Article

Not peer-reviewed version

Representations of the G-drazin Inverse for Certain Anti-triangular Matrices

Huanyin Chen *

Posted Date: 2 September 2024

doi: 10.20944/preprints202409.0090.v1

Keywords: Drazin inverse; g-Drazin inverse; anti-triangular matrix; quadratic equation; Banach algebra doi:10.20944/preprints202409.0090.v1

Article

Representations of the g-Drazin Inverse for Certain Anti-Triangular Matrices

Huanyin Chen

School of Big Data, Fuzhou University of International Studies and Trade, Fuzhou 350202, China; huanyinchenhz@163.com

Abstract: We provide representations for the generalized Drazin inverse of an anti-triangular matrix of the form $\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ in a Banach algebra $\mathcal{A}$, under the condition that $ab = ba$. Specifically, we present the representation of Drazin inverse for these types of anti-triangular matrices in Banach algebras.

Keywords: Drazin inverse; g-Drazin inverse; anti-triangular matrix; quadratic equation; Banach algebra

2020 Mathematics Subject Classication: 15A09; 39B42; 16U90

---

1. Introduction

Let $\mathcal{A}$ be an is a Banach algebra with identity 1. An element $a \in \mathcal{A}$ has generalized Drazin inverse (g-Drazin inverse) if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, ax = xa, a - xa^2 \in \mathcal{A}^{qnil}.$$

If such an x exists, it is unique and is denoted by a^d. Here, $\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid 1 + \lambda x \in \mathcal{A}$ is invertible for all $\lambda \in \mathbb{C}\}$. It is well known that $x \in \mathcal{A}^{qnil}$ if and only if $\lim\limits_{n\to\infty} ||x^n||^{\frac{1}{n}} = 0$. If we replace the quasinilpotent set $\mathcal{A}^{qnil}$ with the set of all nilpotent elements in $\mathcal{A}$, we refer to the unique x as the Drazin inverse of a, and denote it by a^D. Both the Drazin and g-Drazin inverses play significant roles in ring and matrix theory (see [5]).

It is intriguing to investigate the Drazin and g-Drazin inverses of the anti-triangular matrix $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \in M_2(\mathcal{A})$. One motivation for exploring this problem is the quest for a closed-form solution to systems of second-order linear differential equations, which can be expressed in the following vector-valued form: $Ax(t) + Bx(t) + Cx(t) = 0$ where $A, B, C \in \mathbb{C}^{n\times n}$ (with A being potentially singular) and x is an $\mathbb{C}^n$-valued function. Clearly, the solutions to singular systems of differential equations are determined by the Drazin inverse of the aforementioned anti-triangular matrix M (see [2,3]). Although the Drazin and g-Drazin inverses of anti-triangular matrices are valuable tools in the context of differential equations, finding representations for such generalized inverses remains a challenging task.

In 2005, Castro-González and Dopazo gave the representations of the Drazin inverse for a class of complex matrix $\begin{pmatrix} I & F \\ I & 0 \end{pmatrix}$ (see [9] [Theorem 3.3]).

In 2011, Bu et al. investigated the Drazin inverse of the complex matrix $\begin{pmatrix} E & F \\ I & 0 \end{pmatrix}$ under the condition $EF = FE$ (see [2] [Theorem 3.3]).

In 2013, Xu, Song and Zhang studied an expression of the Drazin inverse of the operator matrix $\begin{pmatrix} E & F \\ I & 0 \end{pmatrix} \in M_2(\mathcal{B}(X))$ under the same condition, where $\mathcal{B}(X)$ is the Banach algebra of bounded linear operators on a complex Banach space X (see [16] [Theorem 3.8]).

 doi:10.20944/preprints202409.0090.v1

In 2016, Yu, Wang and Deng characterized the Drazin invertibility of the anti-triangular operator matrix $\begin{pmatrix} E & F \\ I & 0 \end{pmatrix} \in M_2(\mathcal{B}(\mathcal{H}))$ under the conditions $F^{\pi}EF^{D} = 0$, $F^{\pi}EF = F^{\pi}FE$, where $\mathcal{B}(\mathcal{H})$ is the Banach algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$ (see [18] [Theorem 4.1]).

Recently, many authors have explored various conditions under which representations of the Drazin (g-Drazin) inverse of such anti-triangular matrices can be established. For additional references, we direct the reader to [10,11,19–21,23].

The motivation of this paper is to further investigate the representation of the g-Drazin inverse of the anti-triangular matrix $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ in a Banach algebra $\mathcal{A}$. We begin by examining the solvability of a quadratic equation in the Banach algebra $\mathcal{A}$ using Catalan numbers C_n. Next, we study the representation of M under the conditions $ab = ba, a \in \mathcal{A}$ is invertible, $b \in \mathcal{A}^{qnil}$. We then employ the ring of Morita context and the Pierce representation of a Banach algebra element as tools to extend the previous special case to the more general condition $ab = ba, a, b \in \mathcal{A}^{d}$. Consequently, the known results are extended to a broader context within a Banach algebra.

Throughout this paper, all Banach algebras are considered to be complex and possess an identity element. Let $M_2(\mathcal{A})$ be the Banach algebra of all 2×2 matrices over the Banach algebra $\mathcal{A}$. We use $\mathcal{A}^{-1}, \mathcal{A}^{D}$ and $\mathcal{A}^{d}$ to stand for the sets of all invertible, Drazin invertible and g-Drazin invertible elements in $\mathcal{A}$, respectively. For $a \in \mathcal{A}^{d}$, we define $a^{\pi} = 1 - aa^{d}$. Let $a, p^2 = p \in \mathcal{A}$. Then a has the Pierce decomposition given by $pap + pap^{\pi} + p^{\pi}ap + p^{\pi}ap^{\pi}$, which we denote in matrix form as $\begin{pmatrix} pap & pap^{\pi} \\ p^{\pi}ap & p^{\pi}ap^{\pi} \end{pmatrix}_p$.

2. key Lemmas

In this section, we present some necessary lemmas which will be used in the sequel. We start by

Lemma 2.1. *Let $a, b \in \mathcal{A}^{d}$. If $ab = 0$, then $a + b \in \mathcal{A}^{d}$ and*

$$(a+b)^d = \sum_{i=0}^{\infty}(a^d)^{i+1}b^i b^{\pi} + \sum_{i=0}^{\infty} a^i a^{\pi}(b^d)^{i+1}.$$

Proof. See [5] [Lemma 15.2.2]. □

Lemma 2.2. *Let $a, b \in \mathcal{A}^{d}$. If $ab^2 = 0$ and $aba = 0$, then $a + b \in \mathcal{A}^{d}$ and*

$$\begin{aligned}(a+b)^d &= \sum_{i=0}^{\infty}(b^d)^{i+1}a^i a^{\pi} + \sum_{i=0}^{\infty} b^i b^{\pi}(a^d)^{i+1} + \sum_{i=0}^{\infty} b^i b^{\pi}(a^d)^{i+2}b \\ &+ \sum_{i=0}^{\infty}(b^d)^{i+3}a^{i+1}a^{\pi}b - b^d a^d b - (b^d)^2 a a^d b.\end{aligned}$$

Proof. See [17] [Theorem 2.1] and [5] [Corollary 15.2.4]. □

Lemma 2.3. *Let*

$$x = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \text{ or } \begin{pmatrix} b & c \\ 0 & a \end{pmatrix}$$

Then

$$x^d = \begin{pmatrix} a^d & 0 \\ z & b^d \end{pmatrix}, \text{ or } \begin{pmatrix} b^d & z \\ 0 & a^d \end{pmatrix},$$

where $z = \sum_{i=0}^{\infty}(b^d)^{i+2}ca^i a^{\pi} + \sum_{i=0}^{\infty} b^i b^{\pi}c(a^d)^{i+2} - b^d c a^d$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2024 doi:10.20944/preprints202409.0090.v1

Proof. See [5] [Lemma 15.2.1]. □

Lemma 2.4. *Let $\mathcal{A}$ be a Banach algebra and $a \in \mathcal{A}^{-1}, b \in \mathcal{A}^{qnil}$. If $ab = ba$, then the equation $ax + x^2 = b$ has a solution x such that $a + bx \in \mathcal{A}^{-1}, x \in \mathcal{A}^{qnil}$.*

Proof. Let $x = \sum_{i=0}^{\infty} c_i a^{\alpha_i} b^{i+1}$, where $c_i \in \mathbb{C}, \alpha_i \in \mathbb{Z}$. Choose $\alpha_i = -(2i+1)$, Since $ab = ba$, we have

$$\begin{aligned}
ax + x^2 &= \sum_{i=0}^{\infty} c_i a^{\alpha_i+1} b^{i+1} + [\sum_{i=0}^{\infty} c_i a^{\alpha_i} b^{i+1}][\sum_{i=0}^{\infty} c_i a^{\alpha_i} b^{i+1}] \\
&= c_0 a^{\alpha_0+1} b + [c_1 a^{\alpha_1+1} + c_0^2 a^{2\alpha_0}] b^2 \\
&+ [c_2 a^{\alpha_2+1} + c_0 c_1 a^{\alpha_0+\alpha_1} + c_1 c_0 a^{\alpha_1+\alpha_0}] b^3 \\
&+ [c_3 a^{\alpha_3+1} + c_0 c_2 a^{\alpha_0+\alpha_2} + c_1 c_1 a^{\alpha_1+\alpha_1} + c_2 c_0 a^{\alpha_2+\alpha_0}] b^4 \\
&+ \cdots \\
&= c_0 b + [c_1 + c_0^2] a^{-2} b^2 + [c_2 + c_0 c_1 + c_1 c_0] \alpha^{-4} b^3 \\
&+ [c_3 + c_0 c_2 + c_1 c_1 + c_2 c_0] a^{-6} b^4 + \cdots \\
&= b,
\end{aligned}$$

hence, we choose

$$\begin{gathered}
c_0 = 1, c_1 = -1, c_2 = 2, c_3 = -5, c_4 = 14, c_5 = -42, \cdots \\
c_i = -(c_0 c_{i-1} + c_1 c_{i-2} + \cdots + c_{i-1} c_0)(i \in \mathbb{N}).
\end{gathered}$$

Let $\{C_n\}$ be the series of Catalan numbers, i.e.,

$$\begin{gathered}
C_0 = 1, C_1 = 1, c_2 = 2, c_3 = 5, c_4 = 14, c_5 = 42, \cdots, \\
C_n = C_0 C_{n-1} + \cdots + C_{n-1} C_0 (n \in \mathbb{N}).
\end{gathered}$$

Then $c_0 = C_0, c_1 = -C_1$. By induction, we claim that $c_{2n} = C_{2n}, c_{2n+1} = -C_{2n+1}(n \geq 0)$. Hence, $|c_n| = C_n(n \geq 1)$. By using the asymptotic expression of the Catalan numbers C_n, we have

$$\lim_{n\to\infty} C_n / (\frac{4^n}{\sqrt{\pi}(n)^{\frac{3}{2}}}) = 1.$$

Therefore

$$\lim_{n\to\infty} \sqrt[n]{|c_n|} = \lim_{n\to\infty} \frac{4}{\pi^{\frac{1}{2n}} (\sqrt[n]{n})^{\frac{3}{2}}} = 4.$$

Since $b \in \mathcal{A}^{qnil}$, we have $\lim_{n\to\infty} \| b^n \|^{\frac{1}{n}} = 0$. Since

$$\sqrt[n]{\| c_n a^{-(2n+1)} b^{n+1} \|} \leq \sqrt[n]{|c_n|} \, \| a^{-1} \|^{2+\frac{1}{n}} \sqrt[n]{\| b \|} \, \| b^n \|^{\frac{1}{n}},$$

we deduce that

$$\lim_{n\to\infty} \sqrt[n]{\| c_n a^{-(2n+1)} b^{n+1} \|} = 0.$$

This implies that $\sum_{i=0}^{\infty} c_i a^{-(2i+1)} b^{i+1}$ absolutely converges.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2024 doi:10.20944/preprints202409.0090.v1

Accordingly, the equation $ax+x^2=b$ has a solution $x=\sum\limits_{i=0}^{\infty}c_ia^{-(2i+1)}b^{i+1}$, where $c_0=1, c_{k+1}=-\sum\limits_{i=0}^{k}c_ic_{k-i}(k\geq 0)$. Moreover, we verify that

$$\begin{array}{c} c_n=(-1)^nC_n=(-1)^n\frac{(2n)!}{n!(n+1)!},\\ x=[\sum\limits_{i=0}^{\infty}c_ia^{-(2i+1)}b^i]b\in \mathcal{A}^{qnil},\\ a+x=a[1-a^{-1}x]\in \mathcal{A}^{-1}. \end{array}$$

This completes the proof. □

Lemma 2.5. *Let $\mathcal{A}$ be a Banach algebra and $M=\begin{pmatrix} a & b\\ 1 & 0\end{pmatrix}$ with $a\in \mathcal{A}^{-1}, b\in \mathcal{A}^{qnil}$. If $ab=ba$, then $M\in M_2(\mathcal{A})^d$ and*

$$M^d=\begin{pmatrix} (a+x)^{-1}-xy & (a+x)^{-1}x-xyx\\ y & yx\end{pmatrix},$$

where $x=\sum\limits_{i=0}^{\infty}(-1)^i\frac{(2i)!}{i!(i+1)!}a^{-(2i+1)}b^{i+1}, y=\sum\limits_{i=0}^{\infty}(-1)^i(a+x)^{-i-2}x^i$.

Proof. In view of Lemma 2.4, the equation $ax+x^2=b$ has a solution x such that $a+x\in \mathcal{A}^{-1}, x\in \mathcal{A}^{qnil}$. Here,

$$x=\sum_{i=0}^{\infty}(-1)^i\frac{(2i)!}{i!(i+1)!}a^{-(2i+1)}b^{i+1}.$$

It is easy to verify that

$$M=\begin{pmatrix} 1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix} a+x & 0\\ 1 & -x\end{pmatrix}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}.$$

Since $x\in \mathcal{A}^{qnil}$ and $a+x\in \mathcal{A}^{-1}$. Then $\begin{pmatrix} a+x & 0\\ 1 & x\end{pmatrix}$ has g-Drazin inverse. Therefore M has g-Drazin inverse. Exactly, we have

$$\begin{array}{rcl} M^d & = & \begin{pmatrix} 1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix} a+x & 0\\ 1 & -x\end{pmatrix}^d\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\\ & = & \begin{pmatrix} 1 & -x\\ 0 & 1\end{pmatrix}\begin{pmatrix} (a+x)^{-1} & 0\\ y & 0\end{pmatrix}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\\ & = & \begin{pmatrix} (a+x)^{-1}-xy & (a+x)^{-1}x-xyx\\ y & yx\end{pmatrix}, \end{array}$$

where $y=\sum\limits_{i=0}^{\infty}(-1)^i(a+x)^{-i-2}x^i$. □

Lemma 2.6. *Let $\mathcal{A}$ be a Banach algebra and $M=\begin{pmatrix} a & b\\ 1 & 0\end{pmatrix}$ with $a\in \mathcal{A}, b\in \mathcal{A}^{-1}$. Then $M\in M_2(\mathcal{A})^{-1}$ and*

$$M^{-1}=\begin{pmatrix} 0 & 1\\ b^{-1} & -b^{-1}a\end{pmatrix}.$$

Proof. Straightforward. □

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2024 doi:10.20944/preprints202409.0090.v1

Let $p^2 = p \in \mathcal{A}$ and let $\mathcal{A}_1 = p\mathcal{A}p, \mathcal{A}_2 = p^\pi \mathcal{A} p^\pi$. Let T be the ring of Morita context $(\mathcal{A}_1, \mathcal{A}_2, \varphi, \psi)$, i.e.,

$$T = \begin{pmatrix} \mathcal{A}_1 & M_2(p\mathcal{A}p^\pi) \\ M_2(p^\pi \mathcal{A} p) & \mathcal{A}_2 \end{pmatrix}_{(\varphi,\psi)}$$

with the bimodule homomorphisms of the form

$$\varphi : M_2(p\mathcal{A}p^\pi) \times M_2(p^\pi \mathcal{A} p) \to \mathcal{A}_1,$$
$$\psi : M_2(p^\pi \mathcal{A} p) \times M_2(p\mathcal{A}p^\pi) \to \mathcal{A}_2.$$

Then we have a natural isomorphism of rings given by

$$\rho : M_2(\mathcal{A}) \cong T,$$
$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \mapsto \left(\begin{array}{cc:cc} pa_{11}p & pa_{12}p & pa_{11}p^\pi & pa_{12}p^\pi \\ pa_{21}p & pa_{22}p & pa_{21}p^\pi & pa_{22}p^\pi \\ \hdashline p^\pi a_{11}p & p^\pi a_{12}p & p^\pi a_{11}p^\pi & p^\pi a_{12}p^\pi \\ p^\pi a_{21}p & p^\pi a_{22}p & p^\pi a_{21}p^\pi & p^\pi a_{22}p^\pi \end{array}\right)_{(\varphi,\psi)}.$$

Lemma 2.7. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{-1}, b \in \mathcal{A}^d$. If $ab = ba$, then $M \in M_2(\mathcal{A})^d$ and*

$$M^d = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$$

with z_{ij} are formulated by

$$\begin{aligned} z_{11} &= (ab^\pi + x)^{-1} - xy, \\ z_{12} &= bb^d + (ab^\pi + x)^{-1}x - xyx, \\ z_{21} &= b^d, \\ z_{22} &= -ab^d + y + yx, \end{aligned}$$

where

$$\begin{aligned} x &= \sum_{i=0}^{\infty} (-1)^i \frac{(2i)!}{i!(i+1)!} a^{-(2i+1)} b^{i+1} b^\pi, \\ y &= \sum_{i=0}^{\infty} (-1)^i (ab^\pi + x)^{-i-2} x^i. \end{aligned}$$

Proof. Let $p = bb^d$. Since $ab = ba$, we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_p, b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_p \in \mathcal{A}.$$

Then

$$M = \left(\begin{array}{cc:cc} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ \hdashline p & 0 & 0 & 0 \\ 0 & p^\pi & 0 & 0 \end{array}\right) \in M_2(\mathcal{A}).$$

Hence, we have

$$\rho(M) = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}_{(\varphi,\psi)},$$

where

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ p & 0 \end{pmatrix} \in M_2(\mathcal{A}_1), M_2 = \begin{pmatrix} a_2 & b_2 \\ p^\pi & 0 \end{pmatrix} \in M_2(\mathcal{A}_2).$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2024 doi:10.20944/preprints202409.0090.v1

Claim 1. $M_1 \in M_2(\mathcal{A}_1)^d$. Clearly, $a_1 = abb^d, b_1 = b^2 b^d \in \mathcal{A}_1^{-1}$. By Lemma 2.6, we have

$$M_1^d = M_1^{-1} = \begin{pmatrix} 0 & bb^d \\ b_1^{-1} & -b_1^{-1} a_1 \end{pmatrix}.$$

Claim 2. $M_2 \in M_2(\mathcal{A}_2)^d$. Clearly, $a_2 = ab^\pi \in \mathcal{A}_1^{-1}, b_2 = bb^\pi \in \mathcal{A}_2^{qnil}$. By virtue of Lemma 2.5, we have

$$M_2^d = \begin{pmatrix} (a_2 + x)^{-1} - xy & (a_2 + x)^{-1} x - xyx \\ y & yx \end{pmatrix},$$

where $x = \sum_{i=0}^{\infty} (-1)^i \frac{(2i)!}{i!(i+1)!} a_2^{-(2i+1)} b_2^{i+1}, y = \sum_{i=0}^{\infty} (-1)^i (a_2 + x)^{-i-2} x^i$. Therefore $\rho(M) \in T^d$ and

$$[\rho(M)]^d = \left(\begin{array}{cc|cc} 0 & bb^d & 0 & 0 \\ b_1^{-1} & -b_1^{-1} a_1 & 0 & 0 \\ \hline 0 & 0 & (a_2 + x)^{-1} - xy & (a_2 + x)^{-1} x - xyx \\ 0 & 0 & y & yx \end{array}\right)_{(\varphi,\psi)}.$$

Therefore $M \in M_2(\mathcal{A})^d$. Furthermore, we have

$$\begin{aligned} M^d &= \left(\begin{array}{cc|cc} 0 & 0 & bb^d & 0 \\ 0 & (a_2 + x)^{-1} - xy & 0 & (a_2 + x)^{-1} x - xyx \\ \hline b_1^{-1} & 0 & -b_1^{-1} a_1 & 0 \\ 0 & 0 & y & yx \end{array}\right) \\ &= \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \end{aligned}$$

with z_{ij} are formulated by

$$\begin{aligned} z_{11} &= (ab^\pi + x)^{-1} - xy, \\ z_{12} &= bb^d + (ab^\pi + x)^{-1} x - xyx, \\ z_{21} &= b^d, \\ z_{22} &= -ab^d + y + yx, \end{aligned}$$

where

$$\begin{aligned} x &= \sum_{i=0}^{\infty} (-1)^i \frac{(2i)!}{i!(i+1)!} a^{-(2i+1)} b^{i+1} b^\pi, \\ y &= \sum_{i=0}^{\infty} (-1)^i (ab^\pi + x)^{-i-2} x^i. \end{aligned}$$

This completes the proof. □

Lemma 2.8. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ with $a \in \mathcal{A}^{qnil}, b \in \mathcal{A}^d$. If $ab = ba$, then $M \in M_2(\mathcal{A})^d$ and*

$$M^d = \begin{pmatrix} 0 & bb^d \\ b^d & -ab^d \end{pmatrix}.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2024 doi:10.20944/preprints202409.0090.v1

Proof. Let $X = \begin{pmatrix} 0 & bb^d \\ b^d & -ab^d \end{pmatrix}$. One directly verify that

$$\begin{aligned} MX &= \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & bb^d \\ b^d & -ab^d \end{pmatrix} \\ &= \begin{pmatrix} bb^d & 0 \\ 0 & bb^d \end{pmatrix} \\ &= \begin{pmatrix} 0 & bb^d \\ b^d & -ab^d \end{pmatrix}\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \\ &= XM, \\ MX^2 &= (MX)X = X, \\ M-(MX)M &= \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} bb^d & 0 \\ 0 & bb^d \end{pmatrix}\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} a(1-bb^d) & b-b^2b^d \\ 1-bb^d & 0 \end{pmatrix} \in M_2(\mathcal{A})^{qnil}. \end{aligned}$$

Therefore M has g-Drazin inverse and $M^d = X$, as desired. □

3. Main Results

We now present the main results of this paper, which extend [16] [Theorem 3.8] and [18] [Theorem 4.1] to anti-triangular matrices in Banach algebras.

Theorem 3.1. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ with $a, b \in \mathcal{A}^d$. If $ab = ba$, then $M \in M_2(\mathcal{A})^d$ and*

$$M^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta$ are formulated by

$$\begin{aligned} \alpha &= (a^2a^db^\pi + x)^{-1} - xy, \\ \beta &= aa^dbb^d + (a^2a^db^\pi + x)^{-1}x - xyx + a^\pi bb^d, \\ \gamma &= aa^db^d + a^\pi b^d, \\ \delta &= -a^2a^db^d + y + yx - aa^\pi b^d, \end{aligned}$$

where

$$\begin{aligned} x &= \sum_{i=0}^{\infty} (-1)^i \tfrac{(2i)!}{i!(i+1)!} a^{-2i}a^db^{i+1}b^\pi, \\ y &= \sum_{i=0}^{\infty} (-1)^i (a^2a^db^\pi + x)^{-i-2}x^i. \end{aligned}$$

Proof. Let $q = aa^d$. Since $ab = ba$, we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}_q, b = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}_q.$$

Then

$$M = \left(\begin{array}{cc|cc} a_1 & 0 & b_1 & 0 \\ 0 & a_2 & 0 & b_2 \\ \hline p & 0 & 0 & 0 \\ 0 & p^\pi & 0 & 0 \end{array}\right) \in M_2(\mathcal{A}).$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2024 doi:10.20944/preprints202409.0090.v1

By using the isomorphism ρ between the matrix ring $M_2(\mathcal{A})$ and the the ring of Morita context $(\mathcal{A}_1, \mathcal{A}_2, \varphi, \psi)$ mentioned above, we have

$$\rho(M) = \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix}_{(\varphi,\psi)},$$

where

$$M_1 = \begin{pmatrix} a_1 & b_1 \\ q & 0 \end{pmatrix} \in M_2(\mathcal{A}_1), M_2 = \begin{pmatrix} a_2 & b_2 \\ q^\pi & 0 \end{pmatrix} \in M_2(\mathcal{A}_2).$$

Claim 1. $M_1 \in M_2(\mathcal{A}_1)^d$. Obviously, $a_1 \in \mathcal{A}_1^{-1}, b_1 \in \mathcal{A}_1^d$. In view of Lemma 2.7, we have

$$M_1^d = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix}$$

with z_{ij} are formulated by

$$\begin{aligned} z_{11} &= (a^2a^db^\pi + x)^{-1} - xy, \\ z_{12} &= aa^dbb^d + (a^2a^db^\pi + x)^{-1}x - xyx, \\ z_{21} &= aa^db^d, \\ z_{22} &= -a^2a^db^d + y + yx, \end{aligned}$$

where

$$\begin{aligned} x &= \sum_{i=0}^{\infty}(-1)^i\frac{(2i)!}{i!(i+1)!}a^{-2i}a^db^{i+1}b^\pi, \\ y &= \sum_{i=0}^{\infty}(-1)^i(a^2a^db^\pi + x)^{-i-2}x^i. \end{aligned}$$

Claim 2. $M_2 \in M_2(\mathcal{A}_2)^d$. Obviously, $a_2 \in \mathcal{A}_2^{qnil}, b_2 \in \mathcal{A}_2^d$. By virtue of Lemma 2.8, we derive that

$$M_2^d = \begin{pmatrix} 0 & b_2b_2^d \\ b_2^d & -a_2b_2^d \end{pmatrix}.$$

Therefore $\rho(M) \in T^d$ and

$$[\rho(M)]^d = \begin{pmatrix} M_1^d & 0 \\ 0 & M_2^d \end{pmatrix}_{(\varphi,\psi)}.$$

Therefore

$$M^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in M_2(\mathcal{A}),$$

where

$$\begin{aligned} \alpha &= \begin{pmatrix} z_{11} & 0 \\ 0 & 0 \end{pmatrix}_p = z_{11}, \\ \beta &= \begin{pmatrix} z_{12} & 0 \\ 0 & b_2b_2^d \end{pmatrix}_p = z_{12} + a^\pi bb^d, \\ \gamma &= \begin{pmatrix} z_{21} & 0 \\ 0 & b_2^d \end{pmatrix}_p = z_{21} + a^\pi b^d, \\ \delta &= \begin{pmatrix} z_{22} & 0 \\ 0 & -a_2b_2^d \end{pmatrix}_p = z_{22} - aa^\pi b^d. \end{aligned}$$

This completes the proof. □

Corollary 3.2. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ with $a, b \in \mathcal{A}^D$. If $ab = ba$, then $M \in M_2(\mathcal{A})^D$ and*

$$M^D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta$ are formulated by

$$\begin{array}{rcl} \alpha & = & (a^2a^Db^\pi + x)^{-1} - xy, \\ \beta & = & aa^dbb^D + (a^2a^Db^\pi + x)^{-1}x - xyx + a^\pi bb^D, \\ \gamma & = & aa^Db^D + a^\pi b^D, \\ \delta & = & -a^2a^Db^D + y + yx - aa^\pi b^D, \end{array}$$

where

$$\begin{array}{rcl} x & = & \sum\limits_{i=0}^{ind(b)-1} (-1)^i \frac{(2i)!}{i!(i+1)!} a^{-2i} a^D b^{i+1} b^\pi, \\ y & = & \sum\limits_{i=0}^{ind(b)-1} (-1)^i (a^2a^Db^\pi + x)^{-i-2} x^i. \end{array}$$

Proof. Evidently, $z \in \mathcal{A}^D$ if and only if $z \in \mathcal{A}^d$ and $a - a^2a^d \in \mathcal{A}$ is nilpotent. In this case, $z^D = z^d$. Therefore we complete the proof by Theorem 3.1. □

We are now ready to prove:

Theorem 3.3. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ with $a, b, b^\pi a \in \mathcal{A}^d$. If $b^\pi ab^d = 0$ and $b^\pi(ab) = b^\pi(ba)$, then $M \in M_2(\mathcal{A})^d$ and*

$$M^d = \sum_{i=0}^{\infty} P^i[I - PP^d](Q^d)^{i+1},$$

where

$$\begin{array}{rcl} P & = & \begin{pmatrix} b^\pi a & b^\pi b \\ b^\pi & 0 \end{pmatrix}, P^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \\ Q & = & \begin{pmatrix} bb^d a & b^2b^d \\ bb^d & 0 \end{pmatrix}, Q^d = \begin{pmatrix} 0 & bb^d \\ b^d & -b^d a \end{pmatrix} \end{array}$$

with $\alpha, \beta, \gamma, \delta$ are formulated by

$$\begin{array}{rcl} \alpha & = & (b^\pi a^2a^d + x)^{-1} - xy, \\ \beta & = & (b^\pi a^2a^d + x)^{-1}x - xyx, \\ \gamma & = & 0, \\ \delta & = & y + yx, \end{array}$$

where

$$\begin{array}{rcl} x & = & \sum\limits_{i=0}^{\infty} (-1)^i \frac{(2i)!}{i!(i+1)!} a^{-2i} b^\pi a^d b^{i+1}, \\ y & = & \sum\limits_{i=0}^{\infty} (-1)^i (b^\pi a^2a^d + x)^{-i-2} x^i. \end{array}$$

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} b^\pi a & b^\pi b \\ b^\pi & 0 \end{pmatrix}, Q = \begin{pmatrix} bb^d a & b^2b^d \\ bb^d & 0 \end{pmatrix}.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2024 doi:10.20944/preprints202409.0090.v1

Step 1. P has g-Drazin inverse. By hypothesis, we verify that

$$(b^{\pi}a)(b^{\pi}b) = b^{\pi}(ab) = b^{\pi}(ba) = (b^{\pi}b)(b^{\pi}a).$$

In light of Theorem 3.1, we have

$$P^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

with $\alpha, \beta, \gamma, \delta$ are formulated by

$$\begin{array}{rcl} \alpha & = & (b^{\pi}a^2a^d + x)^{-1} - xy, \\ \beta & = & (b^{\pi}a^2a^d + x)^{-1}x - xyx, \\ \gamma & = & 0, \\ \delta & = & y + yx, \end{array}$$

where

$$\begin{array}{rcl} x & = & \sum\limits_{i=0}^{\infty}(-1)^i\frac{(2i)!}{i!(i+1)!}a^{-2i}b^{\pi}a^db^{i+1}, \\ y & = & \sum\limits_{i=0}^{\infty}(-1)^i(b^{\pi}a^2a^d + x)^{-i-2}x^i. \end{array}$$

Step 2. Q has g-Drazin inverse. By virtue of Lemma 2.6,

$$Q^d = \begin{pmatrix} 0 & bb^d \\ b^d & -b^da \end{pmatrix}.$$

Step 3. Since $PQ = 0$, it follows by Lemma 2.1 that

$$\begin{array}{rcl} M^d & = & (P+Q)^d \\ & = & \sum\limits_{i=0}^{\infty}(P^d)^{i+1}Q^iQ^{\pi} + \sum\limits_{i=0}^{\infty}P^iP^{\pi}(Q^d)^{i+1} \\ & = & \sum\limits_{i=0}^{\infty}P^iP^{\pi}(Q^d)^{i+1}. \end{array}$$

This completes the proof. □

Corollary 3.4. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ with $a, b, b^{\pi}a \in \mathcal{A}^D$. If $b^{\pi}abb^D = 0$ and $b^{\pi}(ab) = b^{\pi}(ba)$, then $M \in M_2(\mathcal{A})^D$ and*

$$M^D = \sum_{i=0}^{ind(P)} P^i[I - PP^D](Q^D)^{i+1},$$

where

$$\begin{array}{rcl} P & = & \begin{pmatrix} b^{\pi}a & b^{\pi}b \\ b^{\pi} & 0 \end{pmatrix}, P^D = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \\ Q & = & \begin{pmatrix} bb^Da & b^2b^D \\ bb^D & 0 \end{pmatrix}, Q^D = \begin{pmatrix} 0 & bb^D \\ b^D & -b^Da \end{pmatrix} \end{array}$$

with $\alpha, \beta, \gamma, \delta$ are formulated by

$$\begin{array}{rcl} \alpha & = & (b^{\pi}a^2a^D + x)^{-1} - xy, \\ \beta & = & (b^{\pi}a^2a^D + x)^{-1}x - xyx, \\ \gamma & = & 0, \\ \delta & = & y + yx, \end{array}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2024 doi:10.20944/preprints202409.0090.v1

where

$$
\begin{aligned}
x &= \sum_{i=0}^{ind(b)-1} (-1)^i \frac{(2i)!}{i!(i+1)!} a^{-2i} b^{\pi} a^D b^{i+1}, \\
y &= \sum_{i=0}^{ind(b)-1} (-1)^i (b^{\pi} a^2 a^D + x)^{-i-2} x^i.
\end{aligned}
$$

Proof. It is immediate from Theorem 3.3. □

It is convenient at this stage to derive the following:

Theorem 3.5. *Let $\mathcal{A}$ be a Banach algebra and $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a, d, bc \in \mathcal{A}^d$. If $abc = bca$, $bdc = 0$ and $bd^2 = 0$, then $M \in M_2(\mathcal{A})^d$ and*

$$
\begin{aligned}
M^d &= \sum_{i=0}^{\infty} (Q^d)^{i+1} P^i (I - PP^d) + \sum_{i=0}^{\infty} Q^i Q^{\pi} (P^d)^{i+1} + \sum_{i=0}^{\infty} Q^i (I - QQ^d)(P^d)^{i+2} Q \\
&+ \sum_{i=0}^{\infty} (Q^d)^{i+3} P^{i+1} (I - PP^d) Q - Q^d P^d Q - (Q^d)^2 P P^d Q,
\end{aligned}
$$

where

$$
\begin{aligned}
P &= \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, P^d = \begin{pmatrix} \alpha^2 a + \alpha\beta + \beta\gamma a + \beta\delta & \alpha^2 b + \beta\gamma b \\ c\gamma\alpha a + c\gamma\beta + c\delta\gamma a + c\delta\delta & c\gamma\alpha b + c\delta\gamma b \end{pmatrix}; \\
Q &= \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}, Q^d = \begin{pmatrix} 0 & 0 \\ 0 & d^d \end{pmatrix}
\end{aligned}
$$

with $\alpha, \beta, \gamma, \delta$ are formulated by

$$
\begin{aligned}
\alpha &= (a^2 a^d (bc)^{\pi} + x)^{-1} - xy, \\
\beta &= a a^d bc (bc)^d + (a^2 a^d (bc)^{\pi} + x)^{-1} x - xyx + a^{\pi} bc (bc)^d, \\
\gamma &= a a^d (bc)^d + a^{\pi} (bc)^d, \\
\delta &= -a^2 a^d (bc)^d + y + yx - a a^{\pi} (bc)^d,
\end{aligned}
$$

where

$$
\begin{aligned}
x &= \sum_{i=0}^{\infty} (-1)^i \frac{(2i)!}{i!(i+1)!} a^{-2i} a^d (bc)^{i+1} (bc)^{\pi}, \\
y &= \sum_{i=0}^{\infty} (-1)^i [a^2 a^d (bc)^{\pi} + x]^{-i-2} x^i.
\end{aligned}
$$

Proof. Let $P = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}$. In view of Theorem 3.1, we have

$$
\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix}^d = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
$$

with $\alpha, \beta, \gamma, \delta$ are formulated by

$$
\begin{aligned}
\alpha &= (a^2 a^d (bc)^{\pi} + x)^{-1} - xy, \\
\beta &= a a^d bc (bc)^d + (a^2 a^d (bc)^{\pi} + x)^{-1} x - xyx + a^{\pi} bc (bc)^d, \\
\gamma &= a a^d (bc)^d + a^{\pi} (bc)^d, \\
\delta &= -a^2 a^d (bc)^d + y + yx - a a^{\pi} (bc)^d,
\end{aligned}
$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 2 September 2024 doi:10.20944/preprints202409.0090.v1

where

$$\begin{aligned} x &= \sum_{i=0}^{\infty}(-1)^i\frac{(2i)!}{i!(i+1)!}a^{-2i}a^d(bc)^{i+1}(bc)^{\pi}, \\ y &= \sum_{i=0}^{\infty}(-1)^i[a^2a^d(bc)^{\pi}+x]^{-i-2}x^i. \end{aligned}$$

One easily verifies that

$$\begin{aligned} \begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix} &= \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}, \\ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

By using Cline's formula (see [14] [Theorem 2.2]), P has g-Drazin inverse and

$$\begin{aligned} P^d &= \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}[\begin{pmatrix} a & bc \\ 1 & 0 \end{pmatrix}^d]^2\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^2\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ c\gamma & c\delta \end{pmatrix}\begin{pmatrix} \alpha a+\beta & \alpha b \\ \gamma a+\delta & \gamma b \end{pmatrix} \\ &= \begin{pmatrix} \alpha^2a+\alpha\beta+\beta\gamma a+\beta\delta & \alpha^2b+\beta\gamma b \\ c\gamma\alpha a+c\gamma\beta+c\delta\gamma a+c\delta\delta & c\gamma\alpha b+c\delta\gamma b \end{pmatrix}. \end{aligned}$$

Obviously, we have

$$Q^d = \begin{pmatrix} 0 & 0 \\ 0 & d^d \end{pmatrix}, Q^{\pi} = \begin{pmatrix} 1 & 0 \\ 0 & d^{\pi} \end{pmatrix}.$$

One easily checks that

$$\begin{aligned} PQ^2 &= \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & d^2 \end{pmatrix} = \begin{pmatrix} 0 & bd^2 \\ 0 & 0 \end{pmatrix} = 0, \\ PQP &= \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \\ &= \begin{pmatrix} bdc & 0 \\ 0 & 0 \end{pmatrix} = 0. \end{aligned}$$

According to Lemma 2.2, we derive that

$$\begin{aligned} M^d &= (P+Q)^d \\ &= \sum_{i=0}^{\infty}(Q^d)^{i+1}P^iP^{\pi}+\sum_{i=0}^{\infty}Q^iQ^{\pi}(P^d)^{i+1}+\sum_{i=0}^{\infty}Q^iQ^{\pi}(P^d)^{i+2}Q \\ &+ \sum_{i=0}^{\infty}(Q^d)^{i+3}P^{i+1}P^{\pi}Q-Q^dP^dQ-(Q^d)^2PP^dQ, \end{aligned}$$

as asserted. □

Data Availability Statement: The data used to support the findings of this study are included within the article.

Conflicts of Interest: The authors declare there is no conflicts of interest.

References

1. C. Bu; C. Feng and S. Bai, Representations for the Drazin inverses of the sum of two matrices and some block matrices, *Applied Math. Comput.*, **218**(2012), 20226–20237.
2. C. Bu; K. Zhang and J. Zhao, Representation of the Drazin inverse on solution of a class singular differential equations, *Linear Multilinear Algebra*, **59**(2011), 863–877.
3. S.L. Campbell, The Drazin inverse and systems of second order linear differential equations, *Linear Multilinear Algebra*, **14**(1983), 195–198.
4. H. Chen and M. Sheibani, The g-Drazin inverses of special operator matrices, *Oper. Matrices*, **15**(2021), 151–162.
5. H. Chen and M. Sheibani, *Theory of Clean Rings and Matrices*, Singapore: World Scientific, 2023.
6. H. Chen and M. Sheibani, The generalized Drazin inverse of an operator matrix with commuting entries, *Georgian Math. J.*, **31**(2024), 195–204.
7. H. Chen and M. Sheibani, The g-Drazin inverses of anti-triangular block operator matrices, *Applied Math. Comput.*, **463**(2024) 128368.
8. H. Chen and M. Sheibani, The Drazin inverse for perturbed block-operator matrices, *Filomat*, **38**(2024), 2311–2321.
9. N. Castro-González and E. Dopazo, Representations of the Drazin inverse for a class of block matrices, *Linear Algebra Appl.*, **400**(2005), 253-269.
10. D.S. Cvetković-Ilić, Some results on the $(2,2,0)$ Drazin inverse problem, *Linear Algebra Appl.*, **438**(2013), 4726-4741.
11. C. Deng and Y. Wei, A note on the Drazin inverse of an anti-triangular matrix, *Linear Algebra Appl.*, **431**(2009), 1910-1922.
12. E. Dopazo and M.F. Martinez-Serrano, Further results on the representation of the Drazin inverse of a 2×2 block matrix, *Linear Algebra Appl.*, **432**(2010), 1896–1904.
13. J. Li and H. Wang, Generalized Drazin invertibility of the product and sum of bounded linear operators, Âcta Anal. Funct. Appl., **22**(2020), 33–43.
14. Y. Liao; J. Chen and J. Cui, Cline's formula for the generalized Drazin inverse, *Bull. Malays. Math. Sci. Soc.*, **37**(2014), 37–42.
15. X. Liu; X. Qin and J. Benitez, New additive results for the generalized Drazin inverse in a Banach algebra, *Filomat*, **30**(2016), 2289–2294.
16. Q. Xu; C. Song and L. Zhang, Solvability of certain quadratic operator equations and reprefenttions of Drazin inverses, *Linear Algebra Appl.*, **439**(2013), 291–309.
17. H. Yang and X. Liu, The Drazin inverse of the sum of two matrices and its applications, *J. Comput. Applied Math.*, **235**(2011), 1412–1417.
18. A. Yu; X. Wang and C. Deng, On the Drazin inverse of anti-triangular block matrix, *Linear Algebra Appl.*, **489**(2016), 274–287.
19. D. Zhang; D. Mosić and L. Chen, On the Drazin inverse of anti-triangular block matrices, *Electron. Res. Arch.*, **30**(2022), 2428–2445.
20. D. Zhang; Y. Jin and D. Mosić, Generalizations of certain conditions for Drazin inverse expressions of anti-triangular partitioned matrices, *Aequationes Math.*, **98**(2024), 1081–1098.
21. D. Zhang; Y. Zhao; D. Mosić and V.N. Katsikis, Exact expressions for the Drazin inverse of anti-triangular matrices, *J. Comput. Appl. Math.*, **428**(2023), Article ID 115187, 16 p.
22. H. Zou; D. Mosić and J. Chen, Generalized Drazin invertibility of the product and sum of two elements in a Banach algebra and its applications, *Turk. J. Math.*, **41**(2017), 548–563.
23. H. Zou; J. Chen and D. Mosić, The Drazin invertibility of an anti-triangular matrix over a ring, *Stud. Sci. Math. Hung.*, **54**(2017), 489–508.