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Article

On Duality Principles and Concerned Convex Dual Formulations Applied to a Non-Linear Plate Theory and Related Models

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Abstract: This article develops duality principles applicable to originally non-convex primal variational formulations. More specifically, as a first application, we establish a convex dual approximate variational formulation for a non-linear Kirchhoff-Love plate model. The results are obtained through basic tools of functional analysis, calculus of variations, duality and optimization theory in infinite dimensional spaces. We emphasize such a convex dual approximate formulation obtained may be applied to a large class of similar models in the calculus of variations. Finally, in the last section, we present a duality principle and respective convex dual formulation for a Ginzburg-Landau type equation.

Keywords: duality principle; non-linear plate model; convex dual approximate formulation

MSC: 49N15

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1. Introduction

This article develops a duality principle applicable to a large class of models in the calculus of variations. Specifically in this text, we present applications to the non-linear Kirchhoff-Love plate model.

We emphasize the results on duality theory here addressed and developed are inspired mainly in the approaches of J.J.Telega, W.R. Bielski and co-workers presented in the articles [1–4]. Other main reference is the article by Toland, [5].

Moreover, details on the Sobolev spaces involved may be found in [6].

Similar results and models are addressed in [7–11].

Basic results on convex analysis are addressed in [12]. Other similar results and approaches may be found in [13–15].

Now we start to describe the primal variational formulation for the plate model in question.

Let $\Omega \subset \mathbb{R}^2$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

We assume such a Ω set represents the middle surface of a thin plate with a constant thickness $h > 0$.

Moreover, we suppose such a plate is subject to a external load $(P_\alpha, P) \in L^2(\Omega; \mathbb{R}^3)$ resulting a field of displacements denoted by

$$(u_\alpha, w) = (u_1, u_2, w) \in W_0^{1,2}(\Omega; \mathbb{R}^2) \times W_0^{2,2}(\Omega) = V.$$

Both the load and displacements fields refers to a cartesian system $(0, x_1, x_2, x_3)$ and related canonical basis in $\mathbb{R}^3$.

Finally, we denote $Y_1 = Y_1^* = L^2(\Omega; \mathbb{R}^4)$ and $Y_2 = Y_2^* = L^2(\Omega; \mathbb{R}^2)$.

We also emphasize the boundary conditions in question refer to a clamped plate.

The strain tensors are defined by

$$\gamma_{\alpha\beta}(u) = \frac{u_{\alpha,\beta} + u_{\beta,\alpha}}{2} + \frac{1}{2} w_{,\alpha} w_{,\beta},$$

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and

$$\kappa_{\alpha\beta}(w) = -w_{,\alpha\beta}.$$

The plate total energy functional is defined by

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega H_{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\gamma_{\lambda\mu}(u)\,dx \\ &+\frac{1}{2}\int_\Omega h_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}(u)\kappa_{\lambda\mu}(u)\,dx \\ &-\langle w, P\rangle_{L^2} - \langle u_\alpha, P_\alpha\rangle_{L^2}. \end{aligned} \tag{1}$$

Here $\{H_{\alpha\beta\lambda\mu}\}$ is a fourth order positive definite symmetric constant tensor. Moreover

$$\{h_{\alpha\beta\lambda\mu}\} = \frac{h^2}{12}\{H_{\alpha\beta\lambda\mu}\}$$

and we denote

$$\{\overline{H}_{\alpha\beta\lambda\mu}\} = \{H_{\alpha\beta\lambda\mu}\}^{-1}$$

and

$$\left\{\overline{h}_{\alpha\beta\lambda\mu}\right\} = \{h_{\alpha\beta\lambda\mu}\}^{-1}$$

in an appropriate tensor sense.

2. The Main Duality Principle and Related Convex Dual Approximate Formulation

We start by defining the approximate functional $J_1 : V \to \mathbb{R}$ by

$$J_1(u) = J(u) + \sum_{\alpha=1}^{2}\frac{\varepsilon_1}{2}\int_\Omega (u_\alpha)^2\,dx,$$

and considering an appropriate real constant $K > 0$, the functionals $F_1 : V \to \mathbb{R}$, $F_2 : V \times Y_1^* \to \mathbb{R}$, $F_3 : V \to \mathbb{R}$ and $F_4 : V \to \mathbb{R}$, by

$$\begin{aligned} F_1(u) &= \frac{1}{2}\int_\Omega h_{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}(u)\kappa_{\lambda\mu}(u)\,dx + \sum_{\alpha=1}^{2}\frac{K}{2}\int_\Omega w_{,\alpha}^2\,dx \\ &+\sum_{\alpha=1}^{2}\varepsilon\int_\Omega w_{,\alpha}^2\,dx - \langle w, P\rangle_{L^2} \end{aligned} \tag{2}$$

$$\begin{aligned} F_2(u,N) &= \frac{1}{2}\int_\Omega N_{\alpha\beta}w_{,\alpha}w_{,\beta}\,dx \\ &-\sum_{\alpha=1}^{2}\frac{K}{2}\int_\Omega w_{,\alpha}^2\,dx \end{aligned} \tag{3}$$

$$F_3(u) = \sum_{\alpha=1}^{2}\frac{\varepsilon_1}{2}\int_\Omega u_\alpha^2\,dx \tag{4}$$

$$F_4(u) = \sum_{\alpha=1}^{2}\varepsilon\int_\Omega w_{,\alpha}^2\,dx. \tag{5}$$

Moreover, we define the polar functionals $F_1^* : [Y_2^*]^2 \to \mathbb{R}$, $F_2^* : [Y_2^*]^2 \times Y_1^* \to \mathbb{R}$, $F_3^* : Y_1^* \to \mathbb{R}$ and $F_4^* : [Y_2^*]^2 \to \mathbb{R}$, by

$$F_1^*(R,L) = \sup_{u \in V} \{ \langle w_{,\alpha}, R_\alpha - L_\alpha \rangle_{L^2} - F_1(u) \}, \tag{6}$$

$$\begin{aligned} F_2^*(Q,L) &= \inf_{v \in Y_2} \left\{ \langle v_\alpha, Q_\alpha + L_\alpha \rangle_{L^2} - \frac{1}{2}\int_\Omega N_{\alpha\beta} v_\alpha v_\beta \, dx \right. \\ & \left. + \sum_{\alpha=1}^2 \frac{K}{2} \int_\Omega v_\alpha^2 \, dx + \frac{1}{2}\int_\Omega \overline{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} \, dx \right\} \\ &= \frac{1}{2}\int_\Omega \overline{N_{\alpha\beta}^{(-K)}} (Q_\alpha + L_\alpha)(Q_\beta + L_\beta) \, dx \\ & + \frac{1}{2}\int_\Omega \overline{H}_{\alpha\beta\lambda\mu} N_{\alpha\beta} N_{\lambda\mu} \, dx, \end{aligned} \tag{7}$$

if $N = \{N_{\alpha\beta}\} \in B^*$, where

$$B^* = \{N \in Y_1^* \, : \, \|N_{\alpha\beta}\|_\infty \leq K/8, \; \forall \alpha, \beta \in \{1,2\}\},$$

$$\left\{ N_{\alpha\beta}^{(-K)} \right\} = \{N_{\alpha\beta} - K\delta_{\alpha\beta}\},$$

and

$$\left\{ \overline{N_{\alpha\beta}^{(-K)}} \right\} = \left\{ N_{\alpha\beta}^{(-K)} \right\}^{-1}.$$

Furthermore,

$$\begin{aligned} F_3^*(N) &= \sup_{u \in V} \{ \langle u_\alpha, N_{\alpha\beta,\beta} + P_\alpha \rangle_{L_2} - F_3(u) \} \\ &= \sum_{\alpha=1}^2 \frac{1}{2\varepsilon_1} \int_\Omega (N_{\alpha\beta,\beta} + P_\alpha)^2 \, dx, \end{aligned} \tag{8}$$

$$\begin{aligned} F_4^*(R,Q) &= \sup_{(v_1,v_2) \in Y_2^*} \left\{ \langle (v_1)_\alpha, R_\alpha \rangle_{L^2} - \frac{\varepsilon}{2} \int_\Omega (v_1)_\alpha^2 \, dx \right. \\ & \left. + \langle (v_2)_\alpha, Q_\alpha \rangle_{L^2} - \frac{\varepsilon}{2} \int_\Omega (v_2)_\alpha^2 \, dx \right\} \\ &= \sum_{\alpha=1}^2 \left(\frac{1}{2\varepsilon} \int_\Omega R_\alpha^2 \, dx + \frac{1}{2\varepsilon} \int_\Omega Q_\alpha^2 \, dx \right). \end{aligned} \tag{9}$$

At this point, denoting

$$D^* = \{Q = \{Q_\alpha\} \in Y_2^* \, : \, \|Q_\alpha\| \leq 5, \; \forall \alpha \in \{1,2\}\},$$

we define $J_1^* : (D^*)^2 \times B^* \times Y_2^* \to \mathbb{R}$ by

$$J_1^*(R,Q,N,L) = -F_1^*(R,L) - F_2^*(Q,L,N) - F_3^*(N) + F_4^*(R,Q),$$

and $J_2^* : (D^*)^2 \times B^* \to \mathbb{R}$ by

$$J_2^*(R,Q,N) = \text{sta}_{L \in Y_2^*} J_1^*(R,Q,N,L) = J_1^*(R,Q,N,\hat{L}(R,Q,N)),$$

where $\hat{L} = \hat{L}(R,Q,N) \in Y_2^*$ is the only solution of the linear equation in L

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$$\frac{\partial J_1^*(R,Q,N,\hat{L})}{\partial L} = \mathbf{0}.$$

Moreover, we define $J_3^* : (D^*)^2 \times B^* \to \mathbb{R}$, by

$$\begin{aligned} & J_3^*(R,Q,N) \\ = \; & J_2^*(R,Q,N) \\ & - \sum_{\alpha=1}^{2}\sum_{\beta=1}^{2} \frac{K_1}{2}\left\| \overline{H}_{\alpha\beta\lambda\mu} N_{\lambda\mu} - \frac{(N_{\alpha\rho,\rho}+P_\alpha)_{,\beta} + (N_{\beta\rho,\rho}+P_\alpha)_{,\alpha}}{2\varepsilon_1} - \frac{1}{2}\tilde{v}_\alpha \tilde{v}_\beta \right\|_{0,2}^2, \end{aligned} \tag{10}$$

where

$$\tilde{v}_\alpha = \overline{N_{\alpha\beta}^{(-K)}}(Q_\beta + L_\beta(R,Q,N)), \; \forall \alpha \in \{1,2\}.$$

Here, we assume

$$K_1 \gg \max\{1, K, \max\{\overline{h}_{\alpha\beta\lambda\mu}, \; \alpha,\beta,\lambda,\mu \in \{1,2\}\}\},$$

$$0 < \varepsilon, \varepsilon_1 \ll 1$$

and

$$\frac{1}{\varepsilon} \gg \max\left\{K_1, \frac{1}{\varepsilon_1}\right\}.$$

Observe that

$$\frac{\partial^2 J_3^*(R,Q,N)}{\partial Q_\alpha^2} = \mathcal{O}\left(\frac{1}{\varepsilon}\right) > \mathbf{0},$$

$$\frac{\partial^2 J_3^*(R,Q,N)}{\partial R_\alpha^2} = \mathcal{O}\left(\frac{1}{\varepsilon}\right) > \mathbf{0},$$

$\forall \alpha \in \{1,2\}$.

Thus, considering also the remaining mixed variations in Q_α and R_α, we may infer that

$$\det\left\{\frac{\partial^2 J_3^*(R,Q,N)}{\partial Q_\alpha \, \partial R_\beta}\right\} > \mathbf{0},$$

in $(D^*)^2 \times B^*$.

Moreover, by direct computation, clearly

$$\frac{\partial^2 J_3^*(R,Q,N)}{\partial (N_{\alpha\beta})^2} < \mathbf{0}, \; \forall \alpha,\beta \in \{1,2\}$$

and considering the remaining mixed variations of J_3^* in N_{11}, N_{22}, N_{12} and N_{21} and the concerned remaining minor determinants, we may infer that

$$\det\left\{\frac{\partial^2 J_3^*(R,Q,N)}{\partial N_{\alpha\beta} \, \partial N_{\lambda\mu}}\right\} > \mathbf{0},$$

in $(D^*)^2 \times B^*$.

From such results, we may also infer that J_3^* is convex in (R,Q) and concave in N in $(D^*)^2 \times B^*$.

Let $(\hat{R},\hat{Q},\hat{N},\hat{L}) \in (D^*)^2 \times B^* \times Y_2^*$ be such that

$$\delta J_1^*(\hat{R},\hat{Q},\hat{N},\hat{L}) = \mathbf{0}.$$

Let $u_0 = ((u_0)_\alpha, w_0) \in V$ be such that

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$$(u_0)_\alpha = \frac{\hat{N}_{\alpha\beta,\beta} + P_\alpha}{\varepsilon_1},$$

and

$$(w_0)_{,\alpha} = \frac{\hat{Q}_\alpha}{\varepsilon},$$

$\forall \alpha \in \{1,2\}$.

From standard results in Duality Theory and the Legendre Transform properties, we may obtain

$$\delta J_1(u_0) = \mathbf{0},$$

$$\delta J_2^*(\hat{R}, \hat{Q}, \hat{N}) = \mathbf{0}$$

,

$$\delta J_3^*(\hat{R}, \hat{Q}, \hat{N}) = \mathbf{0}$$

and

$$\begin{aligned} J_1(u_0) &= J_1^*(\hat{R}, \hat{Q}, \hat{N}, \hat{L}) \\ &= J_2^*(\hat{R}, \hat{Q}, \hat{N}) \\ &= J_3^*(\hat{R}, \hat{Q}, \hat{N}). \end{aligned} \tag{11}$$

From such results and the Min-Max Theorem, we have

$$J_3^*(\hat{R}, \hat{Q}, \hat{N}) = \inf_{(R,Q) \in (D^*)^2} \left\{ \sup_{N \in B^*} J_3^*(R, Q, N) \right\}.$$

Joining the pieces, we have got

$$\begin{aligned} J_1(u_0) &= J_1^*(\hat{R}, \hat{Q}, \hat{N}, \hat{L}) \\ &= J_2^*(\hat{R}, \hat{Q}, \hat{N}) \\ &= \inf_{(R,Q) \in (D^*)^2} \left\{ \sup_{N \in B^*} J_3^*(R, Q, N) \right\} \\ &= J_3^*(\hat{R}, \hat{Q}, \hat{N}). \end{aligned} \tag{12}$$

Remark 2.1. *Defining* $J_5^* : (D^*)^2 \to \mathbb{R}$ *by*

$$J_5^*(R, Q) = \sup_{N \in B^*} J_3^*(R, Q, N),$$

we have that such a functional J_5^* *is convex in* $(D^*)^2$ *as the supremum in* $N \in B^*$ *of a family of convex functionals in* (R, Q).

In such a case, we have also obtained

$$\begin{aligned} J_1(u_0) &= J_1^*(\hat{R}, \hat{Q}, \hat{N}, \hat{L}) \\ &= J_2^*(\hat{R}, \hat{Q}, \hat{N}) \\ &= \inf_{(R,Q) \in (D^*)^2} \left\{ \sup_{N \in B^*} J_3^*(R, Q, N) \right\} \\ &= J_3^*(\hat{R}, \hat{Q}, \hat{N}) \\ &= \inf_{(R,Q) \in (D^*)^2} J_5^*(R, Q) \\ &= J_5^*(\hat{R}, \hat{Q}). \end{aligned} \tag{13}$$

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3. One More Duality Principle and Related Convex Dual Formulation

In this section we develop another new duality principle with a related convex dual functional applied to a Ginzburg-Landau type equation.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Consider a functional $J : V \rightarrow \mathbb{R}$ where

$$\begin{aligned} J(u) &= \frac{\gamma}{2}\int_\Omega \nabla u \cdot \nabla u \, dx \\ &+\frac{\alpha}{2}\int_\Omega (u^2-\beta)^2 \, dx - \langle u, f\rangle_{L^2}, \end{aligned} \tag{14}$$

where $\gamma > 0$, $\alpha > 0$, $\beta > 0$ and $f \in L^2(\Omega)$.

Here $u \in V = W_0^{1,2}(\Omega)$ and we denote $Y = Y^* = L^2(\Omega)$.

Define the functionals $F_1 : V \times Y^* \rightarrow \mathbb{R}$, $F_2 : V \times Y^* \rightarrow \mathbb{R}$ and $F_3 : V \rightarrow \mathbb{R}$ by

$$\begin{aligned} F_1(u, v_0^*) &= \frac{\gamma}{4}\int_\Omega \nabla u \cdot \nabla u \, dx + \frac{1}{2}\langle u^2, v_0^*\rangle_{L^2} \\ &+\frac{K}{2}\int_\Omega u^2 \, dx, \end{aligned} \tag{15}$$

$$\begin{aligned} F_2(u, v_0^*) &= \frac{\gamma}{4}\int_\Omega \nabla u \cdot \nabla u \, dx + \frac{1}{2}\langle u^2, v_0^*\rangle_{L^2} \\ &+\frac{K}{2}\int_\Omega u^2 \, dx - \langle u, f\rangle_{L^2} \\ &-\frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx - \beta\int_\Omega v_0^* \, dx, \end{aligned} \tag{16}$$

$$F_3(u) = K\int_\Omega u^2 \, dx, \tag{17}$$

where $K > 0$ is an appropriate real constant.

Define also the polar functionals $F_1^* : [Y^*]^3 \rightarrow \mathbb{R}$, $F_2^* : [Y^*]^3 \rightarrow \mathbb{R}$ and $F_3^* : Y^* \rightarrow \mathbb{R}$ by

$$\begin{aligned} F_1^*(v_1^*, v_0^*, z^*) &= \sup_{u\in V}\{\langle u, v_1^* + z^*/2\rangle_{L^2} - F_1(u, v_0^*)\} \\ &= \frac{1}{2}\int_\Omega \frac{(v_1^* + z^*/2)^2}{((-\gamma\nabla^2 + 2v_0^*)/2 + K)} \, dx \end{aligned} \tag{18}$$

$$\begin{aligned} F_2^*(v_1^*, v_0^*, z^*) &= \inf_{u\in V}\{\langle u, -v_1^* + z^*/2\rangle_{L^2} - F_2(u, v_0^*)\} \\ &= \frac{1}{2}\int_\Omega \frac{(-v_1^* + z^*/2 + f)^2}{((-\gamma\nabla^2 + 2v_0^*)/2 + K)} \, dx \\ &+\frac{1}{2\alpha}\int_\Omega (v_0^*)^2 \, dx + \beta\int_\Omega v_0^* \, dx, \end{aligned} \tag{19}$$

if $v_0^* \in B^*$, where

$$B^* = \{v_0^* \in Y^* \; : \; \|2v_0^*\| \leq K/4\},$$

$$\begin{aligned} F_3^*(z^*) &= \sup_{w\in L^2}\{\langle w, z^*\rangle_{L^2} - F_3(w)\} \\ &= \frac{1}{4K}\int_\Omega (z^*)^2 \, dx. \end{aligned} \tag{20}$$

Moreover, define

$$D^* = \{v_1^* \in Y^* \; : \; \|2v_1^* - f\|_\infty \leq 5\},$$
$$D_1^+ = \{z^* \in Y^* \; : \; z^* f \geq 0, \text{ in } \Omega\},$$

and

$$D_1^* = \{z^* \in D^+ \; : \; \|z^*\|_\infty \leq (5/2)K\}.$$

Assuming $K_1 \gg \max\{\gamma, \alpha, \beta, 1/\alpha, 1, K\}$ and

$$K \gg \alpha,$$

define $J^* : D^* \times B^* \times D_1^* \to \mathbb{R}$ by

$$J^*(v_1^*, v_0^*, z^*) \quad = \quad -F_1^*(v_1^*, v_0^*, z^*) - F_2^*(v_1^*, v_0^*, z^*) + F_3^*(z^*). \tag{21}$$

From the variation of J^* in v_1^*, we have

$$-\frac{v_1^* + z^*/2}{((-\gamma\nabla^2 + 2v_0^*)/2 + K)} + \frac{-v_1^* + z^*/2 + f}{((-\gamma\nabla^2 + 2v_0^*)/2 + K)} = 0, \;\; \text{in } \Omega,$$

so that

$$-2v_1^* + f = 0, \text{ in } \Omega.$$

Moreover, denoting

$$u_0 = \frac{v_1^* + z^*/2}{((-\gamma\nabla^2 + 2v_0^*)/2 + K)},$$

from the variation of J^* in z^*, we obtain

$$u_0 = \frac{z^*}{2K}.$$

From such results, we may also obtain

$$\delta J(u_0) = -\gamma\nabla^2 u_0 + 2\alpha(u_0^2 - \beta)u_0 - f = \mathbf{0},$$

so that

$$-\gamma\nabla^2\left(\frac{z^*}{2K}\right) + 2\alpha\left(\left(\frac{v_1^* + z^*/2}{(-\gamma\nabla^2 + 2v_0^*)/2 + K}\right)^2 - \beta\right)\left(\frac{z^*}{2K}\right) - f = \mathbf{0}$$

in Ω.

With such results in mind, we define also the exactly penalized functional $J_1^* : D^* \times B^* \times D_1^* \to \mathbb{R}$, where

$$\begin{aligned} & J_1^*(v_1^*, v_0^*, z^*) \\ = \;& J^*(v_1^*, v_0^*, z^*) - \frac{K_1}{2}\|2v_1^* - f\|_{0,2}^2 \\ & +30\frac{K}{2}\left\|-\gamma\nabla^2\left(\frac{z^*}{2K}\right) + 2\alpha\left(\left(\frac{v_1^* + z^*/2}{(-\gamma\nabla^2 + 2v_0^*)/2 + K}\right)^2 - \beta\right)\left(\frac{z^*}{2K}\right) - f\right\|_{0,2}^2 \end{aligned} \tag{22}$$

Clearly, we have

$$\frac{\partial^2 J_1^*(v_1^*, v_2^*, v_0^*)}{\partial(v_1^*)^2} = -\mathcal{O}(K_1) < \mathbf{0},$$

and

$$\frac{\partial^2 J_1^*(v_1^*, v_2^*, v_0^*)}{\partial(v_0^*)^2} = -\mathcal{O}(1/\alpha) < \mathbf{0},$$

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so that considering also the mixed variations of J_1^* in v_1^* and v_0^*, we may infer that

$$\det\left\{\frac{\partial^2 J_1^*(v_1^*, v_0^*, z^*)}{\partial v_1^* \partial v_0^*}\right\} > \mathbf{0}, \text{ in } D^* \times B^* \times D_1^*.$$

Hence, J_1^* is concave in (v_1^*, v_0^*) in $D^* \times B^* \times D_1$.

Furthermore,

$$\frac{\partial^2 J_2^*(v_1^*, v_0^*, z^*)}{\partial (z^*)^2} = \mathcal{O}(30/(4K)) > \mathbf{0},$$

in $D^* \times B^* \times D_1^*$, so that J_1^* is convex in z^* in $D^* \times B^* \times D_1^*$.

Let $(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \in D^* \times B^* \times D_1^*$ be such that

$$\delta J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = \mathbf{0}.$$

From this, the last previous results and the Min-Max theorem we may infer that

$$J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) = \inf_{z^* \in D_1^*}\left\{\sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_1^*, v_0^*, z^*)\right\}.$$

Let $u_0 \in V$ be such that

$$u_0 = \frac{\hat{z}^*}{2K}.$$

From fundamentals of duality theory and the Legendre Transform properties, we may obtain

$$\delta J(u_0) = \mathbf{0},$$

and

$$J(u_0) = J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*).$$

Joining the pieces, we have got

$$\begin{aligned} J(u_0) &= J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\ &= \inf_{z^* \in D_1^*}\left\{\sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_1^*, v_0^*, z^*)\right\}. \end{aligned} \tag{23}$$

Remark 3.1. *Defining* $J_3^* : D_1^* \to \mathbb{R}$ *by*

$$J_3^*(z^*) = \sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_1^*, v_0^*, z^*),$$

we have that J_3^* *is convex in* D_1^* *as a supremum of a family of convex functionals in* z^**.*

In such case, we have

$$\begin{aligned} J(u_0) &= J_1^*(\hat{v}_1^*, \hat{v}_0^*, \hat{z}^*) \\ &= \inf_{z^* \in D_1^*}\left\{\sup_{(v_1^*, v_0^*) \in D^* \times B^*} J_1^*(v_1^*, v_0^*, z^*)\right\}. \\ &= \inf_{z^* \in D_1^*} J_3^*(z^*) \\ &= J_3^*(\hat{z}^*). \end{aligned} \tag{24}$$

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The objective of this section is complete.

4. An Approximate Convex Primal Formulation for an Originally Non-Convex Primal One

In this section we obtain an approximate convex primal formulation suitable for an originally non-convex one.

Let $\Omega \subset \mathbb{R}^3$ be an open, bounded and connected set with a regular (Lipschitzian) boundary denoted by $\partial\Omega$.

Let $V = W_0^{1,2}(\Omega)$ and consider a continuously twice Fréchet differentiable functional $J : V \to \mathbb{R}$.

Let $K_1, K_3 \in \mathbb{N}$ be such that $K_1 \gg \max\{1, K_3^2\}$ and let $K > 0$ be such that

$$K = \sqrt{2K_1\pi},$$

so that

$$K^2 = 2K_1\pi.$$

Define

$$V_1 = \{u \in V \ : \ \|u\|_\infty \leq K_3\}.$$

We define the functional $J_1 : V_1 \to \mathbb{R}$ by

$$J_1(u) = J(u) - \frac{1}{2K^{3/2}} \int_\Omega \cos\left(\frac{K^3}{u+K}\right) dx.$$

Observe that

$$\begin{aligned} \frac{\partial J_1(u)}{\partial u} &= \frac{\partial J(u)}{\partial u} + \frac{1}{K^{3/2}} \sin\left(\frac{K^3}{u+K}\right) \frac{-K^3}{(u+K)^2} \\ &= \frac{\partial J(u)}{\partial u} + \mathcal{O}(1/K^{1/2}) \\ &\approx \frac{\partial J(u)}{\partial u}. \end{aligned} \tag{25}$$

Moreover,

$$\begin{aligned} \frac{\partial^2 J_1(u)}{\partial u^2} &= \frac{\partial^2 J(u)}{\partial u^2} + \frac{1}{K^{3/2}} \cos\left(\frac{K^3}{u+K}\right) \left(\frac{-K^3}{(u+K)^2}\right)^2 \\ &\quad + \frac{2}{K^{3/2}} \sin\left(\frac{K^3}{u+K}\right) \left(\frac{K^3}{(u+K)^3}\right) \\ &= \frac{\partial^2 J(u)}{\partial u^2} + \mathcal{O}(K^{1/2}) \\ &> \mathbf{0}, \text{ in } V_1. \end{aligned} \tag{26}$$

Thus, such an approximate functional J_1 is convex V_1 and its first variation is a very close approximation for the first variation of the original functional J.

The objective of this section is complete.

5. Conclusion

In this article, we have developed duality principles and related convex dual variational formulations for originally non-convex primal ones.

We highlight the results here obtained are applicable to a large class of models in the calculus of variations, including other plate and shell non-linear theories, models in superconductivity, phase transition and micro-magnetism, among many others.

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In a near future research we intend to apply such results to some of these mentioned related models.

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