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Hussain Al-Qassem and Mohammed Ali *

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Estimates for Certain Rough Multiple Singular Integrals on Triebel-Lizorkin Space

Hussain Al-Qassem [1] and Mohammed Ali [2,*]

[1] Department of Mathematics and Statistics, Qatar University, Doha, Qatar; husseink@qu.edu.qa
[2] Department of Mathematics and Statistics, Jordan University of Science and Technology, Irbid 22110, Jordan
* Correspondence: myali@just.edu.jo

Abstract: This paper focuses on studying the mapping properties of singular integral operators over product symmetric spaces. The boundedness of such operators is established on Triebel-Lizorkin spaces whenever their rough kernel functions belong to Grafakos and Stefanov class. Our findings generalize, extend and improve some previously known results on singular integral operators as those in [1,2,11].

Keywords: Triebel-Lizorkin space; singular integrals; rough kernels; product spaces

1. Introduction and Main Results

Assume that $\mathbb{R}^s$ ($s = \kappa$ or η) is the $2 \leq s$-Euclidean space and that $\mathbb{S}^{s-1}$ is the unit sphere in $\mathbb{R}^s$ equipped with the normalized Lebesgue surface measure $d\sigma_s(\cdot)$. Also assume that $w' = w/|w|$ for $w \in \mathbb{R}^s \backslash \{0\}$.

Let $\mho$ be an integrable over $\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}$ and satisfy

$$\mho(tu, rv) = \mho(u, v), \quad \forall t, r > 0, \tag{1}$$

$$\int_{\mathbb{S}^{\kappa-1}} \mho(u', v') d\sigma_\kappa(u') = \int_{\mathbb{S}^{\eta-1}} \mho(u', v') d\sigma_\eta(v') = 0. \tag{2}$$

The singular integral operator $T_\mho$ on symmetric spaces $\mathbb{R}^\kappa \times \mathbb{R}^\eta$ is defined, initially for $h \in \mathcal{S}(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$, by

$$\mathbf{T}_\mho h(x, y) = \text{p.v.} \iint_{\mathbb{R}^\kappa \times \mathbb{R}^\eta} h(x-u, y-v) \frac{\mho(u', v')}{|u|^\kappa |v|^\eta} du dv.$$

The study of the boundedness of the operator $\mathbf{T}_\mho$ was started in [1] in which the authors proved the L^p boundedness of $\mathbf{T}_\mho$ for all $p \in (1, \infty)$ if Ω satisfies certain Lipschitz conditions. Subsequently the boundedness of $\mathbf{T}_\mho$ and some of its extensions has been investigated by many researchers. For example, Duoandikoetxea improved the above results in [2] by proving that $\mathbf{T}_\mho$ is bounded on $L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$ under the weaker condition $\mho \in L^q(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$. Later on, the authors of [3], confirmed that $\mathbf{T}_\mho$ is bounded on $L^p(\mathbb{R}^\kappa \times \mathbb{R}^\eta)$ $(1 < p < \infty)$ if $\mho \in L(\log^+ L)^2(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$. In [4] the authors established the L^p boundedness of $\mathbf{T}_\mho$ for $p \in (1, \infty)$ provided that $\mho$ in the block space $B_q^{(0,1)}(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$ for some $q > 1$. Thereafter, the discussion of the mapping properties of $\mathbf{T}_\mho$ and its extensions under various conditions on $\mho$ has received a large amount of attention by many authors, the readers are referred to [1–8].

Our focus in this paper will be in studying the boundedness of $\mathbf{T}_\mho$ whenever $\mho$ belongs to a certain class of functions related to a class of functions introduced by Walsh in [9] and then developed by Grafakos and Stefanov in [10]. To clarify our purpose we recall some definitions and some pertinent results related to our current study. Let $\mathbf{G}_\alpha(\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1})$ (for $\alpha > 0$) be the class of all functions $\mho$ which are integrable over $\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}$ and satisfy the condition on product spaces

$$\sup_{(\xi,\zeta) \in \mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}} \iint_{\mathbb{S}^{\kappa-1} \times \mathbb{S}^{\eta-1}} \log^{\alpha+1}(|\xi \cdot u'|^{-1}) \log^{\alpha+1}(|\zeta \cdot v'|^{-1})$$

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$$\times|\mho(u',v')|d\sigma_\kappa(u')d\sigma_\eta(v')<\infty.$$

By following the same arguments as that employed in [10], we get the following:

$$\begin{aligned}\bigcup_{q>1}L^q\left(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}\right) &\nsubseteq \mathbf{G}_\alpha\left(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}\right) \ \textit{for any}\ \alpha>0,\\ \bigcap_{\alpha>0}\mathbf{G}_\alpha\left(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}\right) &\nsubseteq L(\log^+L)^2(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})\nsubseteq\bigcup_{\alpha>0}\mathbf{G}_\alpha\left(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}\right),\\ \bigcap_{\alpha>0}\mathbf{G}_\alpha\left(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}\right) &\nsubseteq B_q^{(0,1)}(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})\nsubseteq\bigcup_{\alpha>0}\mathbf{G}_\alpha\left(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}\right).\end{aligned}$$

Let us recall the definition of the homogeneous Triebel-Lizorkin space $\dot{F}_p^{\varepsilon,\overrightarrow{\gamma}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$. For $p,\varepsilon\in(1,\infty)$ and $\overrightarrow{\gamma}=(\gamma_1,\gamma_2)\in\mathbb{R}\times\mathbb{R}$, the homogeneous Triebel-Lizorkin space $\dot{F}_p^{\varepsilon,\overrightarrow{\gamma}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ is the class of all tempered distributions h on $\mathbb{R}^\kappa\times\mathbb{R}^\eta$ that satisfy

$$\|h\|_{\dot{F}_p^{\varepsilon,\overrightarrow{\gamma}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}=\left\|\left(\sum_{j,k\in\mathbb{Z}}2^{j\gamma_1\varepsilon}2^{k\gamma_2\varepsilon}|(\mathcal{A}_j\otimes\mathcal{B}_k)*h|^\varepsilon\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}<\infty,$$

where $\widehat{\mathcal{A}}_j(u)=2^{-j\kappa}\mathcal{A}(2^{-j}u)$ for $j\in\mathbb{Z}$, $\widehat{\mathcal{B}}_k(v)=2^{-k\eta}\mathcal{B}(2^{-k}v)$ for $k\in\mathbb{Z}$ and the radial functions $\mathcal{A}\in\mathcal{S}(\mathbb{R}^\kappa)$, $\mathcal{B}\in\mathcal{S}(\mathbb{R}^\eta)$ satisfy the following:

(1) $0\le\mathcal{A}\le1,\ \ 0\le\mathcal{B}\le1$,

(2) $supp\,(\mathcal{A})\subset\left\{u:\frac{1}{2}\le|u|\le2\right\},\ \ supp\,(\mathcal{B})\subset\left\{v:\frac{1}{2}\le|v|\le2\right\}$,

(3) There exists $M>0$ such that $\mathcal{A}(u),\mathcal{B}(v)\ge M$ for all $|u|,|v|\in[\frac{3}{5},\frac{5}{3}]$,

(4) $\sum_{j\in\mathbb{Z}}\mathcal{A}(2^{-j}u)=1$ with $u\neq0$ and $\sum_{k\in\mathbb{Z}}\mathcal{B}(2^{-k}v)=1$ with $v\neq0$.

The authors of [12] proved the following properties:

(i) The Schwartz space $\mathcal{S}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ is dense in $\dot{F}_p^{\varepsilon,\overrightarrow{\gamma}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$,

(ii) $\dot{F}_p^{2,\overrightarrow{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)=L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ for $1<p<\infty$,

(iii) $\dot{F}_p^{\varepsilon_1,\overrightarrow{\gamma}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)\subseteq\dot{F}_p^{\varepsilon_2,\overrightarrow{\gamma}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ if $\varepsilon_1\le\varepsilon_2$.

In [11], Ying showed that if $\mho\in\mathbf{G}_\alpha(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$ for some $\alpha>0$, then $\mathbf{T}_\mho$ is bounded on $L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ for all $p\in(\frac{2+2\alpha}{1+2\alpha},2+2\alpha)$.

In the one parameter setting, the singular operator related to $\mathbf{T}_\mho$ is given by

$$\mathbf{H}_\mho h(x)=\text{p.v.}\int_{\mathbb{R}^\kappa}h(x-u)\frac{\mho(u')}{|u|^\kappa}du.$$

For $\alpha>0$, the class $\mathbf{G}_\alpha(\mathbb{S}^{\kappa-1})$ is the collection of all functions $\mho\in L^1(\mathbb{S}^{\kappa-1})$ which satisfy the Grafakos-Stefanov condition

$$\sup_{\xi\in\mathbb{S}^{\kappa-1}}\int_{\mathbb{S}^{\kappa-1}}|\mho(u')|\log^{\alpha+1}(|\xi\cdot u'|^{-1})d\sigma_\kappa(u')<\infty.$$

In [13], the authors proved that the integral operator $\mathbf{H}_\mho$ is bounded on $\dot{F}_p^{\varepsilon,\gamma_1}(\mathbb{R}^\kappa)$ for $p\in(\frac{2+2\alpha}{1+2\alpha},2+2\alpha)$, $\varepsilon\in(\frac{2+2\alpha}{1+2\alpha},2+2\alpha)$ and $\gamma_1\in\mathbb{R}$.

It is worth mentioning that the Triebel–Lizorkin space $\dot{F}_p^{\varepsilon,\gamma_1}(\mathbb{R}^\kappa)$ covers several classes of many well-known function spaces including Lebesgue spaces $L^p(\mathbb{R}^\kappa)$, the Hardy spaces $H^p(\mathbb{R}^\kappa)$ and the Sobolev spaces $L_p^\alpha(\mathbb{R}^\kappa)$. So it is tacitly that the work on these spaces is more intricate than $L^p(\mathbb{R}^\kappa)$. This

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clearly has instigated many authors to investigate the boundedness of $\mathbf{H}_{\mho}$ and some of its extensions, see for instance [14–26].

In light of the results obtained in [13] regarding the $\dot{F}_p^{\varepsilon,\gamma_1}$ boundedness of the singular integral $\mathbf{H}_{\mho}$ in the one parameter setting whenever $\mho \in \mathbf{G}_\alpha(\mathbb{S}^{\kappa-1})$, and the work done in [11] regarding the L^p boundedness of the singular integral $\mathbf{T}_{\mho}$ in the product domains whenever $\mho \in \mathbf{G}_\alpha(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$, we are motivated to investigate the boundedness of $\mathbf{T}_{\mho}$ on $\dot{F}_p^{\varepsilon,\overrightarrow{\gamma}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ whenever $\mho$ satisfies the Grafakos-Stefanov condition.

The main result of this paper is the following:

Theorem 1. *Suppose that* $\mho \in \mathbf{G}_\alpha(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$ *for some* $\alpha > 0$. *Then* $\mathbf{T}_{\mho}$ *is bounded on* $\dot{F}_p^{\varepsilon,\overrightarrow{\gamma}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ *for* $p \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$, $\varepsilon \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ *and* $\overrightarrow{\gamma} \in \mathbb{R}\times\mathbb{R}$.

2. Auxiliary Lemmas

We devote this section to establishing some preliminary lemmas. For $\mho \in L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$, we consider the sequence of measures $\{\Upsilon_{t,r} : t,r \in \mathbb{R}\}$ and its corresponding maximal operator Υ^* on $\mathbb{R}^\kappa\times\mathbb{R}^\eta$ by

$$\iint_{\mathbb{R}^\kappa\times\mathbb{R}^\eta} h\, d\Upsilon_{t,r} = \iint_{I_{t,r}} h(u,v)\frac{\mho(u',v')}{|u|^\kappa|v|^\eta}dudv$$

and

$$\Upsilon^*(h) = \sup_{t,r\in\mathbb{R}} |\Upsilon_{t,r}| * |h|,$$

where $I_{t,r} = \left\{(u,v) \in \mathbb{R}^\kappa\times\mathbb{R}^\eta : 2^t \le |u| < 2^{t+1}, 2^r \le |v| < 2^{r+1}\right\}$.

By adapting the same argument used in [10] to the product case, it is easy to obtain the following:

Lemma 1. *Let* $\mho \in G_\alpha(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$ *for some* $\alpha > 0$ *and satisfy the conditions* (1)-(2). *Then there is a constant* $C > 0$ *such that the estimates*

$$\begin{aligned}
\left|\hat{\Upsilon}_{t,r}(\xi,\zeta)\right| &\le C, & (3)\\
\left|\hat{\Upsilon}_{t,r}(\xi,\zeta)\right| &\le C\min\left\{|2^t\xi|, \left(\log^+|2^t\xi|\right)^{-(\alpha+1)}\right\}, & (4)\\
\left|\hat{\Upsilon}_{t,r}(\xi,\zeta)\right| &\le C\min\left\{|2^r\zeta|, \left(\log^+|2^r\zeta|\right)^{-(\alpha+1)}\right\} & (5)
\end{aligned}$$

hold for all $t,r \in \mathbb{R}$ *and* $(\xi,\zeta) \in \mathbb{R}^\kappa\times\mathbb{R}^\eta$.

Proof. By the definition of $\hat{\Upsilon}_{t,r}(\xi,\zeta)$, it is easy to see that

$$\left|\hat{\Upsilon}_{t,r}(\xi,\zeta)\right| \le (\log 2)^2\|\mho\|_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}, \tag{6}$$

which proves (3). By a change of variable, we deduce that

$$\left|\hat{\Upsilon}_{t,r}(\xi,\zeta)\right| \le \iint_{\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}} |\mho(u,v)| \int_{2^r}^{2^{r+1}} |J_t(\xi,u,l)|\frac{d\tau}{\tau}d\sigma_\kappa(u)d\sigma_\eta(v), \tag{7}$$

where

$$J_t(\xi,u,l) = \int_1^2 e^{-i(l2^t\xi\cdot u)}\frac{dl}{l}$$

which leads to

$$J_t(\xi,u,l) \le C\left|2^t\xi\left|u\cdot\xi'\right|\right|^{-1/2}.$$

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Hence, by the last estimate and the trivial estimate $|J_t(\xi,u,l)| \leq (\log 2)$ along with the fact that $t/(\log t)^\alpha$ is increasing on $(2^\alpha,\infty)$, we get that

$$|J_t(\xi,u,l)| \leq C\frac{\left(\log 2|\xi'\cdot u|^{-1}\right)^{\alpha+1}}{(\log|2^t\xi|)^{\alpha+1}} \text{ if } |2^t\xi| > 2^\alpha. \tag{8}$$

Thus, the inequalities (7) and (8) give that

$$\begin{aligned}&\left|\hat{\Upsilon}_{t,r}(\xi,\zeta)\right| \\ \leq\ & C\left(\log|2^t\xi|\right)^{-(\alpha+1)}\iint_{\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}}\left(\log\left(2|\xi'\cdot u|^{-1}\right)\right)^{\alpha+1}|\mho(u,v)|d\sigma_\kappa(u)d\sigma_\eta(v),\end{aligned}$$

which in turn implies that

$$\left|\hat{\Upsilon}_{t,r}(\xi,\zeta)\right| \leq C\left(\log|2^t\xi|\right)^{-(\alpha+1)} \text{ if } |2^t\xi| > 2^\alpha. \tag{9}$$

Similarly, we derive that

$$\left|\hat{\Upsilon}_{t,r}(\xi,\zeta)\right| \leq C(\log|2^r\zeta|)^{-(\alpha+1)} \text{ if } |2^r\zeta| > 2^\alpha. \tag{10}$$

Now, by the cancellation property (1), we have

$$\begin{aligned}\left|\hat{\Upsilon}_{t,r}(\xi,\zeta)\right| \leq\ & C \iint_{\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}}|\mho(u,v)|\int_{2^r}^{2^{r+1}}\int_1^2\left|e^{-il2^t\xi\cdot u}-1\right|\frac{dl d\tau}{l\tau}d\sigma_\kappa(u)d\sigma_\eta(v) \\ \leq\ & C|2^t\xi|.\end{aligned} \tag{11}$$

In the same manner, we obtain that

$$\left|\hat{\Upsilon}_{t,r}(\xi,\zeta)\right| \leq C|2^r\zeta|. \tag{12}$$

Therefore, by combining (9) with (11) we get (4), and by combining (10) with (12), we get (5). The lemma is proved. □

The following lemma can be found in [4] (see also [2,3,8]).

Lemma 2. *Let $\mho \in L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$. Then there exists a constant $C_p > 0$ such that*

$$\|\Upsilon^*(f)\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq C_p\|h\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}\|\mho\|_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})} \tag{13}$$

for all $1 < p < \infty$ and $h \in L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$.

Let $\mathcal{A} \in \mathcal{S}(\mathbb{R}^\kappa)$ and $\mathcal{B} \in \mathcal{S}(\mathbb{R}^\eta)$ be radial functions satisfying the following:
(1) $0 \leq \mathcal{A},\ \mathcal{B} \leq 1$,
(2) $supp\,(\mathcal{A}) \subset \left\{u : \frac{1}{2} \leq |u| \leq 2\right\}$, $supp\,(\mathcal{B}) \subset \left\{v : \frac{1}{2} \leq |v| \leq 2\right\}$,
(3) There is a constant $M > 0$ such that $\mathcal{A}(u),\mathcal{B}(v) \geq M$ for all $|u|,|v| \in [\frac{3}{5},\frac{5}{3}]$,
(4) $\int_{\mathbb{R}}\left|\widehat{\mathcal{A}}(2^t u)\right|^2 = 1$ with $u \neq 0$ and $\int_{\mathbb{R}}\left|\widehat{\mathcal{B}}(2^r v)\right|^2 = 1$ with $v \neq 0$.

For simplicity, we denote $\widehat{\mathcal{A}}(tu)$ by $\widehat{\mathcal{A}}_t(u)$ and $\widehat{\mathcal{B}}(rv)$ by $\widehat{\mathcal{B}}_r(v)$. Then it is clear that $\mathcal{A}_{2^t}(u) = 2^{-t\kappa}\mathcal{A}(u/2^t)$ and $\mathcal{B}_{2^r}(v) = 2^{-r\eta}\mathcal{B}(v/2^r)$. Let $\mathcal{W}_{2^t,2^r}(h)(u,v) = (\mathcal{A}_{2^t}\otimes\mathcal{B}_{2^r}) * h(u,v)$. Hence, for any $h \in \mathcal{S}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$, we have

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$$\|h\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \sim \left\|\left(\iint_{\mathbb{R}^+\times\mathbb{R}^+}|(\mathcal{A}_t\otimes\mathcal{B}_r)*h|^\varepsilon\frac{dtdr}{tr}\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}$$
$$\sim \left\|\left(\iint_{\mathbb{R}\times\mathbb{R}}\left|\mathcal{W}_{2^t,2^r}(h)\right|^\varepsilon dtdr\right)^{1/\varepsilon}\right\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}.$$

Let us give the following result regarding the boundedness of the measures $|\Upsilon_{t,r}|*|h|$ on $\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$.

Lemma 3. *Let $\mho\in L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$. Then, the estimate*

$$\||\Upsilon_{t,r}|*|h|\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \le C_p\|h\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}\|\mho\|_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})} \tag{14}$$

holds for all $1<p,\varepsilon<\infty$.

Proof. Let $h\in\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$. Then for any function $f\in\dot{F}_{p'}^{\varepsilon',\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ with $\|f\|_{\dot{F}_{p'}^{\varepsilon',\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}\le 1$, by Hölder's inequality we get

$$\begin{aligned}
&|\langle|\Upsilon_{t,r}|*|h|,f\rangle|\\
\le&\left|\iint_{\mathbb{R}^\kappa\times\mathbb{R}^\eta}\iint_{\mathbb{R}\times\mathbb{R}}|\Upsilon_{t,r}|*\mathcal{W}_{2^{t+n},2^{r+m}}(|h|)\mathcal{W}^*_{2^{t+n},2^{r+m}}(f)(u,v)dndmdudv\right|\\
\le&\left\|\left(\iint_{\mathbb{R}\times\mathbb{R}}\left||\Upsilon_{t,r}|*\mathcal{W}_{2^{t+n},2^{r+m}}(|h|)\right|^\varepsilon dndm\right)^{1/\varepsilon}\right\|_p\\
\times&\left\|\left(\iint_{\mathbb{R}\times\mathbb{R}}\left|\mathcal{W}^*_{2^{t+n},2^{r+m}}(f)\right|^{\varepsilon'}dndm\right)^{1/\varepsilon'}\right\|_{p'}
\end{aligned}$$

which in turn implies

$$\||\Upsilon_{t,r}|*|h|\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \le C\left\|\left(\iint_{\mathbb{R}\times\mathbb{R}}\left||\Upsilon_{t,r}|*\mathcal{W}_{2^{t+n},2^{r+m}}(|h|)\right|^\varepsilon dndm\right)^{1/\varepsilon}\right\|_p. \tag{15}$$

Let us now estimate the L^p-norm of $\left(\iint_{\mathbb{R}\times\mathbb{R}}\left||\Upsilon_{t,r}|*\mathcal{W}_{2^{t+n},2^{r+m}}(|h|)\right|^\varepsilon dndm\right)^{1/\varepsilon}$. Since $p>1$, by duality there exits a function $g\in L^{p'}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ such that $\|g\|_{L^{p'}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}=1$ and

$$\left\|\iint_{\mathbb{R}\times\mathbb{R}}\left||\Upsilon_{t,r}|*\mathcal{W}_{2^{t+n},2^{r+m}}(|h|)\right|dndm\right\|_p = \iint_{\mathbb{R}\times\mathbb{R}}\langle\left||\Upsilon_{t,r}|*\mathcal{W}_{2^{t+n},2^{r+m}}(|h|)\right|,g\rangle dndm$$

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$$
\begin{aligned}
&\leq \iint_{\mathbb{R}\times\mathbb{R}} \langle |\mathcal{W}_{2^{t+n},2^{r+m}}(|h|)(u,v)|, \Upsilon^*(\overline{g})(u,v)\rangle dndm \\
&\leq \left\| \iint_{\mathbb{R}\times\mathbb{R}} \mathcal{W}_{2^{t+n},2^{r+m}}(|h|)dndm \right\|_p \|\Upsilon^*(\overline{g})\|_{p'} \\
&\leq \left\| \iint_{\mathbb{R}\times\mathbb{R}} \mathcal{W}_{2^{t+n},2^{r+m}}(|h|)dndm \right\|_p \|\mho\|_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}, \qquad (16)
\end{aligned}
$$

where $\overline{g}(u,v) = g(-u,-v)$ and the last inequality is obtained by Lemma 2.

By following similar arguments as that employed in the proof of Lemma 2 in [4] we get

$$
\left\| \sup_{t,r\in\mathbb{R}} ||\Upsilon_{t,r}| * \mathcal{W}_{2^{t+n},2^{r+m}}(|h|)| \right\|_p \leq C_p \left\| \sup_{t,r\in\mathbb{R}} |\mathcal{W}_{2^{t+n},2^{r+m}}(|h|)| \right\|_p \|\mho\|_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}. \qquad (17)
$$

By interpolating between (16) and (17) we obtain that

$$
\left\| \left(\iint_{\mathbb{R}\times\mathbb{R}} ||\Upsilon_{t,r}| * \mathcal{W}_{2^{t+n},2^{r+m}}(|h|)|^{\varepsilon} dndm \right)^{1/\varepsilon} \right\|_p \leq \|\mho\|_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})} \|h\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}.
$$

Consequently, by the last inequality and (15), we get (14). □

3. Proof of Theorem 1

Let $\mho \in \mathbf{G}_\alpha(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$ for some $\alpha > 0$. By the translation invariance of $\mathbf{T}_\mho$, it is enough to prove the boundedness of $\mathbf{T}_\mho$ on $\dot{F}_p^{\varepsilon,\vec{\gamma}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ only whenever $\vec{\gamma} = \vec{0}$. It is clear that

$$
\begin{aligned}
\mathbf{T}_\mho h(x,y) &= \iint_{\mathbb{R}\times\mathbb{R}} \Upsilon_{t,r} * h(x,y)dtdr \\
&= \iint_{\mathbb{R}\times\mathbb{R}} \mathcal{Q}_{n,m}(h)dndm, \qquad (18)
\end{aligned}
$$

where

$$
\mathcal{Q}_{n,m}(h) = \iint_{\mathbb{R}\times\mathbb{R}} \mathcal{W}_{2^{t+n},2^{r+m}}(\Upsilon_{t,r} * \mathcal{W}_{2^{t+n},2^{r+m}}(h))dtdr.
$$

Let us estimate the $\|\mathcal{Q}_{n,m}\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}$. By following the same steps in proving (15), we get that

$$
\|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq C \left\| \left(\iint_{\mathbb{R}\times\mathbb{R}} |\Upsilon_{t,r} * \mathcal{W}_{2^{t+n},2^{r+m}}(h)|^{\varepsilon} dtdr \right)^{1/\varepsilon} \right\|_p. \qquad (19)
$$

We need now to consider three cases:

Case 1. $p = 2 = \varepsilon$. In this case, we have $\dot{F}_2^{2,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta) = L^2(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$. So, by invoking Plancherel's theorem, we obtain

$$
\|\mathcal{Q}_{n,m}(h)\|^2_{\dot{F}_2^{2,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}
$$

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$$\begin{aligned}
&\leq C\iint_{\mathbb{R}\times\mathbb{R}}\iint_{\mathbb{R}^\kappa\times\mathbb{R}^\eta}\left|\left(\widehat{\mathcal{A}}(2^{t+n}\xi)\otimes\widehat{\mathcal{B}}(2^{r+m}\zeta)\right)\widehat{\Upsilon}_{t,r}(\xi,\zeta)\hat{f}(\xi,\zeta)\right|^2 d\xi d\zeta drdt\\
&\leq C\iint_{\mathbb{R}\times\mathbb{R}}\iint_{\Delta_{t+n,r+m}}\left|\left(\widehat{\mathcal{A}}(2^{t+n}\xi)\otimes\widehat{\mathcal{B}}(2^{r+m}\zeta)\right)\widehat{\Upsilon}_{t,r}(\xi,\zeta)\hat{h}(\xi,\zeta)\right|^2 d\xi d\zeta drdt\\
&\leq C((1+|n|)(1+|m|))^{-\alpha-1}\|h\|_{L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}, \qquad (20)
\end{aligned}$$

where $\Delta_{t,r}=\left\{(\xi,\zeta)\in\mathbb{R}^\kappa\times\mathbb{R}^\eta:\frac{1}{2}\leq\mathcal{A}(2^t\xi)\leq 2\ and\ \frac{1}{2}\leq\mathcal{B}(2^r\zeta)\leq 2\right\}$.

Case 2. $p=\varepsilon$. By (15), we get that

$$\|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_p^{p,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \qquad (21)$$

$$\begin{aligned}
&\leq C\Big(\iint_{\mathbb{R}\times\mathbb{R}}\iint_{\mathbb{R}^\kappa\times\mathbb{R}^\eta}\Big(\iint_{\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}}\mathrm{M}_{u,v}(\mathcal{W}_{2^{t+n},2^{r+m}}(h)(u,v))\\
&\times\ |\mho(u,v)|\sigma_\kappa(u)d\sigma_\eta(v)\Big)^\varepsilon dxdydtdr\Big)^{1/\varepsilon}\\
&\leq C\Big(\iint_{\mathbb{R}\times\mathbb{R}}\Big(\iint_{\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1}}|\mho(u,v)|\\
&\times\ \left\|\mathrm{M}_{u,v}(\mathcal{W}_{2^{t+n},2^{r+m}}(h)\right\|_p\sigma_\kappa(u)d\sigma_\eta(v)\Big)^p dtdr\Big)^{1/p},
\end{aligned}$$

where

$$\mathrm{M}_{u,v}(h)(u,v)=\sup_{k_1k_2\in\mathbb{R}}\frac{1}{k_1,k_2}\int_0^{k_2}\int_0^{k_1}h(x-tu,y-rv)dtdr$$

which is bounded on $L^p(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ for $1<p<\infty$. Therefore,

$$\|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_p^{p,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}\leq C\|\mho\|_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}\|h\|_{\dot{F}_p^{p,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}. \qquad (22)$$

Case 3. $p>\varepsilon$. By duality, there is a non-negative function ϕ lies in the space $L^{(p/\varepsilon)'}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ such that $\|\phi\|_{L^{(p/\varepsilon)'}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}=1$ and

$$\|\mathcal{Q}_{n,m}(h)\|^\varepsilon_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}$$

$$\begin{aligned}
&\leq C\iint_{\mathbb{R}\times\mathbb{R}}\iint_{\mathbb{R}^\kappa\times\mathbb{R}^\eta}\left|\iint_{I_{t,r}}\frac{\mho(u',v')}{|u|^\kappa|v|^\eta}\mathcal{W}_{2^{t+n},2^{r+m}}(h)(x-u,y-v)dudv\right|^\varepsilon\phi(x,y)dxdy\,dtdr\\
&\leq C\|\mho\|^{\varepsilon/\varepsilon'}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}\iint_{\mathbb{R}\times\mathbb{R}}\iint_{\mathbb{R}^\kappa\times\mathbb{R}^\eta}\iint_{I_{t,r}}\frac{|\mho(u',v')|}{|u|^\kappa|v|^\eta}\\
&\times\ |\mathcal{W}_{2^{t+n},2^{r+m}}(h)(x-u,y-v)|^\varepsilon dudv\phi(x,y)dxdy\,dtdr\\
&\leq C\|\mho\|^{\varepsilon/\varepsilon'}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}\iint_{\mathbb{R}^\kappa\times\mathbb{R}^\eta}\mathrm{Y}^*(\overline{\phi})(x,y)\Big(\iint_{\mathbb{R}\times\mathbb{R}}|\mathcal{W}_{2^{t+n},2^{r+m}}(h)(x,y)|^\varepsilon dtdr\Big)dxdy\\
&\leq C\|\mho\|^{\varepsilon/\varepsilon'}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}\left\|\iint_{\mathbb{R}\times\mathbb{R}}|\mathcal{W}_{2^{t+n},2^{r+m}}(h)(x,y)|^\varepsilon dtdr\right\|_{(p/q)}\|\mathrm{Y}^*(\overline{\phi})\|_{(p/q)'}\\
&\leq C\|\mho\|^{\varepsilon/\varepsilon'+1}_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}\|h\|^\varepsilon_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}.
\end{aligned}$$

Thus, by the last inequality and (22), we obtain that

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$$\|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq C\|\mho\|_{L^1(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})}\|h\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}. \tag{23}$$

for all $p \geq \varepsilon$. Therefor, by employing the duality along with the interpolation, we conclude that the inequality (23) holds for all $1 < p < \infty$ and $1 < \varepsilon < \infty$, which is when interpolated with (20), we get

$$\|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq C\left((1+|n|)(1+|m|)\right)^{-\theta(\alpha+1)}\|h\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \tag{24}$$

for all $\theta \in (0,1)$, $\frac{\theta}{2} < \frac{1}{p} < 1 - \frac{\theta}{2}$ and $\frac{\theta}{2} < \frac{1}{\varepsilon} < 1 - \frac{\theta}{2}$. Since

$$\|\mathbf{T}_\mho(h)\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq C\iint_{\mathbb{R}\times\mathbb{R}}\|\mathcal{Q}_{n,m}(h)\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)}dndm,$$

then by invoking (24) and choosing $\theta > \frac{1}{\alpha+1}$, we end with

$$\|\mathbf{T}_\mho(h)\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \leq C\|h\|_{\dot{F}_p^{\varepsilon,\vec{0}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)} \tag{25}$$

for all $p, \varepsilon \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$.

4. Conclusions

In this work, we proved the boundedness of the singular integrals $\mathbf{T}_\mho$ on Triebel-Lizorkin spaces $\dot{F}_p^{\varepsilon,\vec{\gamma}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ for all $p, \varepsilon \in (\frac{2+2\alpha}{1+2\alpha}, 2+2\alpha)$ whenever the kernel function $\mho$ in $\mathbf{G}_\alpha(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$ for some $\alpha > 0$. The main result in this paper generalizes and improves the main results proved in [1,2,11]. In future work, we aim to prove the boundedness of $\mathbf{T}_\mho$ on $\dot{F}_p^{\varepsilon,\vec{\gamma}}(\mathbb{R}^\kappa\times\mathbb{R}^\eta)$ for a wider range of p provided that $\mho \in \mathbf{G}_\alpha(\mathbb{S}^{\kappa-1}\times\mathbb{S}^{\eta-1})$.

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