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Homogenization of a Thermoelastic Bristly Structure Immersed in a Thermofluid

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Abstract: The effective macroscopic model describing interactions of a viscous compressible heat-conducting fluid and a two-level fine bristly thermoelastic structure is derived from microstructure by means of the Allaire–Briane multi-scale convergence method. This new model naturally generalizes the isothermal formulation earlier constructed by the authors in 2020, see in *Siberian Electronic Mathematical Reports*, vol. 17 (https://doi.org/10.33048/semi.2020.17.100). In applications, the established model can be used, for example, in description, with account of heat transfer phenomemon, of airflow near surface of plant's leaf, in simulation of epithelium surfaces of blood vessels, and in design of biotechnological devices operating in liquids.

Keywords: compressible thermofluid; thermoelastic solid; Stokes–fourier equations; linear thermoelasticity equations; homogenization; periodic structure; biotechnology; bionics

MSC: 35D30; 35Q92; 74F10; 92B05

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1. Introduction

We consider the linearized mathematical model of joint motion of a viscous compressible heat-conducting fluid (thermofluid) and a flat thermoelastic plate with attached bristles. We assume that, in dimensionless variables, the viscous thermofluid and the bristly plate occupy the three-dimensional unit cube $\Omega = \{\boldsymbol{x} \in \mathbb{R}^3 : \ 0 < x_i < 1, \ i = 1,2,3\}$. The flat plate lies at the bottom of the cube and fills in layer $\Omega_{\rm pl} = \{\boldsymbol{x} \in \Omega : \ 0 < x_3 < \Delta\}$ ($\Delta = \text{const} < 1$). The bristles are modeled as thermoelastic cylinders that are very frequently periodically located on the upper surface of the flat plate, orthogonally to this surface. There are cylinders of two different sizes. The shorter and at the same time thinner cylinders are located on the upper surface of the flat plate an order or several orders of magnitude more often than the taller and thicker ones. The heights of the cylinders are fixed and equal to δ_* and δ^*, where $\Delta + \delta_* < \Delta + \delta^* < 1$. The dimensionless distance between the symmetry axes of two adjacent tall cylinders is ε, while the distance between the symmetry axes of two adjacent short cylinders is ε^2. Here $\varepsilon \ll 1$ is a small positive parameter.

The motion of the viscous thermofluid is described by the linearized non-stationary Stokes–Fourier equations and the motion of the thermoelastic bristly plate is governed by the classical non-stationary equations of linear thermoelasticity. On the fluid-solid interface, continuity of velocity, temperature, normal stress, and normal heat flux is prescribed. The system consisting of the Stokes–Fourier equations, the linear thermoelasticity equations, and the fluid-solid interface conditions is endowed with the set of initial (in time t) conditions and boundary conditions on $\partial\Omega$. The velocity field $\boldsymbol{u}_\varepsilon = \boldsymbol{u}_\varepsilon(\boldsymbol{x}, t)$ in the fluid, the displacement field $\boldsymbol{v}_\varepsilon = \boldsymbol{v}_\varepsilon(\boldsymbol{x}, t)$ in the bristly plate, and the distribution of temperature $\theta_\varepsilon = \theta_\varepsilon(\boldsymbol{x}, t)$ in the whole fluid-solid continuum are the sought functions in the system. In this article, the above described dimensionless model incorporating the small parameter ε is called **Model A$_\varepsilon$**. Its precise formulation along with the detailed description of the fine bristly structure is given further in Secs. 2–3. In addition, Sec. 2 provides a result on the existence and uniqueness of weak solutions to Model A$_\varepsilon$ for any fixed $\varepsilon > 0$.

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In the formulation of Model A_ε, each bristle is distinguished. Therefore, this model describes the *microscopic behavior* of the thermomechanical system under study. Such description can be called the 'precise' one. However, Model A_ε is inappropriate for practical analysis: from the computational point of view, the great amount of bristles leads to necessity of use of very fine meshes in numerical analysis, which leads to an inaccessible amount of calculations. This circumstance motivates to substitute 'the precise' Model A_ε by an averaged approximate one that does not contain the small parameter ε. In accord with this, the aim of the present study is to carry out and justify the homogenization procedure, i.e., to pass to the limit in Model A_ε as $\varepsilon \searrow 0$, and to derive a closed system of effective relations that describes the behavior of the thermomechanical system on the *macroscopic level*. To this end, in Sec. 4 we recall and in Sec. 5 we apply the Allaire–Briane multi-scale convergence method and, as the result, derive the three-scale homogenized system of equations and boundary conditions for the set consisting of the respective macroscopic, mesoscopic and microscopic homogenized velocities $\boldsymbol{u}$, $\boldsymbol{u}^{(1)}$, and $\boldsymbol{u}^{(2)}$ and the respective macroscopic, mesoscopic and microscopic homogenized temperatures θ, $\theta^{(1)}$, and $\theta^{(2)}$. We call this three-scale system **Model H-3sc**.

In principle, Model H-3sc serves as a solution to the homogenization problem, since its formulation does not contain the small parameter ε and, therefore, gives an averaged description of the thermomechanical system. At the same time, the system of equations in Model H-3sc is nonclassical and looks very unusual. In order to clarify its physical essence and make possible applications easier, in Secs. 6 and 7, we fulfill the full asymptotic decomposition, which amounts to the gradual scale separation. As the result, we construct the desired effective limit model for the pair of macroscopic velocity and temperature $(\boldsymbol{u}, \theta)$ solely. We state the variational formulation of this model in Sec. 7 and call it **Model H-var**. In Sec. 8, we state its equivalent, in the sense of the theory of distributions, integro-differential formulation and call it **Model H-ID**. This model describes the evolution of the thermomechanical system at the macroscopic level. Model H-ID consists of the classical system of linear thermoelasticity equations for the purely thermoelastic flat plate $\Omega_{\rm pl}$, the two systems of non-classical non-local in time integro-differential Kelvin–Voigt-type equations of thermoviscoelastic layers $\Omega_\theta := \{\boldsymbol{x} \in \Omega\colon\ \Delta < x_3 < \Delta + \delta_*\}$ and $\Omega_\sigma := \{\boldsymbol{x} \in \Omega\colon\ \Delta + \delta_* < x_3 < \Delta + \delta^*\}$, the classical Stokes–Fourier equations of the pure fluid component $\Omega_{\rm fl} := \{\boldsymbol{x} \in \Omega\colon\ \Delta + \delta^* < x_3 < 1\}$, the set of the natural conditions on the discontinuity surfaces $\{\boldsymbol{x} \in \Omega\colon\ x_3 = \Delta\}$, $\{\boldsymbol{x} \in \Omega\colon\ x_3 = \Delta + \delta_*\}$, and $\{\boldsymbol{x} \in \Omega\colon\ x_3 = \Delta + \delta^*\}$, and the set of initial (in time) and boundary (on $\partial\Omega$) conditions for velocity and temperature. After Model H-ID is formulated, in Sec. 9 we propose a procedure for computing effective physical characteristics of the homogenized medium.

Let us remark that the mathematical modeling of bristly structures immersed in liquid or gas on macroscopic scales has a fairly notable history. Almost ninety years ago, in 1938, a rather simple model was proposed in the monograph by S. Goldstein [1][Secs. 53 and 145]. In this model, a laminar flow around a flat plate with a single orthogonally welded pin is considered. The air flows in parallel to the plate. There are found the conditions for the airflow to remain laminar after passing by the pin. The obvious idea that the flow remains laminar again after flowing around another similar pin leads to the conclusion that Goldstein's model can be naturally generalized to cases of any number of pins, that is, to cases of bristly structures. Such a generalization was successfully adapted in a number of works for studying aerodynamics in a neighborhood of a plant leaf with trichomes being taken into account, see, for example, [2,3], [Ch. X]. Trichomes are bristles (fuzz) on a leaf epithelium. It is worth noting that Goldstein's model [1] has its origins in the theory of the wing in aeronautics. As a matter of fact, Goldstein's model and its generalizations [2,3] are strictly restrained to the laminar regimes and are inapplicable for studying more complex situations of flow. A much more general macroscopic model, covering a wide range of interactions between a bristly plate and a fluid flow around it, was constructed by K.-H. Hoffmann, N.D. Botkin and V.N. Starovoitov in 2005 in [4] by the homogenization method starting from the classical Stokes equations of viscous compressible fluid and classical Lamé's equations of linear elasticity. This model is isothermal, and it is based on a microstructure with frequently and periodically arranged cylindrical elastic bristles of the same size.

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In [4], the authors give the full justification of the homogenization procedure and fulfill a series of numerical experiments that show perfect consistency with physical observations.

The study in [4] was motivated by demand in mathematical modeling and design of aptamer-based biosensors: in [4][Introduction] the authors suggested that 'one can impress' (inside the biosensor) 'the aptamer protein layer as a periodic bristle or pin structure on the top of the gold film contacting the liquid'. Since 2005, based on the Hoffmann–Botkin–Starovoitov model (from [4]), several computational algorithms for modeling biosensors have been created and the corresponding numerical experiments have been carried out [5–7]. More specifically, article [5] describes a numerical method and a program for calculating dispersion relations for surface and bulk acoustic waves in multi-layered anisotropic structures that may contain specific bristle-like layers in contact with liquids. This study is of great importance from the point of view of biosensor applications, since the numerical method takes into account the piezoelectric properties of materials, the ability to work with very thin layers and adequate treatment of the interface between a solid bristle-like structure and a liquid. In [6], an interesting approach for simulation of wave patterns in surface acoustic wave biosensors is proposed. This approach relies on [4] and involves a rather subtle application of the differential games theory. Feasibility of this approach is verified in [6] by a set of numerical experiments. The most recent article [7] develops the concepts obtained in [4,5] for investigating the glycocalyx, 'a polysaccharide polymer molecule layer on the endothelium of blood vessels that, according to recent studies, plays an important role in protecting against diseases' (citation from [7]).

At the same time, since the Hoffmann–Botkin–Starovoitov model is based on the most fundamental laws of continuum mechanics and does not contain in its general form any specific relationships related to certain properties of biosensor components, its applicability can be expanded for a much larger number of phenomena related to bristly structures, rather than only with processes in biosensors. In particular, the Hoffmann–Botkin–Starovoitov model can be regarded as a generalization of Goldstein's model [1–3] for (not necessarily laminar) airflow near plant's leafs, in a sense.

Now, let us note that, for many reasons arising from nature and technological demands, it is favorable to consider bristly structures including bristles having two distinct sizes, as in the formulation of Model A_ε in the present article. For example, in the general case, trichomes on the same leaf of a plant may belong to different types depending on length and form, which should be taken into account when studying airflow near the leaf. A characteristic feature is that the number of trichomes of different length has a different order — on a plant leaf, there are quite a lot of short trichomes for one long trichome [8–10]. Another possible application of models of two-level bristly structures interacting with liquids can be found in bionics (or biomimicry). Biomimicry describes the processes in which ideas and concepts developed by nature are translated into technology. According to the observations made in [11], bristles, as a rule, have a strong effect on the wettability of plates: plate surfaces can be superhydrophobic, self-cleaning (superoleophobic) and have low adhesion, which is often very advantageous properties of materials. Two-level (hierarchical) roughness structures are typical for superhydrophobic surfaces in nature. For example, the effect of self-purification in polluted reservoirs using a two-level trichome structure is observed in lotus: in [12][Sec. 42.4.3]BJN, the question of how to create an artificial two-level superhydrophobic surface similar to the surface of a lotus is discussed in detail.

In [13], the Hoffmann–Botkin–Starovoitov model [4] was generalized onto the case of two-level bristly structures in the isothermal case. The model that we construct in the present article is a natural generalization to non-isothermal cases of the Hoffmann–Botkin–Starovoitov isothermal models built earlier for single-level [4] and two-level [13] bristly structures.

In the end of this introduction, firstly, we note that, in the present work, the vast majority of the results and technical calculations that do not directly relate to the temperature function and its limits (as $\varepsilon \searrow 0$) coincide with those obtained in the isothermal case in [13]. Therefore, in this article we do not repeat these calculations and present the corresponding results without proof, limiting ourselves to precise references to the arguments from [13]. Secondly, we would like to notice that the content of

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the present article involves very many notations, which is a usual thing in studies of homogenization problems. Therefore, for convenience of reading we add an appendix in the end of the article, where we aggregate a fairly comprehensive list of used notations.

2. Basic Formulation of 'Thermofluid–Structure' Interactions

According to the fundamentals of continuum mechanics [14][Ch. I], the most general model of joint motion of a heat-conducting elastic body and a viscous thermofluid consists of the mass, linear momentum, and energy balance equations, the first and the second laws of thermodynamics in each phase, individual state equations, determining thermomechanical behavior in each phase, and certain conditions on the fluid-solid interface. Assuming a priori, that perturbations of the considered thermomechanical system are small about some rest state, applying in this view the classical linearization formalism [15][§ 8.1] to the equations of the most general model and passing to the proper dimensionless variables, we arrive eventually at the closed system of linear equations of 'fluid-solid' interactions. The initial-boundary value problem for this system is formulated below and is considered further in the paper.

Model A_ε. (Basic formulation of 'thermofluid – thermoelastic structure' interactions. The dimensionless form.) Let $\Omega = (0,1)^3 \subset \mathbb{R}^3_x$ be divided into two disjoint subdomains Ω^ε_F and Ω^ε_S and the Lipschitz boundary Γ_ε between them, so that the fluid occupies subdomain Ω^ε_F, and the plate with attached bristles occupies subdomain Ω^ε_S. Let $T = \mathrm{const} > 0$ be a given moment of time.

Find a velocity field $\boldsymbol{u}_\varepsilon\colon \Omega^\varepsilon_F \times (0,T) \mapsto \mathbb{R}^3$, a displacement field $\boldsymbol{v}_\varepsilon\colon \Omega^\varepsilon_S \times (0,T) \mapsto \mathbb{R}^3$, and a temperature field $\theta_\varepsilon\colon \Omega \times (0,T) \mapsto \mathbb{R}$ satisfying the equations

$$\alpha_\tau \rho_F \partial_t \boldsymbol{u}_\varepsilon = \mathrm{div}_x \big(2\alpha_\mu \mathbb{D}_x(\boldsymbol{u}_\varepsilon) + (\alpha_\lambda \,\mathrm{div}_x\, \boldsymbol{u}_\varepsilon + \alpha_p \,\mathrm{div}_x\, J_t \boldsymbol{u}_\varepsilon - p^0)\mathbb{I} - \mathbb{B}_F \theta_\varepsilon\big) + \alpha_g \rho_F \boldsymbol{f}, \quad \boldsymbol{x} \in \Omega^\varepsilon_F,\ t \in (0,T), \tag{1a}$$

$$c_F \partial_t \theta_\varepsilon = \mathrm{div}_x(\varkappa_F \nabla_x \theta_\varepsilon) - \mathbb{B}_F : \mathbb{D}_x(\boldsymbol{u}_\varepsilon) + \Psi_F, \quad \boldsymbol{x} \in \Omega^\varepsilon_F,\ t \in (0,T), \tag{1b}$$

$$\alpha_\tau \rho_S \partial^2_t \boldsymbol{v}_\varepsilon = \mathrm{div}_x\big(\boldsymbol{\mathcal{G}} : \nabla_x \boldsymbol{v}_\varepsilon - \mathbb{B}_S \theta_\varepsilon\big) + \alpha_g \rho_S \boldsymbol{f}, \quad \boldsymbol{x} \in \Omega^\varepsilon_S,\ t \in (0,T), \tag{1c}$$

$$c_S \partial_t \theta_\varepsilon = \mathrm{div}_x(\varkappa_S \nabla_x \theta_\varepsilon) - \mathbb{B}_S : \partial_t \mathbb{D}_x(\boldsymbol{v}_\varepsilon) + \Psi_S, \quad \boldsymbol{x} \in \Omega^\varepsilon_S,\ t \in (0,T), \tag{1d}$$

the interface conditions

$$\boldsymbol{u}_\varepsilon = \partial_t \boldsymbol{v}_\varepsilon, \quad \theta^{(f)}_\varepsilon = \theta^{(s)}_\varepsilon, \quad \boldsymbol{x} \in \Gamma_\varepsilon,\ t \in (0,T), \tag{1e}$$

$$\big(2\alpha_\mu \mathbb{D}_x(\boldsymbol{u}_\varepsilon) + (\alpha_\lambda \,\mathrm{div}_x\, \boldsymbol{u}_\varepsilon + \alpha_p \,\mathrm{div}_x\, J_t \boldsymbol{u}_\varepsilon - p^0)\mathbb{I} - \mathbb{B}_F \theta^{(f)}_\varepsilon\big)\boldsymbol{n}_\varepsilon = \big(\boldsymbol{\mathcal{G}} : \nabla_x \boldsymbol{v}_\varepsilon - \mathbb{B}_S \theta^{(s)}_\varepsilon\big)\boldsymbol{n}_\varepsilon, \quad \boldsymbol{x} \in \Gamma_\varepsilon,\ t \in (0,T), \tag{1f}$$

$$\varkappa_F \nabla_x \theta^{(f)}_\varepsilon \cdot \boldsymbol{n}_\varepsilon = \varkappa_S \nabla_x \theta^{(s)}_\varepsilon \cdot \boldsymbol{n}_\varepsilon, \quad \boldsymbol{x} \in \Gamma_\varepsilon,\ t \in (0,T), \tag{1g}$$

the initial conditions

$$\boldsymbol{u}_\varepsilon\big|_{t=0} = \boldsymbol{u}^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega^\varepsilon_F, \tag{1h}$$

$$\boldsymbol{v}_\varepsilon\big|_{t=0} = \boldsymbol{v}^0(\boldsymbol{x}), \quad \partial_t \boldsymbol{v}_\varepsilon\big|_{t=0} = \boldsymbol{w}^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega^\varepsilon_S, \tag{1i}$$

$$\theta_\varepsilon\big|_{t=0} = \theta^0(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1j}$$

and the conditions on the fixed boundary $\partial\Omega$:

$$\boldsymbol{u}_\varepsilon(\boldsymbol{x},t) = \boldsymbol{u}^F(\boldsymbol{x},t), \quad \boldsymbol{x} \in \partial\Omega \cap \partial\Omega^\varepsilon_F,\ t \in (0,T), \tag{1k}$$

$$\boldsymbol{v}_\varepsilon(\boldsymbol{x},t) = 0, \quad \boldsymbol{x} \in \partial\Omega \cap \partial\Omega^\varepsilon_S,\ t \in (0,T), \tag{1l}$$

$$\theta_\varepsilon(\boldsymbol{x},t) = \theta^\star(\boldsymbol{x},t), \quad \boldsymbol{x} \in \partial\Omega,\ t \in (0,T). \tag{1m}$$

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In this formulation, ε is a small positive parameter that characterize the ratio between the characteristic lengths of the microscale and the macroscale. Its precise notion will be given in Sec. 3.

Equations (1a)–(1b) are the linearized non-stationary Stokes–Fourier equations of compressible heat-conducting fluid, equations (1c)–(1d) are the classical equations of linear thermoelasticity. Relations (1e)–(1g) are the continuity equations on the interface Γ_ε for velocity, temperature, and the normal stresses and heat fluxes, respectively. In (1a) and further in the paper, by $\mathbb{D}_x(\boldsymbol{\phi})$ we denote the symmetric part of the gradient of some enough regular vector-function $\boldsymbol{\phi}(\boldsymbol{x})$: $D_{xij}(\boldsymbol{\phi}) = (1/2)(\partial_{x_i}\phi_j + \partial_{x_j}\phi_i)$, $i,j = 1,2,3$. The rest of the notation for differential operators in (1) is quite standard. In (1a) and further in the paper, by J_t we denote the Volterra operator:

$$(J_t\phi)(t) = \int_0^t \phi(s)ds, \quad \forall \phi \in L^1(0,T). \tag{2}$$

By $\mathbb{I}$ we denote the identical transformation in $\mathbb{R}^3$, i.e., $\mathbb{I} = (\delta_{ij})$, where δ_{ij} is Kronecker's symbol. In (1e)–(1g), the following notation for values of temperature θ_ε on interface Γ_ε is used: for any $\boldsymbol{x}_0 \in \Gamma_\varepsilon$ we set

$$\theta_\varepsilon^{(f)}(\boldsymbol{x}_0) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0 \\ \boldsymbol{x}\in\Omega_F^\varepsilon}} \theta_\varepsilon(\boldsymbol{x}), \quad \theta_\varepsilon^{(s)}(\boldsymbol{x}_0) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0 \\ \boldsymbol{x}\in\Omega_S^\varepsilon}} \theta_\varepsilon(\boldsymbol{x}).$$

Vector $\boldsymbol{n}_\varepsilon(\boldsymbol{x}_0)$ is the unit normal to Γ_ε at a point $\boldsymbol{x}_0$, pointing into Ω_F^ε.

Dimensionless coefficients α_τ, α_μ, α_λ, α_p, α_g, $\varkappa_F$, $\varkappa_S$, ρ_F, ρ_S, c_F, and c_S are constant, positive and independent of ε. They are given and relate to the dimensional physical characteristics of the problem via the following identities:

$$\alpha_\tau = \frac{\rho_{sc}\,L_{sc}^2}{\tau_{sc}^2\,p_{sc}}, \quad \alpha_\mu = \frac{\mu}{\tau_{sc}\,p_{sc}}, \quad \alpha_\lambda = \frac{\lambda}{\tau_{sc}\,p_{sc}}, \quad \alpha_p = \frac{1}{\gamma\,p_{sc}}, \quad \alpha_g = \frac{\rho_{sc}\,g\,L_{sc}}{p_{sc}},$$
$$\varkappa_F = \frac{\tau_{sc}\,\vartheta_{sc}^2}{L_{sc}^2\,T_*\,p_{sc}}\varkappa_F', \quad \varkappa_S = \frac{\tau_{sc}\,\vartheta_{sc}^2}{L_{sc}^2\,T_*\,p_{sc}}\varkappa_S', \quad \rho_F = \frac{\rho_F'}{\rho_{sc}}, \quad \rho_S = \frac{\rho_S'}{\rho_{sc}},$$
$$c_F = \frac{\vartheta_{sc}^2}{T_*\,p_{sc}}c_F', \quad c_S = \frac{\vartheta_{sc}^2}{T_*\,p_{sc}}c_S'.$$

Here L_{sc} is the characteristic size of Ω (measured, for example, in meters: m); τ_{sc} (s) is the characteristic duration of physical processes; p_{sc} ($kg\cdot m^{-1}\cdot s^{-2}$) is the atmosphere pressure; g ($m\cdot s^{-2}$) is the acceleration of free fall; ρ_{sc} ($kg\cdot m^{-3}$) is the mean density of air at the temperature 273 K and at the atmosphere pressure; T_* (K) is a reference temperature; ϑ_{sc} (K) is the temperature difference between the boiling- and freezing-points of water at the atmosphere pressure. Dimensional coefficients μ, λ, ρ_F', $\varkappa_F'$, and c_F' in the fluid phase are respective shear and bulk viscosities, mean density, heat conductivity, and specific heat capacity at constant pressure; the dimensional coefficient γ characterizes the compressibility of the fluid. Dimensional coefficients ρ_S', $\varkappa_S'$, and c_S' in the solid phase are respective mean density, heat conductivity and specific heat capacity at constant pressure.

$\boldsymbol{\mathcal{G}} = (\mathcal{G}^{ijkl})$ is the dimensionless elastic stiffness tensor. Its components $\mathcal{G}^{ijkl}$ $(i,j,k,l = 1,2,3)$ can be arbitrary up to base restrictions so that any anisotropic solid can be considered. We have

$$(\boldsymbol{\mathcal{G}} : \nabla_x \boldsymbol{v})_{ij} = \sum_{k,l=1}^{3} \mathcal{G}^{ijkl}\,\partial_{x_l} v_k, \quad i,j = 1,2,3,$$

components $\mathcal{G}^{ijkl}$ are constant, and there are fulfilled the symmetry condition

$$\mathcal{G}^{ijkl} = \mathcal{G}^{ijlk} = \mathcal{G}^{klij} = \mathcal{G}^{jikl}, \quad i,j,k,l = 1,2,3, \tag{3}$$

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and the positive-definiteness condition: there exists a constant $c_{\mathcal{G}} > 0$ such that

$$(\boldsymbol{\mathcal{G}} : \mathbb{X}) : \mathbb{X} \equiv \sum_{i,j,k,l=1}^{3} \mathcal{G}^{ijkl} X_{kl} X_{ij} \geq c_{\mathcal{G}} |\mathbb{X}|^2, \quad \forall \, \mathbb{X} \in \mathbb{R}^{3\times 3}_{\text{symm}}. \tag{4}$$

Here and further, we deal with symmetric 3×3-matrices, say, $\mathbb{X} = (X_{ij})_{i,j=1,2,3}$ such that $X_{ij} = X_{ji}$. We denote the class of these matrices by $\mathbb{R}^{3\times 3}_{\text{symm}}$. Note that demands (2) and (4) perfectly meet the fundamental principles of Newtonian mechanics.

Notation 1. *Above in this section and further in the article, we use the conventional notation for the inner products of fourth-rank tensors and 3×3-matrices and for the dyads of 3×3-matrices and vectors. More precisely, $\boldsymbol{\mathcal{A}} : \mathbb{W}$ is the inner product (convolution) of a fourth-rank tensor $\boldsymbol{\mathcal{A}}$ and a 3×3-matrix $\mathbb{W}$. It is the 3×3-matrix defined by the formula*

$$\boldsymbol{\mathcal{A}} : \mathbb{W} = \left(\sum_{k,l=1,2,3} \mathcal{A}^{ijkl} W_{kl} \right)_{i,j=1,2,3}.$$

The inner product (convolution) of two 3×3-matrices $\mathbb{W}$ and $\mathbb{C}$ is the scalar defined by formula

$$\mathbb{W} : \mathbb{C} = \sum_{i,j=1}^{3} W_{ij} C_{ij}.$$

In particular, we have

$$(\boldsymbol{\mathcal{A}} : \mathbb{W}) : \mathbb{C} = \sum_{i,j,k,l=1}^{3} \mathcal{A}^{ijkl} W_{kl} C_{ij}, \quad |\mathbb{W}|^2 = \mathbb{W} : \mathbb{W} = \sum_{i,j=1}^{3} W_{ij} W_{ij}, \quad \operatorname{tr} \mathbb{W} = \mathbb{W} : \mathbb{I}$$

for all fourth-rank tensors $\boldsymbol{\mathcal{A}}$ and 3×3-matrices $\mathbb{W}$ and $\mathbb{C}$.

The dyad $\mathbb{W} \otimes \mathbb{C}$ of two 3×3-matrices $\mathbb{W}$ and $\mathbb{C}$ is the fourth-rank tensor defined by formula

$$\mathbb{W} \otimes \mathbb{C} = (W_{ij} \, C_{kl})_{i,j,k,l=1,2,3},$$

and the dyad $\boldsymbol{a} \otimes \boldsymbol{b}$ of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the 3×3-matrix defined by the formula

$$\boldsymbol{a} \otimes \boldsymbol{b} = (a_i b_j)_{i,j=1,2,3}.$$

In particular, we have

$$(\mathbb{W} \otimes \mathbb{C}) : \mathbb{A} = (\mathbb{C} : \mathbb{A}) \mathbb{W}, \quad (\boldsymbol{a} \otimes \boldsymbol{b}) \boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{c}) \boldsymbol{a}$$

for all 3×3-matrices $\mathbb{A}$, $\mathbb{W}$, and $\mathbb{C}$, and for all vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

In the formulation of Problem A_ε, the dimensionless elastic stiffness tensor $\boldsymbol{\mathcal{G}}$ relates to the dimensional elastic stiffness tensor $\boldsymbol{\mathcal{G}}'$ via the identity $\boldsymbol{\mathcal{G}} = \boldsymbol{\mathcal{G}}' / p_{sc}$, 3×3-matrices $\mathbb{B}_F = (B^{ij}_F)$ and $\mathbb{B}_S = (B^{ij}_S)$ are the dimensionless matrices characterizing thermal dilatation in the fluid and solid phases, respectively; these matrices are constant, symmetric, and they relate to the dimensional matrices $\mathbb{B}'_F$ and $\mathbb{B}'_S$ via identities

$$\mathbb{B}_F = \frac{\vartheta_{sc}}{p_{sc}} \mathbb{B}'_F, \quad \mathbb{B}_S = \frac{\vartheta_{sc}}{p_{sc}} \mathbb{B}'_S.$$

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The sought dimensionless velocity $\boldsymbol{u}_\varepsilon$, displacement $\boldsymbol{v}_\varepsilon$ and temperature θ_ε relate to the respective dimensional distributions $\boldsymbol{u}'_\varepsilon$, $\boldsymbol{v}'_\varepsilon$ and θ'_ε via identities

$$\boldsymbol{u}_\varepsilon = \frac{\tau_{sc}}{L_{sc}}\boldsymbol{u}'_\varepsilon, \quad \boldsymbol{v}_\varepsilon = \frac{1}{L_{sc}}\boldsymbol{v}'_\varepsilon, \quad \theta_\varepsilon = \frac{\theta'_\varepsilon - T_*}{\vartheta_{sc}}.$$

Finally, $\boldsymbol{f}$ is the dimensionless density of distributed mass forces and Ψ_F and Ψ_S are volumetric dimensionless densities of external heat application in the fluid and the solid phases, respectively. We have

$$\boldsymbol{f} = \frac{\boldsymbol{f}'}{g}, \quad \Psi_F = \frac{\tau_{sc}\,\vartheta_{sc}}{T_*\,p_{sc}}\Psi'_F, \quad \Psi_S = \frac{\tau_{sc}\,\vartheta_{sc}}{T_*\,p_{sc}}\Psi'_S,$$

where $\boldsymbol{f}'$, Ψ'_F and Ψ'_S are the corresponding dimensional thermomechanical characteristics.

The initial distributions p^0, $\boldsymbol{u}^0$, $\boldsymbol{v}^0$, $\boldsymbol{w}^0$, and θ^0 in (1a), (1f) and (1h)–(1j) and the boundary distributions $\boldsymbol{u}^F$ and $\theta^\star$ in (1k) and (1m) are given.

For the further purpose of homogenization, it is necessary to introduce the proper notion of weak solution to Model A_ε, uniform in the whole domain Ω. To this end, let us follow [4][Sec. 2.2] and [13][§ 2].

Using (2), we rewrite equation (1c) and condition $(1i)_1$ equivalently as

$$\alpha_\tau \rho_S \partial_t \boldsymbol{u}_\varepsilon = \operatorname{div}_x\big(\boldsymbol{\mathcal{G}} : J_t \nabla_x \boldsymbol{u}_\varepsilon - \mathbb{B}_S \theta_\varepsilon\big) + \operatorname{div}_x\big(\boldsymbol{\mathcal{G}} : \nabla_x \boldsymbol{v}^0\big) + \alpha_g \rho_S \boldsymbol{f}$$

and equation (1d) as

$$c_S \partial_t \theta_\varepsilon = \operatorname{div}_x(\varkappa_S \nabla_x \theta_\varepsilon) - \mathbb{B}_S : \mathbb{D}_x(\boldsymbol{u}_\varepsilon) + \Psi_S.$$

Here $\boldsymbol{u}_\varepsilon = \partial_t \boldsymbol{v}_\varepsilon$ is the velocity vector.

We accept the following two assumptions on the initial data in Model A_ε.

Assumption 1. *We suppose that the initial data for velocity*

$$\boldsymbol{u}^0(\boldsymbol{x}) = \begin{cases} \boldsymbol{u}^0(\boldsymbol{x}) & \text{for} \quad \boldsymbol{x} \in \Omega^\varepsilon_F, \\ \boldsymbol{w}^0(\boldsymbol{x}) & \text{for} \quad \boldsymbol{x} \in \Omega^\varepsilon_S, \end{cases}$$

given in the whole cube Ω, does not depend on ε. In other words, we impose an independent of ε uniform initial velocity field $\boldsymbol{u}^0$ on Ω.

Assumption 1 is consistent with the requirement that the initial velocity field is continuous in the whole cube Ω.

Assumption 2. *We suppose that the initial distribution of pressure p^0 and the initial displacement $\boldsymbol{v}^0$ are defined in the whole cube Ω, do not depend on ε, and along with $\boldsymbol{u}^0$ and θ^0 satisfy the compatibility conditions*

$$2\alpha_\mu \mathbb{D}_x(\boldsymbol{u}^0) + (\alpha_\lambda \operatorname{div}_x \boldsymbol{u}^0 - p^0)\mathbb{I} - \mathbb{B}_F \theta^0 = \boldsymbol{\mathcal{G}} : \nabla_x \boldsymbol{v}^0 - \mathbb{B}_S \theta^0 \in H^1(\Omega)^{3\times 3}, \tag{5a}$$

$$\varkappa_F \nabla_x \theta^0 - \mathbb{B}_F \boldsymbol{u}^0 = \varkappa_S \nabla_x \theta^0 - \mathbb{B}_S \boldsymbol{u}^0 \in H^1(\Omega)^3. \tag{5b}$$

Relations (5a) and (5b) express, respectively, the smoothness of the initial stress and the initial heat flux in the whole cube Ω.

Now introduce into considerations the characteristic function of domain Ω^ε_F:

$$\chi^\varepsilon(\boldsymbol{x}) = \begin{cases} 1 & \text{for} \quad \boldsymbol{x} \in \Omega^\varepsilon_F, \\ 0 & \text{for} \quad \boldsymbol{x} \in \Omega \setminus \Omega^\varepsilon_F. \end{cases} \tag{6}$$

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Taking into account this notation, we rewrite system (1) as the system of the uniform momentum and energy equations with the discontinuous coefficients in the whole cube Ω:

$$\alpha_\tau \rho_*^\varepsilon \partial_t \boldsymbol{u}_\varepsilon = \operatorname{div}_x \big(\boldsymbol{\mathcal{M}}_\varepsilon^t \nabla_x \boldsymbol{u}_\varepsilon + \mathbb{M}_\varepsilon^0 - \mathbb{B}_*^\varepsilon \theta_\varepsilon \big) + \alpha_g \rho_*^\varepsilon \boldsymbol{f}, \quad \boldsymbol{x} \in \Omega, \ t > 0, \tag{7a}$$

$$c^\varepsilon \partial_t \theta_\varepsilon = \operatorname{div}_x (\varkappa^\varepsilon \nabla_x \theta_\varepsilon) - \mathbb{B}_*^\varepsilon : \mathbb{D}_x(\boldsymbol{u}_\varepsilon) + \Psi^\varepsilon, \quad \boldsymbol{x} \in \Omega, \ t > 0, \tag{7b}$$

supplemented with the set of initial and boundary data:

$$\boldsymbol{u}_\varepsilon \big|_{t=0} = \boldsymbol{u}^0, \quad \theta_\varepsilon \big|_{t=0} = \theta^0, \quad \boldsymbol{x} \in \Omega, \tag{7c}$$

$$\boldsymbol{u}_\varepsilon(\boldsymbol{x}, t) = \boldsymbol{u}^\star(\boldsymbol{x}, t), \quad \theta_\varepsilon(\boldsymbol{x}, t) = \theta^\star(\boldsymbol{x}, t), \quad (\boldsymbol{x}, t) \in \partial\Omega \times (0, T). \tag{7d}$$

In (7a)–(7b) and further, we denote

$$\rho_*^\varepsilon = \chi^\varepsilon \rho_F + (1 - \chi^\varepsilon) \rho_S, \tag{8}$$

$$\boldsymbol{\mathcal{M}}_\varepsilon^t \nabla_x \boldsymbol{u}_\varepsilon = \chi^\varepsilon \big(2\alpha_\mu \mathbb{D}_x(\boldsymbol{u}_\varepsilon) + (\alpha_\lambda \operatorname{div}_x \boldsymbol{u}_\varepsilon + \alpha_p \operatorname{div}_x J_t \boldsymbol{u}_\varepsilon) \mathbb{I} \big) + (1 - \chi^\varepsilon) \boldsymbol{\mathcal{G}} : J_t \nabla_x \boldsymbol{u}_\varepsilon, \tag{9}$$

$$\mathbb{M}_\varepsilon^0 = -\chi^\varepsilon p^0 \, \mathbb{I} + (1 - \chi^\varepsilon) \boldsymbol{\mathcal{G}} : \nabla_x \boldsymbol{v}^0, \tag{10}$$

$$\mathbb{B}_*^\varepsilon = \chi^\varepsilon \mathbb{B}_F + (1 - \chi^\varepsilon) \mathbb{B}_S, \tag{11}$$

$$c^\varepsilon = \chi^\varepsilon c_F + (1 - \chi^\varepsilon) c_S, \tag{12}$$

$$\varkappa^\varepsilon = \chi^\varepsilon \varkappa_F + (1 - \chi^\varepsilon) \varkappa_S, \tag{13}$$

$$\Psi^\varepsilon = \chi^\varepsilon \Psi_F + (1 - \chi^\varepsilon) \Psi_S. \tag{14}$$

In (7d), $\boldsymbol{u}^\star = \boldsymbol{u}^\star(\boldsymbol{x}, t)$ is a given vector-function, which is defined in the whole closed space-time domain $\overline{\Omega} \times [0, T]$ and satisfies the boundary condition

$$\boldsymbol{u}^\star \big|_{\partial\Omega} = \begin{cases} \boldsymbol{u}^F(\boldsymbol{x}, t), & \boldsymbol{x} \in \partial\Omega \cap \partial\Omega_F^\varepsilon, \ t \in (0, T), \\ 0, & \boldsymbol{x} \in \partial\Omega \cap \overline{\partial\Omega_S^\varepsilon}, \ t \in (0, T). \end{cases} \tag{15}$$

Remark 1. *In Sec. 3 further, we will define the geometry of Ω_F^ε and Ω_S^ε such that $\boldsymbol{u}^\star$ and $\boldsymbol{u}^F$ are independent of ε, Ω_F^ε and Ω_S^ε are simply connected, and* meas $(\partial\Omega \cap \partial\Omega_F^\varepsilon) > 0$ *and* meas $(\partial\Omega \cap \partial\Omega_S^\varepsilon) > 0$.

Now, we are in a position to introduce a notion of weak solution to Model A_ε.

Definition 1. *The pair of functions $(\boldsymbol{u}_\varepsilon, \theta_\varepsilon)$ is a* weak solution of Model A_ε, *if it satisfies the regularity demands*

$$\begin{aligned} &\boldsymbol{u}_\varepsilon \in L^\infty(0, T; L^2(\Omega)^3) \cap L^2(0, T; H^1(\Omega_F^\varepsilon)^3), \quad J_t \boldsymbol{u}_\varepsilon \in L^2(0, T; H^1(\Omega)^3), \\ &\theta_\varepsilon \in L^2(0, T; H^1(\Omega)), \end{aligned} \tag{16}$$

the boundary conditions (7d) in the trace sense, and the integral equalities

$$\begin{aligned} &\int_0^T \int_\Omega \big(-\alpha_\tau \rho_*^\varepsilon \boldsymbol{u}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} + \big[\boldsymbol{\mathcal{M}}_\varepsilon^t \nabla_x \boldsymbol{u}_\varepsilon + \mathbb{M}_\varepsilon^0 - \mathbb{B}_*^\varepsilon \theta_\varepsilon \big] : \nabla_x \boldsymbol{\varphi} - \alpha_g \rho_*^\varepsilon \boldsymbol{f} \cdot \boldsymbol{\varphi} \big) d\boldsymbol{x} dt \\ &= \int_\Omega \alpha_\tau \rho_*^\varepsilon(\boldsymbol{x}) \boldsymbol{u}^0(\boldsymbol{x}) \cdot \boldsymbol{\varphi}(\boldsymbol{x}, 0) d\boldsymbol{x} \end{aligned} \tag{17}$$

for all smooth test vector-functions $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{x}, t)$, vanishing in a neighborhood of plane $\{t = T\}$ and boundary $\partial\Omega$, and

$$\int_0^T \int_\Omega \big(-c^\varepsilon \theta_\varepsilon \partial_t \eta + \varkappa^\varepsilon \nabla_x \theta_\varepsilon \cdot \nabla_x \eta + \mathbb{B}_*^\varepsilon : \mathbb{D}_x(\boldsymbol{u}_\varepsilon) \, \eta - \Psi^\varepsilon \eta \big) d\boldsymbol{x} dt = \int_\Omega c^\varepsilon \theta^0(\boldsymbol{x}) \eta(\boldsymbol{x}, 0) d\boldsymbol{x} \tag{18}$$

for all smooth test functions $\eta = \eta(\boldsymbol{x}, t)$ vanishing in the neighborhood of plane $\{t = T\}$ and boundary $\partial\Omega$.

The following result on the well-posedness of Model A_ε and on uniform (in ε) estimates for the family of solutions to Model A_ε is valid.

Proposition 1. *Assume* $\boldsymbol{u}^0 \in L^2(\Omega)^3$, $\boldsymbol{v}^0 \in H^1(\Omega)^3$, $p^0 \in L^2(\Omega)$, $\boldsymbol{f} \in L^2(\Omega \times (0,T))^3$, $\theta^0 \in L^2(\Omega)$, $\boldsymbol{u}^\star \in C^2(\overline{\Omega} \times [0,T])^3$, $\theta^\star \in C^2(\overline{\Omega} \times [0,T])$, *and* $\|\Psi^\varepsilon\|_{L^2(\Omega\times(0,T))} \leq c_\Psi$, *where constant* $c_\Psi > 0$ *does not depend on* ε.

Then, for any fixed $\varepsilon > 0$, *there is a unique weak solution* $(\boldsymbol{u}_\varepsilon, \theta_\varepsilon)$ *to Model* A_ε *in the sense of Definition 1. Moreover, the energy estimate*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\in(0,T)} \big(\|\boldsymbol{u}_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)^3} + \|\theta_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} + \|J_t \nabla_x \boldsymbol{u}_\varepsilon(\cdot,t)\|^2_{L^2(\Omega_S^\varepsilon)^{3\times3}}\big) \\
&\qquad\qquad + \|\nabla_x \boldsymbol{u}_\varepsilon\|^2_{L^2(\Omega_F^\varepsilon\times(0,T))^{3\times3}} + \|\nabla_x \theta_\varepsilon\|^2_{L^2(\Omega\times(0,T))^3} \\
&\leq C_0\big(\|\boldsymbol{f}\|^2_{L^2(\Omega\times(0,T))^3} + c_\Psi^2 + \|\boldsymbol{u}^0\|^2_{L^2(\Omega)^3} + \|\boldsymbol{v}^0\|^2_{H^1(\Omega)^3} \\
&\qquad + \|\theta^0\|^2_{L^2(\Omega)} + \|p^0\|^2_{L^2(\Omega)} + \|\boldsymbol{u}^\star\|^2_{C^2(\overline{\Omega}\times[0,T])^3} + \|\theta^\star\|^2_{C^2(\overline{\Omega}\times[0,T])}\big)
\end{aligned} \tag{19}$$

and the additional estimate

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|(J_t \boldsymbol{u}_\varepsilon)(\cdot,t)\|_{H^1(\Omega)^3} \leq C_1 \tag{20}$$

hold true, where constant $C_0 > 0$ *depends on* T, α_τ, α_μ, α_λ, α_p, α_g, ρ_F, c_F, $\varkappa_F$, $c_{\mathcal{G}}$, ρ_S, c_S, $\varkappa_S$, *and* c_{Korn}, *and constant* $C_1 > 0$ *depends only on* C_0 *and* c_{Korn}. *(Here,* c_{Korn} *is the constant from Korn's inequality [16][Ch. I, § 2, Th. 2.1].) At the same time, both* C_0 *and* C_1 *are independent of* ε.

Proof of Proposition 1 completely repeats the proof of Theorem 3.4 from [17][Secs. 4, 5] with slight natural modifications. □

Generally speaking, estimates (19) and (20) are sufficient to fulfill the homogenization of Model A_ε with the help of the method of three-scale convergence. However, some serious technical difficulties must be overcome in this case. To avoid that, we establish a stronger estimate for $\boldsymbol{u}_\varepsilon$ and θ_ε, using Assumption 2:

Proposition 2. *Let* $\boldsymbol{u}^0 \in H^1(\Omega)^3$, $\theta^0 \in H^1(\Omega)$, $\boldsymbol{f}, \partial_t \boldsymbol{f} \in L^2(\Omega \times (0,T))^3$, $\|\Psi^\varepsilon\|_{L^2(\Omega\times(0,T))} \leq c_\Psi$, $\|\partial_t \Psi^\varepsilon\|_{L^2(\Omega\times(0,T))} \leq c_\Psi$, *and Assumption 2 hold.*

Then the family $\{(\boldsymbol{u}_\varepsilon, \theta_\varepsilon)\}_{\varepsilon>0}$ *of solutions of Model* A_ε *satisfies the estimate*

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t\in(0,T)} \big(\|\partial_t \boldsymbol{u}_\varepsilon(\cdot,t)\|_{L^2(\Omega)^3} + \|\nabla_x \boldsymbol{u}_\varepsilon(\cdot,t)\|_{L^2(\Omega)^{3\times3}} \\
&\qquad + \|\partial_t \theta_\varepsilon(\cdot,t)\|_{L^2(\Omega)} + \|\nabla_x \theta_\varepsilon(\cdot,t)\|_{L^2(\Omega)^3}\big) \leq C_2,
\end{aligned} \tag{21}$$

where C_2 *is a constant independent of* ε.

Proof of this proposition replicates the justification of Theorem 2.6 from [4]. Therefore we give it here rather schematically.

Let us introduce a pair of functions $(\boldsymbol{a}_\varepsilon, \mathfrak{T}_\varepsilon)$ as the weak solution of the problem

$$\alpha_\tau \rho_*^\varepsilon \partial_t \boldsymbol{a}_\varepsilon = \operatorname{div}_x\big(\boldsymbol{\mathcal{M}}_\varepsilon^t \nabla_x \boldsymbol{a}_\varepsilon - \mathbb{B}_*^\varepsilon \mathfrak{T}_\varepsilon\big) + \alpha_g \rho_*^\varepsilon \partial_t \boldsymbol{f} \quad \text{in } \Omega \times (0,T), \tag{22a}$$

$$c^\varepsilon \partial_t \mathfrak{T}_\varepsilon = \operatorname{div}_x(\varkappa^\varepsilon \nabla_x \mathfrak{T}_\varepsilon) - \mathbb{B}_*^\varepsilon : \mathbb{D}_x(\boldsymbol{a}_\varepsilon) + \partial_t \Psi^\varepsilon \quad \text{in } \Omega \times (0,T), \tag{22b}$$

$$\alpha_\tau \rho_*^\varepsilon \boldsymbol{a}_\varepsilon\big|_{t=0} = \operatorname{div}_x\big(\boldsymbol{\mathcal{G}} : \nabla_x \boldsymbol{v}^0 - \mathbb{B}_S \theta^0\big) + \alpha_g \rho_*^\varepsilon \boldsymbol{f}^0 \quad \text{in } \Omega, \tag{22c}$$

$$c^\varepsilon \mathfrak{T}_\varepsilon\big|_{t=0} = \operatorname{div}_x\big(\varkappa_S \nabla_x \theta^0 - \mathbb{B}_S \boldsymbol{u}^0\big) + \Psi^{\varepsilon 0} \quad \text{in } \Omega, \tag{22d}$$

$$\boldsymbol{a}_\varepsilon = \partial_t \boldsymbol{u}^\star, \quad \mathfrak{T}_\varepsilon = \partial_t \theta^\star \quad \text{on } \partial\Omega \times (0,T), \tag{22e}$$

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where $f^0 = f(\cdot, 0)$ and $\Psi^{\varepsilon 0} = \Psi(\cdot, 0)$.

By Theorem 3.4 from [17], the weak solution $(a_\varepsilon, \mathfrak{T}_\varepsilon)$ to problem (22) exists, is unique, and the energy estimates for this problem appears as follows:

$$\begin{aligned} &\operatorname*{ess\,sup}_{t\in(0,T)} \left(\|a_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)^3} + \|\mathfrak{T}_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} + \|J_t\nabla_x a_\varepsilon(\cdot,t)\|^2_{L^2(\Omega^\varepsilon_S)^{3\times3}}\right) \\ &\qquad\qquad + \|\nabla_x a_\varepsilon\|^2_{L^2(\Omega^\varepsilon_F\times(0,T))^{3\times3}} + \|\nabla_x \mathfrak{T}_\varepsilon\|^2_{L^2(\Omega\times(0,T))^3} \le C_3, \end{aligned} \tag{23}$$

which yields

$$\operatorname*{ess\,sup}_{t\in(0,T)} \|(J_t a_\varepsilon)(\cdot,t)\|_{H^1(\Omega)^3} \le C_4, \tag{24}$$

where the positive constants C_3 and C_4 are independent of ε.

In (22c) and (22d) note that

$$\operatorname{div}_x\left(\mathcal{G} : \nabla_x v^0 - \mathbb{B}_S\theta^0\right) = \operatorname{div}_x\left(\chi^\varepsilon(2\alpha_\mu \mathbb{D}_x(u^0) + (\alpha_\lambda \operatorname{div}_x u^0)\mathbb{I}) + \mathbb{M}^0_\varepsilon - \mathbb{B}^\varepsilon_*\theta^0\right),$$

due to (5a), and

$$\operatorname{div}_x\left(\varkappa_S \nabla_x\theta^0 - \mathbb{B}_S u^0\right) = \operatorname{div}_x(\varkappa^\varepsilon\nabla_x\theta^0) - \mathbb{B}^\varepsilon_* : \mathbb{D}_x(u^0),$$

due to (5b) and the symmetry of matrices $\mathbb{B}_F$ and $\mathbb{B}_S$. In this view, we conclude that the pair of functions defined as

$$u_\varepsilon(x,t) = \int_0^t a_\varepsilon(x,s)ds + u^0(x) = (J_t a_\varepsilon)(x,t) + u^0(x),$$

$$\theta_\varepsilon(x,t) = \int_0^t \mathfrak{T}_\varepsilon(x,s)ds + \theta^0(x) = (J_t\mathfrak{T}_\varepsilon)(x,t) + \theta^0(x)$$

is the solution of Model A_ε, and $\partial_t u_\varepsilon = a_\varepsilon$ and $\partial_t\theta_\varepsilon = \mathfrak{T}_\varepsilon$. Now, estimate (21) appears as an immediate consequence of (23) and (24). $\square$

3. Fine geometry of the microstructure

In this section, we precisely define the geometrical forms of Ω^ε_S and Ω^ε_F, following the lines of [13][Sec. 3]. By this we introduce a two-level bristle structure.

Assume that taller bristles are located ε-periodically and shorter bristles are located ε^2-periodically in x_1 and x_2. Parameter ε is small and positive: $\varepsilon > 0$, $\varepsilon \ll 1$. In order to describe exact locations of the bristles, we introduce *mesoscopic variables* $\widehat{y} = (y_1, y_2) \in \mathbb{R}^2$, *microscopic variables* $\widehat{z} = (z_1, z_2) \in \mathbb{R}^2$, and, correspondingly, *the pattern mesoscopic cell* $\Sigma = (0,1)\times(0,1) \subset \mathbb{R}^2_{\widehat{y}}$ and *the pattern microscopic cell* $\Theta = (0,1)\times(0,1) \subset \mathbb{R}^2_{\widehat{z}}$, each consisting of the two nonempty subdomains and the interface between these subdomains:

$$\Sigma = \Sigma_F \cup \Sigma_S \cup \Gamma_\Sigma, \quad \Theta = \Theta_F \cup \Theta_S \cup \Gamma_\Theta.$$

Here Σ_S is the orthogonal projection of a taller bristle onto the flat surface $\{x_3 = \Delta\}$ of plate Ω_{pl} taken in $\varepsilon^{-1} : 1$ scale, i.e., ε^{-1}-times stretched. Analogously, Θ_S is the orthogonal projection of a shorter bristle onto the flat surface $\{x_3 = \Delta\}$ of plate Ω_{pl} taken in $\varepsilon^{-2} : 1$ scale.

We assume that both Σ_S and Θ_S are simply connected sets with smooth boundaries, each of them being locally situated on one side of the boundary. For simplicity, suppose that Σ_S and Θ_S do not have common points with $\partial\Sigma$ and $\partial\Theta$, respectively (see Figures 1 and 2).

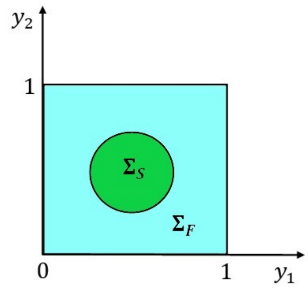

Figure 1. Mesoscopic pattern cell

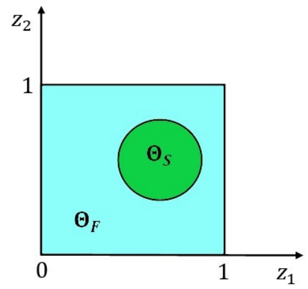

Figure 2. Microscopic pattern cell

Further, let us additionally denote $\widehat{x} = (x_1, x_2)$ and introduce into consideration the characteristic functions $\widehat{\zeta} = \widehat{\zeta}(\widehat{y})$ and $\widehat{\psi} = \widehat{\psi}(\widehat{z})$ of sets Σ_F and Θ_F, respectively:

$$\widehat{\zeta}(\widehat{y}) = \begin{cases} 1 & \text{for } \widehat{y} \in \Sigma_F, \\ 0 & \text{for } \widehat{y} \in \overline{\Sigma} \setminus \Sigma_F, \end{cases} \qquad \widehat{\psi}(\widehat{z}) = \begin{cases} 1 & \text{for } \widehat{z} \in \Theta_F, \\ 0 & \text{for } \widehat{z} \in \overline{\Theta} \setminus \Theta_F. \end{cases} \tag{25}$$

Extend functions $\widehat{\zeta}$ and $\widehat{\psi}$ onto the whole spaces $\mathbb{R}^2_{\widehat{y}}$ and $\mathbb{R}^2_{\widehat{z}}$ 1-periodically.

Using the above introduced constructions, we set up a refined geometrical structure of domains $\Omega_S = \Omega_S^\varepsilon$ and $\Omega_F = \Omega_F^\varepsilon$ as follows.

Let $\Delta = \text{const} > 0$ be the thickness of plate Ω_{pl} without taking bristles in account. Let $\delta^* = \text{const} > 0$ and $\delta_* = \text{const} > 0$ be the heights of taller and shorter bristles, respectively. We assume that $\delta^* + \Delta < 1$. In line with (6), define the characteristic function $\chi(x) = \chi^\varepsilon(x)$ of domain $\Omega_F = \Omega_F^\varepsilon$ as follows:

$$\chi^\varepsilon(x) = \zeta\Big(\frac{\widehat{x}}{\varepsilon}, x_3\Big)\psi\Big(\frac{\widehat{x}}{\varepsilon^2}, x_3\Big), \tag{26a}$$

where

$$\zeta\Big(\frac{\widehat{x}}{\varepsilon}, x_3\Big) = \begin{cases} 1 & \text{for} \quad \delta^* + \Delta < x_3 < 1, \\ \widehat{\zeta}\Big(\frac{\widehat{x}}{\varepsilon}\Big) & \text{for} \quad \Delta \le x_3 \le \delta^* + \Delta, \\ 0 & \text{for} \quad 0 < x_3 < \Delta, \end{cases} \tag{26b}$$

$$\psi\Big(\frac{\widehat{x}}{\varepsilon^2}, x_3\Big) = \begin{cases} 1 & \text{for} \quad \delta_* + \Delta < x_3 < 1, \\ \widehat{\psi}\Big(\frac{\widehat{x}}{\varepsilon^2}\Big) & \text{for} \quad \Delta \le x_3 \le \delta_* + \Delta, \\ 0 & \text{for} \quad 0 < x_3 < \Delta. \end{cases} \tag{26c}$$

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Thus, the structure of Ω_S^ε and Ω_F^ε is introduced. It is loosely shown on Figure 3.

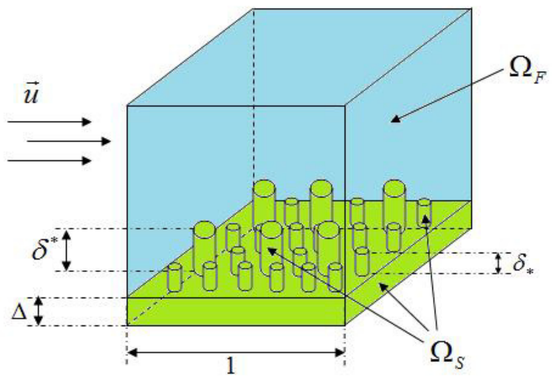

Figure 3. Plate with the bristles and the fluid flow

4. The Allaire–Briane Three-Scale Convergence Method

Our aim now is to pass to the limit in the integral equalities (17)–(18) as $\varepsilon \searrow 0$. This limiting passage is based on the Allaire–Briane three-scale convergence method. We formulate the fundamentals of this method in the form adapted to the problem under consideration, following the lines of [13][Sec. 4].

Proposition 3. (G. Allaire, M. Briane, [18][Th. 2.4]) *Let $\{w_\varepsilon\}_{\varepsilon\searrow 0}$ be a bounded sequence in $L^2(\Omega\times(0,T))$. Then there exist a subsequence from $\{w_\varepsilon\}_{\varepsilon\searrow 0}$ (still denoted by $\{w_\varepsilon\}_{\varepsilon\searrow 0}$) and a function*

$$w = w(\boldsymbol{x},\widehat{\boldsymbol{y}},\widehat{\boldsymbol{z}},t), \quad w\in L^2(\Omega\times\Sigma\times\Theta\times(0,T)),$$

such that the limiting relation

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_\Omega w_\varepsilon(\boldsymbol{x},t)\varphi\Big(\boldsymbol{x},\frac{\widehat{\boldsymbol{x}}}{\varepsilon},\frac{\widehat{\boldsymbol{x}}}{\varepsilon^2},t\Big)d\boldsymbol{x}dt = \int_0^T\int_\Omega\int_\Sigma\int_\Theta w(\boldsymbol{x},\widehat{\boldsymbol{y}},\widehat{\boldsymbol{z}},t)\varphi(\boldsymbol{x},\widehat{\boldsymbol{y}},\widehat{\boldsymbol{z}},t)\,d\boldsymbol{x}d\widehat{\boldsymbol{y}}d\widehat{\boldsymbol{z}}dt \tag{27}$$

holds true for all smooth and 1-periodic in $\widehat{\boldsymbol{y}} = (y_1,y_2)$ and $\widehat{\boldsymbol{z}} = (z_1,z_2)$ test functions $\varphi = \varphi(\boldsymbol{x},\widehat{\boldsymbol{y}},\widehat{\boldsymbol{z}},t)$.

Definition 2. [18][Defn. 2.3] *If (27) holds then we say that sequence $\{w_\varepsilon\}_{\varepsilon\searrow 0}$* converges in the three-scale sense *to w:*

$$w_\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} w \quad \text{in the 3-sc. sense.}$$

Proposition 4. (i) [19][Sec. 3] *The three-scale limit is unique, i.e., if sequence $\{w_\varepsilon\}_{\varepsilon\searrow 0}\subset L^2(\Omega\times(0,T))$ converges in the three-scale sense to functions $w_*, w_{**}\in L^2(\Omega\times\Sigma\times\Theta\times(0,T))$ then $w_* = w_{**}$ a.e. in $\Omega\times\Sigma\times\Theta\times(0,T)$.*

(ii) [18][Th. 2.6] *Let sequences $\{w_\varepsilon\}_{\varepsilon\searrow 0}$ and $\{\nabla_x w_\varepsilon\}_{\varepsilon\searrow 0}$ be bounded in $L^2(\Omega\times(0,T))$ and $L^2(\Omega\times(0,T))^3$, respectively. Then there exist a subsequence from $\{w_\varepsilon\}_{\varepsilon\searrow 0}$ (still denoted by $\{w_\varepsilon\}_{\varepsilon\searrow 0}$) and a triple of functions $w = w(\boldsymbol{x},t)$, $w_1 = w_1(\boldsymbol{x},\widehat{\boldsymbol{y}},t)$ and $w_2 = w_2(\boldsymbol{x},\widehat{\boldsymbol{y}},\widehat{\boldsymbol{z}},t)$ such that*

$$\begin{aligned}&w\in L^2(0,T;H^1(\Omega)),\\ &w_1\in L^2(\Omega\times(0,T);H^1_\sharp(\Sigma)/\mathbb{R}), \quad w_2\in L^2(\Omega\times\Sigma\times(0,T);H^1_\sharp(\Theta)/\mathbb{R}),\end{aligned}$$

and $\quad w_\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} w, \quad \nabla_x w_\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} \nabla_x w + \nabla_{\widehat{y}} w_1 + \nabla_{\widehat{z}} w_2 \quad$ *in the 3-sc. sense.*

(iii) [18][Th. 4.6] *Let sequences $\{w_\varepsilon\}_{\varepsilon\searrow 0}$ and $\{\nabla_x w_\varepsilon\}_{\varepsilon\searrow 0}$ be bounded in $L^2(\Omega\times(0,T))$ and $L^2(\Omega\times(0,T))^3$, respectively. Let χ^ε be defined by formula (26). Then there exist a subsequence from $\{w_\varepsilon\}_{\varepsilon\searrow 0}$ and*

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functions $w = w(\boldsymbol{x}, t)$, $w_1 = w_1(\boldsymbol{x}, \widehat{\boldsymbol{y}}, t)$ *and* $w_2 = w_2(\boldsymbol{x}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}}, t)$, *with the same regularity properties as in assertion (ii), such that*

$$\chi^\varepsilon w_\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} \chi(\boldsymbol{x}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}}) w(\boldsymbol{x}, t) \quad \text{in the 3-sc. sense,}$$
$$\chi^\varepsilon \nabla w_\varepsilon \underset{\varepsilon\searrow 0}{\longrightarrow} \chi(\boldsymbol{x}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}})(\nabla_x w + \nabla_{\widehat{y}} w_1 + \nabla_{\widehat{z}} w_2) \quad \text{in the 3-sc. sense,}$$

where χ *is the characteristic function of set* $\Omega_F \times \Sigma_F \times \Theta_F$, *i.e.,*

$$\chi(\boldsymbol{x}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}}) = \chi(x_3, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}}) = \zeta(\widehat{\boldsymbol{y}}, x_3)\psi(\widehat{\boldsymbol{z}}, x_3), \tag{28a}$$

where

$$\zeta(\widehat{\boldsymbol{y}}, x_3) = \begin{cases} 1 & \text{for} \quad \delta^* + \Delta < x_3 < 1, \\ \widehat{\zeta}(\widehat{\boldsymbol{y}}) & \text{for} \quad \Delta \le x_3 \le \delta^* + \Delta, \\ 0 & \text{for} \quad 0 < x_3 < \Delta, \end{cases} \tag{28b}$$

and

$$\psi(\widehat{\boldsymbol{z}}, x_3) = \begin{cases} 1 & \text{for} \quad \delta_* + \Delta < x_3 < 1, \\ \widehat{\psi}(\widehat{\boldsymbol{z}}) & \text{for} \quad \Delta \le x_3 \le \delta_* + \Delta, \\ 0 & \text{for} \quad 0 < x_3 < \Delta. \end{cases} \tag{28c}$$

In the formulation of Proposition 4 and further in the article, the standard notation for the spaces of periodic functions, which have gradients, is in use:

Notation 2. *By* $H^1_\sharp(\Sigma)/\mathbb{R}$ *and* $H^1_\sharp(\Theta)/\mathbb{R}$ *we denote the spaces of functions belonging to* $H^1(\Sigma)$ *and* $H^1(\Theta)$, *being 1-periodic in* $\widehat{\boldsymbol{y}}$ *and* $\widehat{\boldsymbol{z}}$, *respectively, and satisfying the following normalization conditions:*

$$\int_\Sigma \varphi(\widehat{\boldsymbol{y}}) d\widehat{\boldsymbol{y}} = 0, \quad \int_\Theta \varphi(\widehat{\boldsymbol{z}}) d\widehat{\boldsymbol{z}} = 0.$$

By $\nabla_{\widehat{y}}$ *and* $\nabla_{\widehat{z}}$ *we denote the gradient operators*

$$\nabla_{\widehat{y}} = \begin{pmatrix} \partial_{y_1} \\ \partial_{y_2} \\ 0 \end{pmatrix}, \quad \nabla_{\widehat{z}} = \begin{pmatrix} \partial_{z_1} \\ \partial_{z_2} \\ 0 \end{pmatrix}. \tag{29}$$

5. The Limiting Passage in Model A_ε as $\varepsilon \searrow 0$. Homogenized Three-Scale Equations

Now, with the help of Propositions 1–4, we carry out the homogenization procedure for Model A_ε and derive the closed well-posed homogenized three-scale model. Namely, we establish the following theorem.

Theorem 1. **(i)** *There exist a subsequence* $\{(\boldsymbol{u}_\varepsilon, \theta_\varepsilon)\}_{\varepsilon\searrow 0}$ *from the family of weak solutions of Model* A_ε, *a triplet of vector-functions*

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t), \quad \boldsymbol{u}^{(1)} = \boldsymbol{u}^{(1)}(\boldsymbol{x}, \widehat{\boldsymbol{y}}, t), \quad \boldsymbol{u}^{(2)} = \boldsymbol{u}^{(2)}(\boldsymbol{x}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}}, t)$$

and a triplet of scalar functions

$$\theta = \theta(\boldsymbol{x}, t), \quad \theta^{(1)} = \theta^{(1)}(\boldsymbol{x}, \widehat{\boldsymbol{y}}, t), \quad \theta^{(2)} = \theta^{(2)}(\boldsymbol{x}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}}, t)$$

satisfying the regularity requirements

$$\boldsymbol{u} \in L^\infty(0, T; H^1(\Omega)^3), \quad \partial_t \boldsymbol{u} \in L^\infty(0, T; L^2(\Omega)^3), \tag{30a}$$
$$\boldsymbol{u}^{(1)} \in L^2(\Omega \times (0, T); (H^1_\sharp(\Sigma)/\mathbb{R})^3), \tag{30b}$$

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$$\boldsymbol{u}^{(2)} \in L^2(\Omega \times \Sigma \times (0,T); (H^1_\sharp(\Theta)/\mathbb{R})^3), \tag{30c}$$
$$\theta \in L^\infty(0,T; H^1(\Omega)), \quad \partial_t \theta \in L^\infty(0,T; L^2(\Omega)), \tag{30d}$$
$$\theta^{(1)} \in L^2(\Omega \times (0,T); H^1_\sharp(\Sigma)/\mathbb{R}), \tag{30e}$$
$$\theta^{(2)} \in L^2(\Omega \times \Sigma \times (0,T); H^1_\sharp(\Theta)/\mathbb{R}), \tag{30f}$$

and the limiting relations

$$\boldsymbol{u}_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} \boldsymbol{u} \quad \text{in the 3-sc. sense and strongly in } L^2(\Omega \times (0,T))^3, \tag{31a}$$
$$\nabla_x \boldsymbol{u}_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} \nabla_x \boldsymbol{u} + \nabla_{\hat{y}} \boldsymbol{u}^{(1)} + \nabla_{\hat{z}} \boldsymbol{u}^{(2)} \quad \text{in the 3-sc. sense}, \tag{31b}$$
$$\theta_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} \theta \quad \text{in the 3-sc. sense and strongly in } L^2(\Omega \times (0,T)), \tag{31c}$$
$$\nabla_x \theta_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} \nabla_x \theta + \nabla_{\hat{y}} \theta^{(1)} + \nabla_{\hat{z}} \theta^{(2)} \quad \text{in the 3-sc. sense}. \tag{31d}$$

(ii) *The set of the six limit functions* $(\boldsymbol{u}, \boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}, \theta, \theta^{(1)}, \theta^{(2)})$ *is the unique solution of the limit* Model H-3sc *stated below.*

Model H-3sc. (The homogenized three-scale model.) Find the set of six functions $(\boldsymbol{u}, \boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}, \theta, \theta^{(1)}, \theta^{(2)})$ satisfying the regularity requirements (30), the integral equality

$$\begin{aligned}\int_0^T \int_\Omega \int_\Sigma \int_\Theta \Big\{ & -\alpha_\tau \rho \boldsymbol{u} \cdot \partial_t \boldsymbol{\phi} \\ & + \big[\boldsymbol{\mathcal{M}}^t (\nabla_x \boldsymbol{u} + \nabla_{\hat{y}} \boldsymbol{u}^{(1)} + \nabla_{\hat{z}} \boldsymbol{u}^{(2)}) + \mathbb{M}^0 - \mathbb{B}\theta\big] : (\nabla_x \boldsymbol{\phi} + \nabla_{\hat{y}} \boldsymbol{\phi}_1 + \nabla_{\hat{z}} \boldsymbol{\phi}_2) \\ & - \alpha_g \rho \boldsymbol{f} \cdot \boldsymbol{\phi} \Big\} \, d\hat{z} d\hat{\boldsymbol{y}} d\boldsymbol{x} dt = \int_\Omega \int_\Sigma \int_\Theta \alpha_\tau \rho \boldsymbol{u}^0 \cdot \boldsymbol{\phi}^0 d\hat{z} d\hat{\boldsymbol{y}} d\boldsymbol{x}\end{aligned} \tag{32a}$$

for arbitrary smooth test vector-functions $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{x}, t)$, $\boldsymbol{\phi}_1 = \boldsymbol{\phi}_1(\boldsymbol{x}, \hat{\boldsymbol{y}}, t)$ and $\boldsymbol{\phi}_2 = \boldsymbol{\phi}_2(\boldsymbol{x}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}, t)$ such that $\boldsymbol{\phi}$, $\boldsymbol{\phi}_1$, and $\boldsymbol{\phi}_2$ vanish in a neighborhood of $\partial\Omega$ and section $\{t = T\}$, $\boldsymbol{\phi}_1$ is 1-periodic in $\hat{\boldsymbol{y}}$, and $\boldsymbol{\phi}_2$ is 1-periodic in $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$, the integral equality

$$\begin{aligned}\int_0^T \int_\Omega \int_\Sigma \int_\Theta \Big\{ & -c\theta \partial_t \eta + \varkappa (\nabla_x \theta + \nabla_{\hat{y}} \theta^{(1)} + \nabla_{\hat{z}} \theta^{(2)}) \cdot (\nabla_x \eta + \nabla_{\hat{y}} \eta_1 + \nabla_{\hat{z}} \eta_2) \\ & + \big(\mathbb{B} : (\mathbb{D}_x(\boldsymbol{u}) + \mathbb{D}_{\hat{y}}(\boldsymbol{u}^{(1)}) + \mathbb{D}_{\hat{z}}(\boldsymbol{u}^{(2)}))\big)\eta - \Psi \eta \Big\} \, d\hat{z} d\hat{\boldsymbol{y}} d\boldsymbol{x} dt \\ & = \int_\Omega \int_\Sigma \int_\Theta c\theta^0 \eta^0 d\hat{z} d\hat{\boldsymbol{y}} d\boldsymbol{x}\end{aligned} \tag{32b}$$

for arbitrary smooth test scalar functions $\eta = \eta(\boldsymbol{x}, t)$, $\eta_1 = \eta_1(\boldsymbol{x}, \hat{\boldsymbol{y}}, t)$ and $\eta_2 = \eta_2(\boldsymbol{x}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}, t)$ such that η, η_1, and η_2 vanish in a neighborhood of $\partial\Omega$ and section $\{t = T\}$, η_1 is 1-periodic in $\hat{\boldsymbol{y}}$, and η_2 is 1-periodic in $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$, and the boundary conditions

$$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{u}^\star(\boldsymbol{x}, t), \quad \theta(\boldsymbol{x}, t) = \theta^\star(\boldsymbol{x}, t) \quad \text{on } \partial\Omega \times (0,T) \tag{32c}$$

in the trace sense.

In (32a) and (32b) and further, the following notation is in use.

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Notation 3. *We denote $\boldsymbol{\phi}^0 := \boldsymbol{\phi}|_{t=0}$, $\eta^0 := \eta|_{t=0}$. For $\xi := x, \widehat{y}$ and $\widehat{z}$, we define the linear integro-differential operator $\boldsymbol{\mathcal{M}}^t \nabla_\xi$ by formula*

$$\boldsymbol{\mathcal{M}}^t \nabla_\xi \boldsymbol{w} = \chi\big(2\alpha_\mu \mathbb{D}_\xi(\boldsymbol{w}) + (\alpha_\lambda \operatorname{div}_\xi \boldsymbol{w} + \alpha_p \operatorname{div}_\xi J_t \boldsymbol{w})\mathbb{I}\big) + (1-\chi)\boldsymbol{\mathcal{G}} : J_t \nabla_\xi \boldsymbol{w}, \tag{33}$$

i.e., $\boldsymbol{\mathcal{M}}^t \nabla_\xi$ is defined by formula (9) with χ on the place of χ^ε and ξ on the place of x.

Further, we denote

$$\begin{aligned}
&\rho = \rho(x_3, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}}) := \chi(x_3, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}})\rho_F + \big(1 - \chi(x_3, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}})\big)\rho_S,\\
&\mathbb{M}^0 := -\chi p^0 \mathbb{I} + (1-\chi)\boldsymbol{\mathcal{G}} : \nabla_x \boldsymbol{v}^0,\\
&\mathbb{B} := \chi \mathbb{B}_F + (1-\chi)\mathbb{B}_S, \quad c := \chi c_F + (1-\chi)c_S,\\
&\varkappa := \chi \varkappa_F + (1-\chi)\varkappa_S, \quad \Psi := \chi \Psi_F + (1-\chi)\Psi_S.
\end{aligned}$$

Also, let us write down the explicit expressions for symmetric parts of gradients and for divergences:

$$\mathbb{D}_{\widehat{y}}(\boldsymbol{\varphi}) = \begin{pmatrix} D_{\widehat{y}11}(\boldsymbol{\varphi}) & D_{\widehat{y}12}(\boldsymbol{\varphi}) & 0 \\ D_{\widehat{y}21}(\boldsymbol{\varphi}) & D_{\widehat{y}22}(\boldsymbol{\varphi}) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{D}_{\widehat{z}}(\boldsymbol{\eta}) = \begin{pmatrix} D_{\widehat{z}11}(\boldsymbol{\eta}) & D_{\widehat{z}12}(\boldsymbol{\eta}) & 0 \\ D_{\widehat{z}21}(\boldsymbol{\eta}) & D_{\widehat{z}22}(\boldsymbol{\eta}) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\operatorname{div}_{\widehat{y}} \boldsymbol{\varphi} = \frac{\partial \varphi_1}{\partial y_1} + \frac{\partial \varphi_2}{\partial y_2}, \quad \operatorname{div}_{\widehat{z}} \boldsymbol{\eta} = \frac{\partial \eta_1}{\partial z_1} + \frac{\partial \eta_2}{\partial z_2}$$

where $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\widehat{y})$ and $\boldsymbol{\eta} = \boldsymbol{\eta}(\widehat{z})$ are arbitrary admissible vector-functions.

Proof of Theorem 1. *(i)* The limiting relations (31) in the three-scale sense follow immediately from Propositions 1–4 and the uniform estimates (19)–(21). The limiting relations (31a) and (31c) follow from (21) and the Rellich theorem.

Also, by Propositions 1–4 we have

$$J_t \boldsymbol{u}_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} J_t \boldsymbol{u}, \quad \nabla_x J_t \boldsymbol{u}_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} \nabla_x J_t \boldsymbol{u} + \nabla_{\widehat{y}} J_t \boldsymbol{u}^{(1)} + \nabla_{\widehat{z}} J_t \boldsymbol{u}^{(2)}, \quad \chi^\varepsilon \boldsymbol{u}_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} \chi \boldsymbol{u}, \tag{34a}$$

$$\chi^\varepsilon J_t \boldsymbol{u}_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} \chi J_t \boldsymbol{u}, \quad \chi^\varepsilon \nabla_x J_t \boldsymbol{u}_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} \chi\big(\nabla_x J_t \boldsymbol{u} + \nabla_{\widehat{y}} J_t \boldsymbol{u}^{(1)} + \nabla_{\widehat{z}} J_t \boldsymbol{u}^{(2)}\big), \tag{34b}$$

$$\chi^\varepsilon \nabla_x \boldsymbol{u}_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} \chi\big(\nabla_x \boldsymbol{u} + \nabla_{\widehat{y}} \boldsymbol{u}^{(1)} + \nabla_{\widehat{z}} \boldsymbol{u}^{(2)}\big) \tag{34c}$$

in the 3-sc. sense. Additionally, from Proposition 1 and the Rellich theorem it follows that

$$J_t \boldsymbol{u}_\varepsilon \underset{\varepsilon \searrow 0}{\longrightarrow} J_t \boldsymbol{u} \quad \text{strongly in } L^2(\Omega \times (0,T))^3. \tag{35}$$

(ii) Now insert the test vector-function $\boldsymbol{\varphi}^\varepsilon(\boldsymbol{x}, t)$ of the form

$$\boldsymbol{\varphi}^\varepsilon(\boldsymbol{x}, t) = \boldsymbol{\phi}(\boldsymbol{x}, t) + \varepsilon \boldsymbol{\phi}_1\Big(\boldsymbol{x}, \frac{\widehat{\boldsymbol{x}}}{\varepsilon}, t\Big) + \varepsilon^2 \boldsymbol{\phi}_2\Big(\boldsymbol{x}, \frac{\widehat{\boldsymbol{x}}}{\varepsilon}, \frac{\widehat{\boldsymbol{x}}}{\varepsilon^2}, t\Big) \tag{36}$$

into (17) and the scalar function $\eta^\varepsilon(\boldsymbol{x}, t)$ of the form

$$\eta^\varepsilon(\boldsymbol{x}, t) = \eta(\boldsymbol{x}, t) + \varepsilon \eta_1\Big(\boldsymbol{x}, \frac{\widehat{\boldsymbol{x}}}{\varepsilon}, t\Big) + \varepsilon^2 \eta_2\Big(\boldsymbol{x}, \frac{\widehat{\boldsymbol{x}}}{\varepsilon}, \frac{\widehat{\boldsymbol{x}}}{\varepsilon^2}, t\Big) \tag{37}$$

into (18). In (36), $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{x}, t)$, $\boldsymbol{\phi}_1 = \boldsymbol{\phi}_1(\boldsymbol{x}, \widehat{\boldsymbol{y}}, t)$ and $\boldsymbol{\phi}_2 = \boldsymbol{\phi}_2(\boldsymbol{x}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}}, t)$ are arbitrary smooth test vector-functions vanishing in a neighborhood of $\partial\Omega$ and section $\{t = T\}$ such that $\boldsymbol{\phi}_1$ is 1-periodic in $\widehat{\boldsymbol{y}}$ and $\boldsymbol{\phi}_2$ is 1-periodic in $\widehat{\boldsymbol{y}}$ and $\widehat{\boldsymbol{z}}$. In (37), $\eta = \eta(\boldsymbol{x}, t)$, $\eta_1 = \eta_1(\boldsymbol{x}, \widehat{\boldsymbol{y}}, t)$ and $\eta_2 = \eta_2(\boldsymbol{x}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}}, t)$ are arbitrary smooth test functions vanishing in a neighborhood of $\partial\Omega$ and section $\{t = T\}$ such that η_1 is 1-periodic in $\widehat{\boldsymbol{y}}$, and η_2 is 1-periodic in $\widehat{\boldsymbol{y}}$ and $\widehat{\boldsymbol{z}}$.

With this choice of test functions, passing to the limit as $\varepsilon \searrow 0$ and using relations (31), (34) and (35), from (17) and (18) we derive exactly the variational equations (32a) and (32b). The boundary conditions (32c) (in the sense of traces) clearly follow from the boundary conditions (7d) as $\varepsilon \searrow 0$, due to sufficient regularity of $\boldsymbol{u}_\varepsilon$, θ_ε, $\boldsymbol{u}$, and θ. Thus, the limiting passage as $\varepsilon \searrow 0$ in Model A_ε proves the existence of solutions to Model H-3sc.

The uniqueness assertion for solutions of Model H-3sc is verified standardly for linear problems: the classical energy identity for Model H-3sc yields that the solution is identically equal to zero if $\mathbb{M}^0$, $\boldsymbol{f}$, $\boldsymbol{u}^0$, Ψ, θ^0, $\boldsymbol{u}^\star$, and $\theta^\star$ are identically equal to zero. This proof is similar to justification of Theorem 1 from [13].

Theorem 1 is proved. □

Further, in Sections 6 and 7, we carry out an asymptotic decomposition in Model H-3sc, i.e., we separate the three scales from each other and, as the result, build up an effective macroscopic model.

6. Asymptotic Decomposition I: The Homogenized Two-Scale Model – Model H-2sc

The asymptotic decomposition follows the lines of [13][Secs. 6–12] with necessary modifications, and the most of technical calculations and results in the present article are exactly the same, as in [13]. Due to this, we outline the asymptotic decomposition procedure in the present article in an abridge form, focusing mainly on novel constructions and giving the precise references to the calculations and results from [13].

First, we solve the microstructure and thereby establish the following theorem.

Theorem 2. *Let the set of six functions $\big(\boldsymbol{u}, \boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}, \theta, \theta^{(1)}, \theta^{(2)}\big)$ be the solution of Model H-3sc. Then, its subset — the quadruple $\big(\boldsymbol{u}, \boldsymbol{u}^{(1)}, \theta, \theta^{(1)}\big)$ — serves as a solution to* Model H-2sc *stated below.*

Model H-2sc. (The homogenized two-scale model.) Find two vector-functions $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ and $\boldsymbol{u}^{(1)} = \boldsymbol{u}^{(1)}(\boldsymbol{x}, \widehat{\boldsymbol{y}}, t)$ and two scalar functions $\theta = \theta(\boldsymbol{x}, t)$ and $\theta^{(1)} = \theta^{(1)}(\boldsymbol{x}, \widehat{\boldsymbol{y}}, t)$ satisfying the regularity requirements (30a), (30b), (30d), and (30e), the integral equality

$$
\begin{aligned}
&\int_0^T \int_\Omega \Big\{ -\alpha_\tau \langle \rho \rangle_{\Sigma\times\Theta}(x_3)\, \boldsymbol{u}(\boldsymbol{x}, t) \cdot \partial_t \boldsymbol{\phi}(\boldsymbol{x}, t) \\
&\quad + \int_\Sigma \Big[\boldsymbol{\mathcal{A}}_0(\widehat{\boldsymbol{y}}, x_3) : \Big(\mathbb{D}_x\big(\boldsymbol{u}(\boldsymbol{x}, t)\big) + \mathbb{D}_{\widehat{y}}\big(\boldsymbol{u}^{(1)}(\boldsymbol{x}, \widehat{\boldsymbol{y}}, t)\big) \Big) \\
&\qquad + \boldsymbol{\mathcal{B}}_0(\widehat{\boldsymbol{y}}, x_3) : \Big(\mathbb{D}_x\big((J_t \boldsymbol{u})(\boldsymbol{x}, t)\big) + \mathbb{D}_{\widehat{y}}\big((J_t \boldsymbol{u}^{(1)})(\boldsymbol{x}, \widehat{\boldsymbol{y}}, t)\big) \Big) \\
&\qquad + \int_0^t \boldsymbol{\mathcal{A}}_1(\widehat{\boldsymbol{y}}, t-\tau, x_3) : \Big(\mathbb{D}_x\big(\boldsymbol{u}(\boldsymbol{x}, \tau)\big) + \mathbb{D}_{\widehat{y}}\big(\boldsymbol{u}^{(1)}(\boldsymbol{x}, \widehat{\boldsymbol{y}}, \tau)\big) \Big) d\tau \\
&\qquad + \int_0^t \boldsymbol{\mathcal{B}}_1(\widehat{\boldsymbol{y}}, t-\tau, x_3) : \Big(\mathbb{D}_x\big((J_\tau \boldsymbol{u})(\boldsymbol{x}, \tau)\big) + \mathbb{D}_{\widehat{y}}\big((J_\tau \boldsymbol{u}^{(1)})(\boldsymbol{x}, \widehat{\boldsymbol{y}}, \tau)\big) \Big) d\tau \\
&\qquad + \mathbb{H}_{\theta 1}(\widehat{\boldsymbol{y}}, x_3)\theta(\boldsymbol{x}, t) + \mathbb{H}_{\theta 2}(\widehat{\boldsymbol{y}}, x_3)(J_t \theta)(\boldsymbol{x}, t) \\
&\qquad + \mathbb{F}^0(\widehat{\boldsymbol{y}}, \boldsymbol{x}, t) \Big] : \Big(\mathbb{D}_x\big(\boldsymbol{\phi}(\boldsymbol{x}, t)\big) + \mathbb{D}_{\widehat{y}}\big(\boldsymbol{\phi}_1(\boldsymbol{x}, \widehat{\boldsymbol{y}}, t)\big) \Big) d\widehat{\boldsymbol{y}} \Big\}\, dx dt \\
&= \int_0^T \int_\Omega \alpha_g \langle \rho \rangle_{\Sigma\times\Theta}(x_3) \boldsymbol{f} \cdot \boldsymbol{\phi}\, dx dt + \int_\Omega \alpha_\tau \langle \rho \rangle_{\Sigma\times\Theta}(x_3) \boldsymbol{u}^0(\boldsymbol{x}) \cdot \boldsymbol{\phi}(\boldsymbol{x}, 0) dx
\end{aligned}
\tag{38a}
$$

for arbitrary smooth test vector-functions $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{x}, t)$ and $\boldsymbol{\phi}_1 = \boldsymbol{\phi}_1(\boldsymbol{x}, \widehat{\boldsymbol{y}}, t)$ such that $\boldsymbol{\phi}$ and $\boldsymbol{\phi}_1$ vanish in a neighborhood of $\partial\Omega$ and section $\{t = T\}$ and $\boldsymbol{\phi}_1$ is 1-periodic in $\widehat{\boldsymbol{y}}$, the integral equality

$$
\int_0^T \int_\Omega \Big\{ -\langle c \rangle_{\Sigma\times\Theta}(x_3)\theta(\boldsymbol{x}, t)\partial_t \eta(\boldsymbol{x}, t)
$$

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$$
\begin{aligned}
&+ \int_{\Sigma}\Big[\mathbb{L}_{\varkappa}(\widehat{\boldsymbol{y}},x_3)\big(\nabla_x\theta(\boldsymbol{x},t)+\nabla_{\widehat{y}}\theta^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{y}},t)\big)\cdot\big(\nabla_x\eta(\boldsymbol{x},t)+\nabla_{\widehat{y}}\eta_1(\boldsymbol{x},\widehat{\boldsymbol{y}},t)\big)\\
&\quad+\Big(\mathbb{L}_{B1}(\widehat{\boldsymbol{y}},x_3):\big(\mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},t))+\mathbb{D}_{\widehat{y}}(\boldsymbol{u}^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{y}},t))\big)\\
&\qquad+\int_0^t\mathbb{L}_{B2}(\widehat{\boldsymbol{y}},t-\tau,x_3):\big(\mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},\tau))+\mathbb{D}_{\widehat{y}}(\boldsymbol{u}^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{y}},\tau))\big)d\tau\\
&\qquad+L_\theta(\widehat{\boldsymbol{y}},x_3)\theta(\boldsymbol{x},t)\\
&\qquad+L_p^0(\widehat{\boldsymbol{y}},t,x_3)p^0(\boldsymbol{x})+\mathbb{L}_v^0(\widehat{\boldsymbol{y}},t,x_3):\nabla_x\boldsymbol{v}^0(\boldsymbol{x})\Big)\eta(\boldsymbol{x},t)\Big]d\widehat{\boldsymbol{y}}\Big\}\,d\boldsymbol{x}dt\\
&=\int_0^T\int_\Omega\langle\Psi\rangle_{\Sigma\times\Theta}(\boldsymbol{x},t)\eta(\boldsymbol{x},t)d\boldsymbol{x}dt+\int_\Omega\langle c\rangle_{\Sigma\times\Theta}(x_3)\theta^0(\boldsymbol{x})\eta(\boldsymbol{x},0)d\boldsymbol{x} \qquad (38b)
\end{aligned}
$$

for arbitrary smooth test scalar functions $\eta=\eta(\boldsymbol{x},t)$ and $\eta_1=\eta_1(\boldsymbol{x},\widehat{\boldsymbol{y}},t)$ such that η and η_1 vanish in a neighborhood of $\partial\Omega$ and section $\{t=T\}$ and η_1 is 1-periodic in $\widehat{\boldsymbol{y}}$, and the boundary conditions (32c) in the trace sense.

Notation 4. *In (38) and further in the article, by $\langle\phi\rangle_{\Sigma\times\Theta}$, $\langle\phi\rangle_\Theta$, and $\langle\phi\rangle_\Sigma$ we denote the mean value of 1-periodic functions ϕ on $\Sigma\times\Theta$, Θ, and Σ, respectively:*

$$
\langle\phi\rangle_{\Sigma\times\Theta}=\int_\Sigma\int_\Theta\phi(\widehat{\boldsymbol{y}},\widehat{\boldsymbol{z}})d\widehat{\boldsymbol{z}}d\widehat{\boldsymbol{y}},\quad \langle\phi(\widehat{\boldsymbol{y}},\cdot)\rangle_\Theta=\int_\Theta\phi(\widehat{\boldsymbol{y}},\widehat{\boldsymbol{z}})d\widehat{\boldsymbol{z}},\quad \langle\phi(\cdot,\widehat{\boldsymbol{z}})\rangle_\Sigma=\int_\Sigma\phi(\widehat{\boldsymbol{y}},\widehat{\boldsymbol{z}})d\widehat{\boldsymbol{y}}
$$

for all admissible 1-periodic in $\widehat{\boldsymbol{y}}$ and $\widehat{\boldsymbol{z}}$ functions ϕ.

By $|\Theta_F|$ and $|\Sigma_F|$ we denote the (Lebesgue) measures of Θ_F and Σ_F, respectively:

$$
|\Theta_F|=\int_\Theta\widehat{\psi}(\widehat{\boldsymbol{z}})d\widehat{\boldsymbol{z}}\equiv\int_{\Theta_F}d\widehat{\boldsymbol{z}},\quad |\Sigma_F|=\int_\Sigma\widehat{\zeta}(\widehat{\boldsymbol{y}})d\widehat{\boldsymbol{y}}\equiv\int_{\Sigma_F}d\widehat{\boldsymbol{y}}.
$$

Also, further we use the following notation for the elements of the Cartesian bases in $\mathbb{R}^{3\times3}_{symm}$ and $\mathbb{R}^3$.

Notation 5. *The 3×3-matrix $\mathbb{J}^{mn}$ $(m,n=1,2,3)$ is defined by the formula*

$$
\mathbb{J}^{mn}=\frac{\boldsymbol{e}_m\otimes\boldsymbol{e}_n+\boldsymbol{e}_n\otimes\boldsymbol{e}_m}{2},
$$

or, in the component-wise form,

$$
(\mathbb{J}^{mn})_{ij}=\frac{\delta_{im}\delta_{jn}+\delta_{jm}\delta_{in}}{2},\quad i,j=1,2,3.
$$

By $\boldsymbol{e}_m$ $(m=1,2,3)$ we denote the standard Cartesian basis vectors in $\mathbb{R}^3$. Thus $(\boldsymbol{e}_m)_j=\delta_{mj}$.

The fourth-rank tensors $\boldsymbol{\mathcal{A}}_0$, $\boldsymbol{\mathcal{B}}_0$, $\boldsymbol{\mathcal{A}}_1$, and $\boldsymbol{\mathcal{B}}_1$, the 3×3-matrices $\mathbb{H}_{\theta1}$, $\mathbb{H}_{\theta2}$, $\mathbb{F}^0$, $\mathbb{L}_\varkappa$, $\mathbb{L}_{B1}$, $\mathbb{L}_{B2}$, and $\mathbb{L}_v^0$ and the scalar functions L_θ and L_p^0 are uniquely defined by the solutions of the so-called *cell problems* posed on the microscopic pattern cell Θ. In turn, the cell problems contain the complete information about the homogenized dynamics of the thermomechanical system at the level of the most fine bristle structure. This means that the thermomechanical behavior at the scale of variable $\widehat{\boldsymbol{z}}$ is sublimed in the components of $\boldsymbol{\mathcal{A}}_0$, $\boldsymbol{\mathcal{B}}_0$, $\boldsymbol{\mathcal{A}}_1$, $\boldsymbol{\mathcal{B}}_1$, $\mathbb{H}_{\theta1}$, $\mathbb{H}_{\theta2}$, $\mathbb{F}^0$, $\mathbb{L}_\varkappa$, $\mathbb{L}_{B1}$, $\mathbb{L}_{B2}$, and $\mathbb{L}_v^0$, and in scalars L_θ and L_p^0.

The exact expressions of $\boldsymbol{\mathcal{A}}_0$, $\boldsymbol{\mathcal{B}}_0$, $\boldsymbol{\mathcal{A}}_1$, $\boldsymbol{\mathcal{B}}_1$ and $\mathbb{F}^0$ have already been established in [13][Secs. 6,7] along with the formulations of the corresponding cell problems. The exact expressions of $\mathbb{H}_{\theta1}$, $\mathbb{H}_{\theta2}$, $\mathbb{L}_\varkappa$, $\mathbb{L}_{B1}$, $\mathbb{L}_{B2}$, $\mathbb{L}_v^0$, L_θ, and L_p^0 are novel, as are the formulations of the cell problems associated with these expressions. Now, we present the exact expressions of $\boldsymbol{\mathcal{A}}_0$, $\boldsymbol{\mathcal{B}}_0$, $\boldsymbol{\mathcal{A}}_1$, $\boldsymbol{\mathcal{B}}_1$, $\mathbb{H}_{\theta1}$, $\mathbb{H}_{\theta2}$, $\mathbb{F}^0$, $\mathbb{L}_\varkappa$, $\mathbb{L}_{B1}$, $\mathbb{L}_{B2}$, $\mathbb{L}_v^0$, L_θ, and L_p^0 and the exact formulations of all the cell problems (posed on Θ) along with the results on their well-posedness. After this, we give a proof of Theorem 2.

We have

$$\boldsymbol{\mathcal{A}}_0(\widehat{y},x_3)=\begin{cases}\alpha_\lambda\mathbb{I}\otimes\mathbb{I}+2\alpha_\mu\sum_{m,n=1}^{3}\mathbb{J}^{mn}\otimes\mathbb{J}^{mn} & \text{for } x_3\in(\Delta+\delta^*,1),\\ \widehat{\zeta}(\widehat{y})\Big(\alpha_\lambda\mathbb{I}\otimes\mathbb{I}+2\alpha_\mu\sum_{m,n=1}^{3}\mathbb{J}^{mn}\otimes\mathbb{J}^{mn}\Big) & \text{for } x_3\in(\Delta+\delta_*,\Delta+\delta^*],\\ \widehat{\zeta}(\widehat{y})\boldsymbol{\mathcal{A}}_0^f & \text{for } x_3\in[\Delta,\Delta+\delta_*],\\ 0 & \text{for } x_3\in(0,\Delta),\end{cases}\tag{39a}$$

where

$$\boldsymbol{\mathcal{A}}_0^f\overset{def}{=}\alpha_\lambda\,\mathbb{I}\otimes\Big(|\Theta_F|\,\mathbb{I}+\sum_{m,n=1}^{3}\langle\widehat{\psi}\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{00}^{mn}\rangle_\Theta\,\mathbb{J}^{mn}\Big)$$
$$+\,2\alpha_\mu\sum_{m,n=1}^{3}\Big(|\Theta_F|\,\mathbb{J}^{mn}+\langle\widehat{\psi}\,\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{00}^{mn})\rangle_\Theta\Big)\otimes\mathbb{J}^{mn};\tag{39b}$$

$$\boldsymbol{\mathcal{B}}_0(\widehat{y},x_3)=\begin{cases}\alpha_p\,\mathbb{I}\otimes\mathbb{I} & \text{for}\quad x_3\in(\Delta+\delta^*,1),\\ \widehat{\zeta}(\widehat{y})\alpha_p\,\mathbb{I}\otimes\mathbb{I}+(1-\widehat{\zeta}(\widehat{y}))\boldsymbol{\mathcal{G}} & \text{for}\quad x_3\in(\Delta+\delta_*,\Delta+\delta^*],\\ \widehat{\zeta}(\widehat{y})\boldsymbol{\mathcal{B}}_0^f+(1-|\Theta_F|\widehat{\zeta}(\widehat{y}))\boldsymbol{\mathcal{G}} & \text{for}\quad x_3\in[\Delta,\Delta+\delta_*],\\ \boldsymbol{\mathcal{G}} & \text{for}\quad x_3\in(0,\Delta),\end{cases}\tag{40a}$$

where

$$\boldsymbol{\mathcal{B}}_0^f\overset{def}{=}\alpha_p\,\mathbb{I}\otimes\Big(|\Theta_F|\,\mathbb{I}+\sum_{m,n=1}^{3}\langle\widehat{\psi}\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{00}^{mn}\rangle_\Theta\,\mathbb{J}^{mn}\Big)$$
$$+\sum_{m,n=1}^{3}\Big(\boldsymbol{\mathcal{G}}:\langle(1-\widehat{\psi})\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{00}^{mn})\rangle_\Theta\Big)\otimes\mathbb{J}^{mn};\tag{40b}$$

$$\boldsymbol{\mathcal{A}}_1(\widehat{y},t,x_3)=\begin{cases}\widehat{\zeta}(\widehat{y})\boldsymbol{\mathcal{A}}_1^f(t) & \text{for}\quad x_3\in[\Delta,\Delta+\delta_*],\\ 0 & \text{for}\quad x_3\in(0,\Delta)\cup(\Delta+\delta_*,1),\end{cases}\tag{41a}$$

where

$$\boldsymbol{\mathcal{A}}_1^f(t)\overset{def}{=}\alpha_\lambda\,\mathbb{I}\otimes\sum_{m,n=1}^{3}\langle\widehat{\psi}\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{10}^{mn}\rangle_\Theta(t)\,\mathbb{J}^{mn}+2\alpha_\mu\sum_{m,n=1}^{3}\langle\widehat{\psi}\,\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{10}^{mn})\rangle_\Theta(t)\otimes\mathbb{J}^{mn};\tag{41b}$$

$$\boldsymbol{\mathcal{B}}_1(\widehat{y},t,x_3)=\begin{cases}\widehat{\zeta}(\widehat{y})\boldsymbol{\mathcal{B}}_1^f(t) & \text{for}\quad x_3\in[\Delta,\Delta+\delta_*],\\ 0 & \text{for}\quad x_3\in(0,\Delta)\cup(\Delta+\delta_*,1),\end{cases}\tag{42a}$$

where

$$\boldsymbol{\mathcal{B}}_1^f(t)\overset{def}{=}\alpha_p\,\mathbb{I}\otimes\sum_{m,n=1}^{3}\langle\widehat{\psi}\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{10}^{mn}\rangle_\Theta(t)\,\mathbb{J}^{mn}$$
$$+\sum_{m,n=1}^{3}\Big(\boldsymbol{\mathcal{G}}:\langle(1-\widehat{\psi})\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{10}^{mn})\rangle_\Theta(t)\Big)\otimes\mathbb{J}^{mn};\tag{42b}$$

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$$\mathbb{F}^0(\widehat{\boldsymbol{y}},\boldsymbol{x},t) = \begin{cases} -p^0(\boldsymbol{x})\mathbb{I} \quad \text{for} \quad x_3 \in (\Delta+\delta^*,1), \\ -\widehat{\zeta}(\widehat{\boldsymbol{y}})p^0(\boldsymbol{x})\mathbb{I} + \big(1-\widehat{\zeta}(\widehat{\boldsymbol{y}})\big)\boldsymbol{\mathcal{G}} : \nabla_x \boldsymbol{v}^0 \quad \text{for} \quad x_3 \in (\Delta+\delta_*,\Delta+\delta^*], \\ \widehat{\zeta}(\widehat{\boldsymbol{y}})\Big(p^0(\boldsymbol{x})\mathbb{F}^0_f(t) + \sum\limits_{m,n=1}^{3} \dfrac{\partial v^0_m}{\partial x_n}(\boldsymbol{x})\mathbb{F}^{0mn}_{sol}(t)\Big) + \big(1-|\Theta_F|\widehat{\zeta}(\widehat{\boldsymbol{y}})\big)\boldsymbol{\mathcal{G}} : \nabla_x \boldsymbol{v}^0 \\ \quad \text{for} \quad x_3 \in [\Delta,\Delta+\delta_*], \\ \boldsymbol{\mathcal{G}} : \nabla_x \boldsymbol{v}^0 \quad \text{for} \quad x_3 \in (0,\Delta), \end{cases} \tag{43a}$$

where

$$\mathbb{F}^0_f(t) \stackrel{def}{=} \alpha_\lambda \big\langle \widehat{\psi}\, \mathrm{div}_{\widehat{z}}\, \boldsymbol{Z}_{20} \big\rangle_\Theta(t)\,\mathbb{I} + 2\alpha_\mu \big\langle \widehat{\psi}\, \mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{20}) \big\rangle_\Theta(t) + \alpha_p \,\big\langle \widehat{\psi}\, \mathrm{div}_{\widehat{z}}\, J_t \boldsymbol{Z}_{20} \big\rangle_\Theta(t)\,\mathbb{I} + \boldsymbol{\mathcal{G}} : \big\langle (1-\widehat{\psi})\mathbb{D}_{\widehat{z}}(J_t \boldsymbol{Z}_{20}) \big\rangle_\Theta(t) - |\Theta_F|\,\mathbb{I} \tag{43b}$$

and

$$\mathbb{F}^{0mn}_{sol}(t) \stackrel{def}{=} \alpha_\lambda \big\langle \widehat{\psi}\, \mathrm{div}_{\widehat{z}}\, \boldsymbol{Z}^{mn}_{30} \big\rangle_\Theta(t)\,\mathbb{I} + 2\alpha_\mu \big\langle \widehat{\psi}\, \mathbb{D}_{\widehat{z}}(\boldsymbol{Z}^{mn}_{30}) \big\rangle_\Theta(t) + \alpha_p \,\big\langle \widehat{\psi}\, \mathrm{div}_{\widehat{z}}\, J_t \boldsymbol{Z}^{mn}_{30} \big\rangle_\Theta(t)\,\mathbb{I} + \boldsymbol{\mathcal{G}} : \big\langle (1-\widehat{\psi})\mathbb{D}_{\widehat{z}}(J_t \boldsymbol{Z}^{mn}_{30}) \big\rangle_\Theta(t), \quad m,n = 1,2,3; \tag{43c}$$

$$\mathbb{H}_{\theta 1}(\widehat{\boldsymbol{y}},x_3) = \begin{cases} \widehat{\zeta}(\widehat{\boldsymbol{y}})\mathbb{H}^f_{\theta 1} & \text{for} \quad x_3 \in [\Delta,\Delta+\delta_*], \\ 0 & \text{for} \quad x_3 \in (0,\Delta) \cup (\Delta+\delta_*,1), \end{cases} \tag{44a}$$

where

$$\mathbb{H}^f_{\theta 1} = 2\alpha_\mu \langle \widehat{\psi}\, \mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{40}) \rangle_\Theta + \alpha_\lambda \langle \widehat{\psi}\, \mathrm{div}_{\widehat{z}}\, \boldsymbol{Z}_{40} \rangle_\Theta\, \mathbb{I} - |\Theta_F|\mathbb{B}_F - (1-|\Theta_F|)\mathbb{B}_S; \tag{44b}$$

$$\mathbb{H}_{\theta 2}(\widehat{\boldsymbol{y}},x_3) = \begin{cases} \widehat{\zeta}(\widehat{\boldsymbol{y}})\mathbb{H}^f_{\theta 2} & \text{for} \quad x_3 \in [\Delta,\Delta+\delta_*], \\ 0 & \text{for} \quad x_3 \in (0,\Delta) \cup (\Delta+\delta_*,1), \end{cases} \tag{45a}$$

where

$$\mathbb{H}^f_{\theta 2} = \alpha_p \langle \widehat{\psi}\, \mathrm{div}_{\widehat{z}}\, \boldsymbol{Z}_{40} \rangle_\Theta\, \mathbb{I} + \boldsymbol{\mathcal{G}} : \langle (1-\widehat{\psi})\nabla_{\widehat{z}} \boldsymbol{Z}_{40} \rangle_\Theta; \tag{45b}$$

$$\mathbb{L}_\varkappa(\widehat{\boldsymbol{y}},x_3) = \begin{cases} \varkappa_F \mathbb{I} & \text{for} \quad x_3 \in (\Delta+\delta^*,1), \\ \big(\widehat{\zeta}(\widehat{\boldsymbol{y}})\varkappa_F + (1-\widehat{\zeta}(\widehat{\boldsymbol{y}}))\varkappa_S\big)\mathbb{I} & \text{for} \quad x_3 \in (\Delta+\delta_*,\Delta+\delta^*], \\ \mathbb{L}^\theta_\varkappa(\widehat{\boldsymbol{y}}) & \text{for} \quad x_3 \in [\Delta,\Delta+\delta_*], \\ \varkappa_S \mathbb{I} & \text{for} \quad x_3 \in (0,\Delta), \end{cases} \tag{46a}$$

where

$$\mathbb{L}^\theta_\varkappa(\widehat{\boldsymbol{y}}) = \big(\widehat{\zeta}(\widehat{\boldsymbol{y}})|\Theta_F|\varkappa_F + \big(1-\widehat{\zeta}(\widehat{\boldsymbol{y}})|\Theta_F|\big)\varkappa_S\big)\mathbb{I} + \widehat{\zeta}(\widehat{\boldsymbol{y}})\Big\langle \widehat{\psi}\nabla_{\widehat{z}}\Big(\sum_{n=1}^{3} Z^n_5(\widehat{\boldsymbol{y}},\cdot)\boldsymbol{e}_n\Big)\Big\rangle_\Theta(\varkappa_F - \varkappa_S); \tag{46b}$$

$$\mathbb{L}_{B1}(\widehat{\boldsymbol{y}},x_3) = \begin{cases} \mathbb{B}_F & \text{for} \quad x_3 \in (\Delta+\delta^*,1), \\ \widehat{\zeta}(\widehat{\boldsymbol{y}})\mathbb{B}_F + (1-\widehat{\zeta}(\widehat{\boldsymbol{y}}))\mathbb{B}_S & \text{for} \quad x_3 \in (\Delta+\delta_*,\Delta+\delta^*], \\ \mathbb{L}^\theta_{B1}(\widehat{\boldsymbol{y}}) & \text{for} \quad x_3 \in [\Delta,\Delta+\delta_*], \\ \mathbb{B}_S & \text{for} \quad x_3 \in (0,\Delta), \end{cases} \tag{47a}$$

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where

$$\mathbb{L}^{\theta}_{B1}(\widehat{\boldsymbol{y}}) = \widehat{\zeta}(\widehat{\boldsymbol{y}})|\Theta_F|\mathbb{B}_F + \big(1-\widehat{\zeta}(\widehat{\boldsymbol{y}})|\Theta_F|\big)\mathbb{B}_S + \widehat{\zeta}(\widehat{\boldsymbol{y}})\sum_{m,n=1}^{3}\big(\langle\widehat{\psi}\mathbb{D}_{\widehat{z}}(\mathbf{Z}^{mn}_{00})\rangle_{\Theta} : (\mathbb{B}_F - \mathbb{B}_S)\big)\mathbb{J}^{mn}; \tag{47b}$$

$$\mathbb{L}_{B2}(\widehat{\boldsymbol{y}},t,x_3) = \begin{cases} \widehat{\zeta}(\widehat{\boldsymbol{y}})\mathbb{L}^{f}_{B2}(t) & \text{for} \quad x_3 \in [\Delta, \Delta+\delta_*], \\ 0 & \text{for} \quad x_3 \in (0,\Delta)\cup(\Delta+\delta_*,1), \end{cases} \tag{48a}$$

where

$$\mathbb{L}^{f}_{B2}(t) = \sum_{m,n=1}^{3}\big(\langle\widehat{\psi}\mathbb{D}_{\widehat{z}}(\mathbf{Z}^{mn}_{10}(\cdot,t))\rangle_{\Theta} : (\mathbb{B}_F - \mathbb{B}_S)\big)\mathbb{J}^{mn}; \tag{48b}$$

$$L_{\theta}(\widehat{\boldsymbol{y}},x_3) = \begin{cases} \widehat{\zeta}(\widehat{\boldsymbol{y}})L^{f}_{\theta} & \text{for} \quad x_3 \in [\Delta, \Delta+\delta_*], \\ 0 & \text{for} \quad x_3 \in (0,\Delta)\cup(\Delta+\delta_*,1), \end{cases} \tag{49a}$$

where

$$L^{f}_{\theta} = \langle\widehat{\psi}\mathbb{D}_{\widehat{z}}(\mathbf{Z}_{40})\rangle_{\Theta} : (\mathbb{B}_F - \mathbb{B}_S); \tag{49b}$$

$$L^{0}_{p}(\widehat{\boldsymbol{y}},t,x_3) = \begin{cases} \widehat{\zeta}(\widehat{\boldsymbol{y}})L^{0}_{pf}(t) & \text{for} \quad x_3 \in [\Delta, \Delta+\delta_*], \\ 0 & \text{for} \quad x_3 \in (0,\Delta)\cup(\Delta+\delta_*,1), \end{cases} \tag{50a}$$

where

$$L^{0}_{pf}(t) = \langle\widehat{\psi}\mathbb{D}_{\widehat{z}}(\mathbf{Z}_{20}(\cdot,t))\rangle_{\Theta} : (\mathbb{B}_F - \mathbb{B}_S); \tag{50b}$$

and

$$\mathbb{L}^{0}_{v}(\widehat{\boldsymbol{y}},t,x_3) = \begin{cases} \widehat{\zeta}(\widehat{\boldsymbol{y}})\mathbb{L}^{0}_{vf}(t) & \text{for} \quad x_3 \in [\Delta, \Delta+\delta_*], \\ 0 & \text{for} \quad x_3 \in (0,\Delta)\cup(\Delta+\delta_*,1), \end{cases} \tag{51a}$$

where

$$\mathbb{L}^{0}_{vf}(t) = \sum_{m,n=1}^{3}\big(\langle\widehat{\psi}\mathbb{D}_{\widehat{z}}(\mathbf{Z}^{mn}_{30}(\cdot,t))\rangle_{\Theta} : (\mathbb{B}_F - \mathbb{B}_S)\big)\mathbb{J}^{mn}. \tag{51b}$$

In (39)–(43), (47), (48), (50), and (51), vector-functions $\mathbf{Z}^{ij}_{00}$, $\mathbf{Z}^{ij}_{10}$, $\mathbf{Z}_{20}$, and $\mathbf{Z}^{ij}_{30}$ are the solutions of the following problems set on the microscopic pattern cell Θ:

Problem Z1. ([13][page 1380].) Find a vector-function $\mathbf{Z}^{ij}_{00} = \mathbf{Z}^{ij}_{00}(\widehat{z})$ $(i,j=1,2,3)$, which satisfies the regularity condition

$$\mathbf{Z}^{ij}_{00} \in (H^1_{\sharp}(\Theta)/\mathbb{R})^3, \tag{52}$$

and resolves the Stokes system

$$\int_{\Theta_F}\Big\{\alpha_\lambda\delta_{ij}\mathbb{I} + 2\alpha_\mu\mathbb{J}^{ij} + \alpha_\lambda\mathbb{I}\,\mathrm{div}_{\widehat{z}}\mathbf{Z}^{ij}_{00}(\widehat{z}) + 2\alpha_\mu\mathbb{D}_{\widehat{z}}\big(\mathbf{Z}^{ij}_{00}(\widehat{z})\big)\Big\} : \mathbb{D}_{\widehat{z}}(\boldsymbol{\phi}_{21}(\widehat{z}))d\widehat{z} = 0,$$
$$\forall\,\boldsymbol{\phi}_{21} \in H^1_{\sharp}(\Theta)^3, \tag{53a}$$

$$\mathbf{Z}^{ij}_{00} \in H^1_{\sharp}(\Theta)^3, \tag{53b}$$

and the linear elasticity problem

$$\begin{cases} \mathrm{div}_{\widehat{z}}\Big\{\mathcal{G} : \big(\nabla_{\widehat{z}}\mathbf{Z}^{ij}_{00}(\widehat{z}) + \mathbb{J}^{ij}\big)\Big\} = 0 \text{ on } \Theta_S, \\ \mathbf{Z}^{ij}_{00}(\widehat{z})|_{\partial\Theta_S} = \mathbf{Z}^{ij}_{00}(\widehat{z})|_{\partial\Theta_F\setminus\partial\Theta}, \\ \mathbf{Z}^{ij}_{00} \in H^1_{\sharp}(\Theta_S)^3, \end{cases} \tag{54}$$

in the weak sense, i.e., in the sense of the integral equality

$$\int_{\Theta_S} \Big\{ \mathcal{G} : (\mathbb{J}^{ij} + \nabla_{\widehat{z}} \boldsymbol{Z}_{00}^{ij}(\widehat{z})) \Big\} : \nabla_{\widehat{z}} \boldsymbol{\phi}_{21} d\widehat{z} = 0, \quad \forall\, \boldsymbol{\phi}_{21} \in H^1_\sharp(\Theta)^3. \tag{55}$$

(In (54), $\boldsymbol{Z}_{00}^{ij}(\widehat{z})|_{\partial\Theta_F \setminus \partial\Theta}$ is the trace of the solution of the system (53) on the interface between Θ_F and Θ_S.)

Problem Z2. ([13][pages 1380-1381].) Find a vector-function $\widehat{z} \mapsto \boldsymbol{Z}_{10}^{ij}(\widehat{z}, 0)$ $(i, j = 1, 2, 3)$, defined in Θ_F, which satisfies the regularity condition

$$\boldsymbol{Z}_{10}^{ij}(\cdot, 0) \in (H^1_\sharp(\Theta_F)/\mathbb{R})^3,$$

and the integral equality

$$\int_{\Theta_F} \Big\{ \alpha_p \Big(\delta_{ij} + \mathrm{div}_{\widehat{z}}\, \boldsymbol{Z}_{00}^{ij}(\widehat{z}) \Big) \mathbb{I} + \alpha_\lambda \mathbb{I}\, \mathrm{div}_{\widehat{z}}\, \boldsymbol{Z}_{10}^{ij}(\widehat{z}, 0) + 2\alpha_\mu \mathbb{D}_{\widehat{z}} \big(\boldsymbol{Z}_{10}^{ij}(\widehat{z}, 0) \big) \Big\} : \mathbb{D}_{\widehat{z}}(\boldsymbol{\phi}_{21}(\widehat{z})) d\widehat{z} = 0,$$
$$\forall\, \boldsymbol{\phi}_{21} \in H^1_\sharp(\Theta)^3. \tag{56}$$

In (56), $\boldsymbol{Z}_{00}^{ij}$ is the solution of Problem Z1.

Problem Z3. ([13][page 1381].) Find a vector-function $(\widehat{z}, t) \mapsto \boldsymbol{Z}_{10}^{ij}(\widehat{z}, t)$ $(i, j = 1, 2, 3)$, defined in $\Theta \times (0, T)$, which satisfies the regularity condition

$$\boldsymbol{Z}_{10}^{ij} \in L^\infty(0, T; (H^1_\sharp(\Theta)/\mathbb{R})^3) \cap H^1(0, T; H^1_\sharp(\Theta_F)^3), \tag{57a}$$

the integral equality

$$\int_{\Theta_F} \Big\{ \alpha_\lambda \mathbb{I}\, \mathrm{div}_{\widehat{z}} \frac{\partial \boldsymbol{Z}_{10}^{ij}}{\partial t}(\widehat{z}, t) + 2\alpha_\mu \mathbb{D}_{\widehat{z}} \Big(\frac{\partial \boldsymbol{Z}_{10}^{ij}}{\partial t}(\widehat{z}, t) \Big) + \alpha_p\, \mathbb{I}\, \mathrm{div}_{\widehat{z}}\, \boldsymbol{Z}_{10}^{ij}(\widehat{z}, t) \Big\} : \mathbb{D}_{\widehat{z}}(\boldsymbol{\phi}_{21}(\widehat{z})) d\widehat{z}$$
$$+ \int_{\Theta_S} \Big\{ \mathcal{G} : \nabla_{\widehat{z}} \boldsymbol{Z}_{10}^{ij}(\widehat{z}, t) \Big\} : \nabla_{\widehat{z}} \boldsymbol{\phi}_{21}(\widehat{z}) d\widehat{z} = 0,$$
$$\forall\, \boldsymbol{\phi}_{21} \in H^1_\sharp(\Theta)^3, \quad \text{for a.e. } t \in (0, T], \tag{57b}$$

and the initial condition

$$\boldsymbol{Z}_{10}^{ij}(\widehat{z}, t)|_{t=0} = \boldsymbol{Z}_{10}^{ij}(\widehat{z}, 0) \quad \text{for } \widehat{z} \in \Theta_F, \tag{57c}$$

where $\widehat{z} \mapsto \boldsymbol{Z}_{10}^{ij}(\widehat{z}, 0)$ is the solution of Problem Z2.

Problem Z4. ([13][page 1382].) Find a vector-function $\widehat{z} \mapsto \boldsymbol{Z}_{20}(\widehat{z}, 0)$, defined in Θ_F, which satisfies the regularity condition

$$\boldsymbol{Z}_{20}(\cdot, 0) \in (H^1_\sharp(\Theta_F)/\mathbb{R})^3,$$

and the integral equality

$$\int_{\Theta_F} \Big\{ -\mathbb{I} + \alpha_\lambda \mathbb{I}\, \mathrm{div}_{\widehat{z}}\, \boldsymbol{Z}_{20}(\widehat{z}, 0) + 2\alpha_\mu \mathbb{D}_{\widehat{z}} \big(\boldsymbol{Z}_{20}(\widehat{z}, 0) \big) \Big\} : \mathbb{D}_{\widehat{z}}(\boldsymbol{\phi}_{21}(\widehat{z})) d\widehat{z} = 0, \quad \forall\, \boldsymbol{\phi}_{21} \in H^1_\sharp(\Theta)^3.$$

Problem Z5. ([13][page 1382].) Find a vector-function $(\widehat{z}, t) \mapsto \boldsymbol{Z}_{20}(\widehat{z}, t)$, defined in $\Theta \times (0, T)$, which satisfies the regularity condition

$$\boldsymbol{Z}_{20} \in L^\infty(0, T; (H^1_\sharp(\Theta)/\mathbb{R})^3) \cap H^1(0, T; H^1_\sharp(\Theta_F)^3), \tag{58a}$$

the integral equality

$$\int_{\Theta_F} \Big\{ 2\alpha_\mu \mathbb{D}_{\widehat{z}}\Big(\frac{\partial \mathbf{Z}_{20}}{\partial t}(\widehat{z},t)\Big) + \alpha_\lambda \mathbb{I}\, \mathrm{div}_{\widehat{z}} \frac{\partial \mathbf{Z}_{20}}{\partial t}(\widehat{z},t) + \alpha_p\, \mathbb{I}\, \mathrm{div}_{\widehat{z}}\, \mathbf{Z}_{20}(\widehat{z},t) \Big\} : \mathbb{D}_{\widehat{z}}(\boldsymbol{\phi}_{21}(\widehat{z}))d\widehat{z}$$
$$+ \int_{\Theta_S} \Big\{ \boldsymbol{\mathcal{G}} : \nabla_{\widehat{z}} \mathbf{Z}_{20}(\widehat{z},t) \Big\} : \nabla_{\widehat{z}} \boldsymbol{\phi}_{21}(\widehat{z}) d\widehat{z} = 0,$$
$$\forall\, \boldsymbol{\phi}_{21} \in H^1_\sharp(\Theta)^3, \quad \text{for a.e. } t \in (0,T], \tag{58b}$$

and the initial condition

$$\mathbf{Z}_{20}(\widehat{z},t)\big|_{t=0} = \mathbf{Z}_{20}(\widehat{z},0) \quad \text{for } \widehat{z} \in \Theta_F,$$

where $\widehat{z} \mapsto \mathbf{Z}_{20}(\widehat{z},0)$ is the solution of Problem Z4.

Problem Z6. ([13][page 1382].) Find a vector-function $\widehat{z} \mapsto \mathbf{Z}^{ij}_{30}(\widehat{z},0)$ $(i,j = 1,2,3)$, defined in Θ_F, which satisfies the regularity condition

$$\mathbf{Z}^{ij}_{30}(\cdot,0) \in (H^1_\sharp(\Theta_F)/\mathbb{R})^3,$$

and the integral equality

$$\int_{\Theta_F} \Big\{ 2\alpha_\mu \mathbb{D}_{\widehat{z}}(\mathbf{Z}^{ij}_{30}(\widehat{z},0)) + \alpha_\lambda \mathbb{I}\, \mathrm{div}_{\widehat{z}}\, \mathbf{Z}^{ij}_{30}(\widehat{z},0) \Big\} : \mathbb{D}_{\widehat{z}}(\boldsymbol{\phi}_{21}(\widehat{z}))d\widehat{z}$$
$$+ \int_{\Theta_S} \Big\{ \boldsymbol{\mathcal{G}} : \mathbb{J}^{ij} \Big\} : \nabla_{\widehat{z}} \boldsymbol{\phi}_{21}(\widehat{z}) d\widehat{z} = 0, \quad \forall\, \boldsymbol{\phi}_{21} \in H^1_\sharp(\Theta)^3. \tag{59}$$

Problem Z7. ([13][page 1383].) Find a vector-function $(\widehat{z},t) \mapsto \mathbf{Z}^{ij}_{30}(\widehat{z},t)$ $(i,j = 1,2,3)$, defined in the space-time parallelepiped $\Theta \times (0,T)$, which satisfies the regularity condition

$$\mathbf{Z}^{ij}_{30} \in L^\infty(0,T;(H^1_\sharp(\Theta)/\mathbb{R})^3) \cap H^1(0,T;H^1_\sharp(\Theta_F)^3), \tag{60}$$

the integral equality (58b) (with $\mathbf{Z}^{ij}_{30}$ on the place of $\mathbf{Z}_{20}$) and the initial condition

$$\mathbf{Z}^{ij}_{30}(\widehat{z},t)|_{t=0} = \mathbf{Z}^{ij}_{30}(\widehat{z},0) \quad \text{for } \widehat{z} \in \Theta_F,$$

where the vector-function $\widehat{z} \mapsto \mathbf{Z}^{ij}_{30}(\widehat{z},0)$ is the solution of Problem Z6.

Remark 2. *Each of Problems Z1–Z7 has the unique solution [13][Remarks 6, 15]. Also,* $\mathbf{Z}^{ij}_{k0} = \mathbf{Z}^{ji}_{k0}$ *for all* $i,j = 1,2,3$, $k = 0,1,3$ *[13][Remark 16].*

In (44)–(46) and (49), the vector-function $\mathbf{Z}_{40}$ and the scalar functions Z^i_5 $(i = 1,2,3)$ are the solutions of the following cell problems:

Problem Z8. Find a vector-function $\mathbf{Z}_{40} = \mathbf{Z}_{40}(\widehat{z})$, which satisfies the regularity condition

$$\mathbf{Z}_{40} \in (H^1_\sharp(\Theta)/\mathbb{R})^3, \tag{61a}$$

and resolves the variational equations

$$\int_{\Theta_F} (2\alpha_\mu \mathbb{D}_{\widehat{z}}(\mathbf{Z}_{40}(\widehat{z})) + \alpha_\lambda(\mathrm{div}_{\widehat{z}}\, \mathbf{Z}_{40}(\widehat{z}))\mathbb{I} - (\mathbb{B}_F - \mathbb{B}_S)) : \nabla_{\widehat{z}} \boldsymbol{\phi}_{21}(\widehat{z}) d\widehat{z} = 0,$$
$$\forall\, \boldsymbol{\phi}_{21} \in H^1_\sharp(\Theta)^3, \tag{61b}$$

and

$$\int_{\Theta_S} (\mathcal{G} : \nabla_{\widehat{z}} \mathbf{Z}_{40}(\widehat{z})) : \nabla_{\widehat{z}} \boldsymbol{\phi}_{21}(\widehat{z}) d\widehat{z} + \int_{\Theta_F} \alpha_p (\mathrm{div}_{\widehat{z}} \mathbf{Z}_{40}(\widehat{z})) \mathbb{I} : \nabla_{\widehat{z}} \boldsymbol{\phi}_{21}(\widehat{z}) d\widehat{z} = 0,$$
$$\forall \boldsymbol{\phi}_{21} \in H^1_\sharp(\Theta)^3. \tag{61c}$$

Problem Z9. Find a scalar function $\widehat{z} \mapsto Z_5^i(\widehat{y}, \widehat{z})$ $(i = 1, 2, 3)$, which satisfies the regularity condition

$$Z_5^i(\widehat{y}, \cdot) \in H^1_\sharp(\Theta)/\mathbb{R}, \quad \widehat{y} \in \Sigma, \tag{62a}$$

and resolves the variational equation

$$\int_\Theta \left(\widehat{\zeta}(\widehat{y})\widehat{\psi}(\widehat{z})\varkappa_F + (1 - \widehat{\zeta}(\widehat{y})\widehat{\psi}(\widehat{z}))\varkappa_S\right)\left(\boldsymbol{e}_i + \nabla_{\widehat{z}} Z_5^i(\widehat{y}, \widehat{z})\right) \cdot \nabla_{\widehat{z}} \eta_{21}(\widehat{z}) d\widehat{z} = 0, \quad \forall \eta_{21} \in H^1_\sharp(\Theta). \tag{62b}$$

Variable $\widehat{y}$ plays the role of parameter in this formulation.

These two problems are well-posed:

Proposition 5. *Each of Problems Z8 and Z9 has a unique solution.*

Proof of Proposition 5 is quite standard in the theory of linear elliptic problems, and we present it here rather sketchy. Firstly, by the Lax–Milgram theorem and the Poincaré–Wirtinger inequality, the variational equation (61b) has a solution $\mathbf{Z}_{40} \in H^1_\sharp(\Theta_F)^3$, which is unique up to a constant vector. Secondly, by the classical theory of Sobolev spaces, inclusion $\mathbf{Z}_{40} \in H^1_\sharp(\Theta)^3$ is equivalent to the system consisting of inclusions $\mathbf{Z}_{40} \in H^1_\sharp(\Theta_F)^3$ and $\mathbf{Z}_{40} \in H^1_\sharp(\Theta_S)^3$ and the matching condition

$$\mathbf{Z}_{40}\big|_{\partial\Theta_S} = \mathbf{Z}_{40}\big|_{\partial\Theta_F \setminus \partial\Theta}, \tag{63}$$

where $\mathbf{Z}_{40}\big|_{\partial\Theta_S}$ is the trace of $\mathbf{Z}_{40} \in H^1_\sharp(\Theta_S)^3$ and $\mathbf{Z}_{40}\big|_{\partial\Theta_F\setminus\partial\Theta}$ is the trace of $\mathbf{Z}_{40} \in H^1_\sharp(\Theta_F)^3$ on $\partial\Theta_S$. For the known solution of (61b), by the Lax–Milgram theorem, the system (61c), (63) has a unique solution $\mathbf{Z}_{40} \in H^1_\sharp(\Theta_S)^3$, furthermore, $\mathbf{Z}_{40}$ belongs to $H^1_\sharp(\Theta)^3$.

Shifting, if necessary, the values of $\mathbf{Z}_{40}$ on a constant vector $\boldsymbol{\xi} = -\int_\Theta \mathbf{Z}_{40}(\widehat{z}) d\widehat{z}$, we obtain the unique solution $\mathbf{Z}_{40} \in (H^1_\sharp(\Theta)/\mathbb{R})^3$ to Problem Z8.

Finally, the variational equation (62b) is well-known in the elasticity theory (see, for example, [20]) and its unique solvability again follows from the Lax–Milgram theorem and Poincaré–Wirtinger inequality.

Proposition 5 is proved. □

Remark 3. *From the respective representations* (39b), (40b), (41b), (42b), (43b), (43c), (44b), (45b), (46b), (47b), (48b), (49b), (50b), *and* (51b), *it is clear that tensors* $\boldsymbol{\mathcal{A}}_0^f$, $\boldsymbol{\mathcal{B}}_0^f$, $\boldsymbol{\mathcal{A}}_1^f(t)$, *and* $\boldsymbol{\mathcal{B}}_1^f(t)$, *matrices* $\mathbb{F}_f^0(t)$ *and* $\mathbb{F}_{sol}^{0mn}(t)$, $\mathbb{H}_{\theta 1}^f$, $\mathbb{H}_{\theta 2}^f$, $\mathbb{L}_\varkappa^\theta(\widehat{y})$, $\mathbb{L}_{B1}^\theta(\widehat{y})$, *and* $\mathbb{L}_{B2}^f(t)$, *and scalars* L_θ^f, $L_{pf}^0(t)$, *and* $L_{vf}^0(t)$ *do not depend on* x_3.

Let us turn to justification of Theorem 2.

Proof of Theorem 2. We separate variable $\widehat{z}$ standardly, by the procedure of asymptotic decomposition analogous to the one carried out in [13][Secs. 6, 7]. More certainly, we seek for the representation of $\boldsymbol{u}^{(2)}$ and $\theta^{(2)}$ in the form

$$\boldsymbol{u}^{(2)}(\boldsymbol{x}, \widehat{y}, \widehat{z}, t) = \zeta(\widehat{y}, x_3)\left(\sum_{i,j=1}^{3}\left(\frac{\partial u_i}{\partial x_j}(\boldsymbol{x}, t) + \frac{\partial u_i^{(1)}}{\partial y_j}(\boldsymbol{x}, \widehat{y}, t)\right)\mathbf{Z}_{00}^{ij}(\widehat{z}, x_3)\right)$$

$$+\sum_{i,j=1}^{3}\int_0^t\Big(\frac{\partial u_i}{\partial x_j}(\boldsymbol{x},\tau)+\frac{\partial u_i^{(1)}}{\partial y_j}(\boldsymbol{x},\widehat{\boldsymbol{y}},\tau)\Big)\mathbf{Z}_{10}^{ij}(\widehat{z},t-\tau,x_3)d\tau$$
$$+p^0(\boldsymbol{x})\mathbf{Z}_{20}(\widehat{z},t,x_3)+\sum_{i,j=1}^{3}\frac{\partial v_i^0}{\partial x_j}(\boldsymbol{x})\mathbf{Z}_{30}^{ij}(\widehat{z},t,x_3)+\theta(\boldsymbol{x},t)\mathbf{Z}_{40}(\widehat{z},x_3)\Bigg) \tag{64}$$

and
$$\theta^{(2)}(\boldsymbol{x},\widehat{\boldsymbol{y}},\widehat{z},t)=\sum_{i=1}^{3}\Big(\frac{\partial\theta}{\partial x_i}(\boldsymbol{x},t)+\frac{\partial\theta^{(1)}}{\partial y_i}(\boldsymbol{x},\widehat{\boldsymbol{y}},t)\Big)Z_5^i(\widehat{\boldsymbol{y}},\widehat{z},x_3), \tag{65}$$

where $\mathbf{Z}_{00}^{ij}$, $\mathbf{Z}_{10}^{ij}$, $\mathbf{Z}_{20}$, $\mathbf{Z}_{30}^{ij}$, $\mathbf{Z}_{40}$, and Z_5^i are unknown functions that should be defined as the solutions of the cell problems. In order to formulate the proper cell problems, in (32a) we take

$$\boldsymbol{\phi}(\boldsymbol{x},t):=0,\quad \boldsymbol{\phi}_1(\boldsymbol{x},\widehat{\boldsymbol{y}},t):=0,\quad \boldsymbol{\phi}_2(\boldsymbol{x},\widehat{\boldsymbol{y}},\widehat{z},t):=\frac{\partial\phi_{20}(\boldsymbol{x},\widehat{\boldsymbol{y}},t)}{\partial t}\boldsymbol{\phi}_{21}(\widehat{z}),$$

where ϕ_{20} is an arbitrary smooth 1-periodic in $\widehat{\boldsymbol{y}}$ scalar function, vanishing in a neighborhood of $\partial\Omega$ and the section $\{t=T\}$, and $\boldsymbol{\phi}_{21}$ is an arbitrary 1-periodic smooth vector-function, in (32b) we take

$$\eta(\boldsymbol{x},t):=0,\quad \eta_1(\boldsymbol{x},\widehat{\boldsymbol{y}},t):=0,\quad \eta_2(\boldsymbol{x},\widehat{\boldsymbol{y}},\widehat{z},t):=\eta_{20}(\boldsymbol{x},\widehat{\boldsymbol{y}},t)\eta_{21}(\widehat{z}),$$

where η_{20} is an arbitrary smooth 1-periodic in $\widehat{\boldsymbol{y}}$ scalar function, vanishing in a neighborhood of $\partial\Omega$ and the section $\{t=T\}$, and η_{21} is an arbitrary 1-periodic smooth function. Next, we insert (64) into (32a) and (64) and (65) into (32b). After this, properly collecting terms and performing simple but rather lengthy technical manipulations, we eventually deduce the set of the well-posed cell problems for determining the unique functions $\mathbf{Z}_{00}^{ij}$, $\mathbf{Z}_{10}^{ij}$, $\mathbf{Z}_{20}$, $\mathbf{Z}_{30}^{ij}$, $\mathbf{Z}_{40}$, and Z_5^i, see the above stated Problems Z1–Z9.

Finally, in (32a) we take $\boldsymbol{\phi}_2\equiv 0$ and insert representation (64) for $\boldsymbol{u}^{(2)}$, and in (32b) we take $\eta_2\equiv 0$, and insert (64) and (65) on the respective places of $\boldsymbol{u}^{(2)}$ and $\theta^{(2)}$. By this, we establish exactly the variational equations (38a) and (38b) and the representations (39)–(50) of the two-scale homogenized coefficients, which completes the proof of Theorem 2. □

The two-scale homogenized coefficients in the variational equations (38a) and (38b) have the following main properties.

Proposition 6. (i) *The tensor-valued functions*

$$\boldsymbol{\mathcal{A}}_0=\boldsymbol{\mathcal{A}}_0(\widehat{\boldsymbol{y}},x_3),\ \boldsymbol{\mathcal{B}}_0=\boldsymbol{\mathcal{B}}_0(\widehat{\boldsymbol{y}},x_3),\ \boldsymbol{\mathcal{A}}_1=\boldsymbol{\mathcal{A}}_1(\widehat{\boldsymbol{y}},x_3,t),\ \boldsymbol{\mathcal{B}}_1=\boldsymbol{\mathcal{B}}_1(\widehat{\boldsymbol{y}},x_3,t),$$

and the matrix-valued function $\mathbb{F}^0=\mathbb{F}^0(\widehat{\boldsymbol{y}},\boldsymbol{x},t)$ *satisfy the regularity conditions*

$$\boldsymbol{\mathcal{A}}_0,\ \boldsymbol{\mathcal{B}}_0,\ \boldsymbol{\mathcal{A}}_1,\ \boldsymbol{\mathcal{B}}_1,\ \textit{and}\ \mathbb{F}^0\ \textit{are 1-periodic in}\ \widehat{\boldsymbol{y}}, \tag{66}$$
$$\boldsymbol{\mathcal{A}}_0,\ \boldsymbol{\mathcal{B}}_0\in L^\infty(\Sigma\times(0,1))^{3\times3\times3\times3}, \tag{67}$$
$$\boldsymbol{\mathcal{A}}_1,\ \boldsymbol{\mathcal{B}}_1,\ \partial_t\boldsymbol{\mathcal{A}}_1,\ \partial_t\boldsymbol{\mathcal{B}}_1\in L^\infty(\Sigma\times(0,1)\times(0,T))^{3\times3\times3\times3}, \tag{68}$$
$$\mathbb{F}^0,\ \partial_t\mathbb{F}^0\in L^\infty(\Sigma\times(0,T);L^2(\Omega))^{3\times3}. \tag{69}$$

(ii) *The tensor-valued function* $\boldsymbol{\mathcal{A}}_0$ *satisfies the finiteness property*

$$\boldsymbol{\mathcal{A}}_0=0\ \textit{for}\ (\widehat{\boldsymbol{y}},x_3)\in(\Sigma\times[0,\Delta))\cup(\overline{\Sigma}_S\times[\Delta,\Delta+\delta^*]), \tag{70}$$

the symmetry property

$$\mathcal{A}_0^{ijkl}=\mathcal{A}_0^{jikl}=\mathcal{A}_0^{jilk}=\mathcal{A}_0^{klij},\quad \forall\,(\widehat{\boldsymbol{y}},x_3)\in\Sigma\times[0,1],\quad \forall\,i,j,k,l=1,2,3, \tag{71}$$

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and the uniform positive definiteness property:

$$\begin{aligned}&\textit{there exists a constant } C_5 > 0 \ \textit{ such that } \ \big(\boldsymbol{\mathcal{A}}_0(\widehat{\boldsymbol{y}}, x_3) : \mathbb{X}\big) : \mathbb{X} \geq C_5|\mathbb{X}|^2,\\ &\forall\, \mathbb{X} \in \mathbb{R}^{3\times 3}_{\text{symm}}, \quad \forall\, (\widehat{\boldsymbol{y}}, x_3) \in (\Sigma_F \times [\Delta, \Delta + \delta^*]) \cup (\Sigma \times [\Delta + \delta^*, 1)).\end{aligned} \tag{72}$$

(iii) *The tensor-valued function*

$$\boldsymbol{\mathcal{H}}^s(\widehat{\boldsymbol{y}}, x_3) \overset{def}{=} s\boldsymbol{\mathcal{A}}_0(\widehat{\boldsymbol{y}}, x_3) + \boldsymbol{\mathcal{B}}_0(\widehat{\boldsymbol{y}}, x_3) + s\overline{\boldsymbol{\mathcal{A}}}_1(\widehat{\boldsymbol{y}}, x_3, s) + \overline{\boldsymbol{\mathcal{B}}}_1(\widehat{\boldsymbol{y}}, x_3, s) \tag{73}$$

satisfies the symmetry properties

$$\mathcal{H}^{s\,ijkl} = \mathcal{H}^{s\,jikl} = \mathcal{H}^{s\,jilk}, \quad \forall\, (\widehat{\boldsymbol{y}}, x_3) \in \Sigma \times [0,1],\ \forall\, s > 0, \tag{74}$$

$\forall\, i,j,k,l = 1,2,3$, and

$$(\boldsymbol{\mathcal{H}}^s : \mathbb{X}) : \mathbb{W} = (\boldsymbol{\mathcal{H}}^s : \mathbb{W}) : \mathbb{X}, \quad \forall\, \mathbb{X},\, \mathbb{W} \in \mathbb{R}^{3\times 3}_{\text{symm}}, \tag{75}$$

and the uniform positive definiteness property:

$$\begin{aligned}&\textit{there exists a constant } C_6^s > 0 \ \textit{ such that } \ (\boldsymbol{\mathcal{H}}^s(\widehat{\boldsymbol{y}}, x_3) : \mathbb{X}) : \mathbb{X} \geq C_6^s|\mathbb{X}|^2,\\ &\forall\, \mathbb{X} \in \mathbb{R}^{3\times 3}_{\text{symm}}, \quad \forall\, (\widehat{\boldsymbol{y}}, x_3) \in \Sigma \times [0,1], \quad \forall\, s > 0.\end{aligned} \tag{76}$$

(iv) *For a.e. $(\widehat{\boldsymbol{y}}, \boldsymbol{x}, t) \in \Sigma \times \Omega \times [0,T]$ matrix $\mathbb{F}^0(\widehat{\boldsymbol{y}}, \boldsymbol{x}, t)$ is symmetric.*

(v) *The matrix-valued functions*

$$\begin{aligned}&\mathbb{H}_{\theta 1} = \mathbb{H}_{\theta 1}(\widehat{\boldsymbol{y}}, x_3), \quad \mathbb{H}_{\theta 2} = \mathbb{H}_{\theta 2}(\widehat{\boldsymbol{y}}, x_3), \quad \mathbb{L}_{\varkappa} = \mathbb{L}_{\varkappa}(\widehat{\boldsymbol{y}}, x_3),\\ &\mathbb{L}_{B1} = \mathbb{L}_{B1}(\widehat{\boldsymbol{y}}, x_3), \quad \mathbb{L}_{B2} = \mathbb{L}_{B2}(\widehat{\boldsymbol{y}}, t, x_3), \quad \mathbb{L}_v^0 = \mathbb{L}_v^0(\widehat{\boldsymbol{y}}, t, x_3)\end{aligned}$$

and the scalar functions $L_\theta = L_\theta(\widehat{\boldsymbol{y}}, x_3)$ and $L_p^0 = L_p^0(\widehat{\boldsymbol{y}}, t, x_3)$ satisfy the regularity conditions

$$\mathbb{H}_{\theta 1},\, \mathbb{H}_{\theta 2},\, \mathbb{L}_{\varkappa},\, \mathbb{L}_{B1},\, \mathbb{L}_{B2},\, \mathbb{L}_v^0,\, L_\theta, \textit{ and } L_p^0 \textit{ are 1-periodic in } \widehat{\boldsymbol{y}}, \tag{77}$$

$$\mathbb{H}_{\theta 1},\, \mathbb{H}_{\theta 2},\, \mathbb{L}_{\varkappa},\, L_{B1} \in L^\infty(\Sigma \times (0,1))^{3\times 3}, \tag{78}$$

$$\mathbb{L}_{B2},\, \partial_t \mathbb{L}_{B2},\, \mathbb{L}_v^0,\, \partial_t \mathbb{L}_v^0 \in L^\infty(\Sigma \times (0,T) \times (0,1))^{3\times 3}, \tag{79}$$

$$L_\theta \in L^\infty(\Sigma \times (0,1)), \quad L_p^0,\, \partial_t L_p^0 \in L^\infty(\Sigma \times (0,T) \times (0,1)). \tag{80}$$

(vi) *The matrix-valued function $\mathbb{L}_{\varkappa}$ satisfies the symmetry property*

$$L_{\varkappa}^{ij} = L_{\varkappa}^{ji}, \quad \forall\, (\widehat{\boldsymbol{y}}, x_3) \in \Sigma \times [0,1], \quad \forall\, i,j = 1,2,3, \tag{81}$$

and the uniform positive definiteness property:

$$\begin{aligned}&\textit{there exists a constant } C_7 > 0 \ \textit{ such that } \ \mathbb{L}_{\varkappa}(\widehat{\boldsymbol{y}}, x_3)\boldsymbol{X} \cdot \boldsymbol{X} \geq C_7|\boldsymbol{X}|^2,\\ &\forall\, \boldsymbol{X} \in \mathbb{R}^3, \quad \forall\, (\widehat{\boldsymbol{y}}, x_3) \in \Sigma \times [0,1].\end{aligned} \tag{82}$$

(vii) *Matrices $\mathbb{H}_{\theta 1}$, $\mathbb{H}_{\theta 2}$, $\mathbb{L}_{B1}$, $\mathbb{L}_{B2}$, and $\mathbb{L}_v^0$ are symmetric for all $\widehat{\boldsymbol{y}} \in \Sigma$, $x_3 \in [0,1]$, and $t \in [0,T]$.*

Notation 6. *In (73), by $\overline{\boldsymbol{\mathcal{A}}}_1$ and $\overline{\boldsymbol{\mathcal{B}}}_1$ the respective Laplace transforms in t of $\boldsymbol{\mathcal{A}}_1$ and $\boldsymbol{\mathcal{B}}_1$ are denoted. It is assumed that $\boldsymbol{\mathcal{A}}_1$ and $\boldsymbol{\mathcal{B}}_1$ vanish for $t > T$.*

Recall that the Laplace transform of a function $\varphi(t)$ is defined by the formula

$$\overline{\varphi}(s) = \mathcal{L}[\varphi](s) = \int_0^\infty \varphi(t) e^{-st} dt, \quad s > 0. \tag{83}$$

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Proof of Proposition 6. Assertions (i)–(iv) were proved in [13][Sec. 7]. Proof of assertion (v) is quite analogous to the justification of assertion (i): we have that the periodicity conditions (77) hold since the characteristic functions $\widehat{\zeta}$ and ζ are 1-periodic in $\widehat{\boldsymbol{y}}$ in representations (44)–(50); the regularity properties (78)–(80) hold true due to the regularity properties (52), (57a), (58a), (60), (61a), and (62a) of the solutions of the cell problems. Assertion (vi) directly follows from (46) and (62b) by the same arguments, as in [21][Ch. V, § 3]. Assertion (vii) is evident due to expressions (44), (45), (47), and (48). □

7. Asymptotic Decomposition II: The Effective Macroscopic Model — Variational Formulation

On the second step of the asymptotic decomposition, we separate the intermediate (mesoscopic) level, which is characterized by the sought functions $\boldsymbol{u}^{(1)}$ and $\theta^{(1)}$ and the independent variable $\widehat{\boldsymbol{y}}$. As the result, we derive the variational formulation of the effective macroscopic model describing the dynamics of the 'fluid–structure' interaction.

Theorem 3. *Let the quadruple of functions $(\boldsymbol{u}, \boldsymbol{u}^{(1)}, \theta, \theta^{(1)})$ be the solution of Model H-2sc. Then, its subset — the pair $(\boldsymbol{u}, \theta)$ — serves as a solution to Model H-var stated below.*

Model H-var. (The homogenized model — variational formulation.) Find a velocity field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ and a temperature distribution $\theta = \theta(\boldsymbol{x}, t)$ satisfying the regularity requirements (30a) and (30d), the variational balance of momentum equation

$$
\begin{aligned}
&\int_0^T \int_\Omega \Big\{ -\alpha_\tau \langle \rho \rangle_{\Sigma\times\Theta}(x_3) \boldsymbol{u}(\boldsymbol{x},t) \cdot \partial_t \boldsymbol{\phi}(\boldsymbol{x},t) \\
&\quad + \Big(\boldsymbol{\mathcal{V}}(x_3) : \mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},t)) + \boldsymbol{\mathcal{E}}(x_3) : \mathbb{D}_x\big((J_t\boldsymbol{u})(\boldsymbol{x},t)\big) + \int_0^t \boldsymbol{\mathcal{K}}(t-\tau, x_3) : \mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},\tau))d\tau \\
&\quad + \int_0^t \int_0^{\tau'} \boldsymbol{\mathcal{Q}}(t-\tau', \tau'-\tau, x_3) : \mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},\tau))d\tau d\tau' \\
&\quad + \int_0^t \int_0^{\tau'} \boldsymbol{\mathcal{W}}(t-\tau', \tau'-\tau, x_3) : \mathbb{D}_x\big((J_\tau\boldsymbol{u})(\boldsymbol{x},\tau)\big)d\tau d\tau' \\
&\quad + \mathbb{V}(x_3)\theta(\boldsymbol{x},t) + \mathbb{E}(x_3)(J_t\theta)(\boldsymbol{x},t) \\
&\quad + \int_0^t \mathbb{K}_0(t-\tau, x_3)\theta(\boldsymbol{x},\tau)d\tau + \int_0^t \int_0^{\tau'} \mathbb{K}_1(t-\tau', \tau'-\tau, x_3)\theta(\boldsymbol{x},\tau)d\tau d\tau' \\
&\quad + \int_0^t \mathbb{W}_0(t-\tau, x_3)(J_\tau\theta)(\boldsymbol{x},\tau)d\tau + \int_0^t \int_0^{\tau'} \mathbb{W}_1(t-\tau', \tau'-\tau, x_3)(J_\tau\theta)(\boldsymbol{x},\tau)d\tau \\
&\quad + \mathbb{F}_p(t, x_3)p^0(\boldsymbol{x}) + \boldsymbol{\mathcal{F}}_v(t, x_3) : \mathbb{D}_x(\boldsymbol{v}^0(\boldsymbol{x}))\Big) : \mathbb{D}_x(\boldsymbol{\phi}(\boldsymbol{x},t))\Big\} dx dt \\
&= \int_0^T \int_\Omega \alpha_g \langle \rho \rangle_{\Sigma\times\Theta}(x_3) \boldsymbol{f}(\boldsymbol{x},t) \cdot \boldsymbol{\phi}(\boldsymbol{x},t) dx dt + \int_\Omega \alpha_\tau \langle \rho \rangle_{\Sigma\times\Theta}(x_3) \boldsymbol{u}^0(\boldsymbol{x}) \cdot \boldsymbol{\phi}(\boldsymbol{x},0) dx
\end{aligned} \tag{84a}
$$

for an arbitrary smooth test vector-function $\boldsymbol{\phi} = \boldsymbol{\phi}(\boldsymbol{x}, t)$ vanishing in a neighborhood of boundary $\partial\Omega$ and section $\{t = T\}$, the variational balance of energy equation

$$
\begin{aligned}
&\int_0^T \int_\Omega \Big\{ -\langle c \rangle_{\Sigma\times\Theta}(x_3)\theta(\boldsymbol{x},t)\partial_t\eta(\boldsymbol{x},t) + \mathbb{L}_*(x_3)\nabla_x\theta(\boldsymbol{x},t) \cdot \nabla_x\eta(\boldsymbol{x},t) \\
&\qquad + \Big(\mathbb{L}_{B3}(x_3) : \mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},t)) + \int_0^t \mathbb{L}_{B4}(t-\tau, x_3) : \mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},\tau))d\tau \\
&\qquad\quad + \int_0^t \int_0^{\tau'} \mathbb{L}_{B5}(t-\tau', \tau'-\tau, x_3) : \mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},\tau))d\tau d\tau' \\
&\qquad\quad + L_{\theta 1}(x_3)\theta(\boldsymbol{x},t) + \int_0^t L_{\theta 2}(t-\tau, x_3)\theta(\boldsymbol{x},\tau)d\tau \\
&\qquad\quad + \int_0^t \int_0^{\tau'} L_{\theta 3}(t-\tau', \tau'-\tau, x_3)\theta(\boldsymbol{x},\tau)d\tau d\tau'
\end{aligned}
$$

$$+ \mathbb{L}_v^\star(t,x_3) : \mathbb{D}_x(\boldsymbol{v}^0(\boldsymbol{x})) + L_p^\star(t,x_3)p^0(\boldsymbol{x})\Big)\eta(\boldsymbol{x},t)\Big\}\, d\boldsymbol{x}dt =$$

$$= \int_0^T \int_\Omega \langle\Psi\rangle_{\Sigma\times\Theta}(\boldsymbol{x},t)\eta(\boldsymbol{x},t)d\boldsymbol{x}dt + \int_\Omega \langle c\rangle_{\Sigma\times\Theta}(x_3)\theta^0(\boldsymbol{x})\eta(\boldsymbol{x},0)d\boldsymbol{x} \tag{84b}$$

for an arbitrary smooth test scalar function $\eta = \eta(\boldsymbol{x},t)$ vanishing in a neighborhood of boundary $\partial\Omega$ and section $\{t = T\}$, and the boundary conditions (32c) in the trace sense.

The fourth-rank tensors $\boldsymbol{\mathcal{V}}$, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{K}}$, $\boldsymbol{\mathcal{Q}}$, $\boldsymbol{\mathcal{W}}$, and $\boldsymbol{\mathcal{F}}_v$, the 3×3-matrices $\mathbb{V}$, $\mathbb{E}$, $\mathbb{K}_0$, $\mathbb{K}_1$, $\mathbb{W}_0$, $\mathbb{W}_1$, $\mathbb{F}_p$, $\mathbb{L}_*$, $\mathbb{L}_{B3}$, $\mathbb{L}_{B4}$, $\mathbb{L}_{B5}$, and $\mathbb{L}_v^\star$, and the scalar functions $L_{\theta1}$, $L_{\theta2}$, $L_{\theta3}$, and $L_p^\star$ are uniquely defined by the solutions of the cell problems posed on the mesoscopic pattern cell Σ. These cell problems contain the complete information about the homogenized dynamics of the thermomechanical system at the level of the taller bristles.

The exact expressions of $\boldsymbol{\mathcal{V}}$, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{K}}$, $\boldsymbol{\mathcal{Q}}$, $\boldsymbol{\mathcal{W}}$, $\boldsymbol{\mathcal{F}}_v$, and $\mathbb{F}_p$ have already been established in [13][Secs. 10, 11] along with the formulations of the corresponding cell problems. The exact expressions of $\mathbb{V}$, $\mathbb{E}$, $\mathbb{K}_0$, $\mathbb{K}_1$, $\mathbb{W}_0$, $\mathbb{W}_1$, $\mathbb{L}_*$, $\mathbb{L}_{B3}$, $\mathbb{L}_{B4}$, $\mathbb{L}_{B5}$, $\mathbb{L}_v^\star$, $L_{\theta1}$, $L_{\theta2}$, $L_{\theta3}$, and $L_p^\star$ are novel, as are the formulations of the cell problems associated with these expressions. Now, we present the exact expressions of $\boldsymbol{\mathcal{V}}$, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{K}}$, $\boldsymbol{\mathcal{Q}}$, $\boldsymbol{\mathcal{W}}$, $\boldsymbol{\mathcal{F}}_v$, $\mathbb{V}$, $\mathbb{E}$, $\mathbb{K}_0$, $\mathbb{K}_1$, $\mathbb{W}_0$, $\mathbb{W}_1$, $\mathbb{F}_p$, $\mathbb{L}_*$, $\mathbb{L}_{B3}$, $\mathbb{L}_{B4}$, $\mathbb{L}_{B5}$, $\mathbb{L}_v^\star$, $L_{\theta1}$, $L_{\theta2}$, $L_{\theta3}$, and $L_p^\star$ and the exact formulations of the cell problems (posed on Σ) along with the results on their well-posedness. After this, we prove Theorem 3.

We have

$$\boldsymbol{\mathcal{V}}(x_3) = \begin{cases} \alpha_\lambda \mathbb{I}\otimes\mathbb{I} + 2\alpha_\mu \sum\limits_{m,n=1}^{3} \mathbb{J}^{mn}\otimes\mathbb{J}^{mn} \quad \text{for} \quad x_3 \in (\Delta+\delta^*, 1), \\ |\Sigma_F|\alpha_\lambda \mathbb{I}\otimes\mathbb{I} + 2|\Sigma_F|\,\alpha_\mu \sum\limits_{m,n=1}^{3} \mathbb{J}^{mn}\otimes\mathbb{J}^{mn} + \boldsymbol{\mathcal{V}}^\sigma_{\text{corr}} \\ \qquad \text{for} \quad x_3 \in (\Delta+\delta_*, \Delta+\delta^*], \\ |\Sigma_F|\,|\Theta_F|\,\alpha_\lambda \mathbb{I}\otimes\mathbb{I} + 2|\Sigma_F|\,|\Theta_F|\,\alpha_\mu \sum\limits_{m,n=1}^{3} \mathbb{J}^{mn}\otimes\mathbb{J}^{mn} + \boldsymbol{\mathcal{V}}^\theta_{\text{corr}} \\ \qquad \text{for} \quad x_3 \in [\Delta, \Delta+\delta_*], \\ 0 \quad \text{for} \quad x_3 \in (0,\Delta), \end{cases} \tag{85a}$$

where

$$\boldsymbol{\mathcal{V}}^\sigma_{\text{corr}} = \sum_{i,j=1}^{3} \alpha_\lambda \mathbb{I}\langle\widehat{\zeta}(\cdot)\,\operatorname{div}_{\widehat{y}} \boldsymbol{Y}_0^{ij}(\cdot,x_3)\rangle_\Sigma \otimes \mathbb{J}^{ij}$$
$$+ \sum_{i,j=1}^{3} 2\alpha_\mu \sum_{m,n=1}^{3} \langle\widehat{\zeta}(\cdot) D_{\widehat{y}mn}(\boldsymbol{Y}_0^{ij}(\cdot,x_3))\rangle_\Sigma\, \mathbb{J}^{mn}\otimes\mathbb{J}^{ij}, \quad x_3 \in (\Delta+\delta_*, \Delta+\delta^*], \tag{85b}$$

and

$$\boldsymbol{\mathcal{V}}^\theta_{\text{corr}} = |\Sigma_F|\,\alpha_\lambda \mathbb{I}\otimes \sum_{m,n=1}^{3} \langle\widehat{\psi}\,\operatorname{div}_{\widehat{z}} \boldsymbol{Z}_{00}^{mn}\rangle_\Theta \mathbb{J}^{mn} + 2|\Sigma_F|\alpha_\mu \sum_{m,n=1}^{3} \langle\widehat{\psi}\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{00}^{mn})\rangle_\Theta \otimes \mathbb{J}^{mn}$$
$$+ \sum_{i,j=1}^{3} \Big[\Big(|\Theta_F|\,\alpha_\lambda \mathbb{I}\otimes\mathbb{I} + \alpha_\lambda \mathbb{I}\otimes \sum_{m,n=1}^{3} \langle\widehat{\psi}\,\operatorname{div}_{\widehat{z}} \boldsymbol{Z}_{00}^{mn}\rangle_\Theta \mathbb{J}^{mn}$$
$$+ 2\alpha_\mu \sum_{m,n=1}^{3} \big(|\Theta_F|\,\mathbb{J}^{mn} + \langle\widehat{\psi}\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{00}^{mn})\rangle_\Theta\big)\otimes\mathbb{J}^{mn}\Big) : \langle\widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_0^{ij}(\cdot,x_3))\rangle_\Sigma\Big] \otimes \mathbb{J}^{ij},$$
$$x_3 \in [\Delta, \Delta+\delta_*]; \tag{85c}$$

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$$\boldsymbol{\mathcal{E}}(x_3) = \begin{cases} \alpha_p\, \mathbb{I} \otimes \mathbb{I} \quad \text{for} \quad x_3 \in (\Delta + \delta^*, 1), \\ |\Sigma_F|\, \alpha_p\, \mathbb{I} \otimes \mathbb{I} + (1 - |\Sigma_F|)\boldsymbol{\mathcal{G}} + \boldsymbol{\mathcal{E}}^{\sigma}_{\text{corr}} \;\; \text{for} \quad x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ |\Sigma_F|\, |\Theta_F|\, \alpha_p\, \mathbb{I} \otimes \mathbb{I} + (1 - |\Sigma_F|\, |\Theta_F|)\boldsymbol{\mathcal{G}} + \boldsymbol{\mathcal{E}}^{\theta}_{\text{corr}} \;\; \text{for} \quad x_3 \in [\Delta, \Delta + \delta_*], \\ \boldsymbol{\mathcal{G}} \quad \text{for} \quad x_3 \in (0, \Delta), \end{cases} \tag{86a}$$

where

$$\boldsymbol{\mathcal{E}}^{\sigma}_{\text{corr}} = \sum_{i,j=1}^{3} \big\langle \big[\widehat{\zeta}(\cdot)\alpha_p\, \mathbb{I} \otimes \mathbb{I} + (1 - \widehat{\zeta}(\cdot))\boldsymbol{\mathcal{G}}\big] : \mathbb{D}_{\widehat{y}}\big(\boldsymbol{Y}^{ij}_0(\cdot, x_3)\big) \big\rangle_{\Sigma} \otimes \mathbb{J}^{ij}, \qquad x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \tag{86b}$$

and

$$\begin{aligned} \boldsymbol{\mathcal{E}}^{\theta}_{\text{corr}} = {} & |\Sigma_F|\, \alpha_p\, \mathbb{I} \otimes \sum_{m,n=1}^{3} \langle \widehat{\psi}\, \mathrm{div}_{\widehat{z}}\, \boldsymbol{Z}^{mn}_{00} \rangle_{\Theta} \mathbb{J}^{mn} + |\Sigma_F| \sum_{m,n=1}^{3} \Big(\boldsymbol{\mathcal{G}} : \big\langle (1 - \widehat{\psi})\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}^{mn}_{00}) \big\rangle_{\Theta} \Big) \otimes \mathbb{J}^{mn} \\ & + \sum_{i,j=1}^{3} \Big\langle \Big[\widehat{\zeta}(\cdot) \Big\{ \alpha_p\, \mathbb{I} \otimes \Big(|\Theta_F|\, \mathbb{I} + \sum_{m,n=1}^{3} \langle \widehat{\psi}\, \mathrm{div}_{\widehat{z}}\, \boldsymbol{Z}^{mn}_{00} \rangle_{\Theta} \mathbb{J}^{mn} \Big) \\ & \qquad + \sum_{m,n=1}^{3} \Big(\boldsymbol{\mathcal{G}} : \big\langle (1 - \widehat{\psi})\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}^{mn}_{00}) \big\rangle_{\Theta} \Big) \otimes \mathbb{J}^{mn} \Big\} \\ & + (1 - |\Theta_F|\, \widehat{\zeta}(\cdot))\boldsymbol{\mathcal{G}} \Big] : \mathbb{D}_{\widehat{y}}\big(\boldsymbol{Y}^{ij}_0(\cdot, x_3)\big) \Big\rangle_{\Sigma} \otimes \mathbb{J}^{ij}, \quad x_3 \in [\Delta, \Delta + \delta_*]; \end{aligned} \tag{86c}$$

$$\boldsymbol{\mathcal{K}}(t, x_3) = \begin{cases} 0 \quad \text{for} \quad x_3 \in (\Delta + \delta^*, 1), \\ \boldsymbol{\mathcal{K}}^{\sigma}(t) \quad \text{for} \quad x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \boldsymbol{\mathcal{K}}^{\theta}(t) \quad \text{for} \quad x_3 \in [\Delta, \Delta + \delta_*], \\ 0 \quad \text{for} \quad x_3 \in (0, \Delta), \end{cases} \tag{87a}$$

where

$$\boldsymbol{\mathcal{K}}^{\sigma}(t) = \sum_{i,j=1}^{3} \Big[\Big(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^{3} \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \Big) : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}^{ij}_1(\cdot, t, x_3)) \rangle_{\Sigma} \Big] \otimes \mathbb{J}^{ij}, \qquad x_3 \in (\Delta + \delta_*, \Delta + \delta^*), \tag{87b}$$

and

$$\begin{aligned} \boldsymbol{\mathcal{K}}^{\theta}(t) = {} & \sum_{i,j=1}^{3} \Big[\Big\{ \alpha_\lambda \mathbb{I} \otimes \Big(|\Theta_F|\, \mathbb{I} + \sum_{m,n=1}^{3} \langle \widehat{\psi}\, \mathrm{div}_{\widehat{z}}\, \boldsymbol{Z}^{mn}_{00} \rangle_{\Theta} \mathbb{J}^{mn} \Big) \\ & + 2\alpha_\mu \sum_{m,n=1}^{3} \Big(|\Theta_F|\, \mathbb{J}^{mn} + \langle \widehat{\psi} \mathbb{D}_{\widehat{z}}(\boldsymbol{Z}^{mn}_{00}) \rangle_{\Theta} \Big) \otimes \mathbb{J}^{mn} \Big\} : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}^{ij}_1(\cdot, t, x_3)) \rangle_{\Sigma} \Big] \otimes \mathbb{J}^{ij} \\ & + |\Sigma_F| \Big(\alpha_\lambda \mathbb{I} \otimes \sum_{m,n=1}^{3} \langle \widehat{\psi}(\cdot)\, \mathrm{div}_{\widehat{z}}\, \boldsymbol{Z}^{mn}_{10}(\cdot, t) \rangle_{\Theta} \mathbb{J}^{mn} + 2\alpha_\mu \sum_{m,n=1}^{3} \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{z}}(\boldsymbol{Z}^{mn}_{10}(\cdot, t)) \rangle_{\Theta} \otimes \mathbb{J}^{mn} \Big) \\ & + \sum_{i,j=1}^{3} \Big[\Big(\alpha_\lambda \mathbb{I} \otimes \sum_{m,n=1}^{3} \langle \widehat{\psi}(\cdot)\, \mathrm{div}_{\widehat{z}}\, \boldsymbol{Z}^{mn}_{10}(\cdot, t) \rangle_{\Theta} \mathbb{J}^{mn} \\ & + 2\alpha_\mu \sum_{m,n=1}^{3} \langle \widehat{\psi}(\cdot) \mathbb{D}_{\widehat{z}}(\boldsymbol{Z}^{mn}_{10}(\cdot, t)) \rangle_{\Theta} \otimes \mathbb{J}^{mn} \Big) : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}^{ij}_0(\cdot, x_3)) \rangle_{\Sigma} \Big] \otimes \mathbb{J}^{ij}, \end{aligned}$$

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$$x_3 \in [\Delta, \Delta + \delta_*]; \tag{87c}$$

$$\boldsymbol{\mathcal{Q}}(t,\tau,x_3) = \begin{cases} 0 \quad \text{for} \quad x_3 \in (\Delta + \delta^*, 1), \\ \boldsymbol{\mathcal{Q}}^\sigma(\tau) \quad \text{for} \quad x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \boldsymbol{\mathcal{Q}}_0^\theta(\tau) + \sum_{m,n=1}^{3} \left(\boldsymbol{\mathcal{Q}}_1^\theta(t) : \mathbb{Q}^{mn}(\tau)\right) \otimes \mathbb{J}^{mn} \\ \qquad \text{for} \quad x_3 \in [\Delta, \Delta + \delta_*], \\ 0 \quad \text{for} \quad x_3 \in (0, \Delta), \end{cases} \tag{88a}$$

where

$$\boldsymbol{\mathcal{Q}}^\sigma(\tau) = \sum_{i,j=1}^{3} \Big\langle \Big(\widehat{\zeta}(\cdot)\alpha_p\,\mathbb{I} \otimes \mathbb{I} + (1 - \widehat{\zeta}(\cdot))\boldsymbol{\mathcal{G}}\Big) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_1^{ij}(\cdot,\tau,x_3))\Big\rangle_\Sigma \otimes \mathbb{J}^{ij},$$
$$x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \tag{88b}$$

$$\begin{aligned} \boldsymbol{\mathcal{Q}}_0^\theta(\tau) = \sum_{i,j=1}^{3} \Big\langle \Big[\widehat{\zeta}(\cdot)\Big\{\alpha_p\,\mathbb{I} \otimes \Big(|\Theta_F|\,\mathbb{I} + \sum_{m,n=1}^{3} \langle \widehat{\psi}\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{00}^{mn}\rangle_\Theta \mathbb{J}^{mn}\Big) \\ + \sum_{m,n=1}^{3} \Big(\boldsymbol{\mathcal{G}} : \langle (1 - \widehat{\psi})\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{00}^{mn})\rangle_\Theta\Big) \otimes \mathbb{J}^{mn}\Big\} \\ + (1 - |\Theta_F|\,\widehat{\zeta}(\cdot))\boldsymbol{\mathcal{G}}\Big] : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_1^{ij}(\cdot,\tau,x_3))\Big\rangle_\Sigma \otimes \mathbb{J}^{ij} \\ + |\Sigma_F|\Big\{\alpha_p\,\mathbb{I} \otimes \sum_{m,n=1}^{3} \langle \widehat{\psi}(\cdot)\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{10}^{mn}(\cdot,\tau)\rangle_\Theta \mathbb{J}^{mn} \\ + \sum_{m,n=1}^{3} \Big(\boldsymbol{\mathcal{G}} : \langle (1 - \widehat{\psi}(\cdot))\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{10}^{mn}(\cdot,\tau))\rangle_\Theta\Big) \otimes \mathbb{J}^{mn}\Big\} \\ + \sum_{i,j=1}^{3} \Big[\Big\{\alpha_p\,\mathbb{I} \otimes \sum_{m,n=1}^{3} \langle \widehat{\psi}(\cdot)\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{10}^{mn}(\cdot,\tau)\rangle_\Theta \mathbb{J}^{mn} \\ + \sum_{m,n=1}^{3} \Big(\boldsymbol{\mathcal{G}} : \langle (1 - \widehat{\psi}(\cdot))\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{10}^{mn}(\cdot,\tau))\rangle_\Theta\Big) \otimes \mathbb{J}^{mn}\Big\} : \langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_0^{ij}(\cdot,x_3))\rangle_\Sigma\Big] \otimes \mathbb{J}^{ij}, \end{aligned}$$
$$x_3 \in [\Delta, \Delta + \delta_*], \tag{88c}$$

$$\begin{aligned} \boldsymbol{\mathcal{Q}}_1^\theta(t) = \alpha_\lambda \mathbb{I} \otimes \sum_{m,n=1}^{3} \langle \widehat{\psi}(\cdot)\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{10}^{mn}(\cdot,t)\rangle_\Theta \mathbb{J}^{mn} \\ + 2\alpha_\mu \sum_{m,n=1}^{3} \langle \widehat{\psi}(\cdot)\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{10}^{mn}(\cdot,t))\rangle_\Theta \otimes \mathbb{J}^{mn}, \quad x_3 \in [\Delta, \Delta + \delta_*], \end{aligned} \tag{88d}$$

and

$$\mathbb{Q}^{mn}(\tau) = \langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_1^{mn}(\cdot,\tau,x_3))\rangle_\Sigma, \quad x_3 \in [\Delta, \Delta + \delta_*]; \tag{88e}$$

$$\boldsymbol{\mathcal{W}}(t,\tau,x_3) = \begin{cases} 0 \quad \text{for} \quad x_3 \in (0,1) \setminus [\Delta, \Delta + \delta_*], \\ \sum_{m,n=1}^{3} \Big(\boldsymbol{\mathcal{W}}_1^\theta(t) : \mathbb{Q}^{mn}(\tau)\Big) \otimes \mathbb{J}^{mn} \quad \text{for} \quad x_3 \in [\Delta, \Delta + \delta_*], \end{cases} \tag{89a}$$

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where

$$\mathcal{W}_1^{\theta}(t) = \alpha_p\,\mathbb{I}\otimes\sum_{m,n=1}^{3}\langle\widehat{\psi}(\cdot)\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{10}^{mn}(\cdot,t)\rangle_{\Theta}\mathbb{J}^{mn}$$
$$+\sum_{m,n=1}^{3}\Big(\boldsymbol{\mathcal{G}}:\langle(1-\widehat{\psi}(\cdot))\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{10}^{mn}(\cdot,t))\rangle_{\Theta}\Big)\otimes\mathbb{J}^{mn},\quad x_3\in[\Delta,\Delta+\delta_*], \tag{89b}$$

and $\mathbb{Q}^{pq}(\tau)$ is given by formula (88e);

$$\boldsymbol{\mathcal{F}}_v(t,x_3) = \begin{cases} 0 \quad \text{for} \quad x_3\in(\Delta+\delta^*,1), \\ (1-|\Sigma_F|)\boldsymbol{\mathcal{G}}+\boldsymbol{\mathcal{F}}^{\sigma}_{\text{v-corr}}(t) \quad \text{for} \quad x_3\in(\Delta+\delta_*,\Delta+\delta^*], \\ (1-|\Sigma_F|\,|\Theta_F|)\boldsymbol{\mathcal{G}}+\boldsymbol{\mathcal{F}}^{\theta}_{\text{v-corr}}(t) \quad \text{for} \quad x_3\in[\Delta,\Delta+\delta_*], \\ \boldsymbol{\mathcal{G}} \quad \text{for} \quad x_3\in(0,\Delta), \end{cases} \tag{90a}$$

where

$$\boldsymbol{\mathcal{F}}^{\sigma}_{\text{v-corr}}(t) = \sum_{i,j=1}^{3}\Big[\Big(\alpha_\lambda\mathbb{I}\otimes\mathbb{I}+2\alpha_\mu\sum_{m,n=1}^{3}\mathbb{J}^{mn}\otimes\mathbb{J}^{mn}\Big):\langle\widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_3^{ij}(\cdot,t,x_3))\rangle_{\Sigma}\Big]\otimes\mathbb{J}^{ij}$$
$$+\sum_{i,j=1}^{3}\Big\langle\Big(\widehat{\zeta}(\cdot)\alpha_p\,\mathbb{I}\otimes\mathbb{I}+(1-\widehat{\zeta}(\cdot))\boldsymbol{\mathcal{G}}\Big):\mathbb{D}_{\widehat{y}}((J_t\boldsymbol{Y}_3^{ij})(\cdot,t,x_3))\Big\rangle_{\Sigma}\otimes\mathbb{J}^{ij},$$
$$x_3\in(\Delta+\delta_*,\Delta+\delta^*], \tag{90b}$$

and

$$\boldsymbol{\mathcal{F}}^{\theta}_{\text{v-corr}}(t) = \sum_{i,j=1}^{3}\Big[\Big\{\alpha_\lambda\mathbb{I}\otimes\Big(|\Theta_F|\,\mathbb{I}+\sum_{m,n=1}^{3}\langle\widehat{\psi}\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{00}^{mn}\rangle_{\Theta}\mathbb{J}^{mn}\Big)$$
$$+2\alpha_\mu\sum_{m,n=1}^{3}\Big(|\Theta_F|\,\mathbb{J}^{mn}+\langle\widehat{\psi}\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{00}^{mn})\rangle_{\Theta}\Big)\otimes\mathbb{J}^{mn}\Big\}:\langle\widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_3^{ij}(\cdot,t,x_3))\rangle_{\Sigma}\Big]\otimes\mathbb{J}^{ij}$$
$$+\sum_{i,j=1}^{3}\Big\langle\Big[\widehat{\zeta}(\cdot)\Big\{\alpha_p\,\mathbb{I}\otimes\Big(|\Theta_F|\,\mathbb{I}+\sum_{m,n=1}^{3}\langle\widehat{\psi}\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{00}^{mn}\rangle_{\Theta}\mathbb{J}^{mn}\Big)$$
$$+\sum_{m,n=1}^{3}\Big(\boldsymbol{\mathcal{G}}:\langle(1-\widehat{\psi})\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{00}^{mn})\rangle_{\Theta}\Big)\otimes\mathbb{J}^{mn}\Big\}$$
$$+(1-|\Theta_F|\,\widehat{\zeta}(\cdot))\boldsymbol{\mathcal{G}}\Big]:\mathbb{D}_{\widehat{y}}((J_t\boldsymbol{Y}_3^{ij})(\cdot,t,x_3))\Big\rangle_{\Sigma}\otimes\mathbb{J}^{ij}$$
$$+\sum_{i,j=1}^{3}\int_0^t\Big[\Big(\alpha_\lambda\mathbb{I}\otimes\sum_{m,n=1}^{3}\langle\widehat{\psi}(\cdot)\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{10}^{mn}(\cdot,t-\tau)\rangle_{\Theta}\mathbb{J}^{mn}$$
$$+2\alpha_\mu\sum_{m,n=1}^{3}\langle\widehat{\psi}(\cdot)\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{10}^{mn}(\cdot,t-\tau))\rangle_{\Theta}\otimes\mathbb{J}^{mn}\Big):\langle\widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_3^{ij}(\cdot,\tau,x_3))\rangle_{\Sigma}\Big]\otimes\mathbb{J}^{ij}d\tau$$
$$+\sum_{i,j=1}^{3}\int_0^t\Big[\Big\{\alpha_p\,\mathbb{I}\otimes\sum_{m,n=1}^{3}\langle\widehat{\psi}(\cdot)\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{10}^{mn}(\cdot,t-\tau)\rangle_{\Theta}\mathbb{J}^{mn}$$
$$+\sum_{m,n=1}^{3}\Big(\boldsymbol{\mathcal{G}}:\langle(1-\widehat{\psi}(\cdot))\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{10}^{mn}(\cdot,t-\tau))\rangle_{\Theta}\Big)\otimes\mathbb{J}^{mn}\Big\}$$
$$:\langle\widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}((J_\tau\boldsymbol{Y}_3^{ij})(\cdot,\tau,x_3))\rangle_{\Sigma}\Big]\otimes\mathbb{J}^{ij}d\tau$$
$$+|\Sigma_F|\sum_{i,j=1}^{3}\Big(\alpha_\lambda\langle\widehat{\psi}(\cdot)\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{30}^{ij}(\cdot,t)\rangle_{\Theta}\mathbb{I}+2\alpha_\mu\langle\widehat{\psi}(\cdot)\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{30}^{ij}(\cdot,t))\rangle_{\Theta}$$

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$$+\alpha_p \langle \widehat{\psi}(\cdot)\,\mathrm{div}_{\widehat{z}}(J_t \boldsymbol{Z}^{ij}_{30})(\cdot,t)\rangle_\Theta \mathbb{I} + \boldsymbol{\mathcal{G}} : \langle (1-\widehat{\psi}(\cdot))\mathbb{D}_{\widehat{z}}((J_t \boldsymbol{Z}^{ij}_{30})(\cdot,t))\rangle_\Theta \Big) \otimes \mathbb{J}^{ij},$$
$$x_3 \in (\Delta, \Delta+\delta_*]; \tag{90c}$$

$$\mathbb{F}_p(t,x_3) = \begin{cases} -\mathbb{I} \quad \text{for} \quad x_3 \in (\Delta+\delta^*,1), \\ -|\Sigma_F|\,\mathbb{I} + \mathbb{F}^\sigma_{\text{p-corr}}(t) \quad \text{for} \quad x_3 \in (\Delta+\delta_*,\Delta+\delta^*], \\ -|\Sigma_F|\,|\Theta_F|\,\mathbb{I} + \mathbb{F}^\theta_{\text{p-corr}}(t) \quad \text{for} \quad x_3 \in [\Delta,\Delta+\delta_*], \\ 0 \quad \text{for} \quad x_3 \in (0,\Delta), \end{cases} \tag{91a}$$

where

$$\mathbb{F}^\sigma_{\text{p-corr}}(t) = \Big(\alpha_\lambda \mathbb{I}\otimes\mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^{3} \mathbb{J}^{mn}\otimes\mathbb{J}^{mn}\Big) : \langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_2(\cdot,t,x_3))\rangle_\Sigma$$
$$+ \langle [\widehat{\zeta}(\cdot)\alpha_p\,\mathbb{I}\otimes\mathbb{I} + (1-\widehat{\zeta}(\cdot))\boldsymbol{\mathcal{G}}] : \mathbb{D}_{\widehat{y}}((J_t\boldsymbol{Y}_2)(\cdot,t,x_3))\rangle_\Sigma,$$
$$x_3 \in (\Delta+\delta_*,\Delta+\delta^*], \tag{91b}$$

and

$$\mathbb{F}^\theta_{\text{p-corr}}(t) = \Big[\alpha_\lambda \mathbb{I}\otimes\Big(|\Theta_F|\,\mathbb{I} + \sum_{m,n=1}^{3}\langle\widehat{\psi}\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}^{mn}_{00}\rangle_\Theta \mathbb{J}^{mn}\Big)$$
$$+ 2\alpha_\mu \sum_{m,n=1}^{3}\Big(|\Theta_F|\,\mathbb{J}^{mn} + \langle\widehat{\psi}\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}^{mn}_{00})\rangle_\Theta\Big)\otimes\mathbb{J}^{mn}\Big] : \langle\widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_2(\cdot,t,x_3))\rangle_\Sigma$$
$$+ \Big\langle \Big[\widehat{\zeta}(\cdot)\Big\{\alpha_p\,\mathbb{I}\otimes\Big(|\Theta_F|\,\mathbb{I} + \sum_{m,n=1}^{3}\langle\widehat{\psi}\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}^{mn}_{00}\rangle_\Theta\mathbb{J}^{mn}\Big)$$
$$+ \sum_{m,n=1}^{3}\Big(\boldsymbol{\mathcal{G}} : \langle(1-\widehat{\psi})\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}^{mn}_{00})\rangle_\Theta\Big)\otimes\mathbb{J}^{mn}\Big\}$$
$$+ (1-|\Theta_F|\widehat{\zeta}(\cdot))\boldsymbol{\mathcal{G}}\Big] : \mathbb{D}_{\widehat{y}}((J_t\boldsymbol{Y}_2)(\cdot,t,x_3))\Big\rangle_\Sigma$$
$$+ \int_0^t \Big[\alpha_\lambda\mathbb{I}\otimes\sum_{m,n=1}^{3}\langle\widehat{\psi}(\cdot)\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}^{mn}_{10}(\cdot,t-\tau)\rangle_\Theta\mathbb{J}^{mn}$$
$$+ 2\alpha_\mu\sum_{m,n=1}^{3}\langle\widehat{\psi}(\cdot)\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}^{mn}_{10}(\cdot,t-\tau))\rangle_\Theta\otimes\mathbb{J}^{mn}\Big] : \langle\widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_2(\cdot,\tau,x_3))\rangle_\Sigma d\tau$$
$$+ \int_0^t \Big[\alpha_p\,\mathbb{I}\otimes\sum_{m,n=1}^{3}\langle\widehat{\psi}(\cdot)\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}^{mn}_{10}(\cdot,t-\tau)\rangle_\Theta\mathbb{J}^{mn}$$
$$+ \sum_{m,n=1}^{3}\Big(\boldsymbol{\mathcal{G}} : \langle(1-\widehat{\psi}(\cdot))\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}^{mn}_{10}(\cdot,t-\tau))\rangle_\Theta\Big)\otimes\mathbb{J}^{mn}\Big] :$$
$$: \langle\widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}((J_\tau\boldsymbol{Y}_2)(\cdot,\tau,x_3))\rangle_\Sigma d\tau$$
$$+ |\Sigma_F|\Big(\alpha_\lambda\langle\widehat{\psi}(\cdot)\,\mathrm{div}_{\widehat{z}}\,\boldsymbol{Z}_{20}(\cdot,t)\rangle_\Theta\mathbb{I} + 2\alpha_\mu\langle\widehat{\psi}(\cdot)\mathbb{D}_{\widehat{z}}(\boldsymbol{Z}_{20}(\cdot,t))\rangle_\Theta$$
$$+ \alpha_p\,\langle\widehat{\psi}(\cdot)\,\mathrm{div}_{\widehat{z}}(J_t\boldsymbol{Z}_{20})(\cdot,t)\rangle_\Theta\mathbb{I} + \boldsymbol{\mathcal{G}} : \langle(1-\widehat{\psi}(\cdot))\mathbb{D}_{\widehat{z}}((J_t\boldsymbol{Z}_{20})(\cdot,t))\rangle_\Theta\Big),$$
$$x_3 \in (\Delta,\Delta+\delta_*]; \tag{91c}$$

$$\mathbb{V}(x_3) = \begin{cases} 0 \quad \text{for} \quad x_3 \in (0,\Delta)\cup(\Delta+\delta^*,1), \\ \mathbb{V}^\sigma \quad \text{for} \quad x_3 \in (\Delta+\delta_*,\Delta+\delta^*], \\ \mathbb{V}^\theta \quad \text{for} \quad x_3 \in [\Delta,\Delta+\delta_*], \end{cases} \tag{92a}$$

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where

$$\mathbb{V}^{\sigma} = \alpha_\lambda \big\langle \widehat{\zeta}(\cdot)\, \mathrm{div}_{\widehat{y}}\, \boldsymbol{Y}_4(\cdot, x_3) \big\rangle_\Sigma \mathbb{I} + 2\alpha_\mu \big\langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_4(\cdot, x_3)) \big\rangle_\Sigma \quad \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \tag{92b}$$

$$\mathbb{V}^{\theta} = \boldsymbol{\mathcal{A}}_0^f : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_4(\cdot, x_3)) \rangle_\Sigma + |\Sigma_F|\, \mathbb{H}_{\theta 1}^f \quad \text{for } x_3 \in [\Delta, \Delta + \delta_*]; \tag{92c}$$

$$\mathbb{E}(x_3) = \begin{cases} 0 & \text{for} \quad x_3 \in (0, \Delta) \cup (\Delta + \delta^*, 1), \\ \mathbb{E}^{\sigma} & \text{for} \quad x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \mathbb{E}^{\theta} & \text{for} \quad x_3 \in [\Delta, \Delta + \delta_*], \end{cases} \tag{93a}$$

where

$$\mathbb{E}^{\sigma} = \alpha_p \, \big\langle \widehat{\zeta}(\cdot)\, \mathrm{div}_{\widehat{y}}\, \boldsymbol{Y}_4(\cdot, x_3) \big\rangle_\Sigma \mathbb{I} + \boldsymbol{\mathcal{G}} : \big\langle (1 - \widehat{\zeta}(\cdot)) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_4(\cdot, x_3)) \big\rangle_\Sigma \quad \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \tag{93b}$$

$$\mathbb{E}^{\theta} = \boldsymbol{\mathcal{B}}_0^f : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_4(\cdot, x_3)) \rangle_\Sigma + \boldsymbol{\mathcal{G}} : \langle (1 - |\Theta_F|\, \widehat{\zeta}(\cdot)) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_4(\cdot, x_3)) \rangle_\Sigma + |\Sigma_F|\, \mathbb{H}_{\theta 2}^f \quad \text{for } x_3 \in [\Delta, \Delta + \delta_*]; \tag{93c}$$

$$\mathbb{K}_0(t, x_3) = \begin{cases} 0 & \text{for} \quad x_3 \in (0, \Delta) \cup (\Delta + \delta^*, 1), \\ \mathbb{K}_0^{\sigma}(t) & \text{for} \quad x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \mathbb{K}_0^{\theta}(t) & \text{for} \quad x_3 \in [\Delta, \Delta + \delta_*], \end{cases} \tag{94a}$$

where

$$\mathbb{K}_0^{\sigma}(t) = \alpha_\lambda \big\langle \widehat{\zeta}(\cdot)\, \mathrm{div}_{\widehat{y}}\, \boldsymbol{Y}_5(\cdot, t, x_3) \big\rangle_\Sigma \mathbb{I} + 2\alpha_\mu \big\langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_5(\cdot, t, x_3)) \big\rangle_\Sigma \quad \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \tag{94b}$$

$$\mathbb{K}_0^{\theta}(t) = \boldsymbol{\mathcal{A}}_0^f : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_5(\cdot, t, x_3)) \rangle_\Sigma + \boldsymbol{\mathcal{A}}_1^f(t) : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_4(\cdot, x_3)) \rangle_\Sigma \quad \text{for } x_3 \in [\Delta, \Delta + \delta_*]; \tag{94c}$$

$$\mathbb{K}_1(t, \tau, x_3) = \begin{cases} 0 & \text{for} \quad x_3 \in (0, \Delta) \cup (\Delta + \delta_*, 1), \\ \mathbb{K}_1^{\theta}(t, \tau) & \text{for} \quad x_3 \in [\Delta, \Delta + \delta_*], \end{cases} \tag{95a}$$

where

$$\mathbb{K}_1^{\theta}(t, \tau) = \boldsymbol{\mathcal{A}}_1^f(t) : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_5(\cdot, \tau, x_3)) \rangle_\Sigma \quad \text{for } x_3 \in [\Delta, \Delta + \delta_*]; \tag{95b}$$

$$\mathbb{W}_0(t, x_3) = \begin{cases} 0 & \text{for} \quad x_3 \in (0, \Delta) \cup (\Delta + \delta^*, 1), \\ \mathbb{W}_0^{\sigma}(t) & \text{for} \quad x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \mathbb{W}_0^{\theta}(t) & \text{for} \quad x_3 \in [\Delta, \Delta + \delta_*], \end{cases} \tag{96a}$$

where

$$\mathbb{W}_0^{\sigma}(t) = \alpha_p \, \big\langle \widehat{\zeta}(\cdot)\, \mathrm{div}_{\widehat{y}}\, \boldsymbol{Y}_5(\cdot, t, x_3) \big\rangle_\Sigma \mathbb{I} + \boldsymbol{\mathcal{G}} : \big\langle (1 - \widehat{\zeta}(\cdot)) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_5(\cdot, t, x_3)) \big\rangle_\Sigma \quad \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \tag{96b}$$

$$\mathbb{W}_0^{\theta}(t) = \boldsymbol{\mathcal{B}}_0^f : \langle \widehat{\zeta}(\cdot) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_5(\cdot, t, x_3)) \rangle_\Sigma + \boldsymbol{\mathcal{G}} : \langle (1 - |\Theta_F|\, \widehat{\zeta}(\cdot)) \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_5(\cdot, t, x_3)) \rangle_\Sigma$$

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$$+\boldsymbol{\mathcal{B}}_1^f(t) : \langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_4(\cdot, x_3))\rangle_\Sigma \quad \text{for } x_3 \in [\Delta, \Delta + \delta_*]; \tag{96c}$$

$$\mathbb{W}_1(t,\tau,x_3) = \begin{cases} 0 & \text{for} \quad x_3 \in (0,\Delta) \cup (\Delta + \delta_*, 1), \\ \mathbb{W}_1^\theta(t,\tau) & \text{for} \quad x_3 \in [\Delta, \Delta + \delta_*], \end{cases} \tag{97a}$$

where

$$\mathbb{W}_1^\theta(t,\tau) = \boldsymbol{\mathcal{B}}_1^f(t) : \langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_5(\cdot,\tau,x_3))\rangle_\Sigma \quad \text{for } x_3 \in [\Delta, \Delta + \delta_*]; \tag{97b}$$

$$\mathbb{L}_*(x_3) = \begin{cases} \varkappa_F\,\mathbb{I} & \text{for} \quad x_3 \in (\Delta + \delta^*, 1), \\ \mathbb{L}_*^\sigma & \text{for} \quad x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \mathbb{L}_*^\theta & \text{for} \quad x_3 \in [\Delta, \Delta + \delta_*], \\ \varkappa_S\,\mathbb{I} & \text{for} \quad x_3 \in (0, \Delta), \end{cases} \tag{98a}$$

where

$$\mathbb{L}_*^\sigma = (|\Sigma_F|\varkappa_F + (1 - |\Sigma_F|)\varkappa_S)\mathbb{I} + \Big\langle \widehat{\zeta}(\cdot)\nabla_{\widehat{y}} \sum_{n=1}^{3} Y_6(\cdot, x_3)\boldsymbol{e}_n \Big\rangle_\Sigma^t (\varkappa_F - \varkappa_S) \quad \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \tag{98b}$$

$$\mathbb{L}_*^\theta = \Big\langle \mathbb{L}_\varkappa^\theta(\cdot)\Big(\mathbb{I} + \Big(\nabla_{\widehat{y}} \sum_{n=1}^{3} Y_6(\cdot, x_3)\boldsymbol{e}_n\Big)^t\Big)\Big\rangle_\Sigma \quad \text{for } x_3 \in [\Delta, \Delta + \delta_*]; \tag{98c}$$

$$\mathbb{L}_{B3}(x_3) = \begin{cases} \mathbb{B}_F & \text{for} \quad x_3 \in (\Delta + \delta^*, 1), \\ \mathbb{L}_{B3}^\sigma & \text{for} \quad x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \mathbb{L}_{B3}^\theta & \text{for} \quad x_3 \in [\Delta, \Delta + \delta_*], \\ \mathbb{B}_S & \text{for} \quad x_3 \in (0, \Delta), \end{cases} \tag{99a}$$

where

$$\mathbb{L}_{B3}^\sigma = |\Sigma_F|\mathbb{B}_F + (1 - |\Sigma_F|)\mathbb{B}_S + \sum_{m,n=1}^{3} \left(\langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_0^{mn}(\cdot, x_3))\rangle_\Sigma : (\mathbb{B}_F - \mathbb{B}_S)\right)\mathbb{J}^{mn} \quad \text{for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \tag{99b}$$

$$\mathbb{L}_{B3}^\theta = \langle \mathbb{L}_{B1}^\theta(\cdot)\rangle_\Sigma + \sum_{m,n=1}^{3} \langle \mathbb{L}_{B1}^\theta(\cdot) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_0^{mn}(\cdot, x_3))\rangle_\Sigma\,\mathbb{J}^{mn} \quad \text{for } x_3 \in [\Delta, \Delta + \delta_*]; \tag{99c}$$

$$\mathbb{L}_{B4}(t,x_3) = \begin{cases} 0 & \text{for} \quad x_3 \in (0,\Delta) \cup (\Delta + \delta^*, 1), \\ \mathbb{L}_{B4}^\sigma(t) & \text{for} \quad x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \\ \mathbb{L}_{B4}^\theta(t) & \text{for} \quad x_3 \in [\Delta, \Delta + \delta_*], \end{cases} \tag{100a}$$

where

$$\mathbb{L}_{B4}^\sigma(t) = \sum_{m,n=1}^{3} \left(\langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_1^{mn}(\cdot, t, x_3))\rangle_\Sigma : (\mathbb{B}_F - \mathbb{B}_S)\right)\mathbb{J}^{mn} \text{ for } x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \tag{100b}$$

$$\mathbb{L}_{B4}^\theta(t) = \sum_{m,n=1}^{3} \langle \mathbb{L}_{B1}^\theta(\cdot) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_1^{mn}(\cdot, t, x_3))\rangle_\Sigma\,\mathbb{J}^{mn} + |\Sigma_F|\,\mathbb{L}_{B2}^f(t) + \sum_{m,n=1}^{3} \left(\langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_0^{mn}(\cdot, x_3))\rangle_\Sigma : \mathbb{L}_{B2}^f(t)\right)\mathbb{J}^{mn} \quad \text{for } x_3 \in [\Delta, \Delta + \delta_*]; \tag{100c}$$

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$$\mathbb{L}_{B5}(t,\tau,x_3) = \begin{cases} 0 \quad \text{for} \quad x_3 \in (0,\Delta) \cup (\Delta+\delta_*,1), \\ \mathbb{L}^{\theta}_{B5}(t,\tau) \quad \text{for} \quad x_3 \in [\Delta,\Delta+\delta_*], \end{cases} \tag{101a}$$

where

$$\mathbb{L}^{\theta}_{B5}(t,\tau) = \sum_{m,n=1}^{3} \left(\left\langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\mathbf{Y}^{mn}_1(\cdot,\tau,x_3)) \right\rangle_{\Sigma} : \mathbb{L}^{f}_{B2}(t) \right) \mathbb{J}^{mn} \quad \text{for } x_3 \in [\Delta,\Delta+\delta_*]; \tag{101b}$$

$$\mathbb{L}^{\star}_{v}(t,x_3) = \begin{cases} 0 \quad \text{for} \quad x_3 \in (0,\Delta) \cup (\Delta+\delta^*,1), \\ \mathbb{L}^{\star\sigma}_{v}(t) \quad \text{for} \quad x_3 \in (\Delta+\delta_*,\Delta+\delta^*], \\ \mathbb{L}^{\star\theta}_{v}(t) \quad \text{for} \quad x_3 \in [\Delta,\Delta+\delta_*], \end{cases} \tag{102a}$$

where

$$\mathbb{L}^{\star\sigma}_{v}(t) = \sum_{m,n=1}^{3} \left(\left\langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\mathbf{Y}^{mn}_3(\cdot,t,x_3)) \right\rangle_{\Sigma} : (\mathbb{B}_F - \mathbb{B}_S) \right) \mathbb{J}^{mn} \text{ for } x_3 \in (\Delta+\delta_*,\Delta+\delta^*], \tag{102b}$$

$$\begin{aligned} \mathbb{L}^{\star\theta}_{v}(t) = & \sum_{m,n=1}^{3} \left\langle \mathbb{L}^{\theta}_{B1}(\cdot) : \mathbb{D}_{\widehat{y}}(\mathbf{Y}^{mn}_3(\cdot,t,x_3)) \right\rangle_{\Sigma} \mathbb{J}^{mn} \\ & + \int_0^t \sum_{m,n=1}^{3} \left(\left\langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\mathbf{Y}^{mn}_3(\cdot,\tau,x_3)) \right\rangle_{\Sigma} : \mathbb{L}^{f}_{B2}(t-\tau) \right) \mathbb{J}^{mn} d\tau \text{ for } x_3 \in [\Delta,\Delta+\delta_*]; \end{aligned} \tag{102c}$$

$$L_{\theta 1}(x_3) = \begin{cases} 0 \quad \text{for} \quad x_3 \in (0,\Delta) \cup (\Delta+\delta^*,1), \\ L^{\sigma}_{\theta 1} \quad \text{for} \quad x_3 \in (\Delta+\delta_*,\Delta+\delta^*], \\ L^{\theta}_{\theta 1} \quad \text{for} \quad x_3 \in [\Delta,\Delta+\delta_*], \end{cases} \tag{103a}$$

where

$$L^{\sigma}_{\theta 1} = \left\langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\mathbf{Y}_4(\cdot,x_3)) \right\rangle_{\Sigma} : (\mathbb{B}_F - \mathbb{B}_S) \text{ for } x_3 \in (\Delta+\delta_*,\Delta+\delta^*], \tag{103b}$$

$$L^{\theta}_{\theta 1} = \left\langle \mathbb{L}^{\theta}_{B1}(\cdot) : \mathbb{D}_{\widehat{y}}(\mathbf{Y}_4(\cdot,x_3)) \right\rangle_{\Sigma} + |\Sigma_F| L^{f}_{\theta} \text{ for } \quad x_3 \in [\Delta,\Delta+\delta_*] \tag{103c}$$

$$L_{\theta 2}(t,x_3) = \begin{cases} 0 \quad \text{for} \quad x_3 \in (0,\Delta) \cup (\Delta+\delta^*,1), \\ L^{\sigma}_{\theta 2}(t) \quad \text{for} \quad x_3 \in (\Delta+\delta_*,\Delta+\delta^*], \\ L^{\theta}_{\theta 2}(t) \quad \text{for} \quad x_3 \in [\Delta,\Delta+\delta_*], \end{cases} \tag{104a}$$

where

$$L^{\sigma}_{\theta 2}(t) = \left\langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\mathbf{Y}_5(\cdot,t,x_3)) \right\rangle_{\Sigma} : (\mathbb{B}_F - \mathbb{B}_S) \text{ for } x_3 \in (\Delta+\delta_*,\Delta+\delta^*], \tag{104b}$$

$$\begin{aligned} L^{\theta}_{\theta 2}(t) = & \left\langle \mathbb{L}^{\theta}_{B1}(\cdot) : \mathbb{D}_{\widehat{y}}(\mathbf{Y}_5(\cdot,t,x_3)) \right\rangle_{\Sigma} \\ & + \left\langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\mathbf{Y}_4(\cdot,x_3)) \right\rangle_{\Sigma} : \mathbb{L}^{f}_{B2}(t) \text{ for } x_3 \in [\Delta,\Delta+\delta_*]; \end{aligned} \tag{104c}$$

$$L_{\theta 3}(t,\tau,x_3) = \begin{cases} 0 \quad \text{for} \quad x_3 \in (0,\Delta) \cup (\Delta+\delta_*,1), \\ L^{\theta}_{\theta 3}(t,\tau) \quad \text{for} \quad x_3 \in [\Delta,\Delta+\delta_*], \end{cases} \tag{105a}$$

where

$$L^{\theta}_{\theta 3}(t,\tau) = \left\langle \widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\mathbf{Y}_5(\cdot,\tau,x_3)) \right\rangle_{\Sigma} : \mathbb{L}^{f}_{B2}(t) \quad \text{for } x_3 \in [\Delta,\Delta+\delta_*]; \tag{105b}$$

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$$L_p^{\star}(t,x_3)=\begin{cases}0 \quad \text{for} \quad x_3\in(0,\Delta)\cup(\Delta+\delta^*,1),\\ L_p^{\star\sigma}(t) \quad \text{for} \quad x_3\in(\Delta+\delta_*,\Delta+\delta^*],\\ L_p^{\star\theta}(t) \quad \text{for} \quad x_3\in[\Delta,\Delta+\delta_*],\end{cases} \tag{106a}$$

where

$$L_p^{\star\sigma}(t)=\langle\widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_2(\cdot,t,x_3))\rangle_\Sigma:(\mathbb{B}_F-\mathbb{B}_S) \text{ for } x_3\in(\Delta+\delta_*,\Delta+\delta^*], \tag{106b}$$

$$L_p^{\star\theta}(t)=\langle\mathbb{L}_{B1}^{\theta}(\cdot):\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_2(\cdot,t,x_3))\rangle_\Sigma + \int_0^t\langle\widehat{\zeta}(\cdot)\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_2(\cdot,\tau,x_3))\rangle_\Sigma:\mathbb{L}_{B2}^{f}(t-\tau)d\tau \text{ for } x_3\in[\Delta,\Delta+\delta_*]. \tag{106c}$$

In (85)–(91), (99)–(102), and (106), vector-functions $\boldsymbol{Y}_0^{ij}$, $\boldsymbol{Y}_1^{ij}$, $\boldsymbol{Y}_2$, and $\boldsymbol{Y}_3^{ij}$ are the solutions of the following problems set on the mesoscopic pattern cell Σ:

Problem Y1. ([13][page 1411].) Find a vector-function $\widehat{\boldsymbol{y}}\mapsto\boldsymbol{Y}_0^{ij}(\widehat{\boldsymbol{y}},x_3)$ $(i,j=1,2,3)$ defined in the pattern cell Σ for all $x_3\in[\Delta,\Delta+\delta^*]$ and satisfying the regularity condition

$$\boldsymbol{Y}_0^{ij}(\cdot,x_3)\in(H_\sharp^1(\Sigma)/\mathbb{R})^3, \quad x_3\in[\Delta,\Delta+\delta^*], \tag{107a}$$

and the integral equalities

$$\int_{\Sigma_F}\Big[\Big(\alpha_\lambda\mathbb{I}\otimes\mathbb{I}+2\alpha_\mu\sum_{m,n=1}^{3}\mathbb{J}^{mn}\otimes\mathbb{J}^{mn}\Big):(\mathbb{J}^{ij}+\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_0^{ij}(\widehat{\boldsymbol{y}},x_3)))\Big]:\mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}}))d\widehat{\boldsymbol{y}}=0, \quad \forall\,\boldsymbol{\phi}_{12}\in H_\sharp^1(\Sigma)^3, \quad \forall\,x_3\in(\Delta+\delta_*,\Delta+\delta^*]; \tag{107b}$$

$$\int_{\Sigma_F}\Big[\boldsymbol{\mathcal{A}}_0^f:(\mathbb{J}^{ij}+\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_0^{ij}(\widehat{\boldsymbol{y}},x_3)))\Big]:\mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}}))d\widehat{\boldsymbol{y}}=0, \ \forall\,\boldsymbol{\phi}_{12}\in H_\sharp^1(\Sigma)^3, \ \forall\,x_3\in[\Delta,\Delta+\delta_*]; \tag{107c}$$

$$\int_{\Sigma}\Big[(1-\widehat{\zeta}(\widehat{\boldsymbol{y}}))\boldsymbol{\mathcal{G}}:\Big(\mathbb{J}^{ij}+\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_0^{ij}(\widehat{\boldsymbol{y}},x_3))\Big)\Big]:\mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}}))d\widehat{\boldsymbol{y}}=0, \quad \forall\,\boldsymbol{\phi}_{12}\in H_\sharp^1(\Sigma)^3, \quad \forall\,x_3\in(\Delta+\delta_*,\Delta+\delta^*]; \tag{107d}$$

and

$$\int_{\Sigma}\Big[(1-\widehat{\zeta}(\widehat{\boldsymbol{y}}))\boldsymbol{\mathcal{G}}:\Big(\mathbb{J}^{ij}+\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_0^{ij}(\widehat{\boldsymbol{y}},x_3))\Big)\Big]:\mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}}))d\widehat{\boldsymbol{y}}=0, \quad \forall\,\boldsymbol{\phi}_{12}\in H_\sharp^1(\Sigma)^3, \quad \forall\,x_3\in[\Delta,\Delta+\delta_*]. \tag{107e}$$

Problem Y2. ([13][page 1412].) Find a vector-function $\widehat{\boldsymbol{y}}\mapsto\boldsymbol{Y}_2(\widehat{\boldsymbol{y}},0,x_3)$ defined in Σ_F for all $x_3\in[\Delta,\Delta+\delta^*]$ and satisfying the regularity condition

$$\boldsymbol{Y}_2(\cdot,0,x_3)\in(H_\sharp^1(\Sigma_F)/\mathbb{R})^3, \quad x_3\in[\Delta,\Delta+\delta^*], \tag{108a}$$

and the integral equalities

$$\int_{\Sigma_F}\Big[\Big(\alpha_\lambda\mathbb{I}\otimes\mathbb{I}+2\alpha_\mu\sum_{m,n=1}^{3}\mathbb{J}^{mn}\otimes\mathbb{J}^{mn}\Big):\mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_2(\widehat{\boldsymbol{y}},0,x_3))-\mathbb{I}\Big]:\mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}}))d\widehat{\boldsymbol{y}}=0, \quad \forall\,\boldsymbol{\phi}_{12}\in H_\sharp^1(\Sigma)^3, \quad \forall\,x_3\in(\Delta+\delta_*,\Delta+\delta^*]; \tag{108b}$$

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and

$$\int_{\Sigma_F}\Big[\mathcal{A}_0^f : \mathbb{D}_{\hat{y}}(Y_2(\hat{y},0,x_3)) + \mathbb{F}_f^0(0)\Big] : \mathbb{D}_{\hat{y}}(\phi_{12}(\hat{y}))d\hat{y} = 0,$$
$$\forall\, \phi_{12} \in H_\sharp^1(\Sigma)^3, \quad \forall\, x_3 \in [\Delta, \Delta+\delta_*]. \tag{108c}$$

Problem Y3. ([13][page 1412].) Find a vector-function $\hat{y} \mapsto Y_3^{ij}(\hat{y},0,x_3)$ $(i,j = 1,2,3)$ defined in Σ_F for all $x_3 \in [\Delta, \Delta+\delta^*]$ and satisfying the regularity condition

$$Y_3(\cdot,0,x_3) \in (H_\sharp^1(\Sigma_F)/\mathbb{R})^3, \quad x_3 \in [\Delta, \Delta+\delta^*], \tag{109a}$$

and the integral equalities

$$\int_{\Sigma_F}\Big[\Big(\alpha_\lambda \mathbb{I}\otimes\mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^{3} \mathbb{J}^{mn}\otimes\mathbb{J}^{mn}\Big) : \mathbb{D}_{\hat{y}}(Y_3^{ij}(\hat{y},0,x_3))\Big] : \mathbb{D}_{\hat{y}}(\phi_{12}(\hat{y}))d\hat{y}$$
$$+ \int_{\partial\Sigma_S}\Big[(\mathcal{G} : \mathbb{J}^{ij})\,\phi_{12}(\sigma_{\hat{y}})\Big]\cdot n_\Sigma(\sigma_{\hat{y}})d\sigma_{\hat{y}} = 0,$$
$$\forall\, \phi_{12} \in H_\sharp^1(\Sigma)^3, \quad \forall\, x_3 \in (\Delta+\delta_*, \Delta+\delta^*]; \tag{109b}$$

and

$$\int_{\Sigma_F}[\mathcal{A}_0^f : \mathbb{D}_{\hat{y}}(Y_3^{ij}(\hat{y},0,x_3)) + \mathbb{F}_{sol}^{0ij}(0) - |\Theta_F|\,\mathcal{G} : \mathbb{J}^{ij}] : \mathbb{D}_{\hat{y}}(\phi_{12}(\hat{y}))d\hat{y} = 0,$$
$$\forall\, \phi_{12} \in H_\sharp^1(\Sigma)^3, \quad \forall\, x_3 \in [\Delta, \Delta+\delta_*]. \tag{109c}$$

In (109b), $n_\Sigma = (n_{\Sigma 1}, n_{\Sigma 2}, 0)$ is the unit outward normal to the boundary of domain $\Sigma_S \subset \Sigma$, and $d\sigma_{\hat{y}}$ stands for an infinitesimal element of $\partial\Sigma_S$.

Problem Y4. ([13][page 1414].) Find a vector-function $\hat{y} \mapsto Y_1^{ij}(\hat{y},0,x_3)$ $(i,j = 1,2,3)$ defined in the set Σ_F for all $x_3 \in [\Delta, \Delta+\delta^*]$ and satisfying the regularity condition

$$Y_1^{ij}(\cdot,0,x_3) \in (H_\sharp^1(\Sigma_F)/\mathbb{R})^3, \quad \forall\, x_3 \in [\Delta, \Delta+\delta^*], \tag{110a}$$

and the integral equalities

$$\int_{\Sigma_F}\Big\{\alpha_p\,(\mathbb{I}\otimes\mathbb{I}) : \Big(\mathbb{J}^{ij} + \mathbb{D}_{\hat{y}}(Y_0^{ij}(\hat{y},x_3))\Big)$$
$$+ \Big(\alpha_\lambda \mathbb{I}\otimes\mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^{3} \mathbb{J}^{mn}\otimes\mathbb{J}^{mn}\Big) : \mathbb{D}_{\hat{y}}(Y_1^{ij}(\hat{y},0,x_3))\Big\} : \mathbb{D}_{\hat{y}}(\phi_{12}(\hat{y}))d\hat{y} = 0,$$
$$\forall\, \phi_{12} \in H_\sharp^1(\Sigma)^3, \quad \forall\, x_3 \in (\Delta+\delta_*, \Delta+\delta^*], \quad \forall\, i,j = 1,2,3, \tag{110b}$$

and

$$\int_{\Sigma_F}\Big[(\mathcal{B}_0^f + (1-|\Theta_F|)\mathcal{G}) : \Big(\mathbb{J}^{ij} + \mathbb{D}_{\hat{y}}(Y_0^{ij}(\hat{y},x_3))\Big)$$
$$+ \mathcal{A}_0^f : \mathbb{D}_{\hat{y}}(Y_1^{ij}(\hat{y},0,x_3))\Big] : \mathbb{D}_{\hat{y}}(\phi_{12}(\hat{y}))d\hat{y} = 0,$$
$$\forall\, \phi_{12} \in H_\sharp^1(\Sigma)^3, \quad \forall\, x_3 \in [\Delta, \Delta+\delta_*], \quad \forall\, i,j = 1,2,3. \tag{110c}$$

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Problem Y5. ([13][page 1415].) Find a vector-function $(\widehat{y}, t) \mapsto \boldsymbol{Y}_1^{ij}(\widehat{y}, t, x_3)$ $(i, j = 1, 2, 3)$ defined in the set $\Sigma \times (0, T]$ for all $x_3 \in [\Delta, \Delta + \delta^*]$ and satisfying the regularity condition

$$\boldsymbol{Y}_1^{ij}(\cdot, \cdot, x_3) \in L^\infty(0, T; (H^1_\sharp(\Sigma)/\mathbb{R})^3) \cap H^1(0, T; H^1_\sharp(\Sigma_F)^3), \quad \forall\, x_3 \in [\Delta, \Delta + \delta^*], \tag{111a}$$

and the integral equalities

$$\begin{aligned}\int_\Sigma \Big[\widehat{\zeta}(\widehat{y}) \Big\{ & \alpha_p \, (\mathbb{I} \otimes \mathbb{I}) : (\mathbb{J}^{ij} + \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_0^{ij}(\widehat{y}, x_3))) \\ & + \Big(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^{3} \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \Big) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_1^{ij}(\widehat{y}, t, x_3)) \\ & + \alpha_p \, (\mathbb{I} \otimes \mathbb{I}) : \mathbb{D}_{\widehat{y}}((J_t \boldsymbol{Y}_1^{ij})(\widehat{y}, t, x_3)) \Big\} \\ & + (1 - \widehat{\zeta}(\widehat{y})) \boldsymbol{\mathcal{G}} : \mathbb{D}_{\widehat{y}}((J_t \boldsymbol{Y}_1^{ij})(\widehat{y}, t, x_3)) \Big] : \mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{y})) d\widehat{y} = 0, \\ & \quad \forall\, \boldsymbol{\phi}_{12} \in H^1_\sharp(\Sigma)^3, \ \forall\, t \in [0, T], \ \forall\, x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \ \forall\, i, j = 1, 2, 3, \end{aligned} \tag{111b}$$

and

$$\begin{aligned}\int_\Sigma \Big[& \widehat{\zeta}(\widehat{y}) (\boldsymbol{\mathcal{B}}_0^f + (1 - |\Theta_F|) \boldsymbol{\mathcal{G}}) : (\mathbb{J}^{ij} + \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_0^{ij}(\widehat{y}, x_3))) \\ & + \widehat{\zeta}(\widehat{y}) \Big\{ \boldsymbol{\mathcal{A}}_0^f : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_1^{ij}(\widehat{y}, t, x_3)) + \boldsymbol{\mathcal{B}}_0^f : \mathbb{D}_{\widehat{y}}((J_t \boldsymbol{Y}_1^{ij})(\widehat{y}, t, x_3)) \\ & + \int_0^t [\boldsymbol{\mathcal{A}}_1^f(t - \tau) + (J_t \boldsymbol{\mathcal{B}}_1^f)(t - \tau)] : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_1^{ij}(\widehat{y}, \tau, x_3)) d\tau \Big\} \\ & + (1 - |\Theta_F| \, \widehat{\zeta}(\widehat{y})) \boldsymbol{\mathcal{G}} : \mathbb{D}_{\widehat{y}}((J_t \boldsymbol{Y}_1^{ij})(\widehat{y}, t, x_3)) \Big] : \mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{y})) d\widehat{y} = 0, \\ & \forall\, \boldsymbol{\phi}_{12} \in H^1_\sharp(\Sigma)^3, \quad \forall\, t \in [0, T], \quad \forall\, x_3 \in [\Delta, \Delta + \delta_*], \quad \forall\, i, j = 1, 2, 3. \end{aligned} \tag{111c}$$

In (110b), (110c), (111b), and (111c), function $\boldsymbol{Y}_0^{ij}$ $(i, j = 1, 2, 3)$ is the solution of Problem Y1.

Problem Y6. ([13][pages 1415–1416].) Find a vector-function $(\widehat{y}, t) \mapsto \boldsymbol{Y}_2(\widehat{y}, t, x_3)$ defined in the set $\Sigma \times (0, T]$ for all $x_3 \in [\Delta, \Delta + \delta^*]$ and satisfying the regularity condition

$$\boldsymbol{Y}_2(\cdot, \cdot, x_3) \in L^\infty(0, T; (H^1_\sharp(\Sigma)/\mathbb{R})^3) \cap H^1(0, T; H^1_\sharp(\Sigma_F)^3), \quad \forall\, x_3 \in [\Delta, \Delta + \delta^*], \tag{112a}$$

and the integral equalities

$$\begin{aligned}\int_\Sigma \Big[\widehat{\zeta}(\widehat{y}) \Big\{ & \Big(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^{3} \mathbb{J}^{mn} \otimes \mathbb{J}^{mn} \Big) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_2(\widehat{y}, t, x_3)) \\ & + \alpha_p \, (\mathbb{I} \otimes \mathbb{I}) : \mathbb{D}_{\widehat{y}}((J_t \boldsymbol{Y}_2)(\widehat{y}, t, x_3)) \Big\} \\ & + (1 - \widehat{\zeta}(\widehat{y})) \boldsymbol{\mathcal{G}} : \mathbb{D}_{\widehat{y}}((J_t \boldsymbol{Y}_2)(\widehat{y}, t, x_3)) - \widehat{\zeta}(\widehat{y}) \mathbb{I} \Big] : \mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{y})) d\widehat{y} = 0, \\ & \quad \forall\, \boldsymbol{\phi}_{12} \in H^1_\sharp(\Sigma)^3, \quad \forall\, t \in (0, T], \quad \forall\, x_3 \in (\Delta + \delta_*, \Delta + \delta^*], \end{aligned} \tag{112b}$$

and

$$\begin{aligned}\int_\Sigma \Big[\widehat{\zeta}(\widehat{y}) \Big\{ & \boldsymbol{\mathcal{A}}_0^f : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_2(\widehat{y}, t, x_3)) + \boldsymbol{\mathcal{B}}_0^f : \mathbb{D}_{\widehat{y}}((J_t \boldsymbol{Y}_2)(\widehat{y}, t, x_3)) \\ & + \int_0^t \Big(\boldsymbol{\mathcal{A}}_1^f(t - \tau) + (J_t \boldsymbol{\mathcal{B}}_1^f)(t - \tau) \Big) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_2(\widehat{y}, \tau, x_3)) d\tau \Big\} \end{aligned}$$

$$
\begin{aligned}
&+ (1-|\Theta_F|\,\widehat{\zeta}(\widehat{\boldsymbol{y}}))\boldsymbol{\mathcal{G}} : \mathbb{D}_{\widehat{y}}\big((J_t\boldsymbol{Y}_2)(\widehat{\boldsymbol{y}},t,x_3)\big)\Big] : \mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}}))d\widehat{\boldsymbol{y}} \\
&\qquad + \int_\Sigma \widehat{\zeta}(\widehat{\boldsymbol{y}})\mathbb{F}_f^0(t) : \mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}}))d\widehat{\boldsymbol{y}} = 0, \\
&\forall\, \boldsymbol{\phi}_{12} \in H^1_\sharp(\Sigma)^3, \quad \forall\, t \in (0,T], \quad \forall\, x_3 \in [\Delta, \Delta+\delta_*].
\end{aligned} \tag{112c}
$$

Problem Y7. ([13][page 1416].) Find a vector-function $(\widehat{\boldsymbol{y}},t) \mapsto \boldsymbol{Y}_3^{ij}(\widehat{\boldsymbol{y}},t,x_3)$ $(i,j=1,2,3)$ defined in the set $\Sigma \times (0,T]$ for all $x_3 \in [\Delta, \Delta+\delta^*]$ and satisfying the regularity condition

$$
\boldsymbol{Y}_3^{ij}(\cdot,\cdot,x_3) \in L^\infty(0,T;(H^1_\sharp(\Sigma)/\mathbb{R})^3) \cap H^1(0,T;H^1_\sharp(\Sigma_F)^3), \quad \forall\, x_3 \in [\Delta, \Delta+\delta^*], \tag{113a}
$$

and the integral equalities

$$
\begin{aligned}
&\int_\Sigma \Big[\widehat{\zeta}(\widehat{\boldsymbol{y}})\Big\{\Big(\alpha_\lambda \mathbb{I}\otimes\mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^3 \mathbb{J}^{mn}\otimes\mathbb{J}^{mn}\Big) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_3^{ij}(\widehat{\boldsymbol{y}},t,x_3)) \\
&\quad + \alpha_p\,(\mathbb{I}\otimes\mathbb{I}) : \mathbb{D}_{\widehat{y}}\big((J_t\boldsymbol{Y}_3^{ij})(\widehat{\boldsymbol{y}},t,x_3)\big)\Big\} \\
&\quad + (1-\widehat{\zeta}(\widehat{\boldsymbol{y}}))\boldsymbol{\mathcal{G}} : \mathbb{D}_{\widehat{y}}\big((J_t\boldsymbol{Y}_3^{ij})(\widehat{\boldsymbol{y}},t,x_3)\big) - \widehat{\zeta}(\widehat{\boldsymbol{y}})\boldsymbol{\mathcal{G}} : \mathbb{J}^{ij}\Big] : \mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}}))d\widehat{\boldsymbol{y}} = 0, \\
&\quad \forall\, \boldsymbol{\phi}_{12} \in H^1_\sharp(\Sigma)^3, \quad \forall\, t \in (0,T], \quad \forall\, x_3 \in (\Delta+\delta_*, \Delta+\delta^*],
\end{aligned} \tag{113b}
$$

and

$$
\begin{aligned}
&\int_\Sigma \Big[\widehat{\zeta}(\widehat{\boldsymbol{y}})\Big\{\boldsymbol{\mathcal{A}}_0^f : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_3^{ij}(\widehat{\boldsymbol{y}},t,x_3)) + \boldsymbol{\mathcal{B}}_0^f : \mathbb{D}_{\widehat{y}}\big((J_t\boldsymbol{Y}_3^{ij})(\widehat{\boldsymbol{y}},t,x_3)\big) \\
&\quad + \int_0^t \Big(\boldsymbol{\mathcal{A}}_1^f(t-\tau) + (J_t\boldsymbol{\mathcal{B}}_1^f)(t-\tau)\Big) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_3^{ij}(\widehat{\boldsymbol{y}},\tau,x_3))d\tau\Big\} \\
&\quad + (1-|\Theta_F|\,\widehat{\zeta}(\widehat{\boldsymbol{y}}))\boldsymbol{\mathcal{G}} : \mathbb{D}_{\widehat{y}}\big((J_t\boldsymbol{Y}_3^{ij})(\widehat{\boldsymbol{y}},t,x_3)\big)\Big] : \mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}}))d\widehat{\boldsymbol{y}} \\
&\qquad + \int_\Sigma \widehat{\zeta}(\widehat{\boldsymbol{y}})\Big(\mathbb{F}_{sol}^{0ij}(t) - |\Theta_F|\,\boldsymbol{\mathcal{G}} : \mathbb{J}^{ij}\Big) : \mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}}))d\widehat{\boldsymbol{y}} = 0, \\
&\quad \forall\, \boldsymbol{\phi}_{12} \in H^1_\sharp(\Sigma)^3, \quad \forall\, t \in (0,T], \quad \forall\, x_3 \in [\Delta, \Delta+\delta_*].
\end{aligned} \tag{113c}
$$

Remark 4. *Each of Problems Y1–Y7 has the unique solution [13][Propos. 8, 9, 11] and each of the vector-functions $\boldsymbol{Y}_0^{ij}$, $\boldsymbol{Y}_1^{ij}$, $\boldsymbol{Y}_2$, and $\boldsymbol{Y}_3^{ij}$ does not vary with change of x_3 on segments $(\Delta, \Delta+\delta_*)$ and $(\Delta+\delta_*, \Delta+\delta^*)$ [13][Remark 25], i.e.,*

$$
\frac{\partial \boldsymbol{Y}_0^{ij}}{\partial x_3} = \frac{\partial \boldsymbol{Y}_1^{ij}}{\partial x_3} = \frac{\partial \boldsymbol{Y}_2}{\partial x_3} = \frac{\partial \boldsymbol{Y}_3^{ij}}{\partial x_3} = 0 \quad \textit{for } x_3 \in (\Delta, \Delta+\delta_*) \cup (\Delta+\delta_*, \Delta+\delta^*).
$$

Also, $\boldsymbol{Y}_k^{ij} = \boldsymbol{Y}_k^{ji}$ for all $k = 0,1,3$, and $i,j = 1,2,3$ [13][Remark 24].

In (92)–(98) and (103)–(105), vector-functions $\boldsymbol{Y}_4$ and $\boldsymbol{Y}_5$ and the scalar functions Y_6^i $(i=1,2,3)$ are the solutions of the following cell problems:

Problem Y8. Find a vector-function $\widehat{\boldsymbol{y}} \mapsto \boldsymbol{Y}_4(\widehat{\boldsymbol{y}},x_3)$ defined in the pattern cell Σ for all $x_3 \in [\Delta, \Delta+\delta^*, 1]$ and satisfying the regularity condition

$$
\boldsymbol{Y}_4(\cdot,x_3) \in (H^1_\sharp(\Sigma)/\mathbb{R})^3, \quad x_3 \in [\Delta, \Delta+\delta^*], \tag{114a}
$$

and the integral equality

$$
\int_\Sigma \big[\boldsymbol{\mathcal{A}}_0(\widehat{\boldsymbol{y}},x_3) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_4(\widehat{\boldsymbol{y}},x_3)) + \mathbb{H}_{\theta 1}(\widehat{\boldsymbol{y}},x_3)\big] : \mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}}))d\widehat{\boldsymbol{y}} = 0,
$$

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$$\forall\, \boldsymbol{\phi}_{12} \in H^1_\sharp(\Sigma)^3, \quad x_3 \in [\Delta, \Delta + \delta^*]. \tag{114b}$$

Problem Y9. Find a vector-function $(\widehat{y}, t) \mapsto \boldsymbol{Y}_5(\widehat{\boldsymbol{y}}, t, x_3)$ defined in the set $\Sigma \times (0, T]$ for all $x_3 \in [\Delta, \Delta + \delta^*]$ and satisfying the regularity condition

$$\boldsymbol{Y}_5(\cdot, x_3) \in L^\infty(0, T; (H^1_\sharp(\Sigma)/\mathbb{R})^3) \cap H^1(0, T; H^1_\sharp(\Sigma_F)^3), \quad x_3 \in [\Delta, \Delta + \delta^*], \tag{115a}$$

and the integral equality

$$\begin{aligned}
\int_\Sigma \Big[& \boldsymbol{\mathcal{A}}_0(\widehat{\boldsymbol{y}}, x_3) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_5(\widehat{\boldsymbol{y}}, t, x_3)) + \boldsymbol{\mathcal{B}}_0(\widehat{\boldsymbol{y}}, x_3) : \mathbb{D}_{\widehat{y}}((J_t \boldsymbol{Y}_5(\widehat{\boldsymbol{y}}, t, x_3)) \\
& + \int_0^t (\boldsymbol{\mathcal{A}}_1(\widehat{\boldsymbol{y}}, t-\tau, x_3) + (J_t \boldsymbol{\mathcal{B}}_1)(\widehat{\boldsymbol{y}}, t-\tau, x_3)) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_5(\widehat{\boldsymbol{y}}, \tau, x_3)) d\tau \\
& + \boldsymbol{\mathcal{B}}_0(\widehat{\boldsymbol{y}}, x_3) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_4(\widehat{\boldsymbol{y}}, x_3)) + \boldsymbol{\mathcal{A}}_1(\widehat{\boldsymbol{y}}, t, x_3) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_4(\widehat{\boldsymbol{y}}, x_3)) \\
& + (J_t \boldsymbol{\mathcal{B}}_1)(\widehat{\boldsymbol{y}}, t, x_3) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_4(\widehat{\boldsymbol{y}}, x_3)) + \mathbb{H}_{\theta 2}(\widehat{\boldsymbol{y}}, x_3) \Big] : \mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}})) d\widehat{\boldsymbol{y}} = 0, \\
& \forall\, \boldsymbol{\phi}_{12} \in H^1_\sharp(\Sigma)^3, \quad x_3 \in [\Delta, \Delta + \delta^*],
\end{aligned} \tag{115b}$$

where $\boldsymbol{Y}_4$ is the solution of Problem Y8.

Problem Y10. Find a scalar function $\widehat{\boldsymbol{y}} \mapsto Y_6^i(\cdot, x_3)$ $(i = 1, 2, 3)$ defined in the pattern cell Σ for all $x_3 \in [\Delta, \Delta + \delta^*]$ and satisfying the regularity condition

$$Y_6^i(\cdot, x_3) \in H^1_\sharp(\Sigma)/\mathbb{R}, \quad \forall\, x_3 \in [\Delta, \Delta + \delta^*], \tag{116a}$$

and the integral equality

$$\int_\Sigma \mathbb{L}_\varkappa(\widehat{\boldsymbol{y}}, x_3)(\boldsymbol{e}_i + \nabla_{\widehat{y}} Y_6^i(\widehat{\boldsymbol{y}}, x_3)) \cdot \nabla_{\widehat{y}} \eta_{12}(\widehat{\boldsymbol{y}}) d\widehat{\boldsymbol{y}} = 0, \quad \forall\, \eta_{12} \in H^1_\sharp(\Sigma), \,\forall\, x_3 \in [\Delta, \Delta + \delta^*]. \tag{116b}$$

Problems Y8–Y10 are well-posed:

Proposition 7. *Each of Problems Y8, Y9, and Y10 has a unique solution.*

Proof. Due to the positive definiteness, symmetry and sufficient regularity of $\boldsymbol{\mathcal{A}}_0$ and $\mathbb{L}_\varkappa$ and due to the sufficient regularity of $\mathbb{H}_{\theta 1}$ (see Proposition 6), Problems Y8 and Y10 have the unique solutions by the Lax–Milgram theorem.

Applying the Laplace transform in t to (115b) and multiplying the resulting equation by s, where $s > 0$ is the dual to t, we establish the variational equation

$$\begin{aligned}
\int_\Sigma \Big[& \boldsymbol{\mathcal{H}}^s(\widehat{\boldsymbol{y}}, x_3) : \mathbb{D}_{\widehat{y}}(\overline{\boldsymbol{Y}}_5(\widehat{\boldsymbol{y}}, s, x_3)) \\
& + (\boldsymbol{\mathcal{B}}_0(\widehat{\boldsymbol{y}}, x_3) + s\overline{\boldsymbol{\mathcal{A}}}_1(\widehat{\boldsymbol{y}}, s, x_3) + \overline{\boldsymbol{\mathcal{B}}}_1(\widehat{\boldsymbol{y}}, s, x_3)) : \mathbb{D}_{\widehat{y}}(\boldsymbol{Y}_4(\widehat{\boldsymbol{y}}, x_3)) \\
& \qquad + \mathbb{H}_{\theta 2}(\widehat{\boldsymbol{y}}, x_3) \Big] : \mathbb{D}_{\widehat{y}}(\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}})) d\widehat{\boldsymbol{y}} = 0, \\
& \forall\, \boldsymbol{\phi}_{12} \in H^1_\sharp(\Sigma)^3, \quad x_3 \in [\Delta, \Delta + \delta^*], \quad s > 0.
\end{aligned} \tag{117}$$

Here, $\boldsymbol{\mathcal{H}}^s$ is defined by formula (73), and Notation 6 for the Laplace transform is used. Since $\boldsymbol{\mathcal{H}}^s$ is positive definite and symmetric (see assertion (iii) of Proposition 6) and $\boldsymbol{\mathcal{H}}^s$, $\boldsymbol{\mathcal{B}}_0$, $\boldsymbol{\mathcal{A}}_1$, $\boldsymbol{\mathcal{B}}_1$, and $\boldsymbol{Y}_4$ are sufficiently regular (see Proposition 6 and formula (114a)), the variational equation (117) has a unique solution $\overline{\boldsymbol{Y}}_5$ for $s > 0$ by the Lax–Milgram theorem. In turn, due to one-on-oneness of the Laplace transform $\mathcal{L}$, we conclude that $\boldsymbol{Y}_5 = \mathcal{L}^{-1}(\overline{\boldsymbol{Y}}_5)$ is the unique solution of the variational equation (115b) on segment $\{\Delta \leq x_3 \leq \Delta + \delta^*\}$, which completes the proof of the unique solvability of Problem Y9. □

Remark 5. *Due to Remark 3, the vector-functions $\mathbf{Y}_4$ and $\mathbf{Y}_5$ and the scalar functions Y_6^i $(i = 1, 2, 3)$ do not vary with change of x_3 on segments $(\Delta, \Delta + \delta_*)$ and $(\Delta + \delta_*, \Delta + \delta^*)$, i.e.,*

$$\frac{\partial \mathbf{Y}_4}{\partial x_3} = \frac{\partial \mathbf{Y}_5}{\partial x_3} = 0, \quad \frac{\partial Y_6^i}{\partial x_3} = 0 \quad \text{for } x_3 \in (\Delta, \Delta + \delta_*) \cup (\Delta + \delta_*, \Delta + \delta^*).$$

Remark 6. *Due to Remarks 3–5, from the explicit representations (85)–(106) it is clear that tensors $\boldsymbol{\mathcal{V}}^{\sigma}_{\text{corr}}$, $\boldsymbol{\mathcal{V}}^{\theta}_{\text{corr}}$, $\boldsymbol{\mathcal{E}}^{\sigma}_{\text{corr}}$, $\boldsymbol{\mathcal{E}}^{\theta}_{\text{corr}}$, matrices $\mathbb{V}^{\sigma}$, $\mathbb{V}^{\theta}$, $\mathbb{E}^{\sigma}$, $\mathbb{E}^{\theta}$, $\mathbb{L}^{\sigma}_{*}$, $\mathbb{L}^{\theta}_{*}$, $\mathbb{L}^{\sigma}_{B3}$, $\mathbb{L}^{\theta}_{B3}$, and scalars $L^{\sigma}_{\theta 1}$, $L^{\theta}_{\theta 1}$ are constant and tensors $\boldsymbol{\mathcal{K}}^{\sigma}(t)$, $\boldsymbol{\mathcal{K}}^{\theta}(t)$, $\boldsymbol{\mathcal{Q}}^{\sigma}(\tau)$, $\boldsymbol{\mathcal{Q}}^{\theta}_0(\tau)$, $\boldsymbol{\mathcal{Q}}^{\theta}_1(t)$, $\boldsymbol{\mathcal{W}}^{\theta}_1(t)$, $\boldsymbol{\mathcal{F}}^{\sigma}_{v\text{-corr}}(t)$, $\boldsymbol{\mathcal{F}}^{\theta}_{v\text{-corr}}(t)$, matrices $\mathbb{Q}^{pq}(\tau)$, $\mathbb{F}^{\sigma}_{p\text{-corr}}(t)$, $\mathbb{F}^{\theta}_{p\text{-corr}}(t)$, $\mathbb{K}^{\sigma}_0(t)$, $\mathbb{K}^{\theta}_0(t)$, $\mathbb{K}^{\theta}_1(t,\tau)$, $\mathbb{W}^{\sigma}_0(t)$, $\mathbb{W}^{\theta}_0(t)$, $\mathbb{L}^{\sigma}_{B4}(t)$, $\mathbb{L}^{\theta}_{B4}(t)$, $\mathbb{L}^{\theta}_{B5}(t,\tau)$, $\mathbb{L}^{\star\sigma}_{v}(t)$, $\mathbb{L}^{\star\theta}_{v}(t)$, and scalars $L^{\sigma}_{\theta 2}(t)$, $L^{\theta}_{\theta 2}(t)$, $L^{\theta}_{\theta 3}(t,\tau)$, $L^{\star\sigma}_{p}(t)$, $L^{\star\theta}_{p}(t)$ depend only on t and/or τ.*

Let us turn to justification of Theorem 3.

Proof of Theorem 3. We separate variable $\widehat{\boldsymbol{y}}$ standardly, by the procedure of asymptotic decomposition analogous to the one carried out in [13][Secs. 10, 11]. More certainly, we seek for the representation of $\boldsymbol{u}^{(1)}$ and $\theta^{(1)}$ in the form

$$\begin{aligned}\boldsymbol{u}^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{y}},t) &= \sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j}(\boldsymbol{x},t)\mathbf{Y}_0^{ij}(\widehat{\boldsymbol{y}},x_3) + \sum_{i,j=1}^{3} \int_0^t \frac{\partial u_i}{\partial x_j}(\boldsymbol{x},t-\tau)\mathbf{Y}_1^{ij}(\widehat{\boldsymbol{y}},\tau,x_3)d\tau \\ &+ p^0(\boldsymbol{x})\mathbf{Y}_2(\widehat{\boldsymbol{y}},t,x_3) + \sum_{i,j=1}^{3} \frac{\partial v_i^0}{\partial x_j}(\boldsymbol{x})\mathbf{Y}_3^{ij}(\widehat{\boldsymbol{y}},t,x_3) \\ &+ \theta(\boldsymbol{x},t)\mathbf{Y}_4(\widehat{\boldsymbol{y}},x_3) + \int_0^t \theta(\boldsymbol{x},t-\tau)\mathbf{Y}_5(\widehat{\boldsymbol{y}},\tau,x_3)d\tau \end{aligned} \tag{118}$$

and

$$\theta^{(1)}(\boldsymbol{x},\widehat{\boldsymbol{y}},t) = \sum_{i=1}^{3} Y_6^i(\widehat{\boldsymbol{y}},x_3)\frac{\partial \theta(\boldsymbol{x},t)}{\partial x_i}, \tag{119}$$

where $\mathbf{Y}_0^{ij}$, $\mathbf{Y}_1^{ij}$, $\mathbf{Y}_2$, $\mathbf{Y}_3^{ij}$, $\mathbf{Y}_4$, $\mathbf{Y}_5$, and Y_6^i are unknown functions that should be defined as the solutions of the cell problems. In order to formulate the proper cell problems, in (38a) we take

$$\boldsymbol{\phi}(\boldsymbol{x},t) := 0, \quad \boldsymbol{\phi}_1(\boldsymbol{x},\widehat{\boldsymbol{y}},t) := \frac{\partial \phi_{10}(t)}{\partial t}\phi_{11}(\boldsymbol{x})\boldsymbol{\phi}_{12}(\widehat{\boldsymbol{y}}),$$

where ϕ_{10} is an arbitrary smooth function vanishing in a neighborhood of section $\{t = T\}$, ϕ_{11} is an arbitrary smooth function vanishing in a neighborhood of $\partial\Omega$, and $\boldsymbol{\phi}_{12}$ is an arbitrary 1-periodic smooth vector-function, in (38b) we take

$$\eta(\boldsymbol{x},t) := 0, \quad \eta_1(\boldsymbol{x},\widehat{\boldsymbol{y}},t) := \eta_{11}(\boldsymbol{x},t)\eta_{12}(\widehat{\boldsymbol{y}}),$$

where η_{11} is an arbitrary smooth function vanishing in a neighborhood of $\partial\Omega$ and section $\{t = T\}$ and η_{12} is an arbitrary 1-periodic smooth function. Next, we insert (118) into (38a) and (118) and (119) into (38b). After this, properly collecting terms and performing simple but rather lengthy technical manipulations, we eventually deduce the set of the well-posed cell problems for determining the unique functions $\mathbf{Y}_0^{ij}$, $\mathbf{Y}_1^{ij}$, $\mathbf{Y}_2$, $\mathbf{Y}_3^{ij}$, $\mathbf{Y}_4$, $\mathbf{Y}_5$, and Y_6^i, see the above stated Problems Y1–Y10.

Finally, in (38a) we take $\boldsymbol{\phi}_1 \equiv 0$ and insert representation (118) for $\boldsymbol{u}^{(1)}$ and in (38b) we take $\eta_1 \equiv 0$ and insert (118) and (119) on the respective places of $\boldsymbol{u}^{(1)}$ and $\theta^{(1)}$. By this, we establish exactly the variational equations (84a) and (84b), which completes the proof of Theorem 3. □

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The effective homogenized coefficients in the variational equations (84a) and (84b) have the following main properties.

Proposition 8. **(i)** *The tensor-valued functions* $\boldsymbol{\mathcal{V}} = \boldsymbol{\mathcal{V}}(x_3)$, $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}(x_3)$, $\boldsymbol{\mathcal{K}} = \boldsymbol{\mathcal{K}}(t,x_3)$, $\boldsymbol{\mathcal{Q}} = \boldsymbol{\mathcal{Q}}(t,\tau,x_3)$, $\boldsymbol{\mathcal{W}} = \boldsymbol{\mathcal{W}}(t,\tau,x_3)$, *and* $\boldsymbol{\mathcal{F}}_v = \boldsymbol{\mathcal{F}}_v(t,x_3)$ *and the matrix-valued function* $\mathbb{F}_p = \mathbb{F}_p(t,x_3)$ *satisfy the following regularity conditions:*

$$\boldsymbol{\mathcal{V}},\, \boldsymbol{\mathcal{E}} \in L^\infty(0,1)^{3\times3\times3\times3}, \tag{120}$$

$$\boldsymbol{\mathcal{K}},\, \boldsymbol{\mathcal{F}}_v,\, \partial_t\boldsymbol{\mathcal{K}},\, \partial_t\boldsymbol{\mathcal{F}}_v \in L^\infty((0,T)\times(0,1))^{3\times3\times3\times3}, \tag{121}$$

$$\boldsymbol{\mathcal{Q}},\, \boldsymbol{\mathcal{W}},\, \partial_t\boldsymbol{\mathcal{Q}} \in L^\infty((0,T)\times(0,T)\times(0,1))^{3\times3\times3\times3}, \tag{122}$$

$$\mathbb{F}_p,\, \partial_t\mathbb{F}_p \in L^\infty((0,T)\times(0,1))^{3\times3}. \tag{123}$$

(ii) *The tensor-valued function* $\boldsymbol{\mathcal{V}}$ *satisfies the finiteness property*

$$\boldsymbol{\mathcal{V}} = 0 \quad \textit{for} \quad x_3 \in (0,\Delta), \tag{124}$$

the symmetry property

$$\mathcal{V}^{ijkl} = \mathcal{V}^{jikl} = \mathcal{V}^{jilk} = \mathcal{V}^{klij}, \quad \forall\, x_3 \in [0,1], \quad \forall\, i,j,k,l = 1,2,3, \tag{125}$$

and the uniform positive definiteness property:

$$\textit{there exists a constant } C_8 > 0 \textit{ such that } \big(\boldsymbol{\mathcal{V}}(x_3) : \mathbb{X}\big) : \mathbb{X} \geq C_8|\mathbb{X}|^2, \quad \forall\, \mathbb{X} \in \mathbb{R}^{3\times3}_{\text{symm}}, \quad \forall\, x_3 \in [\Delta,1]. \tag{126}$$

(iii) *The tensor-valued function*

$$\begin{aligned}\boldsymbol{\mathfrak{H}}^s(x_3) :=& s\boldsymbol{\mathcal{V}}(x_3) + \boldsymbol{\mathcal{E}}(x_3) + s\overline{\boldsymbol{\mathcal{K}}}(s,x_3) + 1_{x_3\in(\Delta+\delta_*,\Delta+\delta^*]}(\boldsymbol{x})\overline{\boldsymbol{\mathcal{Q}}}^{\sigma}(s) \\ &+ 1_{x_3\in(\Delta,\Delta+\delta_*]}(\boldsymbol{x})\Big\{\overline{\boldsymbol{\mathcal{Q}}}^{\theta}_0(s) + \sum_{m,n=1}^{3}\Big[\Big(s\overline{\boldsymbol{\mathcal{Q}}}^{\theta}_1(s) + \overline{\boldsymbol{\mathcal{W}}}^{\theta}_1(s)\Big) : \overline{\mathbb{Q}}^{mn}(s)\Big] \otimes \mathbb{J}^{mn}\Big\}\end{aligned} \tag{127}$$

satisfies the symmetry properties

$$\mathfrak{H}^{s\,ijkl} = \mathfrak{H}^{s\,jikl} = \mathfrak{H}^{s\,jilk}, \quad \forall\, x_3 \in [0,1], \quad \forall\, s > 0,\ \forall\, i,j,k,l = 1,2,3, \tag{128}$$

$$(\boldsymbol{\mathfrak{H}}^s : \mathbb{X}) : \mathbb{W} = (\boldsymbol{\mathfrak{H}}^s : \mathbb{W}) : \mathbb{X}, \quad \forall\, \mathbb{X}, \mathbb{W} \in \mathbb{R}^{3\times3}_{\text{symm}}, \tag{129}$$

and the uniform positive definiteness property:

$$\textit{there exists a constant } C^s_9 > 0 \textit{ such that } (\boldsymbol{\mathfrak{H}}^s(x_3) : \mathbb{X}) : \mathbb{X} \geq C^s_9|\mathbb{X}|^2, \quad \forall\, \mathbb{X} \in \mathbb{R}^{3\times3}_{\text{symm}}, \quad \forall\, x_3 \in [0,1], \quad \forall\, s > 0. \tag{130}$$

(iv) *For a.e.* $(t,x_3) \in (0,T)\times(0,1)$, *tensor* $\boldsymbol{\mathcal{F}}_v(t,x_3)$ *satisfies the symmetry properties*

$$\mathcal{F}^{ijkl}_v = \mathcal{F}^{jikl}_v = \mathcal{F}^{jilk}_v, \quad \forall\, i,j,k,l = 1,2,3, \tag{131}$$

$$(\boldsymbol{\mathcal{F}}_v : \mathbb{X}) : \mathbb{W} = (\boldsymbol{\mathcal{F}}_v : \mathbb{W}) : \mathbb{X}, \quad \forall\, \mathbb{X}, \mathbb{W} \in \mathbb{R}^{3\times3}_{\text{symm}}. \tag{132}$$

(v) *For a.e.* $(t,x_3) \in (0,T)\times(0,1)$, *matrix* $\mathbb{F}_p(t,x_3)$ *is symmetric.*
(vi) *The matrix-valued functions* $\mathbb{V} = \mathbb{V}(x_3)$, $\mathbb{E} = \mathbb{E}(x_3)$, $\mathbb{K}_0 = \mathbb{K}_0(t,x_3)$, $\mathbb{K}_1 = \mathbb{K}_1(t,\tau,x_3)$, $\mathbb{W}_0 = \mathbb{W}_0(t,x_3)$, $\mathbb{W}_1 = \mathbb{W}_1(t,\tau,x_3)$, $\mathbb{L}_* = \mathbb{L}_*(x_3)$, $\mathbb{L}_{B3} = \mathbb{L}_{B3}(x_3)$, $\mathbb{L}_{B4} = \mathbb{L}_{B4}(t,x_3)$, $\mathbb{L}_{B5} = \mathbb{L}_{B5}(t,\tau,x_3)$,

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$\mathbb{L}_v^\star = \mathbb{L}_v^\star(t,x_3)$ *and the scalar functions* $L_{\theta 1} = L_{\theta 1}(x_3)$, $L_{\theta 2} = L_{\theta 2}(t,x_3)$, $L_{\theta 3} = L_{\theta 3}(t,\tau,x_3)$, $L_p^\star = L_p^\star(t,x_3)$ *satisfy the following regularity conditions:*

$$\mathbb{V},\ \mathbb{E},\ \mathbb{L}_*,\ \mathbb{L}_{B3} \in L^\infty(0,1)^{3\times 3}, \tag{133}$$
$$\mathbb{K}_0,\ \mathbb{W}_0,\ \mathbb{L}_{B4},\ \mathbb{L}_v^\star \in L^\infty((0,T)\times(0,1))^{3\times 3}, \tag{134}$$
$$\mathbb{K}_1,\ \mathbb{W}_1,\ \mathbb{L}_{B5} \in L^\infty((0,T)\times(0,T)\times(0,1))^{3\times 3}, \tag{135}$$
$$L_{\theta 1} \in L^\infty(0,1),\ L_{\theta 2},\ L_p^\star \in L^\infty((0,T)\times(0,1)), \tag{136}$$
$$L_{\theta 3} \in L^\infty((0,T)\times(0,T)\times(0,1)). \tag{137}$$

(vii) *The matrix-valued function* $\mathbb{L}_*$ *satisfies the symmetry property*

$$L_*^{ij} = L_*^{ji}, \quad \forall\, x_3 \in [0,1], \quad \forall\, i,j = 1,2,3, \tag{138}$$

and the uniform positive definiteness property:

$$\begin{aligned} &\text{there exists a constant } C_{10} > 0 \text{ such that} \\ &\mathbb{L}_*(x_3)\boldsymbol{\xi}\cdot\boldsymbol{\xi} \geq C_{10}|\boldsymbol{\xi}|^2, \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^3, \quad \forall\, x_3 \in [\Delta,1]. \end{aligned} \tag{139}$$

(viii) *Matrices* $\mathbb{V}$, $\mathbb{E}$, $\mathbb{K}_0$, $\mathbb{K}_1 = \mathbb{K}_1$, $\mathbb{W}_0$, $\mathbb{W}_1$, $\mathbb{L}_{B3}$, $\mathbb{L}_{B4}$, $\mathbb{L}_{B5}$, *and* $\mathbb{L}_v^\star$ *are symmetric.*

Notation 7. *In* (127), *by* $1_{x_3\in\mathcal{I}}$ *we denote the indicator function of the set* $\mathcal{I}$*:*

$$1_{x_3\in\mathcal{I}}(\boldsymbol{x}) = \begin{cases} 1 & \text{if} \quad x_3 \in \mathcal{I}, \\ 0 & \text{if} \quad x_3 \notin \mathcal{I}. \end{cases}$$

Proof. Assertions (i)–(v) of Proposition 8 were proved in [13][Sec. 11, Propos. 12].

Proof of assertion (vi) is quite analogous to the justification of assertion (i): we have that the regularity properties (133)–(137) hold true due to the regularity properties (107a), (111a), (112a), (113a), (114a), (116a) of the solutions of the cell problems (Problems Y1, Y5–Y10) and due to the regularity properties of the tensor-valued functions $\boldsymbol{\mathcal{A}}_0$, $\boldsymbol{\mathcal{A}}_1$, $\boldsymbol{\mathcal{B}}_0$, $\boldsymbol{\mathcal{B}}_1$, see assertion (i) of Proposition 6.

Assertion (vii) directly follows from representations (46), (98) and the integral equalities (46b), (116b) by the same arguments, as in [21][Ch. V, § 3].

Further, notice that tensors $\boldsymbol{\mathcal{A}}_0^f$, $\boldsymbol{\mathcal{A}}_1^f(t)$, $\boldsymbol{\mathcal{B}}_0^f$, and $\boldsymbol{\mathcal{B}}_1^f(t)$ are symmetric due to the respective representations (39b), (41b), (40b), and (42b) and the symmetry property $\boldsymbol{Z}_{k0}^{ij} = \boldsymbol{Z}_{k0}^{ji}$, see Remark 2. Now, the symmetry of $\mathbb{V}$ follows from representation (92) and the symmetry of $\boldsymbol{\mathcal{A}}_1^f(t)$ and $\mathbb{H}_{\theta 1}^f$ (see assertion (vii) of Proposition 6); the symmetry of $\mathbb{E}$ follows from representation (93) and the symmetry of $\boldsymbol{\mathcal{B}}_0^f$, $\boldsymbol{\mathcal{G}}$, and $\mathbb{H}_{\theta 2}^f$ (see assertion (vii) of Proposition 6); the symmetry of $\mathbb{K}_0$ follows from representation (94) and the symmetry of $\boldsymbol{\mathcal{A}}_0^f$ and $\boldsymbol{\mathcal{A}}_1^f(t)$; the symmetry of $\mathbb{K}_1$ follows from representation (95) and the symmetry of $\boldsymbol{\mathcal{A}}_1^f(t)$; the symmetry of $\mathbb{W}_0$ follows from representation (96) and the symmetry of $\boldsymbol{\mathcal{B}}_0^f$ and $\boldsymbol{\mathcal{G}}$; the symmetry of $\mathbb{W}_1$ follows from (97) and the symmetry of $\boldsymbol{\mathcal{B}}_1^f(t)$; the symmetry of $\mathbb{L}_{B3}$ follows from (99), the symmetry property $\boldsymbol{Y}_0^{ij} = \boldsymbol{Y}_0^{ji}$, see Remark 4, and the symmetry of $\mathbb{L}_{B1}^\theta$, see assertion (vii) of Proposition 6; the symmetry of $\mathbb{L}_{B4}$ follows from (100), the symmetry property $\boldsymbol{Y}_k^{ij} = \boldsymbol{Y}_k^{ji}$ $(k = 0,1)$, see Remark 4, and the symmetry of $\mathbb{L}_{B1}^\theta$ and $\mathbb{L}_{B2}^f$, see assertion (vii) of Proposition 6; the symmetry of $\mathbb{L}_{B5}$ follows from (101), the symmetry property $\boldsymbol{Y}_1^{ij} = \boldsymbol{Y}_1^{ji}$ and the symmetry of $\mathbb{L}_{B2}^f$; the symmetry of $\mathbb{L}_v^\star$ follows from (102), the symmetry property $\boldsymbol{Y}_3^{ij} = \boldsymbol{Y}_3^{ji}$, see Remark 4, and the symmetry of $\mathbb{L}_{B2}^f(t)$. Proposition 8 is proved. □

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8. The Effective Macroscopic Model — Integro-Differential Formulation

Analogously to [13][Sec. 12], we deduce that, in the sense of the theory of distributions, Model H-var is equivalent to the following initial-boundary value problem.

Model H-ID. (The homogenized model — integro-differential formulation.) Find a macroscopic velocity field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ and a macroscopic temperature distribution $\theta = \theta(\boldsymbol{x}, t)$ satisfying the following set of equations and initial, boundary and interfacial conditions.

■ In $\Omega_{\text{fl}} = \{\boldsymbol{x} \in \Omega : \ \Delta + \delta^* < x_3 < 1\}$, for $t > 0$, the fluid motion is governed by the Stokes–Fourier system

$$\alpha_\tau \rho_F \partial_t \boldsymbol{u} = \operatorname{div}_x\big(2\alpha_\mu \mathbb{D}_x(\boldsymbol{u}) + (\alpha_\lambda \operatorname{div}_x \boldsymbol{u} + \alpha_p \operatorname{div}_x J_t\boldsymbol{u} - p^0)\mathbb{I} - \mathbb{B}_F\theta\big) + \alpha_g \rho_F \boldsymbol{f}, \tag{140a}$$

$$c_F \partial_t \theta = \operatorname{div}_x(\varkappa_F \nabla_x \theta) - \mathbb{B}_F : \mathbb{D}_x(\boldsymbol{u}) + \Psi_F. \tag{140b}$$

■ In $\Omega_\sigma = \{\boldsymbol{x} \in \Omega : \ \Delta + \delta_* < x_3 < \Delta + \delta^*\}$, for $t > 0$, the fluid-structure interactions are governed by the equations of linear thermoviscoelasticity with memory effects: the balance of momentum equation

$$\begin{aligned}\alpha_\tau \rho_\sigma \partial_t \boldsymbol{u} = {} & \operatorname{div}_x\big(\boldsymbol{\mathcal{P}}^\sigma : \mathbb{D}_x(\boldsymbol{u}) + \boldsymbol{\mathcal{G}}^\sigma : \mathbb{D}_x(J_t\boldsymbol{u}) + \mathbb{V}^\sigma\theta + \mathbb{E}^\sigma J_t\theta\big) \\ & + \operatorname{div}_x \int_0^t \boldsymbol{\mathcal{R}}_0^\sigma(t-\tau) : \mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},\tau))d\tau + \operatorname{div}_x \int_0^t \mathbb{R}_1^\sigma(t-\tau)\theta(\boldsymbol{x},\tau)d\tau \\ & + \operatorname{div}_x\big(\mathbb{F}_p^\sigma(t)p^0 + \boldsymbol{\mathcal{F}}_v^\sigma(t) : \mathbb{D}_x(\boldsymbol{v}^0)\big) + \alpha_g \rho_\sigma \boldsymbol{f}\end{aligned} \tag{140c}$$

and the energy equation

$$\begin{aligned}c_\sigma \partial_t \theta = {} & \operatorname{div}_x(\mathbb{L}_*^\sigma \nabla_x \theta) - \mathbb{L}_{B3}^\sigma : \mathbb{D}_x(\boldsymbol{u}) - L_{\theta 1}^\sigma \theta \\ & - \int_0^t \mathbb{L}_{B4}^\sigma(t-\tau) : \mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},\tau))d\tau - \int_0^t L_{\theta 2}^\sigma(t-\tau)\theta(\boldsymbol{x},\tau)d\tau \\ & - L_p^{\star\sigma} p^0 - \mathbb{L}_v^{\star\sigma} : \mathbb{D}_x(\boldsymbol{v}^0) + \Psi_\sigma.\end{aligned} \tag{140d}$$

In (140c) and (140d), $\rho_\sigma = |\Sigma_F|\rho_F + (1 - |\Sigma_F|)\rho_S$ is the constant mean density and $c_\sigma = |\Sigma_F|c_F + (1 - |\Sigma_F|)c_S$ is the constant mean heat capacity of the homogenized thermoviscoelastic medium occupying layer Ω_σ;

$$\boldsymbol{\mathcal{P}}^\sigma = |\Sigma_F|\big(\alpha_\lambda \mathbb{I} \otimes \mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^{3} \mathbb{J}^{mn} \otimes \mathbb{J}^{mn}\big) + \boldsymbol{\mathcal{V}}_{\text{corr}}^\sigma$$

is the constant effective instantaneous viscous stress tensor,

$$\boldsymbol{\mathcal{G}}^\sigma = |\Sigma_F|\alpha_p \mathbb{I} \otimes \mathbb{I} + (1 - |\Sigma_F|)\boldsymbol{\mathcal{G}} + \boldsymbol{\mathcal{E}}_{\text{corr}}^\sigma$$

is the constant effective instantaneous elastic stiffness tensor, $\mathbb{V}^\sigma$ and $\mathbb{E}^\sigma$ are two constant matrices corresponding to the effective instantaneous thermal dilatation, $\mathbb{L}_*^\sigma$ is the matrix of effective heat conductivity, the components of matrix $\mathbb{L}_{B3}^\sigma$ and the scalar coefficient $L_{\theta 1}^\sigma$ are the effective coefficients characterizing irreversible heat generation due to the combined effect of thermal dilatation and viscosity friction in Ω_σ; the components of tensor

$$\boldsymbol{\mathcal{R}}_0^\sigma(t) = \boldsymbol{\mathcal{K}}^\sigma(t) + (J_t \boldsymbol{\mathcal{Q}}^\sigma)(t)$$

and matrices

$$\mathbb{R}_1^\sigma(t) = \mathbb{K}_0^\sigma(t) + (J_t \mathbb{W}_0^\sigma)(t), \quad \mathbb{L}_{B4}^\sigma(t)$$

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and scalar $L^{\sigma}_{\theta 2}(t)$ are the time-dependent relaxation kernels determining influence of thermomechanical history of the medium occupying Ω_σ during the time period $(0, t)$ on the current state at the time moment t; the components of tensor

$$\mathcal{F}^{\sigma}_{v}(t) = (1 - |\Sigma_F|)\mathcal{G} + \mathcal{F}^{\sigma}_{\text{v-corr}}(t)$$

and matrices

$$\mathbb{F}^{\sigma}_{p}(t) = -|\Sigma_F|\mathbb{I} - \mathbb{F}^{\sigma}_{\text{p-corr}}(t), \quad \mathbb{L}^{\star\sigma}_{v}(t)$$

and scalar $L^{\star\sigma}_{p}(t)$ are the additional time-dependent effective coefficients arising from the initial fluid-structure balance of stress and heat in Ω_σ.

■ In $\Omega_\theta = \{x \in \Omega\colon\ \Delta < x_3 < \Delta + \delta_*\}$, for $t > 0$, the fluid-structure interactions are governed by the equations of linear thermoviscoelasticity with memory effects: the balance of momentum equation

$$\begin{aligned}
\alpha_\tau \rho_\theta \partial_t \boldsymbol{u} = {} & \operatorname{div}_x\big(\mathcal{P}^{\theta} : \mathbb{D}_x(\boldsymbol{u}) + \mathcal{G}^{\theta} : \mathbb{D}_x(J_t \boldsymbol{u}) + \mathbb{V}^{\theta}\theta + \mathbb{E}^{\theta} J_t \theta\big) \\
& + \operatorname{div}_x \int_0^t \mathcal{R}^{\theta}_{1}(t-\tau) : \mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},\tau))d\tau + \operatorname{div}_x \int_0^t \mathbb{R}^{\theta}_{1}(t-\tau)\theta(\boldsymbol{x},\tau)d\tau \\
& + \operatorname{div}_x \int_0^t \int_0^{\tau'} \mathcal{R}^{\theta}_{2}(t-\tau', \tau'-\tau) : \mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},\tau))d\tau d\tau' \\
& + \operatorname{div}_x \int_0^t \int_0^{\tau'} \mathbb{R}^{\theta}_{2}(t-\tau', \tau'-\tau)\theta(\boldsymbol{x},\tau)d\tau d\tau' \\
& + \operatorname{div}_x\big(\mathbb{F}^{\theta}_{p}(t)p^0 + \mathcal{F}^{\theta}_{v}(t) : \mathbb{D}_x(\boldsymbol{v}^0)\big) + \alpha_g \rho_\theta \boldsymbol{f}
\end{aligned} \tag{140e}$$

and the energy equation

$$\begin{aligned}
c_\theta \partial_t \theta = {} & \operatorname{div}_x(\mathbb{L}^{\theta}_{*}\nabla_x\theta) - \mathbb{L}^{\theta}_{B3} : \mathbb{D}_x(\boldsymbol{u}) - L^{\theta}_{\theta 1}\theta \\
& - \int_0^t \mathbb{L}^{\theta}_{B4}(t-\tau) : \mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},\tau))d\tau - \int_0^t L^{\theta}_{\theta 2}(t-\tau)\theta(\boldsymbol{x},\tau)d\tau \\
& - \int_0^t \int_0^{\tau'} \mathbb{L}^{\theta}_{B5}(t-\tau', \tau'-\tau) : \mathbb{D}_x(\boldsymbol{u}(\boldsymbol{x},\tau))d\tau d\tau' \\
& - \int_0^t \int_0^{\tau'} L^{\theta}_{\theta 3}(t-\tau', \tau'-\tau)\theta(\boldsymbol{x},\tau)d\tau d\tau' - L^{\star\theta}_{p} p^0 - \mathbb{L}^{\star\theta}_{v} : \mathbb{D}_x(\boldsymbol{v}^0) + \Psi_\theta.
\end{aligned} \tag{140f}$$

In (140e) and (140f), $\rho_\theta = |\Sigma_F||\Theta_F|\rho_F + (1 - |\Sigma_F||\Theta_F|)\rho_S$ is the constant mean density and $c_\theta = |\Sigma_F||\Theta_F|c_F + (1 - |\Sigma_F||\Theta_F|)c_S$ is the constant mean heat capacity of the homogenized thermoviscoelastic medium occupying the layer Ω_θ;

$$\mathcal{P}^{\theta} = |\Sigma_F||\Theta_F|\big(\alpha_\lambda \mathbb{I}\otimes\mathbb{I} + 2\alpha_\mu \sum_{m,n=1}^{3} \mathbb{J}^{mn}\otimes\mathbb{J}^{mn}\big) + \mathcal{V}^{\theta}_{\text{corr}}$$

is the constant effective instantaneous viscous stress tensor,

$$\mathcal{G}^{\theta} = |\Sigma_F||\Theta_F|\alpha_p \mathbb{I}\otimes\mathbb{I} + (1 - |\Sigma_F||\Theta_F|)\mathcal{G} + \mathcal{E}^{\theta}_{\text{corr}}$$

is the constant effective instantaneous elastic stiffness tensor, $\mathbb{V}^{\theta}$ and $\mathbb{E}^{\theta}$ are two constant matrices corresponding to the effective instantaneous thermal dilatation, $\mathbb{L}^{\theta}_{*}$ is the matrix of effective heat conductivity, the components of matrix $\mathbb{L}^{\theta}_{B3}$ and the scalar coefficient $L^{\theta}_{\theta 1}$ are the effective coefficients characterizing irreversible heat generation due to the combined effect of thermal dilatation and viscosity friction in Ω_θ; the components of tensors

$$\mathcal{R}^{\theta}_{1}(t) = \mathcal{K}^{\theta}(t) + (J_t \mathcal{Q}^{\theta}_{0})(t),$$

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$$\boldsymbol{\mathcal{R}}_2^\theta(t_1,t_2) = \sum_{p,q=1}^{3} \left(\boldsymbol{\mathcal{Q}}_1^\theta(t_1) : \mathbb{Q}^{pq}(t_2) + \boldsymbol{\mathcal{W}}_1^\theta(t_1) : (J_{t_2}\mathbb{Q}^{pq})(t_2) \right) \otimes \mathbb{J}^{pq},$$

and matrices

$$\mathbb{R}_1^\theta(t) = \mathbb{K}_0^\theta(t) + (J_t\mathbb{W}_0^\theta)(t), \quad \mathbb{R}_2^\theta(t_1,t_2) = \mathbb{K}_1^\theta(t_1,t_2) + (J_{t_2}\mathbb{W}_1^\theta)(t_1,t_2),$$

$\mathbb{L}_{B4}^\theta(t)$, and $\mathbb{L}_{B5}^\theta(t_1,t_2)$, and scalars $L_{\theta 2}^\theta(t)$ and $L_{\theta 3}^\theta(t_1,t_2)$ are the time-dependent relaxation kernels determining influence of thermomechanical history of the medium occupying Ω_θ during the time period $(0,t)$ on the current state at the time moment t; the components of tensor

$$\boldsymbol{\mathcal{F}}_v^\theta(t) = (1 - |\Sigma_F||\Theta_F|)\boldsymbol{\mathcal{G}} + \boldsymbol{\mathcal{F}}_{\text{v-corr}}^\theta(t)$$

and matrices

$$\mathbb{F}_p^\theta(t) = -|\Sigma_F||\Theta_F|\mathbb{I} - \mathbb{F}_{\text{p-corr}}^\theta(t), \quad \mathbb{L}_v^{\star\theta}(t)$$

and scalar $L_p^{\star\theta}(t)$ are the additional time-dependent effective coefficients arising from the initial fluid-structure balance of stress and heat in Ω_θ.

■ In $\Omega_{\text{pl}} = \{\boldsymbol{x} \in \Omega\colon\ 0 < x_3 < \Delta\}$, for $t > 0$, the motion of the elastic heat-conducting plate is governed by the classical equations of linear thermoelasticity: the balance of momentum equation

$$\alpha_\tau \rho_S \partial_t \boldsymbol{u} = \operatorname{div}_x \left(\boldsymbol{\mathcal{G}} : \nabla_x J_t \boldsymbol{u} - \mathbb{B}_S \theta + \boldsymbol{\mathcal{G}} : \nabla_x \boldsymbol{v}^0 \right) + \alpha_g \rho_S \boldsymbol{f} \tag{140g}$$

and the energy equation

$$c_S \partial_t \theta = \operatorname{div}_x(\varkappa_S \nabla_x \theta) - \mathbb{B}_S : \partial_t \mathbb{D}_x(\boldsymbol{u}) + \Psi_S. \tag{140h}$$

■ The macroscopic velocity and temperature satisfy the initial conditions

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}^0(\boldsymbol{x}), \quad \theta(\boldsymbol{x},0) = \theta^0(\boldsymbol{x}) \quad \boldsymbol{x} \in \Omega, \tag{140i}$$

i.e., the initial conditions (7c).

■ On the outer boundary of Ω, the macroscopic velocity and temperature satisfy the conditions

$$\boldsymbol{u} = \boldsymbol{u}^\star, \quad \theta = \theta^\star \quad \text{for } \boldsymbol{x} \in \partial\Omega,\ t \in (0,T), \tag{140j}$$

i.e., the boundary conditions (7d), which are inherited from Problems A_ε, H-3sc, and H-2sc, successively.

■ On interfaces $\{\boldsymbol{x} \in \Omega\colon\ x_3 = \Delta + \delta^*\}$, $\{\boldsymbol{x} \in \Omega\colon\ x_3 = \Delta + \delta_*\}$ and $\{\boldsymbol{x} \in \Omega\colon\ x_3 = \Delta\}$, the standard matching relations hold, which are the classical conditions of continuity of velocity, temperature, normal stress, and normal heat flux.

Remark 7. *As in [13][Remark 28], we note that $\boldsymbol{\mathcal{P}}^\sigma$ and $\boldsymbol{\mathcal{P}}^\theta$ are the restrictions of $\boldsymbol{\mathcal{V}}$ to the segments $\{\Delta + \delta_* < x_3 \le \Delta + \delta^*\}$ and $\{\Delta \le x_3 \le \Delta + \delta_*\}$, respectively; and $\boldsymbol{\mathcal{G}}^\sigma$ and $\boldsymbol{\mathcal{G}}^\theta$ are the restrictions of $\boldsymbol{\mathcal{E}}$ to the segments $\{\Delta + \delta_* < x_3 \le \Delta + \delta^*\}$ and $\{\Delta \le x_3 \le \Delta + \delta_*\}$, respectively.*

In equations (140a)–(140j), tensors $\boldsymbol{\mathcal{P}}^\sigma$, $\boldsymbol{\mathcal{G}}^\sigma$, $\boldsymbol{\mathcal{R}}_0^\sigma$, $\boldsymbol{\mathcal{F}}_v^\sigma$, $\boldsymbol{\mathcal{P}}^\theta$, $\boldsymbol{\mathcal{G}}^\theta$, $\boldsymbol{\mathcal{R}}_1^\theta$, $\boldsymbol{\mathcal{R}}_2^\theta$, $\boldsymbol{\mathcal{F}}_v^\theta$, and $\boldsymbol{\mathcal{G}}$, matrices $\mathbb{B}_F$, $\mathbb{V}^\sigma$, $\mathbb{E}^\sigma$, $\mathbb{R}_1^\sigma$, $\mathbb{F}_p^\sigma$, $\mathbb{L}_*^\sigma$, $\mathbb{L}_{B3}^\sigma$, $\mathbb{L}_{B4}^\sigma$, $\mathbb{L}_v^{\star\sigma}$, $\mathbb{V}^\theta$, $\mathbb{E}^\theta$, $\mathbb{R}_1^\theta$, $\mathbb{R}_2^\theta$, $\mathbb{F}_p^\theta$, $\mathbb{L}_*^\theta$, $\mathbb{L}_{B3}^\theta$, $\mathbb{L}_{B4}^\theta$, $\mathbb{L}_{B5}^\theta$, $\mathbb{L}_v^{\star\theta}$, and $\mathbb{B}_S$, scalar coefficients α_τ, ρ_F, α_μ, α_λ, α_p, α_g, c_F, $\varkappa_F$, ρ_σ, c_σ, $L_{\theta 1}^\sigma$, $L_p^{\star\sigma}$, ρ_θ, c_θ, $L_{\theta 1}^\theta$, $L_p^{\star\theta}$, ρ_S, c_S, and $\varkappa_S$, scalar relaxation tensors $L_{\theta 2}^\sigma$, $L_{\theta 2}^\theta$, and $L_{\theta 3}^\theta$, the distributed mass force $\boldsymbol{f}$, the volumetric densities of external heat application Ψ_F, Ψ_σ, Ψ_θ, Ψ_S, the initial functions $\boldsymbol{v}^0$, p^0, $\boldsymbol{u}^0$, and θ^0, and the boundary functions $\boldsymbol{u}^\star$ and $\theta^\star$ are considered to be given. Thus we obtain that Model H-ID is the integro-differential closed effective macroscopic

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homogenized model of '*the compressible thermofluid – two-level fine thermoelastic structure*' interactions, where $(\boldsymbol{u}, \theta)$ is the pair of sought functions.

We naturally formulate the notion of weak solutions of Model H-ID as follows.

Definition 3. *A pair of functions $(\boldsymbol{u}, \theta)$ is called* a weak solution to Model H-ID *if it is a solution of Problem H-var.*

Theorem 3 immediately yields the following assertion.

Corollary 1. *Assume that $\boldsymbol{\mathcal{P}}^{\sigma}$, $\boldsymbol{\mathcal{G}}^{\sigma}$, $\boldsymbol{\mathcal{R}}_0^{\sigma}$, $\boldsymbol{\mathcal{F}}_v^{\sigma}$, $\boldsymbol{\mathcal{P}}^{\theta}$, $\boldsymbol{\mathcal{G}}^{\theta}$, $\boldsymbol{\mathcal{R}}_1^{\theta}$, $\boldsymbol{\mathcal{R}}_2^{\theta}$, $\boldsymbol{\mathcal{F}}_v^{\theta}$, $\mathbb{V}^{\sigma}$, $\mathbb{E}^{\sigma}$, $\mathbb{R}_1^{\sigma}$, $\mathbb{F}_p^{\sigma}$, $\mathbb{L}_*^{\sigma}$, $\mathbb{L}_{B3}^{\sigma}$, $\mathbb{L}_{B4}^{\sigma}$, $\mathbb{L}_v^{\star\sigma}$, $\mathbb{V}^{\theta}$, $\mathbb{E}^{\theta}$, $\mathbb{R}_1^{\theta}$, $\mathbb{R}_2^{\theta}$, $\mathbb{F}_p^{\theta}$, $\mathbb{L}_*^{\theta}$, $\mathbb{L}_{B3}^{\theta}$, $\mathbb{L}_{B4}^{\theta}$, $\mathbb{L}_{B5}^{\theta}$, $\mathbb{L}_v^{\star\theta}$, ρ_{σ}, c_{σ}, $L_{\theta 1}^{\sigma}$, $L_p^{\star\sigma}$, ρ_{θ}, c_{θ}, $L_{\theta 1}^{\theta}$, $L_p^{\star\theta}$, $L_{\theta 2}^{\sigma}$, $L_{\theta 2}^{\theta}$, and $L_{\theta 3}^{\theta}$ are defined from the data of micro- and mesoscopic structure in accordance with formulas (39)–(51), (85)–(106) and solutions of Problems Z1–Z9 and Y1–Y10.*

Then, for any given $\boldsymbol{v}^0 \in H^1(\Omega)^3$, $p^0 \in H^1(\Omega)$, $\boldsymbol{u}^0 \in H^1(\Omega)^3$, $\theta^0 \in H^1(\Omega)$, $\boldsymbol{u}^{\star} \in C^2(\overline{\Omega} \times [0,T])^3$, $\theta^{\star} \in C^2(\overline{\Omega} \times [0,T])$, $\boldsymbol{f} \in L^2(\Omega \times (0,T))^3$, Ψ_F, Ψ_S, Ψ_{σ}, $\Psi_{\theta} \in L^2(\Omega \times (0,T))$ such that $\boldsymbol{u}^0(\boldsymbol{x}) = \boldsymbol{u}^{\star}(\boldsymbol{x}, 0)$, $\theta^0(\boldsymbol{x}) = \theta^{\star}(\boldsymbol{x}, 0)$ for $\boldsymbol{x} \in \partial\Omega$, $\partial_t \boldsymbol{f} \in L^2(\Omega \times (0,T))^3$, and $\partial_t \Psi_F$, $\partial_t \Psi_S$, $\partial_t \Psi_{\sigma}$, $\partial_t \Psi_{\theta} \in L^2(\Omega \times (0,T))$, there exists a unique weak solution of Model H-ID in the sense of Definition 3.

9. Concluding Remarks and Discussion

On the base of the constructions made in Secs. 5–8, we can now propose the following algorithm of determining the effective macroscopic physical characteristics of the reciprocal motion of the fine two-level thermoelastic bristle structure and the viscous compressible thermofluid, starting from the microstructure.

- **(i)** Using the given data of the microstructure, solve Problems Z1–Z9 to find $\boldsymbol{Z}_{00}^{ij}$, $\boldsymbol{Z}_{10}^{ij}$, $\boldsymbol{Z}_{20}$, $\boldsymbol{Z}_{30}^{ij}$, $\boldsymbol{Z}_{40}$, and Z_5^i.
- **(ii)** Inserting the solutions of Problems Z1–Z9 into (39b), (40b), (41b), (42b), (43b), (43c), (44b), (45b), (46b), (47b), (48b), (53b), (49b), and (50b), find tensors $\boldsymbol{\mathcal{A}}_0^f$, $\boldsymbol{\mathcal{B}}_0^f$, $\boldsymbol{\mathcal{A}}_1^f$, and $\boldsymbol{\mathcal{B}}_1^f$, matrices $\mathbb{F}_f^0$, $\mathbb{F}_{sol}^{0\,ij}$, $\mathbb{H}_{\theta 1}^f$, $\mathbb{H}_{\theta 2}^f$, $\mathbb{L}_{\varkappa}^{\theta}$, $\mathbb{L}_{B1}^{\theta}$, $\mathbb{L}_{B2}^f$, and $\mathbb{L}_{vf}^0$, and scalars L_{θ}^f and L_{pf}^0, respectively.
- **(iii)** Using tensors $\boldsymbol{\mathcal{A}}_0^f$, $\boldsymbol{\mathcal{B}}_0^f$, $\boldsymbol{\mathcal{A}}_1^f$, and $\boldsymbol{\mathcal{B}}_1^f$ and matrices $\mathbb{F}_f^0$, $\mathbb{F}_{sol}^{0\,ij}$, $\mathbb{H}_{\theta 1}^f$, $\mathbb{H}_{\theta 2}^f$, and $\mathbb{L}_{\varkappa}^{\theta}$, obtained on the previous step, solve Problems Y1–Y10 to find $\boldsymbol{Y}_0^{ij}$, $\boldsymbol{Y}_1^{ij}$, $\boldsymbol{Y}_2$, $\boldsymbol{Y}_3^{ij}$, $\boldsymbol{Y}_4$, $\boldsymbol{Y}_5$, and Y_6^i.
- **(iv)** Inserting the solutions of Problems Y1–Y10 into (85)–(106), calculate the homogenized macroscopic tensors $\boldsymbol{\mathcal{V}}$, $\boldsymbol{\mathcal{E}}$, $\boldsymbol{\mathcal{K}}$, $\boldsymbol{\mathcal{Q}}$, $\boldsymbol{\mathcal{W}}$, $\boldsymbol{\mathcal{F}}_v$, matrices $\mathbb{F}_p$, $\mathbb{V}$, $\mathbb{E}$, $\mathbb{K}_0$, $\mathbb{K}_1$, $\mathbb{W}_0$, $\mathbb{W}_1$, $\mathbb{L}_*$, $\mathbb{L}_{B3}$, $\mathbb{L}_{B4}$, $\mathbb{L}_{B5}$, $\mathbb{L}_v^{\star}$, and scalars $L_{\theta 1}$, $L_{\theta 2}$, $L_{\theta 3}$, and $L_p^{\star}$, respectively.
- **(v)** Provided with the data obtained on the previous step, solve Problem H-var (and, equivalently in the sense of distributions, Problem H-ID) to find the macroscopic velocity distribution $\boldsymbol{u}$ and temperature distribution θ.

This five-step algorithm is quite possible to implement numerically. In contrast, Model A_{ε} with a small ε is inaccessible for practical analysis due to the enormous amount of necessary calculations. The interim homogenized Models H-3sc and H-2sc also worth consideration in line with a possible numerical analysis.

Both from a purely theoretical point of view and from the point of view of promising applications in technology, Model H-ID serves as a generalization of previously constructed models [1,2,4,13] from the isothermal case to the case when the effect of heat transfer is taken into account. As it is mentioned in Sec. 1, application of Model H-ID can most likely be found in describing the airflow near the surface of a plant leaf, in modeling the epithelial surfaces of blood vessels, in modeling superhydrophobic and superoleophobic surfaces, as well as in designing biotechnological devices operating in liquids.

In future works, it would be useful and interesting to consider isothermal and non-isothermal Model A_{ε}-type systems taking into account transfer of admixture by a free flow of liquid (air) and sedimentation of the admixture particles on the surface of a two-level fine bristly structure. In

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such systems, the laws of balance of the admixture concentration in an open liquid (air) and on the solid-liquid (solid-air) interface can be taken in accordance with W. Hornung and W. Jaeger [22]. To homogenize the balance of the sedimented admixture on the solid-liquid (solid-air) interface, the Allaire–Damlamian–Hornung modified version of two-scale convergence [23] can be generalized and used. By this, one could make a further extension of study of bristly structures in interaction with liquids and gases based on [13] and the present article.

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10. Nomenclature

In this appendix, we provide a fairly comprehensive list of notations used in the article.

Roman Symbols		
Notation	Description	Introduced in
$\boldsymbol{\mathcal{A}}_0$, $\boldsymbol{\mathcal{A}}_1$	effective two-scale tensors	(39a), (41a)
$\boldsymbol{\mathcal{A}}_0^f$, $\boldsymbol{\mathcal{A}}_1^f$	tensors derived from microstructure	(39b), (41b)
$\boldsymbol{\mathcal{B}}_0$, $\boldsymbol{\mathcal{B}}_1$	effective two-scale tensors	(40a), (42a)
$\boldsymbol{\mathcal{B}}_0^f$, $\boldsymbol{\mathcal{B}}_1^f$	tensors derived from microstructure	(40b), (42b)
$\mathbb{B}$	homogenized three-scale dimensionless thermal dilatation matrix	Notat. 3
$\mathbb{B}_F$, $\mathbb{B}_S$	dimensionless thermal dilatation matrices in fluid and solid, resp.	(1a), (1c)
$\mathbb{B}_*^\varepsilon$	uniform notation for dimensionless thermal dilatation in Ω	(11)
c	homogenized three-scale dimensionless specific heat capacity	Notat. 3
c_F, c_S	dimensionless specific heat capacities in fluid and solid, resp.	(1b), (1d)
c^ε	uniform notation for dimensionless specific heat capacity in Ω	(12)
$\mathbb{D}_x$, $\mathbb{D}_{\widehat{y}}$, $\mathbb{D}_{\widehat{z}}$	symmetric parts of the gradients ∇_x $\nabla_{\widehat{y}}$, and $\nabla_{\widehat{z}}$, resp.	Secs. 2, 5
$\boldsymbol{\mathcal{E}}$	effective instantaneous elasticity tensor	(86a)
$\boldsymbol{\mathcal{E}}^\theta_{\mathrm{corr}}$	instantaneous elasticity corrector term on $[\Delta, \Delta + \delta_*]$	(86c)
$\boldsymbol{\mathcal{E}}^\sigma_{\mathrm{corr}}$	instantaneous elasticity corrector term on $(\Delta + \delta_*, \Delta + \delta^*]$	(86b)
$\mathbb{E}$	matrix corresponding to the effective instantaneous thermal dilatation	(93a)
$\mathbb{E}^\theta$	restriction of $\mathbb{V}$ to $[\Delta, \Delta + \delta_*]$	(93c)
$\mathbb{E}^\sigma$	restriction of $\mathbb{V}$ to $(\Delta + \delta_*, \Delta + \delta^*]$	(93b)
$\boldsymbol{e}_m$	Cartesian basis vector in $\mathbb{R}^3$	Notat. 5

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$\boldsymbol{\mathcal{F}}_v$	tensor derived from the meso- and microstructures	(90a)
$\boldsymbol{\mathcal{F}}_v^\theta$	restriction of $\boldsymbol{\mathcal{F}}_v$ to $[\Delta, \Delta+\delta_*]$	(90a), Model H-ID
$\boldsymbol{\mathcal{F}}_v^\sigma$	restriction of $\boldsymbol{\mathcal{F}}_v$ to $(\Delta+\delta_*, \Delta+\delta^*]$	(90a), Model H-ID
$\boldsymbol{\mathcal{F}}_{\text{v-corr}}^\theta$	tensor derived from the meso- and microstructures	(90c)
$\boldsymbol{\mathcal{F}}_{\text{v-corr}}^\sigma$	tensor derived from the meso- and microstructures	(90b)
$\mathbb{F}_p$	matrix derived from the meso- and microstructures	(91a)
$\mathbb{F}_p^\theta$	restriction of $\mathbb{F}_p$ to $[\Delta, \Delta+\delta_*]$	(91a), Model H-ID
$\mathbb{F}_p^\sigma$	restriction of $\mathbb{F}_p$ to $(\Delta+\delta_*, \Delta+\delta^*]$	(91a), Model H-ID
$\mathbb{F}^0$	effective two-scale matrix	(43a)
$\mathbb{F}_f^0$, $\mathbb{F}_{sol}^{0mn}$	matrices derived from microstructure	(43b), (43c)
$\mathbb{F}_{\text{p-corr}}^\theta$	tensor derived from the meso- and microstructures	(91c)
$\mathbb{F}_{\text{p-corr}}^\sigma$	tensor derived from the meso- and microstructures	(91b)
f	distributed mass force	(1a)
$\boldsymbol{\mathcal{G}}$	elastic stiffness tensor	(1c)
$\boldsymbol{\mathcal{G}}^\theta$	effective elastic stiffness tensor in Ω_θ	(140e)
$\boldsymbol{\mathcal{G}}^\sigma$	effective elastic stiffness tensor in Ω_σ	(140c)
$\boldsymbol{\mathcal{H}}^s$	Laplace image of the principle two-scale stress tensor multiplied by s	(73)
$\mathfrak{H}^s$	Laplace image of the principle effective stress tensor multiplied by s	(127)
$\mathbb{H}_{\theta 1}$, $\mathbb{H}_{\theta 2}$	effective two-scale matrices	(44a), (45a)
$\mathbb{H}_{\theta 1}^f$, $\mathbb{H}_{\theta 2}^f$	effective constant matrices corresponding to thermal dilatation	(44b), (45b)
$\mathbb{I}$	unit 3×3-matrix	Sec. 2
J_t, J_τ	Volterra operator (primitive of function)	(2)
$\mathbb{J}^{mn}$	3×3-matrix $(1/2)(\boldsymbol{e}_m \otimes \boldsymbol{e}_n + \boldsymbol{e}_n \otimes \boldsymbol{e}_m)$	Notat. 5
$\boldsymbol{\mathcal{K}}$	effective relaxation tensor	(87a)
$\boldsymbol{\mathcal{K}}^\theta$	restriction of $\boldsymbol{\mathcal{K}}$ to $[\Delta, \Delta+\delta_*]$	(87c)
$\boldsymbol{\mathcal{K}}^\sigma$	restriction of $\boldsymbol{\mathcal{K}}$ to $(\Delta+\delta_*, \Delta+\delta^*]$	(87b)
$\mathcal{L}$	Laplace transform	(83)
$\mathbb{K}_0$, $\mathbb{K}_1$	effective macroscopic matrices corresponding to thermal memory effects	(94a), (95a)
$\mathbb{K}_0^\theta$, $\mathbb{K}_1^\theta$	restriction of $\mathbb{K}_0$, $\mathbb{K}_1$, resp., to $[\Delta, \Delta+\delta_*]$	(94c), (95b)
$\mathbb{K}_0^\sigma$	restriction of $\mathbb{K}_0$ to $(\Delta+\delta_*, \Delta+\delta^*]$	(94b)
$\mathbb{L}_*$	matrix of effective macroscopic heat conductivity	(98a)
$\mathbb{L}_*^\theta$	restriction of $\mathbb{L}_*$ to $[\Delta, \Delta+\delta_*]$	(98c)
$\mathbb{L}_*^\sigma$	restriction of $\mathbb{L}_*$ to $(\Delta+\delta_*, \Delta+\delta^*]$	(98b)
$\mathbb{L}_{B1}$, $\mathbb{L}_{B2}$, $\mathbb{L}_v^0$	effective two-scale matrices corresponding to thermal dilatation	(47a), (48a), (51a)
$\mathbb{L}_{B1}^\theta$	restriction of $\mathbb{L}_{B1}$ to $[\Delta, \Delta+\delta_*]$	(47b)
$\mathbb{L}_{B2}^f$, $\mathbb{L}_{vf}^0$	restrictions of $\mathbb{L}_{B2}$, $\mathbb{L}_v^0$, resp., to $\Sigma_F \times [\Delta, \Delta+\delta_*]$	(48b), (51b)

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$\mathbb{L}_{B3}$, $\mathbb{L}_{B4}$, $\mathbb{L}_{B5}$, $\mathbb{L}_v^\star$	effective macroscopic matrices corresponding to thermal dilatation	(99a), (100a), (101a), (102a)
$\mathbb{L}_{B3}^\theta$, $\mathbb{L}_{B4}^\theta$, $\mathbb{L}_{B5}^\theta$, $\mathbb{L}_v^{\star\theta}$	restrictions of $\mathbb{L}_{B3}$, $\mathbb{L}_{B4}$, $\mathbb{L}_{B5}$, $\mathbb{L}_v^\star$, resp., to $[\Delta, \Delta+\delta_*]$	(99c), (100c), (101b), (102c)
$\mathbb{L}_{B3}^\sigma$, $\mathbb{L}_{B4}^\sigma$, $\mathbb{L}_v^{\star\sigma}$	restrictions of $\mathbb{L}_{B3}$, $\mathbb{L}_{B4}$, $\mathbb{L}_{B5}$, $\mathbb{L}_v^\star$, resp., to $(\Delta+\delta_*, \Delta+\delta^*]$	(99b), (100b), (102b)
$\mathbb{L}_{B2}^f$, $\mathbb{L}_{vf}^0$	restrictions of $\mathbb{L}_{B2}$, $\mathbb{L}_v^0$, resp., to $\Sigma_F \times [\Delta, \Delta+\delta_*]$	(48b), (51b)
$\mathbb{L}_\varkappa$	effective two-scale heat conductivity matrix	(46a)
$\mathbb{L}_\varkappa^\theta$	restriction of $\mathbb{L}_\varkappa$ to $[\Delta, \Delta+\delta_*]$	(46b)
L_θ, L_p^0	effective two-scale scalar coefficients corresponding to thermal dilatation	(49a), (50a)
L_θ^f, L_{pf}^0	restrictions of L_θ, L_p^0, resp., to $\Sigma_F \times [\Delta, \Delta+\delta_*]$	(49b), (50b)
$L_{\theta 1}$, $L_p^\star$	effective macroscopic scalar coefficients	(103a), (106a)
$L_{\theta 1}^\theta$, $L_p^{\star\theta}$	restrictions of $L_{\theta 1}$, $L_p^\star$, resp., to $[\Delta, \Delta+\delta_*]$	(103c), (106c)
$L_{\theta 1}^\sigma$, $L_p^{\star\sigma}$	restrictions of $L_{\theta 1}$, $L_p^\star$, resp., to $(\Delta+\delta_*, \Delta+\delta^*]$	(103b), (106b)
$L_{\theta 2}$, $L_{\theta 3}$	macroscopic relaxation kernels	(104a), (105a)
$L_{\theta 2}^\theta$, $L_{\theta 3}^\theta$	restrictions of $L_{\theta 2}$, $L_{\theta 3}$, resp., to $[\Delta, \Delta+\delta_*]$	(104c), (105b)
$L_{\theta 2}^\sigma$	restriction of $L_{\theta 2}$ to $(\Delta+\delta_*, \Delta+\delta^*]$	(104b)
$\boldsymbol{\mathcal{M}}^t$	homogenized three-scale uniform stress tensor	(33)
$\boldsymbol{\mathcal{M}}_\varepsilon^t$	uniform stress tensor in microscopic description	(9)
$\mathbb{M}^0$	homogenized three-scale partial initial data for stress	(32a)
$\mathbb{M}_\varepsilon^0$	partial initial data for stress in microscopic description	(10)
$\boldsymbol{n}_\varepsilon$	unit normal to the fluid-structure interface	(1f), (1g)
$\boldsymbol{n}_\Sigma$	unit outward normal to $\partial\Sigma_S$	(109b)
$\boldsymbol{\mathcal{P}}^\theta$	instantaneous viscous stress tensor in Ω_θ	(140e)
$\boldsymbol{\mathcal{P}}^\sigma$	instantaneous viscous stress tensor in Ω_σ	(140c)
p^0	initial pressure distribution	(1a)
$\boldsymbol{\mathcal{Q}}$	effective relaxation tensor	(88a)
$\boldsymbol{\mathcal{Q}}_0^\theta$, $\boldsymbol{\mathcal{Q}}_1^\theta$	tensors derived from the meso- and microstructures	(88c), (88d)
$\boldsymbol{\mathcal{Q}}^\sigma$	restriction of $\boldsymbol{\mathcal{Q}}$ to $(\Delta+\delta_*, \Delta+\delta^*]$	(88b)
$\mathbb{Q}^{mn}$	matrix derived from the meso- and microstructures	(88e)
$\boldsymbol{\mathcal{R}}_1^\theta$, $\boldsymbol{\mathcal{R}}_2^\theta$	viscoelastic relaxation tensors in Ω_θ	(140e)
$\boldsymbol{\mathcal{R}}_0^\sigma$	viscoelastic relaxation tensor in Ω_σ	(140c)
$\mathbb{R}_1^\theta$, $\mathbb{R}_2^\theta$	heat relaxation matrices in Ω_θ	(140c)
$\mathbb{R}_1^\sigma$	heat relaxation matrix in Ω_σ	(140c)
s	argument of the Laplace transform image	(73), (83)
tr	trace of a matrix	Notat. 1
$\boldsymbol{u}$, $\boldsymbol{u}_\varepsilon$	macroscopic velocity vector	Sec. 1
$\boldsymbol{u}^0$	initial macroscopic velocity vector	(1h)
$\boldsymbol{u}^{(1)}$	mesoscopic velocity vector	Sec. 1
$\boldsymbol{u}^{(2)}$	microscopic velocity vector	Sec. 1
$\boldsymbol{u}^\star$	boundary macroscopic velocity distribution	(7d), (15)
$\boldsymbol{\mathcal{V}}$	effective instantaneous viscosity tensor	(85a)

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$\boldsymbol{\mathcal{V}}^{\theta}_{\rm corr}$	instantaneous viscosity corrector term on $[\Delta, \Delta + \delta_*]$	(85c)
$\boldsymbol{\mathcal{V}}^{\sigma}_{\rm corr}$	instantaneous viscosity corrector term on $(\Delta + \delta_*, \Delta + \delta^*]$	(85b)
$\mathbb{V}$	matrix corresponding to the effective instantaneous thermal dilatation	(92a)
$\mathbb{V}^{\theta}$	restriction of $\mathbb{V}$ to $[\Delta, \Delta + \delta_*]$	(92c)
$\mathbb{V}^{\sigma}$	restriction of $\mathbb{V}$ to $(\Delta + \delta_*, \Delta + \delta^*]$	(92b)
$\boldsymbol{v}_\varepsilon$	macroscopic displacement vector	Sec. 1
$\boldsymbol{v}^0$	initial macroscopic displacement vector	(1i)
$\boldsymbol{\mathcal{W}}$	effective relaxation tensor	(89a)
$\boldsymbol{\mathcal{W}}^{\theta}_1$	tensor derived from the meso- and microstructures	(89b)
$\mathbb{W}_0$, $\mathbb{W}_1$	effective macroscopic matrices corresponding to thermal memory effects	(96a), (97a)
$\mathbb{W}^{\theta}_0$, $\mathbb{W}^{\theta}_1$	restriction of $\mathbb{W}_0$, $\mathbb{W}_1$, resp., to $[\Delta, \Delta + \delta_*]$	(96c), (97b)
$\mathbb{W}^{\sigma}_0$	restriction of $\mathbb{W}_0$ to $(\Delta + \delta_*, \Delta + \delta^*]$	(96b)
$\boldsymbol{x}$	macroscopic position vector	Sec. 1
$\widehat{\boldsymbol{x}}$	vector (x_1, x_2)	Sec. 3
$\boldsymbol{Y}^{ij}_0$, $\boldsymbol{Y}^{ij}_1$, $\boldsymbol{Y}_2$, $\boldsymbol{Y}^{ij}_3$, $\boldsymbol{Y}_4$, $\boldsymbol{Y}_5$, Y^{i}_6	solutions of the mesoscopic cell problems	Probl. Y1–Y10
$\widehat{\boldsymbol{y}}$	mesoscopic position vector	Sec. 3
$\boldsymbol{Z}^{ij}_{00}$, $\boldsymbol{Z}^{ij}_{10}$, $\boldsymbol{Z}_{20}$, $\boldsymbol{Z}^{ij}_{30}$, $\boldsymbol{Z}_{40}$, Z^{i}_5	solutions of the microscopic cell problems	Probl. Z1–Z9
$\widehat{\boldsymbol{z}}$	microscopic position vector	Sec. 3

Greek Symbols

Notation	Description	Introduced in
α_τ, α_g, α_λ, α_μ, α_p	positive dimensionless ratios	(1), Sec. 2
Γ^ε	fluid-structure interface	Model A_ε
Δ	thickness of elastic plate $\Omega_{\rm pl}$	Sec. 1
δ_{ij}	Kronecker's symbol	Sec. 2
δ_*	height of a shorter bristle	Sec. 1
δ^*	height of a taller bristle	Sec. 1
ε	small characteristic parameter of the periodic structure	Sec. 1
ζ	extension of $\widehat{\zeta}$	(26b)
$\widehat{\zeta}$	characteristic function of Σ_F	$(25)_1$
θ, θ_ε	macroscopic temperature	Sec. 1
θ^0	initial temperature	(1j)
$\theta^{(1)}$	mesoscopic temperature	Sec. 1
$\theta^{(2)}$	microscopic temperature	Sec. 1
$\theta^\star$	boundary temperature distribution	(1m)
Θ	pattern microscopic cell	Sec. 3
Θ_F, Θ_S	liquid and solid parts of Θ, resp.	Sec. 3
$\varkappa$	homogenized three-scale dimensionless	Notat. 3

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	heat conductivity	
$\varkappa_F$	dimensionless heat conductivity in fluid	(1b)
$\varkappa_S$	dimensionless heat conductivity in solid	(1d)
$\varkappa^\varepsilon$	uniform notation for dimensionless heat conductivity in Ω	(13)
ρ	homogenized three-scale density	Sec. 5
ρ_F	fluid density	Sec. 2
ρ_S	density of the elastic body	Sec. 2
ρ_θ, ρ_σ	mean densities in Ω_θ and Ω_σ, resp.	(140e), (140c)
ρ^ε_*	uniform notation for density in Ω	(8)
$\langle\rho\rangle_{\Sigma\times\Theta}$	mean value of ρ on $\Sigma\times\Theta$	(38)
Σ	pattern mesoscopic cell	Sec. 3
Σ_F, Σ_S	liquid and elastic parts of Σ, resp.	Sec. 3
χ	homogenized three-scale characteristic function of the fluid domain	(28a)
χ^ε	characteristic function of the fluid domain in microscopic description	(26a)
ψ	extension of $\widehat{\psi}$	(26c)
$\widehat{\psi}$	characteristic function of Θ_F	$(25)_2$
Ψ	homogenized three-scale volumetric dimensionless density of external heat application	Notat. 3
Ψ_F, Ψ_S	volumetric dimensionless densities of external heat application in fluid and solid, resp.	(1b), (1d)
Ψ^ε	uniform notation for volumetric dimensionless density of external heat application in Ω	(14)
$\Omega=(0,1)^3$	domain of dimensionless macroscopic positions	Sec. 1
Ω^ε_F, Ω^ε_S	fluid domain and elastic body, resp.	Model A_ε, Sec. 3
Ω_{fl}	fluid layer above all bristles	Sec. 1
Ω_{pl}	elastic plate without bristles	Sec. 1
Ω_θ	spatial layer, where the shorter bristles locate	Sec. 1
Ω_σ	spatial layer, where the taller bristles locate	Sec. 1

Some operators and binary operations

Notation	Description	Introduced in
$\mathrm{div}_{\widehat{y}}$, $\mathrm{div}_{\widehat{z}}$	divergence operators	Sec. 5
$:$	inner product (convolution)	Notat. 1
$\otimes$	dyad	Notat. 1
$\nabla_{\widehat{y}}$, $\nabla_{\widehat{z}}$	gradient operators	(29)
$\bar{\varphi}$	Laplace transform of a function φ: $\bar{\varphi}=\mathcal{L}[\varphi]$	(83)
$\langle\varphi\rangle_\Theta$, $\langle\varphi\rangle_\Sigma$, $\langle\varphi\rangle_{\Sigma\times\Theta}$	mean value of a function $\varphi=\varphi(\widehat{y},\widehat{z})$ on Θ, Σ, and $\Sigma\times\Theta$, resp.	Notat. 4

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