  $x > x_1$ ,

$$\left| \frac{c_2 - c_4}{\varepsilon} + \int_0^x e^{\Lambda_p(s)-\Lambda_q(s)} \left( \frac{c_1 - c_3}{\varepsilon} + \int_0^s e^{-\Lambda_p(u)} du \right) ds \right| e^{\Lambda_q(x)} \varepsilon > e^{m_q} x \varepsilon.$$

So we get  $\lim_{x \rightarrow \infty} |\eta(x) - y(x)| = \infty$ . This indicates that the equation (1) is not Ulam stable.

If  $\int_0^\omega p(x) dx \neq 0$  and  $\int_0^\omega (p(x) + f(x)) dx = 0$  (i.e.  $\int_0^\omega q(x) dx = 0$ ). Let  $p(x) = \bar{p} + \tilde{p}(x)$ , where  $\bar{p} = \frac{1}{\omega} \int_0^\omega p(x) dx$ . Let us first consider  $\int_0^\omega p(x) dx > 0$ . Then  $\int_0^\omega \tilde{p}(x) dx = 0$  and  $\bar{p} > 0$ . We have that there exist two numbers  $m_{\bar{p}}$  and  $M_{\bar{p}}$  such that

$$m_{\bar{p}} \leq \Lambda_{\bar{p}}(x) \leq M_{\bar{p}}$$

for  $x \in \mathbb{R}$ .

If  $c_1 \geq c_3$ , for  $s > 0$ ,

$$\begin{aligned} e^{\Lambda_p(s)-\Lambda_q(s)} \left( \frac{c_1 - c_3}{\varepsilon} + \int_0^s e^{-\Lambda_p(u)} du \right) &\geq e^{\bar{p}s + m_{\bar{p}} - M_q} \int_0^s e^{-\bar{p}u - M_{\bar{p}}} du \\ &= e^{\bar{p}s + m_{\bar{p}} - M_q - M_{\bar{p}}} \frac{1}{\bar{p}} (1 - e^{-\bar{p}s}) \\ &= \frac{1}{\bar{p}} e^{m_{\bar{p}} - M_q - M_{\bar{p}}} (e^{\bar{p}s} - 1). \end{aligned}$$

This implies that, for  $x > 0$  and  $c_1 \geq c_3$ ,

$$\begin{aligned} \frac{c_2 - c_4}{\varepsilon} + \int_0^x e^{\Lambda_p(s)-\Lambda_q(s)} \left( \frac{c_1 - c_3}{\varepsilon} + \int_0^s e^{-\Lambda_p(u)} du \right) ds \\ \geq e^{m_{\bar{p}} - M_q - M_{\bar{p}}} \left( \frac{1}{\bar{p}^2} (e^{\bar{p}x} - 1) - \frac{1}{\bar{p}} x \right) + \frac{c_2 - c_4}{\varepsilon}. \end{aligned}$$

Then there exists a  $x_1 > 0$  such that, for  $x \geq x_1$ ,

$$e^{m_{\bar{p}} - M_q - M_{\bar{p}}} \left( \frac{1}{\bar{p}^2} (e^{\bar{p}x} - 1) - \frac{1}{\bar{p}} x \right) + \frac{c_2 - c_4}{\varepsilon} > x.$$

Hence, for  $x > x_1$ ,

$$\left| \frac{c_2 - c_4}{\varepsilon} + \int_0^x e^{\Lambda_p(s)-\Lambda_q(s)} \left( \frac{c_1 - c_3}{\varepsilon} + \int_0^s e^{-\Lambda_p(u)} du \right) ds \right| e^{\Lambda_q(x)} \varepsilon > e^{m_q} x \varepsilon.$$

So we get  $\lim_{x \rightarrow \infty} |\eta(x) - y(x)| = \infty$ .

If  $c_1 < c_3$ , for  $s < 0$ ,

$$\begin{aligned} -e^{\Lambda_p(s)-\Lambda_q(s)} \left( \frac{c_1 - c_3}{\varepsilon} + \int_0^s e^{-\Lambda_p(u)} du \right) &= e^{\Lambda_p(s)-\Lambda_q(s)} \left( \frac{c_3 - c_1}{\varepsilon} + \int_s^0 e^{-\Lambda_p(u)} du \right) \\ &\geq e^{\bar{p}s+m_{\bar{p}}-M_q} \int_s^0 e^{-\bar{p}u-M_{\bar{p}}} du \\ &= e^{\bar{p}s+m_{\bar{p}}-M_q-M_{\bar{p}}} \frac{1}{\bar{p}} (e^{-\bar{p}s} - 1) \\ &= e^{m_{\bar{p}}-M_q-M_{\bar{p}}} \frac{1}{\bar{p}} (1 - e^{\bar{p}s}). \end{aligned}$$

This implies that, for  $x < 0$  and  $c_1 < c_3$ ,

$$\begin{aligned} \frac{c_2 - c_4}{\varepsilon} + \int_0^x e^{\Lambda_p(s)-\Lambda_q(s)} \left( \frac{c_1 - c_3}{\varepsilon} + \int_0^s e^{-\Lambda_p(u)} du \right) ds \\ \frac{c_2 - c_4}{\varepsilon} - \int_x^0 e^{\Lambda_p(s)-\Lambda_q(s)} \left( \frac{c_1 - c_3}{\varepsilon} + \int_0^s e^{-\Lambda_p(u)} du \right) ds \\ \geq \int_x^0 e^{m_{\bar{p}}-M_q-M_{\bar{p}}} \frac{1}{\bar{p}} (1 - e^{\bar{p}s}) ds + \frac{c_2 - c_4}{\varepsilon} \\ = e^{m_{\bar{p}}-M_q-M_{\bar{p}}} \left( -\frac{1}{\bar{p}^2} (1 - e^{\bar{p}x}) - \frac{1}{\bar{p}} x \right) + \frac{c_2 - c_4}{\varepsilon}. \end{aligned}$$

Then there exists a  $x_2 < 0$  such that, for  $x < x_2$ ,

$$e^{m_{\bar{p}}-M_q-M_{\bar{p}}} \left( -\frac{1}{\bar{p}^2} (1 - e^{\bar{p}x}) - \frac{1}{\bar{p}} x \right) + \frac{c_2 - c_4}{\varepsilon} > e^{m_{\bar{p}}-M_q-M_{\bar{p}}} \left( -\frac{1}{2\bar{p}} x \right).$$

Hence, for  $x < x_2$ ,

$$\left| \frac{c_2 - c_4}{\varepsilon} + \int_0^x e^{\Lambda_p(s)-\Lambda_q(s)} \left( \frac{c_1 - c_3}{\varepsilon} + \int_0^s e^{-\Lambda_p(u)} du \right) ds \right| e^{\Lambda_q(x)} \varepsilon > e^{m_q+m_{\bar{p}}-M_q-M_{\bar{p}}} \left( -\frac{1}{2\bar{p}} x \right) \varepsilon.$$

So we get  $\lim_{x \rightarrow -\infty} |\eta(x) - y(x)| = \infty$ .

Hence, for  $\int_0^\omega p(x) dx > 0$  and  $\int_0^\omega q(x) dx = 0$ , the equation (1) is not Ulam stable.

For the situation of  $\int_0^\omega p(x) dx < 0$  and  $\int_0^\omega q(x) dx = 0$  and the situation of  $\int_0^\omega p(x) dx = 0$  and  $\int_0^\omega q(x) dx \neq 0$  are similar to above discussion, so we omit them. This completes the proof.  $\square$

To this extent, we can use Theorem 2.1 and Theorem 2.3 to obtain the following necessary and sufficient conditions.

**Theorem 8.** Suppose that the Riccati equation

$$p' = p^2 + fp - f' + g$$

has a  $\omega$ -periodic solution  $p(x)$ . Then (1) has US on  $\mathbb{R}$  if and only if  $\int_0^\omega p(x) dx \int_0^\omega (p(x) + f(x)) dx \neq 0$ .

**Remark 1.** Consider the differential equation

$$y'' - (1 + \cos x)y' + \left( \frac{1}{4} + \frac{1}{2} \cos x \right) y = 0. \quad (1.3)$$

We have  $f(x) = -(1 + \cos x)$  and  $g(x) = \frac{1}{4} + \frac{1}{2} \cos x$ . Since  $-\frac{1}{4} \leq g(x) \leq \frac{3}{4}$ , Theorem 1.4 can not be used to (1.3). Meanwhile, the theorems of [11] can not be used to (1.3) yet because (1.3) has damping coefficient. The Riccati equation

$$p' = p^2 + fp - f' + g = p^2 - (1 + \cos x)p + \frac{1}{4} + \frac{1}{2} \cos x - \sin x$$

has a  $2\pi$ -periodic solution  $p(x) = \frac{1}{2} + \cos x$ . Let  $q(x) = \frac{1}{2}$ . By using Theorem 2.1, we know that (1.3) has US on  $\mathbb{R}$ . Accordingly, the above theorems are all novel.

**Remark 2.** The results of above theorems have symmetry. In fact, we can consider two symmetric equations

$$y'' - (p + q)y' + (pq - q')y = 0$$

and

$$y'' - (p + q)y' + (pq - p')y = 0.$$

They have the same conclusions under the framework of Theorem 2.1, Theorem 2.3 and Theorem 2.4.

**Remark 3.** For nonhomogeneous equation

$$y''(x) + f(x)y'(x) + g(x)y = h(x),$$

we can obtain the analogues by using similar technologies as in [11].

## 2. Minimal US Constant for Constant Coefficient Differential Equation

In this section, we consider the constant coefficient differential equation

$$y'' - (p + q)y' + pqy = 0, \quad (2.1)$$

where  $p, q \in \mathbb{R}$ . We have the following theorem.

**Theorem 9.** If  $pq \neq 0$ , then (2.1) has US with minimum US constant  $\frac{1}{|pq|}$  on  $\mathbb{R}$ .

**Proof.** According to the definitions of  $\Lambda_p, \Lambda_q, \Gamma_p$  and  $\Gamma_q$ , we have

$$\Lambda_p(x) = px, \quad \Lambda_q(x) = qx, \quad \Gamma_p(x) = -\frac{1}{p}e^{-px}, \quad \Gamma_q(x) = -\frac{1}{q}e^{-qx}.$$

Let  $k$  be US constant. If  $p > 0$  and  $q > 0$ , we use Theorem 2.1 to get that  $k = \frac{1}{pq}$ . Similarly, we obtain that  $k = \frac{1}{-pq}$  if  $p > 0$  and  $q < 0$  or  $p < 0$  and  $q > 0$  and  $k = \frac{1}{pq}$  if  $p < 0$  and  $q < 0$ . So (2.1) has US with US constant  $\frac{1}{|pq|}$  on  $\mathbb{R}$ .

Next, we will use the method of contradiction to prove its minimization. Assume that there exists US constant  $k_0$  satisfying  $0 < k_0 < \frac{1}{|pq|}$ . Let  $y \in C^2(\mathbb{R}, \mathbb{R})$  be the solution of the equation (2.1). And let  $\eta \in C^2(\mathbb{R}, \mathbb{R})$  be the solution of the equation

$$\eta'' - (p + q)\eta' + pq\eta = \varepsilon.$$

We have

$$(\eta'' - y'') - (p + q)(\eta' - y') + pq(\eta - y) = \varepsilon.$$

Since  $k_0$  is US constant, so

$$|\eta - y| \leq k_0\varepsilon. \quad (2.2)$$

Denote  $\eta - y$  by  $z$ . We get

$$z'' - (p + q)z' + pqz = \varepsilon.$$

Then

$$\begin{aligned} z'' - (p + q)z' &= -pqz + \varepsilon \\ &\geq -|pq|k_0\varepsilon + \varepsilon \\ &= |pq|\varepsilon\left(\frac{1}{|pq|} - k_0\right) \\ &\triangleq M > 0. \end{aligned}$$

Hence,

$$z' - (p + q)z = \int_{x_0}^x M ds - z'(x_0) + (p + q)z(x_0).$$

This deduces that

$$\begin{aligned} z' &= (p + q)z + Mx - Mx_0 - z'(x_0) + (p + q)z(x_0) \\ &\geq -|p + q|k_0\varepsilon + Mx - Mx_0 - z'(x_0) + (p + q)z(x_0). \end{aligned}$$

Put  $N = -|p + q|k_0\varepsilon - Mx_0 - z'(x_0) + (p + q)z(x_0)$ . So we obtain that  $z' \geq Mx + N$ . Thus, we get

$$z(x) \geq \frac{1}{2}Mx^2 - \frac{1}{2}Mx_0^2 + Nx - Nx_0 + z(x_0),$$

which implies  $\lim_{x \rightarrow \infty} z(x) = \infty$ , i.e.,  $\lim_{x \rightarrow \infty} (\eta(x) - y(x)) = \infty$ . This contradicts (2.2) on  $\mathbb{R}$ . This ends the proof.  $\square$

**Remark 4.** Let  $p = -q = \lambda$ , then the equation (2.1) becomes to

$$y'' - \lambda^2 y = 0.$$

At this situation the minimum US constant is  $\frac{1}{|pq|} = \frac{1}{\lambda^2}$  on  $\mathbb{R}$ . This proclaims that above theorem extend Theorem 7 of [11].

## Funding

The first author was supported by Projects of Talents Recruitment of GDUPT (2022rcyj2012) and the Characteristic Innovation Project of Guangdong Province Ordinary University (2023KTSCX089). The second author was supported by Maoming Science and technology projects (2024063).

## Competing Interests

The authors declare that they have no conflict of interest or competing interests.

## Author Contributions

Both authors contributed equally to the results in this paper.

## References

1. Ulam, S. M. A Collection of Mathematical Problems. Intersci. Tracts Pure Appl. Math., vol. 8, Interscience Publishers, New York, London, 1960.
2. Ulam, S. M. Problems in Modern Mathematics. Science Editions John Wiley & Sons, Inc., New York, 1964.

3. Alsina, C.; Ger, R. On some inequalities and stability results related to the exponential function. *J. Inequal. Appl.* 1998, 2 (4), 373-380.
4. Onitsuka, M.; Shoji, T. Hyers-Ulam stability of first-order homogeneous linear differential equations with a real-valued coefficient. *Appl. Math. Lett.* 2017, 63, 102-108.
5. Jung, S. M. Hyers-Ulam stability of linear differential equations of first order. *Appl. Math. Lett.* 2004, 17 (10), 1135-1140.
6. Jung, S. M. Hyers-Ulam stability of linear differential equations of first order, III. *J. Math. Anal. Appl.* 2005, 311 (1), 139-146.
7. Jung, S. M. Hyers-Ulam stability of linear differential equations of first order. II. *Appl. Math. Lett.* 2006, 19 (9), 854-858.
8. Jung, S. M.; Nam, Y. W. Hyers-Ulam stability of the first order inhomogeneous matrix difference equation. *J. Comput. Anal. Appl.* 2017, 23 (8), 1368-1383.
9. Fukutaka, R.; Onitsuka, M. Best constant in Hyers-Ulam stability of first-order homogeneous linear differential equations with a periodic coefficient. *J. Math. Anal. Appl.* 2019, 473, 1432-1446.
10. Fukutaka, R.; Onitsuka, M. A necessary and sufficient condition for Hyers-Ulam stability of first-order periodic linear differential equations. *Appl. Math. Lett.* 2020, 100, 106040.
11. Fukutaka, R.; Onitsuka, M. Ulam stability for a class of Hill's equations. *Symmetry*. 2019, 11, 1483.
12. Cădariu, L.; Popa, D.; Raşa, I. Ulam stability of a second linear differential operator with nonconstant coefficients. *Symmetry*. 2020, 12, 1451.
13. Murali, R.; Selvan, A. P. Hyers-Ulam stability of  $n$ th order linear differential equation. *Proyecciones*. 2019, 38 (3), 553-566.
14. Murali, R.; Selvan, A. P. Hyers-Ulam-Rassias stability for the linear ordinary differential equation of third order. *Kragujevac J. Math.* 2018, 42 (4), 579-590.
15. Murali, R.; Park, C.; Selvan, A. P. Hyers-Ulam stability for an  $n$ th order differential equation using fixed point approach. *J. Appl. Anal. Comput.* 2021, 11 (2), 614-631.
16. Murali, R.; Selvan, A. P.; Park, C. Ulam stability of linear differential equations using Fourier transform. *AIMS Math.* 2020, 5 (2), 766-780.
17. Shen, Y.; Li, Y. A general method for the Ulam stability of linear differential equations. *Bull. Malays. Math. Sci. Soc.* 2019, 42 (6), 3187-3211.
18. Unyong, B.; Govindan, V.; Bowmiya, S.; Rajchakit, G.; Gunasekaran, N.; Vadivel, R.; Lim, C.; Agarwal, P. Generalized linear differential equation using Hyers-Ulam stability approach. *AIMS Math.* 2021 6 (2), 1607-1623.
19. Yang, Y.; Meng, F. A kind of stricter Hyers-Ulam stability of second order linear differential equations of Carathéodory type. *Appl. Math. Lett.* 2021, 115, 106946.

**Disclaimer/Publisher's Note:** The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.