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An Integral Related with the Riemann Hypothesis

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An Integral Related with the Riemann Hypothesis

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Abstract: In this note we prove that the Riemann hypothesis is false. The proof is by contradiction based on a criterion of Hu.

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1. Introduction

The infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s = \sigma + it$ is a complex number, converges for $\sigma > 1$. The Riemann zeta function is its meromorphic continuation to the whole complex plane. It is well known that the Riemann zeta function $\zeta(s)$ has zeros at negative even integers which are called trivial zeros. The Riemann hypothesis asserts that all nontrivial zeros satisfy $\sigma = 1/2$. The Riemann zeta function is a main subject in number theory, for its basic properties and other advanced aspects one may refer to [2,4–6].

Hu [3] showed the following integral equivalence: the Riemann hypothesis is true if and only if

$$\frac{1}{\pi}\int_0^{\infty} \frac{\log\left|\frac{\zeta(1/2+it)}{\zeta(1/2)}\right|}{t^2} dt = \frac{\zeta'(1/2)}{2\zeta(1/2)} - 2. \tag{1.1}$$

While it is widely believed the Riemann hypothesis would be true, in this note we are going to prove that this is not the case.

Theorem 1. *The Riemann hypothesis is false.*

We will prove this by contradiction. That is, we suppose on the contrary that the Riemann hypothesis is true, then we deduce (1.1) does not hold.

2. The Proof

Let $\Gamma(s)$ be the gamma function. We recall the basics of the Riemann xi function from [2]. The Riemann xi function is

$$\xi(s) = (s-1)\pi^{-s/2}\Gamma(1+s/2)\zeta(s) \tag{2.1}$$

and it has a product expression

$$\xi(s) = \frac{1}{2}\prod_{\rho}\left(1 - \frac{s}{\rho}\right) \tag{2.2}$$

where ρ runs over the nontrivial zeros of the Riemann zeta function. Taking logarithmic derivatives of both (2.1) and (2.2) gives that, for s not a nontrivial zeta zero,

$$\sum_{\rho} \frac{1}{s-\rho} = \frac{1}{s-1} - \frac{1}{2}\log\pi + \frac{\zeta'(s)}{\zeta(s)} + \frac{1}{2}\psi(1+s/2) \tag{2.3}$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$ is the digamma function.

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Remark 2. For $a \neq 0, -1, -2, \cdots$, in the sequel we will use the following *regularization* of the digamma function:

$$\sum_{n=0}^{\infty} \frac{1}{n+a} = -\psi(a). \tag{2.4}$$

The nature of the regularization is to subtract the same constant infinity. It is well known [1] that for $a \neq 0, -1, -2, \cdots$,

$$\psi(a) = -\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+a} \right) \tag{2.5}$$

where $\gamma = 0.5772\cdots$ is the Euler constant. Thus for another $b \neq 0, -1, -2, \cdots$, we have

$$\begin{aligned} \psi(a) - \psi(b) &= \left(-\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+a} \right) \right) - \left(-\gamma + \sum_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+b} \right) \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n+b} - \sum_{n=0}^{\infty} \frac{1}{n+a}, \end{aligned}$$

and in this sense we write the regularization (2.4).

Hu [3] proved the following criterion of the Riemann hypothesis.

Theorem 3. *The Riemann hypothesis is equivalent to*

$$\frac{1}{\pi} \int_0^{\infty} \frac{\log \left| \frac{\zeta(1/2+it)}{\zeta(1/2)} \right|}{t^2} dt = \frac{\zeta'(1/2)}{2\zeta(1/2)} - 2 = \frac{\pi}{8} + \frac{\gamma}{4} + \frac{1}{4} \log 8\pi - 2 = -0.6569\cdots. \tag{2.6}$$

Proof. Take $a = 1/2$ in [3, p.499, Theorem 7.27] or see [3, p.496, Theorem 7.26]. □

We now return to the proof of Theorem 1. For the computation we need the following.

Lemma 4. *Let $a < 0$ be a negative real number. Then*

$$\int_0^{\infty} \frac{\log\left(1 + \frac{t^2}{a}\right)}{t^2} dt = \frac{i\pi}{\sqrt{-a}}. \tag{2.7}$$

Proof. This can be computed by *Mathematica*. □

We now proceed to the proof of Theorem 1.

Proof of Theorem 1. Since $|\zeta(1/2+it)| = |\zeta(1/2-it)|$ we have

$$\frac{1}{\pi} \int_0^{\infty} \frac{\log \left| \frac{\zeta(1/2+it)}{\zeta(1/2)} \right|}{t^2} dt = \frac{1}{2\pi} \int_0^{\infty} \frac{\log \left| \frac{\zeta(1/2+it)}{\zeta(1/2)} \right| + \log \left| \frac{\zeta(1/2-it)}{\zeta(1/2)} \right|}{t^2} dt. \tag{2.8}$$

It follows from (2.1) and (2.2) that

$$\zeta(s) = \pi^{s/2} \frac{\prod_\rho \left(1 - \frac{s}{\rho}\right)}{2(s-1)\Gamma(1+s/2)}. \tag{2.9}$$

Therefore we have

$$\left| \frac{\zeta(1/2+it)}{\zeta(1/2)} \right| = \left| \pi^{\frac{it}{2}} \prod_\rho \left(1 + \frac{it}{\frac{1}{2} - \rho} \right) \cdot \frac{1}{1-2it} \cdot \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{5}{4} + \frac{1}{2}it)} \right|, \tag{2.10}$$

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$$\left|\frac{\zeta(1/2-it)}{\zeta(1/2)}\right| = \left|\pi^{\frac{-it}{2}} \prod_{\rho}\left(1-\frac{it}{\frac{1}{2}-\rho}\right) \cdot \frac{1}{1+2it} \cdot \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{5}{4}-\frac{1}{2}it)}\right|. \tag{2.11}$$

Since $|\Gamma(a+ib)| = |\Gamma(a-ib)|$ for real a, b and [7]

$$\frac{|\Gamma(a)|^2}{|\Gamma(a+ib)|^2} = \prod_{n=0}^{\infty}\left(1+\frac{b^2}{(a+n)^2}\right), \tag{2.12}$$

we have

$$\left|\frac{\zeta(1/2+it)}{\zeta(1/2)}\right| \cdot \left|\frac{\zeta(1/2-it)}{\zeta(1/2)}\right| = \left|\prod_{\rho}\left(1+\frac{t^2}{(\frac{1}{2}-\rho)^2}\right)\right| \cdot \frac{1}{1+4t^2} \cdot \prod_{n=0}^{\infty}\left(1+\frac{t^2}{4(\frac{5}{4}+n)^2}\right). \tag{2.13}$$

We denote

$$S_1 = \int_0^{\infty} \frac{\log\left|\prod_{\rho}\left(1+\frac{t^2}{(\frac{1}{2}-\rho)^2}\right)\right|}{t^2} dt,$$

$$S_2 = \int_0^{\infty} \frac{\log\left(\frac{1}{1+4t^2}\right)}{t^2} dt,$$

$$S_3 = \int_0^{\infty} \frac{\log\left(\prod_{n=0}^{\infty}\left(1+\frac{t^2}{4(\frac{5}{4}+n)^2}\right)\right)}{t^2} dt.$$

Then

$$\int_0^{\infty} \frac{\log\left|\frac{\zeta(1/2+it)}{\zeta(1/2)}\right| + \log\left|\frac{\zeta(1/2-it)}{\zeta(1/2)}\right|}{t^2} dt = S_1 + S_2 + S_3.$$

We compute S_1, S_2, S_3 term by term. We have

$$S_1 = \mathrm{Re} \int_0^{\infty} \frac{\log \prod_{\rho}\left(1+\frac{t^2}{(\frac{1}{2}-\rho)^2}\right)}{t^2} dt = \sum_{\rho} \mathrm{Re} \int_0^{\infty} \frac{\log\left(1+\frac{t^2}{(\frac{1}{2}-\rho)^2}\right)}{t^2} dt. \tag{2.14}$$

Suppose on the contrary that the Riemann hypothesis is true, then $(1/2-\rho)^2 < 0$ is a negative number for every ρ. By Lemma 4 we have

$$\int_0^{\infty} \frac{\log\left(1+\frac{t^2}{(\frac{1}{2}-\rho)^2}\right)}{t^2} dt = \frac{i\pi}{\sqrt{-(1/2-\rho)^2}},$$

which is pure imaginary (for every ρ), therefore the sum

$$S_1 = 0. \tag{2.15}$$

For S_2 we have

$$S_2 = -2\pi. \tag{2.16}$$

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For S_3 we have

$$S_3 = \sum_{n=0}^{\infty} \int_0^{\infty} \frac{\log\left(1 + \frac{t^2}{4(\frac{5}{4}+n)^2}\right)}{t^2} dt = \sum_{n=0}^{\infty} \frac{\pi}{2(\frac{5}{4}+n)} = -\frac{\pi}{2}\psi(\frac{5}{4}), \tag{2.17}$$

where in the last step we have used the regularization of the digamma function (2.4). Finally by (2.8) we have

$$\frac{1}{\pi}\int_0^{\infty} \frac{\log\left|\frac{\zeta(1/2+it)}{\zeta(1/2)}\right|}{t^2} dt = \frac{1}{2\pi}(S_1 + S_2 + S_3) = -1 - \frac{1}{4}\psi(\frac{5}{4}) = -0.943\cdots, \tag{2.18}$$

which is inconsistent with (2.6). The proof is complete. □

Remark 5. In the last step of (2.17) we have used the regularization of the digamma function and there are two reasons. The first is that the integral in (2.6) is convergent and thus a finite number. The second is that

$$S_3 = \int_0^{\infty} \frac{\log \frac{|\Gamma(5/4)|^2}{|\Gamma(5/4+it/2)|^2}}{t^2} dt = \int_0^{\infty} \frac{\log|\Gamma(5/4)|^2}{t^2} dt - \int_0^{\infty} \frac{\log|\Gamma(5/4+it/2)|^2}{t^2} dt.$$

But

$$\int_0^{\infty} \frac{\log|\Gamma(5/4)|^2}{t^2} dt = \infty, \quad \int_0^{\infty} \frac{\log|\Gamma(5/4+it/2)|^2}{t^2} dt = \infty. \tag{2.19}$$

By Remark 2 we need use the regularization of the digamma function.

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References

1. M. Abramowitz and I. A. Stegun, Handbook of mathematical functions, with formulas, graphs, and mathematical tables. New York: Dover Publications, 1965.
2. H. M. Edwards, Riemann's Zeta Function. Academic Press. 1974.
3. P. C. Hu and C. C. Yang, Value distribution theory related to number theory, Birkhäuser, 2006.
4. A. Ivić. The Riemann Zeta Function, New York: John Wiley Sons, 1985.
5. A. A. Karatsuba and S. M. Voronin, The Riemann Zeta-Function. Berlin: W. de Gruyter.1992.
6. E. C. Titchmarsh, The Theory of the Riemann Zeta Function, 2nd revised edition. Oxford University Press.1986.
7. Wikipedia, Gamma function, https://en.wikipedia.org/wiki/Gamma_function#Properties