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Article

Energy of the Moving Particles in 3-Body Problem of Classical Electrodynamics

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Abstract: In previous papers we have derived equations of motion with radiation terms and proved the existence-uniqueness of periodic solution of the 3-body problem of classical electrodynamics. The number of equations of motion – 12 is more than the number of unknown functions – 9. It turns out that 9 equations are independent and the rest 3 ones are consequences of them. These three equations have a solution and they give the expressions for the energy of the moving particles. The main goal of the present paper is to estimate the energy of rounding about the nuclei particles and compare it with the values given by quantum mechanics.

Keywords: classical electrodynamics; 3-body problem; energy estimation of the rounding particles

1. Introduction

The main purpose of the present paper is to continue the investigation of the 3-body problem of classical electrodynamics in the 3D Kepler formulation from [1,2]. The number of equations of motion for 3 bodies in the Minkowski space is 12, while the unknown functions are 9. It is proved in [2] that the last three equations are consequences from the first 9 once. These three equations (two in the Kepler formulation) give estimations of the energy of the moving particles. Our main goal here is to calculate this energy and compare it with the values from quantum mechanics [3,4]. The results obtained refer to the helium atom where both electrons circle on the first (ground) stationary state.

The paper consists of six sections and References. Section 1 is an Introduction. In Section 2, a derivation of the explicit form of the equations of motion for the three-body problem is made. Every vector equation contains in the right-hand side the Lorentz force and a radiation term. The Lorentz force is a sum of two terms each of which shows the influence of the other two particles on the first one. In Section 3, the formalism of the transition to the Euclidean coordinates is described, introducing delays depending on the unknown trajectories. In Section 4, we obtain the radiation part of the energy terms in the energy equations. In Section 5 we obtain inequalities which allow us to estimate the energy of the moving particles. Section 6 is a Conclusion. We calculate the energy of the moving electrons rounding the nuclei. We establish that with the natural parameters of electron motion, the energies coincide with the values obtained from quantum mechanics [3,4].

The equations of motion for the 3-body problem introduced in [2] are:

$$m_1 \frac{d\lambda_r^{(1)}}{ds_1} = \frac{e_1}{c^2} \left(F_{rl}^{(12)} \lambda_l^{(1)} + F_{rl}^{(13)} \lambda_l^{(1)} + F_{rl}^{(1)rad} \lambda_l^{(1)} \right)$$

$$m_2 \frac{d\lambda_r^{(2)}}{ds_2} = \frac{e_2}{c^2} \left(F_{rl}^{(21)} \lambda_l^{(2)} + F_{rl}^{(23)} \lambda_l^{(2)} + F_{rl}^{(2)rad} \lambda_l^{(2)} \right) \quad (1)$$

$$m_3 \frac{d\lambda_r^{(3)}}{ds_3} = \frac{e_3}{c^2} \left(F_{rl}^{(31)} \lambda_l^{(3)} + F_{rl}^{(32)} \lambda_l^{(3)} + F_{rl}^{(3)rad} \lambda_l^{(3)} \right)$$

$$(r=1,2,3,4).$$

As usually Latin indices run from 1 to 4, while Greek – from 1 to 3.

Let us note the number of equations in (1) is 12, while the unknown functions (trajectories) are 9. It is proved in [2] that every 4-th equation is a consequence of the previous three ones. In this way one obtains that the independent equations in (1) are 9 in number.

2. Derivation the Explicit Form of the Energy Equation

We follow the procedure introduced in [2]. By $(x_1^{(k)}(t), x_2^{(k)}(t), x_3^{(k)}(t), x_4^{(k)} = ict)$ ($k=1,2,3$) we denote the space-time coordinates of the charged particles. The dot product in the Minkowski space is $\langle a, b \rangle_4 = a_r b_r = \sum_{r=1}^4 a_r b_r$, while in the 3-dimensional subspace — $\langle a, b \rangle = a_\alpha b_\alpha = \sum_{\alpha=1}^3 a_\alpha b_\alpha$; c is the vacuum speed of light; m_k ($k=1,2,3$) are the proper masses of the particles; e_k ($k=1,2,3$) — their charges. The elements of proper times are ds_k ($k=1,2,3$) and the unit tangent vectors to the world lines are $\lambda^{(k)} = (\lambda_1^{(k)}, \lambda_2^{(k)}, \lambda_3^{(k)}, \lambda_4^{(k)}) = \left(\frac{u_1^{(1)}(t)}{\Delta_k}, \frac{u_2^{(1)}(t)}{\Delta_k}, \frac{u_3^{(1)}(t)}{\Delta_k}, \frac{ic}{\Delta_k} \right)$, where $\vec{u}^{(k)} = (u_1^{(k)}(t), u_2^{(k)}(t), u_3^{(k)}(t)) = (\dot{x}_1^{(k)}(t), \dot{x}_2^{(k)}(t), \dot{x}_3^{(k)}(t))$, $\Delta_k = \sqrt{c^2 - \langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t) \rangle}$.

To compare denotations with originally accepted ones we recall that (cf. [5]):

$$\gamma_k = \frac{1}{\sqrt{1 - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle / c^2}}; \quad \frac{d}{ds_k} = \frac{\gamma_k}{c} \frac{d}{dt}; \quad \lambda_\alpha^{(k)} = \frac{\gamma_k u_\alpha^{(k)}(t)}{c} = \frac{u_\alpha^{(k)}(t)}{\Delta_k} (\alpha=1,2,3); \quad \lambda_4^{(k)} = i\gamma_k = \frac{ic}{\Delta_k}.$$

The components of accelerations are $\frac{d\lambda^{(k)}}{ds_k} = \left(\frac{\gamma_k}{c} \frac{d}{dt} \frac{\gamma_k \lambda_\alpha^{(k)}}{c}, i \frac{\gamma_k}{c} \frac{d\gamma_k}{dt} \right) = \left(\frac{1}{\Delta_k} \frac{d}{dt} \frac{u_\alpha^{(k)}(t)}{\Delta_k}, \frac{ic}{\Delta_k} \frac{d}{dt} \frac{1}{\Delta_k} \right).$

The isotropic vectors $\xi^{(kn)}$ ($k=1,2,3; n \neq k$) are obtained as in [2] (cf. [5,6]). Namely, fix any event on the world line of the k -th particle and draw the light cone into the past. This cone intersects the world line of the n -th particle in any other (past) event. Then

$$\xi^{(kn)} = (\xi_1^{(kn)}, \xi_2^{(kn)}, \xi_3^{(kn)}, \xi_4^{(kn)}) = (x_1^{(k)}(t) - x_1^{(n)}(t - \tau_{kn}), x_2^{(k)}(t) - x_2^{(n)}(t - \tau_{kn}), x_3^{(k)}(t) - x_3^{(n)}(t - \tau_{kn}), ic\tau_{kn}).$$

Since $\langle \xi^{(kn)}, \xi^{(kn)} \rangle_4 = 0$, the retarded functions $\tau_{kn}(t)$ should satisfy the following 6 functional equations

$$\tau_{kn}(t) = \frac{1}{c} \sqrt{\langle \xi^{(kn)}, \xi^{(kn)} \rangle_4} \equiv \frac{1}{c} \sqrt{\sum_{\alpha=1}^3 [x_\alpha^{(k)}(t) - x_\alpha^{(n)}(t - \tau_{kn}(t))]^2}.$$

For the velocities vectors we have

$$\lambda^{(n)} = \left(\frac{u_1^{(n)}(t - \tau_{kn})}{c}, \frac{u_2^{(n)}(t - \tau_{kn})}{c}, \frac{u_3^{(n)}(t - \tau_{kn})}{c}, \frac{ic}{\Delta_{kn}} \right) = \left(\frac{\vec{u}^{(n)}(t - \tau_{kn})}{c}, \frac{ic}{\Delta_{kn}} \right),$$

where $\Delta_{kn} = \sqrt{c^2 - \langle \vec{u}^{(n)}(t - \tau_{kn}), \vec{u}^{(n)}(t - \tau_{kn}) \rangle}$; $\frac{d}{ds_k} = \frac{1}{\Delta_k} \frac{d}{dt}$; $\frac{d}{ds_n} = \frac{1}{\Delta_{kn}} \frac{d}{dt_{kn}} = \frac{1}{\Delta_{kn}} \frac{dt}{dt_{kn}} \frac{d}{dt}$,

$$\frac{dt}{dt_{kn}} = \frac{c^2 \tau_{kn} - \langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \rangle}{c^2 \tau_{kn} - \langle \vec{\xi}^{(kn)}, \vec{u}^{(k)} \rangle} = D_{kn} .$$

To derive the explicit form of the radiation terms (cf. [7]), we consider a charge $e_k (k=1,2,3)$ describing any curve $L_k (k=1,2,3)$ in the space-time. Let $A_k(x_1^{(k)}(t), x_2^{(k)}(t), x_3^{(k)}(t), ict)$ be any event and let

$A_k^{ret}(x_1^{(k)}(\tilde{t}_k), x_2^{(k)}(\tilde{t}_k), x_3^{(k)}(\tilde{t}_k), ict_{\tilde{t}_k})$, $\tilde{t}_k < t$ be the intersection of $L_k (k=1,2,3)$ with the null-cone drawn into the past from $A_k (k=1,2,3)$, and $A_k^{adv}(x_1^{(k)}(\hat{t}_k), x_2^{(k)}(\hat{t}_k), x_3^{(k)}(\hat{t}_k), ict_{\hat{t}_k})$, $\hat{t}_k > t$ be the intersection of $L_k (k=1,2,3)$ with the null-cone drawn into the future from $A_k (k=1,2,3)$. Let $A_k^{ret} A_k$ and $A_k^{adv} A_k$ be the isotropic vectors

$$\xi^{(k)ret} = (\xi_1^{(k)ret}, \xi_2^{(k)ret}, \xi_3^{(k)ret}, \xi_4^{(k)ret}) , \quad \xi^{(k)adv} = (\xi_1^{(k)adv}, \xi_2^{(k)adv}, \xi_3^{(k)adv}, \xi_4^{(k)adv}) .$$

Then we set

$$\tau_k^{ret}(t) = t - \tilde{t}_k \Rightarrow \tilde{t}_k = t - \tau_k^{ret}(t) \quad \text{and} \quad \tau_k^{adv}(t) = \hat{t}_k - t \Rightarrow \hat{t}_k = t + \tau_k^{adv}(t) \quad \text{and obtain}$$

$$\xi^{(k)ret} = (\xi_1^{(k)ret}, \xi_2^{(k)ret}, \xi_3^{(k)ret}, ict_{\tau_k^{ret}}) = (\vec{\xi}^{(k)ret}, ict_{\tau_k^{ret}}) = (x_1^{(k)}(t) - x_1^{(k)}(t - \tau_k^{ret}), x_2^{(k)}(t) - x_2^{(k)}(t - \tau_k^{ret}), x_3^{(k)}(t) - x_3^{(k)}(t - \tau_k^{ret}), ict_{\tau_k^{ret}}) ;$$

$$\xi^{(k)adv} = (\xi_1^{(k)adv}, \xi_2^{(k)adv}, \xi_3^{(k)adv}, ict_{\tau_k^{adv}}) = (\vec{\xi}^{(k)adv}, ict_{\tau_k^{adv}}) =$$

$$= (x_1^{(k)}(t) - x_1^{(k)}(t + \tau_k^{adv}), x_2^{(k)}(t) - x_2^{(k)}(t + \tau_k^{adv}), x_3^{(k)}(t) - x_3^{(k)}(t + \tau_k^{adv}), ict_{\tau_k^{adv}}) .$$

The velocity vectors to the world-line $L_k (k=1,2,3)$ at A_k^{ret} and A_k^{adv} are:

$$\lambda^{(k)ret} = (\lambda_1^{(k)ret}, \lambda_2^{(k)ret}, \lambda_3^{(k)ret}, \lambda_4^{(k)ret}) = \left(\frac{u_1^{(k)}(\tilde{t}_k)}{\Delta_k^{ret}}, \frac{u_2^{(k)}(\tilde{t}_k)}{\Delta_k^{ret}}, \frac{u_3^{(k)}(\tilde{t}_k)}{\Delta_k^{ret}}, \frac{ic}{\Delta_k^{ret}} \right) = \left(\frac{\vec{u}^{(k)}(\tilde{t}_k)}{\Delta_k^{ret}}, \frac{ic}{\Delta_k^{ret}} \right) ;$$

$$\Delta_k^{ret} = \sqrt{c^2 - \langle \vec{u}^{(k)}(\tilde{t}_k), \vec{u}^{(k)}(\tilde{t}_k) \rangle} ;$$

$$\lambda^{(k)adv} = (\lambda_1^{(k)adv}, \lambda_2^{(k)adv}, \lambda_3^{(k)adv}, \lambda_4^{(k)adv}) = \left(\frac{u_1^{(k)}(\hat{t}_k)}{\Delta_k^{adv}}, \frac{u_2^{(k)}(\hat{t}_k)}{\Delta_k^{adv}}, \frac{u_3^{(k)}(\hat{t}_k)}{\Delta_k^{adv}}, \frac{ic}{\Delta_k^{adv}} \right) = \left(\frac{\vec{u}^{(k)}(\hat{t}_k)}{\Delta_k^{adv}}, \frac{ic}{\Delta_k^{adv}} \right) ;$$

$$\Delta_k^{adv} = \sqrt{c^2 - \langle \vec{u}^{(k)}(\hat{t}_k), \vec{u}^{(k)}(\hat{t}_k) \rangle} .$$

Following the Dirac physical assumptions we define the radiation term as a half of the difference between both retarded and advanced potentials (cf. [7])

$$F_{mn}^{(k)rad} = \frac{1}{2} \left[\left(\frac{\partial A_n^{(k)ret}}{\partial x_m^{(k)ret}} - \frac{\partial A_m^{(n)ret}}{\partial x_n^{(k)ret}} \right) - \left(\frac{\partial A_n^{(k)adv}}{\partial x_m^{(k)adv}} - \frac{\partial A_m^{(n)adv}}{\partial x_n^{(k)adv}} \right) \right] , \quad \text{where} \quad A_n^{(k)ret} = - \frac{e_k \lambda_n^{(k)ret}}{\langle \lambda^{(k)ret}, \xi^{(k)ret} \rangle_4} \quad \text{and}$$

$$A_n^{(k)adv} = - \frac{e_k \lambda_n^{(k)adv}}{\langle \lambda^{(k)adv}, \xi^{(k)adv} \rangle_4} .$$

Then the equations of motion become ($r=1,2,3,4$):

$$m_1 \frac{d\lambda_r^{(1)}}{ds_1} = \frac{e_1}{c^2} \left(F_{rl}^{(12)} \lambda_l^{(1)} + F_{rl}^{(13)} \lambda_l^{(1)} + \frac{1}{2} \left[\frac{\partial A_l^{(1)ret}}{\partial x_r^{(1)ret}} - \frac{\partial A_r^{(1)ret}}{\partial x_l^{(1)ret}} - \left(\frac{\partial A_l^{(1)adv}}{\partial x_r^{(1)adv}} - \frac{\partial A_r^{(1)adv}}{\partial x_l^{(1)adv}} \right) \right] \lambda_l^{(1)} \right) ,$$

$$m_2 \frac{d\lambda_r^{(2)}}{ds_2} = \frac{e_2}{c^2} \left(F_{rl}^{(21)} \lambda_l^{(2)} + F_{rl}^{(23)} \lambda_l^{(2)} + \frac{1}{2} \left[\frac{\partial A_l^{(2)ret}}{\partial x_r^{(2)ret}} - \frac{\partial A_r^{(2)ret}}{\partial x_l^{(2)ret}} - \left(\frac{\partial A_l^{(2)adv}}{\partial x_r^{(2)adv}} - \frac{\partial A_r^{(2)adv}}{\partial x_l^{(2)adv}} \right) \right] \lambda_l^{(2)} \right) ,$$

$$m_3 \frac{d\lambda_r^{(3)}}{ds_3} = \frac{e_3}{c^2} \left(F_{rl}^{(31)} \lambda_l^{(3)} + F_{rl}^{(32)} \lambda_l^{(3)} + \frac{1}{2} \left[\frac{\partial A_l^{(3)ret}}{\partial x_r^{(3)ret}} - \frac{\partial A_r^{(3)ret}}{\partial x_l^{(3)ret}} - \left(\frac{\partial A_l^{(3)adv}}{\partial x_r^{(3)adv}} - \frac{\partial A_r^{(3)adv}}{\partial x_l^{(3)adv}} \right) \right] \lambda_l^{(3)} \right).$$

In the present paper we investigate only the fourth, eight and twelfth equations ($k=1,2,3$):

$$\begin{aligned} \frac{d\lambda_4^{(k)}}{ds_k} = & \sum_{n=1, n \neq k}^3 \frac{e_k e_n}{m_k c^2} \left\{ \frac{\xi_4^{(kn)} \langle \lambda^{(k)}, \lambda^{(n)} \rangle_4 - \lambda_4^{(n)} \langle \lambda^{(k)}, \xi^{(kn)} \rangle_4}{\langle \lambda^{(n)}, \xi^{(kn)} \rangle_4^3} \left(1 + \left\langle \frac{d\lambda^{(n)}}{ds_n}, \xi^{(kn)} \right\rangle_4 \right) + \right. \\ & \left. + \frac{1}{\langle \lambda^{(n)}, \xi^{(kn)} \rangle_4^2} \left[\frac{d\lambda_4^{(n)}}{ds_n} \langle \lambda^{(k)}, \xi^{(kn)} \rangle_4 - \xi_4^{(kn)} \left\langle \lambda^{(k)}, \frac{d\lambda^{(n)}}{ds_n} \right\rangle_4 \right] \right\} + \\ & + \frac{e_k^2}{2m_k c^2} \left[\frac{\xi_4^{(k)ret} \langle \lambda^{(k)}, \lambda^{(k)ret} \rangle_4 - \lambda_4^{(k)ret} \langle \xi^{(k)ret}, \lambda^{(k)} \rangle_4}{\langle \lambda^{(k)ret}, \xi^{(k)ret} \rangle_4^3} \left(1 + \left\langle \xi^{(k)ret}, \frac{d\lambda^{(k)ret}}{ds_{ret}} \right\rangle_4 \right) + \right. \\ & \left. + \frac{1}{\langle \lambda^{(k)ret}, \xi^{(k)ret} \rangle_4^2} \left(\langle \xi^{(k)ret}, \lambda^{(k)} \rangle_4 \frac{d\lambda_4^{(k)ret}}{ds_{ret}} - \left\langle \lambda^{(k)}, \frac{d\lambda^{(k)ret}}{ds_{ret}} \right\rangle_4 \xi_4^{(k)ret} \right) \right] - \\ & - \frac{e_k^2}{2m_k c^2} \left[\frac{\xi_4^{(k)adv} \langle \lambda^{(k)}, \lambda^{(k)adv} \rangle_4 - \lambda_4^{(k)adv} \langle \xi^{(k)adv}, \lambda^{(k)} \rangle_4}{\langle \lambda^{(k)adv}, \xi^{(k)adv} \rangle_4^3} \left(1 + \left\langle \xi^{(k)adv}, \frac{d\lambda^{(k)adv}}{ds_{adv}} \right\rangle_4 \right) + \right. \\ & \left. + \frac{1}{\langle \lambda^{(k)adv}, \xi^{(k)adv} \rangle_4^2} \left(\langle \xi^{(k)adv}, \lambda^{(k)} \rangle_4 \frac{d\lambda_4^{(k)adv}}{ds_{adv}} - \left\langle \lambda^{(k)}, \frac{d\lambda^{(k)adv}}{ds_{adv}} \right\rangle_4 \xi_4^{(k)adv} \right) \right]. \end{aligned}$$

3. Transition to the Euclidean Coordinates

Following reasoning from [7] the relations $\langle \xi^{(k)ret}, \xi^{(k)ret} \rangle_4 = 0$, $\langle \xi^{(k)adv}, \xi^{(k)adv} \rangle_4 = 0$ yield

$$\tau_k^{ret} = \frac{1}{c} \sqrt{\langle \vec{\xi}^{(k)ret}, \vec{\xi}^{(k)ret} \rangle}, \tau_k^{adv} = \frac{1}{c} \sqrt{\langle \vec{\xi}^{(k)adv}, \vec{\xi}^{(k)adv} \rangle}.$$

For the retarded part of radiation term we differentiate $t - \tilde{t}_k = \frac{1}{c} \sqrt{\sum_{\gamma=1}^3 [x_\gamma^{(k)}(t) - x_\gamma^{(k)}(\tilde{t}_k)]^2}$ with

respect to $\tilde{t}_k$ and obtain $D_k^{ret} = \frac{dt}{d\tilde{t}_k} = \frac{c^2 \tau_k^{ret} - \langle \vec{\xi}^{(k)ret}, \vec{u}^{(k)}(t - \tau_k^{ret}) \rangle}{c^2 \tau_k^{ret} - \langle \vec{\xi}^{(k)ret}, \vec{u}^{(k)}(t) \rangle}$ and in a similar way for

advanced term we obtain $D_k^{adv} = \frac{dt}{d\tilde{t}_k} = \frac{c^2 \tau_k^{adv} - \langle \vec{\xi}^{(k)adv}, \vec{u}^{(k)}(t + \tau_k^{adv}) \rangle}{c^2 \tau_k^{adv} - \langle \vec{\xi}^{(k)adv}, \vec{u}^{(k)}(t) \rangle}$.

Further on,

$$\frac{d\lambda_\alpha^{(k)}}{ds_k} = \frac{1}{\Delta_k} \frac{d}{dt} \left(\frac{u_\alpha^{(k)}(t)}{\Delta_k} \right) = \frac{\dot{u}_\alpha^{(k)}(t)}{\Delta_k^2} + \frac{u_\alpha^{(k)}(t) \langle \vec{u}^{(k)}(t), \dot{\vec{u}}^{(k)}(t) \rangle}{\Delta_k^4}; \quad \frac{d\lambda_4^{(k)}}{ds_k} = \frac{ic}{\Delta_k} \frac{d}{dt} \left(\frac{1}{\Delta_k} \right) = \frac{ic \langle \vec{u}^{(k)}(t), \dot{\vec{u}}^{(k)}(t) \rangle}{\Delta_k^4};$$

$$\begin{aligned}
\frac{d\lambda_{\alpha}^{(n)}}{ds_n} &= D_{kn} \left(\frac{\dot{u}_{\alpha}^{(n)}(t - \tau_{kn})}{\Delta_{kn}^2} + \frac{u_{\alpha}^{(n)}(t - \tau_{kn}) \langle \vec{u}^{(n)}(t - \tau_{kn}), \dot{\vec{u}}^{(n)}(t - \tau_{kn}) \rangle}{\Delta_{kn}^4} \right); \\
\frac{d\lambda_4^{(n)}}{ds_n} &= \frac{icD_{kn} \langle \vec{u}^{(n)}(t - \tau_{kn}), \dot{\vec{u}}^{(n)}(t - \tau_{kn}) \rangle}{\Delta_{kn}^4}; \\
\langle \lambda^{(k)}, \lambda^{(n)} \rangle_4 &= \frac{\langle \vec{u}^{(k)}(t), \vec{u}^{(n)}(t - \tau_{kn}) \rangle - c^2}{\Delta_k \Delta_{kn}}; \quad \langle \lambda^{(k)}, \xi^{(kn)} \rangle_4 = \frac{\langle \vec{u}^{(k)}(t), \vec{\xi}^{(kn)} \rangle - c^2 \tau_{kn}}{\Delta_k}; \\
\langle \lambda^{(n)}, \xi^{(kn)} \rangle_4 &= \frac{\langle \vec{u}^{(n)}(t - \tau_{kn}), \vec{\xi}^{(kn)} \rangle - c^2 \tau_{kn}}{\Delta_{kn}}; \\
\left\langle \xi^{(kn)}, \frac{d\lambda^{(n)}}{ds_n} \right\rangle_4 &= D_{kn} \left(\frac{\langle \vec{\xi}^{(kn)}, \dot{\vec{u}}^{(n)}(t - \tau_{kn}) \rangle}{\Delta_{kn}^2} + \frac{\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)}(t - \tau_{kn}) \rangle - c^2 \tau_{kn}}{\Delta_{kn}^4} \langle \vec{u}^{(n)}(t - \tau_{kn}), \dot{\vec{u}}^{(n)}(t - \tau_{kn}) \rangle \right); \\
\left\langle \lambda^{(k)}, \frac{d\lambda^{(n)}}{ds_n} \right\rangle_4 &= \frac{D_{kn}}{\Delta_k} \left(\frac{\langle \vec{u}^{(k)}(t), \dot{\vec{u}}^{(n)}(t - \tau_{kn}) \rangle}{\Delta_{kn}^2} + \frac{\langle \vec{u}^{(k)}(t), \vec{u}^{(n)}(t - \tau_{kn}) \rangle - c^2 \tau_{kn}}{\Delta_{kn}^4} \langle \vec{u}^{(n)}(t - \tau_{kn}), \dot{\vec{u}}^{(n)}(t - \tau_{kn}) \rangle \right); \\
\langle \lambda^{(k)}, \lambda^{(k)ret} \rangle_4 &= \frac{\langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t - \tau_k^{ret}) \rangle - c^2}{\Delta_k \Delta_k^{ret}}; \quad \langle \lambda^{(k)}, \lambda^{(k)adv} \rangle_4 = \frac{\langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t + \tau_k^{adv}) \rangle - c^2}{\Delta_k \Delta_k^{adv}}; \\
\langle \xi^{(k)ret}, \lambda^{(k)} \rangle_4 &= \frac{\langle \vec{u}^{(k)}(t), \vec{\xi}^{(k)ret} \rangle - c^2 \tau_k^{ret}}{\Delta_k}; \quad \langle \xi^{(k)adv}, \lambda^{(k)} \rangle_4 = \frac{\langle \vec{u}^{(k)}(t), \vec{\xi}^{(k)adv} \rangle - c^2 \tau_k^{adv}}{\Delta_k}; \\
\langle \xi^{(k)ret}, \lambda^{(k)ret} \rangle_4 &= \frac{\langle \vec{u}^{(k)ret}(t - \tau_k^{ret}), \vec{\xi}^{(k)ret} \rangle - c^2 \tau_k^{ret}}{\Delta_k^{ret}}; \quad \langle \xi^{(k)adv}, \lambda^{(k)adv} \rangle_4 = \frac{\langle \vec{u}^{(k)adv}(t + \tau_k^{adv}), \vec{\xi}^{(k)adv} \rangle - c^2 \tau_k^{adv}}{\Delta_k^{adv}}; \\
\frac{d}{ds_{ret}} &= \frac{1}{\Delta_k^{ret}} \frac{d}{d\tilde{t}} = \frac{1}{\Delta_k^{ret}} \frac{dt}{d\tilde{t}_k} \frac{d}{dt} = \frac{D_k^{ret}}{\Delta_k^{ret}} \frac{d}{dt}; \quad \frac{d}{ds_{adv}} = \frac{1}{\Delta_k^{adv}} \frac{d}{d\tilde{t}} = \frac{1}{\Delta_k^{adv}} \frac{dt}{d\tilde{t}_k} \frac{d}{dt} = \frac{D_k^{adv}}{\Delta_k^{adv}} \frac{d}{dt}; \\
\frac{d\lambda_{\alpha}^{(k)ret}}{ds_{ret}} &= D_k^{ret} \left(\frac{\dot{u}_{\alpha}^{(k)}(t - \tau_k^{ret})}{(\Delta_k^{ret})^2} + \frac{u_{\alpha}^{(k)}(t - \tau_k^{ret}) \langle \vec{u}^{(k)}(t - \tau_k^{ret}), \dot{\vec{u}}^{(k)}(t - \tau_k^{ret}) \rangle}{(\Delta_k^{ret})^4} \right); \\
\frac{d\lambda_4^{(k)ret}}{ds_{ret}} &= \frac{icD_k^{ret} \langle \vec{u}^{(k)}(t - \tau_k^{ret}), \dot{\vec{u}}^{(k)}(t - \tau_k^{ret}) \rangle}{(\Delta_k^{ret})^4}; \\
\frac{d\lambda_{\alpha}^{(k)adv}}{ds_{adv}} &= D_k^{adv} \left(\frac{\dot{u}_{\alpha}^{(k)}(t + \tau_k^{adv})}{(\Delta_k^{adv})^2} + \frac{u_{\alpha}^{(k)}(t + \tau_k^{adv}) \langle \vec{u}^{(k)}(t + \tau_k^{adv}), \dot{\vec{u}}^{(k)}(t + \tau_k^{adv}) \rangle}{(\Delta_k^{adv})^4} \right);
\end{aligned}$$

$$\frac{d\lambda_4^{(k)adv}}{ds_{adv}} = \frac{icD_k^{adv} \langle \vec{u}^{(k)}(t + \tau_k^{adv}), \dot{\vec{u}}^{(k)}(t + \tau_k^{adv}) \rangle}{(\Delta_k^{adv})^4};$$

$$\left\langle \vec{\xi}^{(k)ret}, \frac{d\lambda^{(k)ret}}{ds_{ret}} \right\rangle_4 = D_k^{ret} \left(\frac{\langle \vec{\xi}^{(k)ret}, \dot{\vec{u}}^{(k)}(t - \tau_k^{ret}) \rangle}{(\Delta_k^{ret})^2} + \frac{\langle \vec{\xi}^{(k)ret}, \vec{u}^{(k)}(t - \tau_k^{ret}) \rangle - c^2 \tau_k^{ret}}{(\Delta_k^{ret})^4} \langle \vec{u}^{(k)}(t - \tau_k^{ret}), \dot{\vec{u}}^{(k)}(t - \tau_k^{ret}) \rangle \right);$$

$$\left\langle \lambda^{(k)}, \frac{d\lambda^{(k)ret}}{ds_{ret}} \right\rangle_4 = \frac{D_k^{ret}}{\Delta_k} \left(\frac{\langle \vec{u}^{(k)}(t), \dot{\vec{u}}^{(k)}(t - \tau_k^{ret}) \rangle}{(\Delta_k^{ret})^2} + \frac{\langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t - \tau_k^{ret}) \rangle - c^2 \tau_k^{ret}}{(\Delta_k^{ret})^4} \langle \vec{u}^{(k)}(t), \dot{\vec{u}}^{(k)}(t - \tau_k^{ret}) \rangle \right);$$

$$\left\langle \vec{\xi}^{(k)adv}, \frac{d\lambda^{(k)adv}}{ds_{adv}} \right\rangle_4 = D_k^{adv} \left(\frac{\langle \vec{\xi}^{(k)adv}, \dot{\vec{u}}^{(k)}(t + \tau_k^{adv}) \rangle}{(\Delta_k^{adv})^2} + \frac{\langle \vec{\xi}^{(k)adv}, \vec{u}^{(k)}(t + \tau_k^{adv}) \rangle - c^2 \tau_k^{adv}}{(\Delta_k^{adv})^4} \langle \vec{u}^{(k)}(t + \tau_k^{adv}), \dot{\vec{u}}^{(k)}(t + \tau_k^{adv}) \rangle \right);$$

$$\left\langle \lambda^{(k)}, \frac{d\lambda^{(k)adv}}{ds_{adv}} \right\rangle_4 = \frac{D_k^{adv}}{\Delta_k} \left(\frac{\langle \vec{u}^{(k)}(t), \dot{\vec{u}}^{(k)}(t + \tau_k^{adv}) \rangle}{(\Delta_k^{adv})^2} + \frac{\langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t + \tau_k^{adv}) \rangle - c^2 \tau_k^{adv}}{(\Delta_k^{adv})^4} \langle \vec{u}^{(k)}(t), \dot{\vec{u}}^{(k)}(t + \tau_k^{adv}) \rangle \right);$$

$$H_{kn} = 1 + D_{kn} \left(\frac{\langle \vec{\xi}^{(kn)}, \dot{\vec{u}}^{(n)} \rangle}{\Delta_{kn}^2} + \frac{\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \rangle - c^2 \tau_{kn}}{\Delta_{kn}^4} \langle \vec{u}^{(n)}, \dot{\vec{u}}^{(n)} \rangle \right), (k = 1, 2, 3), n \neq k;$$

$$H_k^{ret} = 1 + D_k^{ret} \left(\frac{\langle \vec{\xi}^{(k)ret}, \dot{\vec{u}}^{(k)ret} \rangle}{(\Delta_k^{ret})^2} + \frac{\langle \vec{\xi}^{(k)ret}, \vec{u}^{(k)ret} \rangle - c^2 \tau_k^{ret}}{(\Delta_k^{ret})^4} \langle \vec{u}^{(k)ret}, \dot{\vec{u}}^{(k)ret} \rangle \right)$$

$$H_k^{adv} = 1 + D_k^{adv} \left(\frac{\langle \vec{\xi}^{(k)adv}, \dot{\vec{u}}^{(k)adv} \rangle}{(\Delta_k^{adv})^2} + \frac{\langle \vec{\xi}^{(k)adv}, \vec{u}^{(k)adv} \rangle - c^2 \tau_k^{adv}}{(\Delta_k^{adv})^4} \langle \vec{u}^{(k)adv}, \dot{\vec{u}}^{(k)adv} \rangle \right).$$

Then the energy equations become ($k = 1, 2, 3$):

$$\frac{\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)} \rangle}{\Delta_k^4} = \sum_{n=1, n \neq k}^3 \frac{e_k e_n}{m_k c^2} \left[\frac{\left(c^2 - \langle \vec{u}^{(k)}, \vec{u}^{(n)} \rangle \right) \tau_{kn} - \left(c^2 \tau_{kn} - \langle \vec{\xi}^{(kn)}, \vec{u}^{(k)} \rangle \right)}{\Delta_k \Delta_{kn} \left(\frac{\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \rangle - c^2 \tau_{kn}}{\Delta_{kn}} \right)^3} H_{kn} + \right. \quad (2)$$

$$\left. + \Delta_{kn}^2 \left[\frac{\langle \vec{\xi}^{(kn)}, \vec{u}^{(k)} \rangle - c^2 \tau_{kn}}{\Delta_k} \frac{D_{kn}}{\Delta_{kn}^4} \langle \vec{u}^{(n)}, \dot{\vec{u}}^{(n)} \rangle - \frac{\tau_{kn} D_{kn}}{\Delta_k} \left(\frac{\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(n)} \rangle}{\Delta_{kn}^2} + \frac{\langle \vec{u}^{(k)}, \vec{u}^{(n)} \rangle - c^2}{\Delta_{kn}^4} \langle \vec{u}^{(n)}, \dot{\vec{u}}^{(n)} \rangle \right) \right] \right]$$

$$\begin{aligned}
& + \frac{e_k^2}{2m_k c^2} \left\{ \frac{\tau_k^{ret} \left(\frac{\langle \vec{u}^{(k)}, \vec{u}^{(k)ret} \rangle - c^2}{\Delta_k \Delta_{(k)ret}} - \frac{\langle \vec{u}^{(k)}, \vec{\xi}^{(k)ret} \rangle - c^2 \tau_k^{ret}}{\Delta_{(k)ret} \Delta_k} \right)}{\left(\frac{\langle \vec{u}^{(k)ret}, \vec{\xi}^{(k)ret} \rangle - c^2 \tau_k^{ret}}{\Delta_{(k)ret}} \right)^3} H_k^{ret} + \right. \\
& + \Delta_{(k)ret}^2 \left[\frac{\left(\frac{\langle \vec{\xi}^{(k)ret}, \vec{u}^{(k)} \rangle - c^2 \tau_k^{ret}}{\Delta_k} \frac{D_k^{ret} \langle \vec{u}^{(k)ret}, \dot{\vec{u}}^{(k)ret} \rangle}{\Delta_{(k)ret}^4} \right)}{\left(c^2 \tau_k^{ret} - \langle \vec{\xi}^{(k)ret}, \vec{u}^{(k)ret} \rangle \right)^2} - \frac{\tau_k^{ret} D_k^{ret}}{\Delta_k} \left(\frac{\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)ret} \rangle}{\Delta_{(k)ret}^2} + \frac{\langle \vec{u}^{(k)}, \vec{u}^{(k)ret} \rangle - c^2}{\Delta_{(k)ret}^4} \langle \vec{u}^{(k)ret}, \dot{\vec{u}}^{(k)ret} \rangle \right)}{\left(c^2 \tau_k^{ret} - \langle \vec{\xi}^{(k)ret}, \vec{u}^{(k)ret} \rangle \right)^2} \right] \left. \right\} - \\
& - \frac{e_k^2}{2m_k c^2} \left\{ \frac{\tau_k^{adv} \left(\frac{\langle \vec{u}^{(k)}, \vec{u}^{(k)adv} \rangle - c^2}{\Delta_k \Delta_{(k)adv}} - \frac{\langle \vec{u}^{(k)}, \vec{\xi}^{(k)adv} \rangle - c^2 \tau_k^{adv}}{\Delta_{(k)adv} \Delta_k} \right)}{\left(\frac{\langle \vec{u}^{(k)adv}, \vec{\xi}^{(k)adv} \rangle - c^2 \tau_k^{adv}}{\Delta_{(k)adv}} \right)^3} H_k^{adv} + \right. \\
& + \Delta_{(k)adv}^2 \left[\frac{\left(\frac{\langle \vec{\xi}^{(k)adv}, \vec{u}^{(k)} \rangle - c^2 \tau_k^{adv}}{\Delta_k} \frac{D_k^{adv} \langle \vec{u}^{(k)adv}, \dot{\vec{u}}^{(k)adv} \rangle}{\Delta_{(k)adv}^4} \right)}{\left(c^2 \tau_k^{adv} - \langle \vec{\xi}^{(k)adv}, \vec{u}^{(k)adv} \rangle \right)^2} - \frac{\tau_k^{adv} D_k^{adv}}{\Delta_k} \left(\frac{\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)adv} \rangle}{\Delta_{(k)adv}^2} + \frac{\langle \vec{u}^{(k)}, \vec{u}^{(k)adv} \rangle - c^2}{\Delta_{(k)adv}^4} \langle \vec{u}^{(k)adv}, \dot{\vec{u}}^{(k)adv} \rangle \right)}{\left(c^2 \tau_k^{adv} - \langle \vec{\xi}^{(k)adv}, \vec{u}^{(k)adv} \rangle \right)^2} \right] \left. \right\}
\end{aligned}$$

or

$$\frac{m_k \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)} \rangle}{\Delta_k^3} = \quad (3)$$

$$\begin{aligned}
& = \sum_{n=1, n \neq k}^3 \frac{e_k e_n}{c^2} \left\{ \frac{\langle \vec{\xi}^{(kn)}, \vec{u}^{(k)} \rangle - \tau_{kn} \langle \vec{u}^{(k)}, \vec{u}^{(n)} \rangle}{\left(\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \rangle - c^2 \tau_{kn} \right)^3} \Delta_{kn}^2 H_{kn} + \frac{\left(\langle \vec{\xi}^{(kn)}, \vec{u}^{(k)} \rangle - c^2 \tau_{kn} \right) D_{kn} \langle \vec{u}^{(n)}, \dot{\vec{u}}^{(n)} \rangle}{\Delta_{kn}^2 \left(c^2 \tau_{kn} - \langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \rangle \right)^2} - \tau_{kn} D_{kn} \frac{\Delta_{kn}^2 \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(n)} \rangle + \left(\langle \vec{u}^{(k)}, \vec{u}^{(n)} \rangle - c^2 \right) \langle \vec{u}^{(n)}, \dot{\vec{u}}^{(n)} \rangle}{\Delta_{kn}^2 \left(c^2 \tau_{kn} - \langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \rangle \right)^2} \right\} + \\
& + \frac{e_k^2}{2c^2} \left\{ \frac{\langle \vec{u}^{(k)}, \vec{\xi}^{(k)ret} \rangle - \tau_k^{ret} \langle \vec{u}^{(k)}, \vec{u}^{(k)ret} \rangle}{\left(\langle \vec{u}^{(k)ret}, \vec{\xi}^{(k)ret} \rangle - c^2 \tau_k^{ret} \right)^3} (\Delta_k^{ret})^2 H_k^{ret} + \frac{\left(\langle \vec{u}^{(k)}, \vec{\xi}^{(k)ret} \rangle - c^2 \tau_k^{ret} \right) D_k^{ret} \langle \vec{u}^{(k)ret}, \dot{\vec{u}}^{(k)ret} \rangle}{(\Delta_k^{ret})^2 \left(c^2 \tau_k^{ret} - \langle \vec{u}^{(k)ret}, \vec{\xi}^{(k)ret} \rangle \right)^2} - \tau_k^{ret} D_k^{ret} \frac{(\Delta_k^{ret})^2 \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)ret} \rangle + \left(\langle \vec{u}^{(k)}, \vec{u}^{(k)ret} \rangle - c^2 \right) \langle \vec{u}^{(k)ret}, \dot{\vec{u}}^{(k)ret} \rangle}{(\Delta_k^{ret})^2 \left(c^2 \tau_k^{ret} - \langle \vec{u}^{(k)ret}, \vec{\xi}^{(k)ret} \rangle \right)^2} \right\} - \\
& - \frac{e_k^2}{2c^2} \left\{ \frac{\langle \vec{u}^{(k)}, \vec{\xi}^{(k)adv} \rangle - \tau_k^{adv} \langle \vec{u}^{(k)}, \vec{u}^{(k)adv} \rangle}{\left(\langle \vec{u}^{(k)adv}, \vec{\xi}^{(k)adv} \rangle - c^2 \tau_k^{adv} \right)^3} (\Delta_k^{adv})^2 H_k^{adv} + \frac{\left(\langle \vec{u}^{(k)}, \vec{\xi}^{(k)adv} \rangle - c^2 \tau_k^{adv} \right) D_k^{adv} \langle \vec{u}^{(k)adv}, \dot{\vec{u}}^{(k)adv} \rangle}{(\Delta_k^{adv})^2 \left(c^2 \tau_k^{adv} - \langle \vec{u}^{(k)adv}, \vec{\xi}^{(k)adv} \rangle \right)^2} - \tau_k^{adv} D_k^{adv} \frac{(\Delta_k^{adv})^2 \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)adv} \rangle + \left(\langle \vec{u}^{(k)}, \vec{u}^{(k)adv} \rangle - c^2 \right) \langle \vec{u}^{(k)adv}, \dot{\vec{u}}^{(k)adv} \rangle}{(\Delta_k^{adv})^2 \left(c^2 \tau_k^{adv} - \langle \vec{u}^{(k)adv}, \vec{\xi}^{(k)adv} \rangle \right)^2} \right\}.
\end{aligned}$$

4. Transformation of the Radiation Part of the Energy Terms

Here we follow the Dirac assumption $\tau_k^{ret} = \tau_k^{adv} = \tau$, τ is a small parameter. This assumption is motivated by the fact $\tau = \tau_0 \sqrt{1 - \beta^2}$, $\left(\tau_0 = r_e / c \approx 10^{-24} \text{ sec} \right)$. Since $u_\alpha^{(k)}(t)$ are infinitely smooth functions, using the Taylor expansions we have:

$$\xi_\alpha^{(k)adv} = x_\alpha^{(k)}(t + \tau) - x_\alpha^{(k)}(t) = \tau u_\alpha^{(k)}(t) + O(\tau^2) \Rightarrow \xi_\alpha^{(k)adv} \approx \tau u_\alpha^{(k)}(t); \quad \xi_\alpha^{(k)ret} = x_\alpha^{(k)}(t) - x_\alpha^{(k)}(t - \tau) \approx u_\alpha^{(k)} \tau;$$

$$u_{\alpha}^{(k)}(t+\tau) = u_{\alpha}^{(k)}(t) + O(\tau); \quad u_{\alpha}^{(k)}(t-\tau) = u_{\alpha}^{(k)}(t) - O(\tau); \quad u_{\alpha}^{(k)}(t)u_{\alpha}^{(k)}(t+\tau) = \left(u_{\alpha}^{(k)}(t)\right)^2 + O(\tau);$$

$$u_{\alpha}^{(k)}(t)u_{\alpha}^{(k)}(t-\tau) = \left(u_{\alpha}^{(k)}(t)\right)^2 - O(\tau);$$

$$\langle \vec{u}^{(k)}, \vec{u}^{(k)adv} \rangle = \sum_{\gamma=1}^3 u_{\gamma}^{(k)}(t)u_{\gamma}^{(k)}(t+\tau) \approx \sum_{\gamma=1}^3 u_{\gamma}^{(k)}(t)u_{\gamma}^{(k)}(t) = \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle;$$

$$\langle \vec{u}^{(k)adv}, \vec{u}^{(k)adv} \rangle \approx \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle; \quad \langle \vec{u}^{(k)ret}, \vec{u}^{(k)ret} \rangle \approx \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle;$$

$$c^2 \tau_k^{ret} - \langle \vec{\xi}^{(k)ret}, \vec{u}^{(k)ret} \rangle = c^2 \tau - \tau \langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t-\tau) \rangle \approx \tau \left(c^2 - \langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t) \rangle \right);$$

$$c^2 \tau_k^{adv} - \langle \vec{\xi}^{(k)adv}, \vec{u}^{(k)adv} \rangle = c^2 \tau - \tau \langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t+\tau) \rangle \approx \tau \left(c^2 - \langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t) \rangle \right);$$

$$D_{kn} = \frac{c^2 \tau_{kn} - \langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \rangle}{c^2 \tau_{kn} - \langle \vec{\xi}^{(kn)}, \vec{u}^{(k)} \rangle} \approx 1; \quad D_k^{ret} = \frac{c \tau_k^{ret} - \langle \vec{\xi}^{(k)ret}, \vec{u}^{(k)}(t-\tau_k^{ret}) \rangle}{c \tau_k^{ret} - \langle \vec{\xi}^{(k)ret}, \vec{u}^{(k)}(t) \rangle} \approx \frac{\tau \left(c^2 - \langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t) \rangle \right)}{\tau \left(c^2 - \langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t) \rangle \right)} = 1;$$

$$D_k^{adv} = \frac{c \tau_k^{adv} - \langle \vec{\xi}^{(k)adv}, \vec{u}^{(k)}(t+\tau_k^{adv}) \rangle}{c \tau_k^{adv} - \langle \vec{\xi}^{(k)adv}, \vec{u}^{(k)}(t) \rangle} \approx \frac{\tau \left(c^2 - \langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t) \rangle \right)}{\tau \left(c^2 - \langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t) \rangle \right)} = 1;$$

$$H_k^{ret} \approx 1 + \left(\frac{\langle \tau \vec{u}^{(k)}(t), \dot{\vec{u}}^{(k)}(t-\tau) \rangle}{c^2 - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle} - \frac{\tau \left(c^2 - \langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t) \rangle \right) \langle \vec{u}^{(k)}(t), \dot{\vec{u}}^{(k)}(t-\tau) \rangle}{\left(c^2 - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle \right)^2} \right) = 1;$$

$$H_k^{adv} = 1 + \left(\frac{\langle \tau \vec{u}^{(k)}(t), \dot{\vec{u}}^{(k)}(t+\tau) \rangle}{c^2 - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle} - \frac{\tau \left(c^2 - \langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t) \rangle \right) \langle \vec{u}^{(k)}(t), \dot{\vec{u}}^{(k)}(t+\tau) \rangle}{\left(c^2 - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle \right)^2} \right) = 1.$$

Then the radiation part becomes

$$\begin{aligned} R_{(k)}^{rad} &= \frac{e_k^2}{2c^2} \left\{ \frac{\langle \vec{u}^{(k)}, \vec{\xi}^{(k)ret} \rangle - \tau_k^{ret} \langle \vec{u}^{(k)}, \vec{u}^{(k)ret} \rangle}{\left(\langle \vec{u}^{(k)ret}, \vec{\xi}^{(k)ret} \rangle - c^2 \tau_k^{ret} \right)^3} (\Delta_k^{ret})^2 H_k^{ret} + \frac{\left(\langle \vec{u}^{(k)}, \vec{\xi}^{(k)ret} \rangle - c^2 \tau_k^{ret} \right) D_k^{ret} \langle \vec{u}^{(k)ret}, \dot{\vec{u}}^{(k)ret} \rangle}{(\Delta_k^{ret})^2 \left(c^2 \tau_k^{ret} - \langle \vec{u}^{(k)ret}, \vec{\xi}^{(k)ret} \rangle \right)^2} - \tau_k^{ret} D_k^{ret} \frac{(\Delta_k^{ret})^2 \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)ret} \rangle + \left(\langle \vec{u}^{(k)}, \vec{u}^{(k)ret} \rangle - c^2 \right) \langle \vec{u}^{(k)ret}, \dot{\vec{u}}^{(k)ret} \rangle}{\Delta_k^{ret} \left(c^2 \tau_k^{ret} - \langle \vec{u}^{(k)ret}, \vec{\xi}^{(k)ret} \rangle \right)^2} \right. \\ &\quad \left. - \frac{\langle \vec{u}^{(k)}, \vec{\xi}^{(k)adv} \rangle - \tau_k^{adv} \langle \vec{u}^{(k)}, \vec{u}^{(k)adv} \rangle}{\left(\langle \vec{u}^{(k)adv}, \vec{\xi}^{(k)adv} \rangle - c^2 \tau_k^{adv} \right)^3} (\Delta_k^{adv})^2 H_k^{adv} - \frac{\left(\langle \vec{u}^{(k)}, \vec{\xi}^{(k)adv} \rangle - c^2 \tau_k^{adv} \right) D_k^{adv} \langle \vec{u}^{(k)adv}, \dot{\vec{u}}^{(k)adv} \rangle}{(\Delta_k^{adv})^2 \left(c^2 \tau_k^{adv} - \langle \vec{u}^{(k)adv}, \vec{\xi}^{(k)adv} \rangle \right)^2} + \tau_k^{adv} D_k^{adv} \frac{(\Delta_k^{adv})^2 \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)adv} \rangle + \left(\langle \vec{u}^{(k)}, \vec{u}^{(k)adv} \rangle - c^2 \right) \langle \vec{u}^{(k)adv}, \dot{\vec{u}}^{(k)adv} \rangle}{(\Delta_k^{adv})^2 \left(c^2 \tau_k^{adv} - \langle \vec{u}^{(k)adv}, \vec{\xi}^{(k)adv} \rangle \right)^2} \right\} = \\ &= \frac{e_k^2}{2c^2} \left\{ \frac{\tau \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \tau \rangle}{\left(c^2 \tau - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \tau \rangle \right)^3} (\Delta_k^{ret})^2 + \frac{\left(\langle \vec{u}^{(k)}, \vec{u}^{(k)} \tau \rangle - c^2 \tau \right) \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)ret} \rangle}{(\Delta_k^{ret})^2 \left(c^2 \tau - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \tau \rangle \right)^2} - \tau \frac{(\Delta_k^{ret})^2 \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)ret} \rangle + \left(\langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle - c^2 \right) \langle \vec{u}^{(k)ret}, \dot{\vec{u}}^{(k)ret} \rangle}{(\Delta_k^{ret})^2 \left(c^2 \tau - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \tau \rangle \right)^2} \right. \\ &\quad \left. - \frac{\tau \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \tau \rangle}{\left(c^2 \tau - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \tau \rangle \right)^3} (\Delta_k^{adv})^2 - \frac{\left(\langle \vec{u}^{(k)}, \vec{u}^{(k)} \tau \rangle - c^2 \tau \right) \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)adv} \rangle}{(\Delta_k^{adv})^2 \left(c^2 \tau - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \tau \rangle \right)^2} + \tau \frac{(\Delta_k^{adv})^2 \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)adv} \rangle + \left(\langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle - c^2 \right) \langle \vec{u}^{(k)adv}, \dot{\vec{u}}^{(k)adv} \rangle}{(\Delta_k^{adv})^2 \left(c^2 \tau - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \tau \rangle \right)^2} \right\} = \\ &= \frac{e_k^2}{2c^2} \left\{ - \frac{\left(c^2 - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle \right) \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)ret} \rangle}{\Delta_k^2 \tau \left(c^2 - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle \right)^2} - \tau \frac{\Delta_k^2 \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)ret} \rangle - \left(c^2 - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle \right) \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)ret} \rangle}{\Delta_k^2 \tau^2 \left(c^2 - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle \right)^2} - \right. \\ &\quad \left. - \frac{\left(c^2 - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle \right) \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)adv} \rangle}{\Delta_k^2 \tau \left(c^2 - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle \right)^2} + \tau \frac{\Delta_k^2 \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)adv} \rangle - \left(c^2 - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle \right) \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)adv} \rangle}{\Delta_k^2 \tau^2 \left(c^2 - \langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle \right)^2} \right\} = \\ &= \frac{e_k^2}{2c^2 \Delta_k^4} \left\{ \frac{\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)adv} \rangle}{\tau} - \frac{\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)ret} \rangle}{\tau} + \frac{\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)adv} \rangle - \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)ret} \rangle}{\tau} - \frac{\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)ret} \rangle - \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)ret} \rangle}{\tau} \right\} = \frac{e_k^2}{c^2 \Delta_k^4} \left\langle \vec{u}^{(k)}, \frac{\dot{\vec{u}}^{(k)adv} - \dot{\vec{u}}^{(k)ret}}{2\tau} \right\rangle \approx \frac{e_k^2}{c^2 \Delta_k^4} \langle \vec{u}^{(k)}, \ddot{\vec{u}}^{(k)} \rangle. \end{aligned}$$

Besides

$$H_{21} = 1 + \frac{\langle \vec{\xi}^{(21)}, \dot{\vec{u}}^{(1)} \rangle}{\Delta_{21}^2} + \frac{\left(\langle \vec{\xi}^{(21)}, \vec{u}^{(1)} \rangle - c^2 \tau_{21} \right) \langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \rangle}{\Delta_{21}^4} = 1,$$

$$H_{23} = 1 + \frac{\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \rangle}{\Delta_{23}^2} + \frac{\left(\langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \rangle - c^2 \tau_{23} \right) \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \rangle}{\Delta_{23}^4}.$$

It is known that (cf. [5]):

$$E_k = m_k c^2 / \sqrt{1 - \frac{\langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle}{c^2}} = \frac{m_k c^3}{\Delta_k}, \quad \frac{dE_k}{dt} = -\frac{1}{2} \frac{m_k c^3 (-2 \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)} \rangle)}{\Delta_k^3} = \frac{m_k c^3 \langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)} \rangle}{\Delta_k^3}.$$

Then from (3) we obtain

$$\begin{aligned} \frac{dE_2}{dt} &= c e_2 e_1 \left[\frac{\tau_{21} \langle \vec{u}^{(2)}, \vec{u}^{(1)} \rangle - \langle \vec{\xi}^{(21)}, \vec{u}^{(2)} \rangle}{\left(c^2 \tau_{21} - \langle \vec{\xi}^{(21)}, \vec{u}^{(1)} \rangle \right)^3} \Delta_{21}^2 + \frac{\left(\langle \xi^{(21)}, \vec{u}^{(2)} \rangle - \tau_{21} \langle \vec{u}^{(2)}, \vec{u}^{(1)} \rangle \right) \langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \rangle - \tau_{21} \Delta_{21}^2 \langle \vec{u}^{(2)}, \dot{\vec{u}}^{(1)} \rangle}{\Delta_{21}^2 \left(c^2 \tau_{21} - \langle \vec{\xi}^{(21)}, \vec{u}^{(1)} \rangle \right)^2} \right] + \\ &+ c e_2 e_3 \left[\frac{\tau_{23} \langle \vec{u}^{(2)}, \vec{u}^{(3)} \rangle - \langle \vec{\xi}^{(23)}, \vec{u}^{(2)} \rangle}{\left(c^2 \tau_{23} - \langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \rangle \right)^3} \Delta_{23}^2 H_{23} + \frac{\left(\langle \xi^{(23)}, \vec{u}^{(2)} \rangle - \tau_{23} \langle \vec{u}^{(2)}, \vec{u}^{(3)} \rangle \right) \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \rangle - \tau_{23} \Delta_{23}^2 \langle \vec{u}^{(2)}, \dot{\vec{u}}^{(3)} \rangle}{\Delta_{23}^2 \left(c^2 \tau_{23} - \langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \rangle \right)^2} \right] + \\ &+ \frac{c e_2^2 \langle \vec{u}^{(2)}, \ddot{\vec{u}}^{(2)} \rangle}{\Delta_2^4}; \\ \frac{dE_3}{dt} &= c e_3 e_1 \left[\frac{\tau_{21} \langle \vec{u}^{(3)}, \vec{u}^{(1)} \rangle - \langle \vec{\xi}^{(31)}, \vec{u}^{(3)} \rangle}{\left(c^2 \tau_{31} - \langle \vec{\xi}^{(31)}, \vec{u}^{(1)} \rangle \right)^3} \Delta_{31}^2 + \frac{\left(\langle \xi^{(31)}, \vec{u}^{(3)} \rangle - \tau_{31} \langle \vec{u}^{(3)}, \vec{u}^{(1)} \rangle \right) \langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \rangle - \tau_{31} \Delta_{31}^2 \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(1)} \rangle}{\Delta_{31}^2 \left(c^2 \tau_{31} - \langle \vec{\xi}^{(31)}, \vec{u}^{(1)} \rangle \right)^2} \right] + \\ &+ c e_3 e_2 \left[\frac{\tau_{32} \langle \vec{u}^{(3)}, \vec{u}^{(2)} \rangle - \langle \vec{\xi}^{(32)}, \vec{u}^{(3)} \rangle}{\left(c^2 \tau_{32} - \langle \vec{\xi}^{(32)}, \vec{u}^{(3)} \rangle \right)^3} \Delta_{32}^2 H_{32} + \frac{\left(\langle \xi^{(32)}, \vec{u}^{(3)} \rangle - \tau_{32} \langle \vec{u}^{(3)}, \vec{u}^{(2)} \rangle \right) \langle \vec{u}^{(2)}, \dot{\vec{u}}^{(2)} \rangle - \tau_{32} \Delta_{32}^2 \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(2)} \rangle}{\Delta_{32}^2 \left(c^2 \tau_{32} - \langle \vec{\xi}^{(32)}, \vec{u}^{(3)} \rangle \right)^2} \right] + \\ &+ \frac{c e_3^2 \langle \vec{u}^{(3)}, \ddot{\vec{u}}^{(3)} \rangle}{\Delta_3^4}. \end{aligned}$$

In the Kepler formulation one obtains $\vec{u}^{(1)} = \vec{0}$ and $\Delta_{31} = \sqrt{c^2 - \langle \vec{u}(t - \tau_{31}), \vec{u}(t - \tau_{31}) \rangle} \approx c \sqrt{1 - \frac{\langle \vec{u}(t - \tau_{31}), \vec{u}(t - \tau_{31}) \rangle}{c^2}} \approx c$, because we consider the case $|\langle \vec{u}(t - \tau_{31}), \vec{u}(t - \tau_{31}) \rangle| \leq \bar{c}^2 < c^2$ and then $\beta = \bar{c} / c = 1/137$ - Sommerfeld fine structure constant (cf. [3]).

Further on,

$$\xi^{(21)} = (\xi_1^{(21)}, \xi_2^{(21)}, \xi_3^{(21)}, \xi_4^{(21)}) = (x_1^{(2)}(t), x_2^{(2)}(t), x_3^{(2)}(t), i c \tau_{21}),$$

$$\xi^{(23)} = (\xi_1^{(23)}, \xi_2^{(23)}, \xi_3^{(23)}, \xi_4^{(23)}) = (x_1^{(2)}(t) - x_1^{(3)}(t - \tau_{23}), x_2^{(2)}(t) - x_2^{(3)}(t - \tau_{23}), x_3^{(2)}(t) - x_3^{(3)}(t - \tau_{23}), i c \tau_{23}),$$

$$\xi^{(31)} = (\xi_1^{(31)}, \xi_2^{(31)}, \xi_3^{(31)}, \xi_4^{(31)}) = (x_1^{(3)}(t), x_2^{(3)}(t), x_3^{(3)}(t), i c \tau_{31}),$$

$$\xi^{(32)} = (\xi_1^{(32)}, \xi_2^{(32)}, \xi_3^{(32)}, \xi_4^{(32)}) = (x_1^{(3)}(t) - x_1^{(2)}(t - \tau_{32}), x_2^{(3)}(t) - x_2^{(2)}(t - \tau_{32}), x_3^{(3)}(t) - x_3^{(2)}(t - \tau_{32}), i c \tau_{32}),$$

$$\left| \langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \rangle \right| \leq \sqrt{\langle \vec{\xi}^{(23)}, \vec{\xi}^{(23)} \rangle} \sqrt{\langle \vec{u}^{(3)}, \vec{u}^{(3)} \rangle} = c \tau_{23} \bar{c},$$

$$\begin{aligned}
H_{23} &\approx 1 + \frac{\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \rangle}{c^2} + \frac{\left(\langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \rangle - c^2 \tau_{23} \right) \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \rangle}{c^4}, \\
H_{32} &\approx 1 + \frac{\langle \vec{\xi}^{(32)}, \dot{\vec{u}}^{(2)} \rangle}{c^2} + \frac{\left(\langle \vec{\xi}^{(32)}, \vec{u}^{(2)} \rangle - c^2 \tau_{32} \right) \langle \vec{u}^{(2)}, \dot{\vec{u}}^{(2)} \rangle}{c^4}, \\
\frac{dE_2}{dt} &= e_2 e_1 \frac{-\langle \vec{x}^{(2)}, \vec{u}^{(2)} \rangle}{c^3 \tau_{21}^3} + \\
&+ e_2 e_3 \left[\frac{\tau_{23} \langle \vec{u}^{(2)}, \vec{u}^{(3)} \rangle - \langle \vec{\xi}^{(23)}, \vec{u}^{(2)} \rangle}{\left(c^2 \tau_{23} - \langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \rangle \right)^3} c^3 \left(1 + \frac{\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \rangle}{c^2} + \frac{\left(\langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \rangle - c^2 \tau_{23} \right) \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \rangle}{c^4} \right) + \right. \\
&\left. + \frac{\left(\langle \vec{\xi}^{(23)}, \vec{u}^{(2)} \rangle - \tau_{23} \langle \vec{u}^{(2)}, \vec{u}^{(3)} \rangle \right) \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \rangle - \tau_{23} c^2 \langle \vec{u}^{(2)}, \dot{\vec{u}}^{(3)} \rangle}{c \left(c^2 \tau_{23} - \langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \rangle \right)^2} \right] + \frac{e_2^2 \langle \vec{u}^{(2)}, \ddot{\vec{u}}^{(2)} \rangle}{c^3}, \\
\frac{dE_3}{dt} &= e_3 e_1 \frac{-\langle \vec{x}^{(3)}, \vec{u}^{(3)} \rangle}{c^3 \tau_{31}^3} + e_3 e_2 \left[\frac{\tau_{32} \langle \vec{u}^{(3)}, \vec{u}^{(2)} \rangle - \langle \vec{\xi}^{(32)}, \vec{u}^{(3)} \rangle}{\left(c^2 \tau_{32} - \langle \vec{\xi}^{(32)}, \vec{u}^{(3)} \rangle \right)^3} c^3 \left(1 + \frac{\langle \vec{\xi}^{(32)}, \dot{\vec{u}}^{(2)} \rangle}{c^2} + \frac{\left(\langle \vec{\xi}^{(32)}, \vec{u}^{(2)} \rangle - c^2 \tau_{32} \right) \langle \vec{u}^{(2)}, \dot{\vec{u}}^{(2)} \rangle}{c^4} \right) + \right. \\
&\left. + \frac{\left(\langle \vec{\xi}^{(32)}, \vec{u}^{(3)} \rangle - \tau_{32} \langle \vec{u}^{(3)}, \vec{u}^{(2)} \rangle \right) \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \rangle - \tau_{32} c^2 \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(2)} \rangle}{c \left(c^2 \tau_{32} - \langle \vec{\xi}^{(32)}, \vec{u}^{(3)} \rangle \right)^2} \right] + \frac{e_3^2 \langle \vec{u}^{(3)}, \ddot{\vec{u}}^{(3)} \rangle}{c^3}.
\end{aligned}$$

5. Energy Inequalities

We recall that $\|\vec{u}^{(k)}\| = \sqrt{\langle \vec{u}^{(k)}, \vec{u}^{(k)} \rangle}$, $(k=2,3)$, $\rho_k(t) = \sqrt{\langle \vec{x}^{(k)}, \vec{x}^{(k)} \rangle}$, $(k=2,3)$,
 $c\tau_{kn} = \sqrt{\langle \vec{\xi}^{(kn)}, \vec{\xi}^{(kn)} \rangle}$.

Then

$$\begin{aligned}
|E_2(t)| &\leq |e_2 e_1| \int_0^t \frac{\rho_2 \|\vec{u}^{(2)}\|}{c^3 \tau_{21}^3} ds + \\
&+ e_2 e_3 \int_0^t \left[\frac{\tau_{23} \|\vec{u}^{(2)}\| \|\vec{u}^{(3)}\| + \sqrt{\langle \vec{\xi}^{(23)}, \vec{\xi}^{(23)} \rangle} \|\vec{u}^{(2)}\|}{\left(c^2 \tau_{23} - \langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \rangle \right)^3} c^3 \left(1 + \frac{\sqrt{\langle \vec{\xi}^{(23)}, \vec{\xi}^{(23)} \rangle} \|\vec{u}^{(3)}\|}{c^2} + \frac{\left(\sqrt{\langle \vec{\xi}^{(23)}, \vec{\xi}^{(23)} \rangle} \|\vec{u}^{(3)}\| + c^2 \tau_{23} \right) \|\vec{u}^{(3)}\| \|\dot{\vec{u}}^{(3)}\|}{c^4} \right) + \right. \\
&\left. + \frac{\left(\sqrt{\langle \vec{\xi}^{(23)}, \vec{\xi}^{(23)} \rangle} \|\vec{u}^{(2)}\| + \tau_{23} \|\vec{u}^{(2)}\| \|\vec{u}^{(3)}\| \right) \|\vec{u}^{(3)}\| \|\dot{\vec{u}}^{(3)}\| + \tau_{23} c^2 \|\vec{u}^{(2)}\| \|\dot{\vec{u}}^{(3)}\|}{c \left(c^2 \tau_{23} - \langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \rangle \right)^2} \right] ds + \int_0^t \frac{e_2^2 \|\vec{u}^{(2)}\| \sqrt{\langle \ddot{\vec{u}}^{(2)}, \ddot{\vec{u}}^{(2)} \rangle}}{c^3} ds \leq \\
&\leq |e_2 e_1| \int_0^t \frac{\rho_2 \bar{c}}{c^3 \tau_{21}^3} ds + |e_2 e_3| \int_0^t \left[\frac{\tau_{23} (\bar{c}^2 + c \bar{c})}{\left(c^2 \tau_{23} - c \tau_{23} \bar{c} \right)^3} c^3 \left(1 + \frac{c \tau_{23} \|\dot{\vec{u}}^{(3)}\|}{c^2} + \frac{\tau_{23} (c \bar{c} + c^2) \bar{c} \|\dot{\vec{u}}^{(3)}\|}{c^4} \right) + \right. \\
&\left. + \frac{(c \tau_{23} \bar{c} + \tau_{23} \bar{c}^2) \bar{c} \|\dot{\vec{u}}^{(3)}\| + \tau_{23} c^2 \bar{c} \|\dot{\vec{u}}^{(3)}\|}{c \left(c^2 \tau_{23} - c \tau_{23} \bar{c} \right)^2} \right] ds + \int_0^t \frac{e_2^2 \bar{c} \|\ddot{\vec{u}}^{(2)}\|}{c^3} ds \equiv \\
&\equiv \beta \int_0^t \left[|e_2 e_1| \frac{\rho_2}{c^2 \tau_{21}^3} + |e_2 e_3| \frac{(1+\beta)}{(1-\beta)^3} \left(\frac{1}{c \tau_{23}^2} + \frac{2 \|\dot{\vec{u}}^{(3)}\|}{c^2 \tau_{23}} \right) + \frac{e_2^2 \|\ddot{\vec{u}}^{(2)}\|}{c^2} \right] ds,
\end{aligned}$$

$$\begin{aligned}
|E_3(t)| &\leq |e_3 e_1| \int_0^t \frac{\rho_3 \bar{c}}{c^3 \tau_{31}^3} ds + |e_3 e_2| \int_0^t \left[\frac{\tau_{32} \bar{c}^2 + c \tau_{32} \bar{c}}{(c^2 \tau_{32} - c \tau_{32} \bar{c})^3} c^3 \left(1 + \frac{c^3 + c \bar{c}^2 + \bar{c} c^2}{c^4} \tau_{32} \|\dot{u}^{(2)}\| \right) \right. \\
&\quad \left. + \frac{(c \tau_{32} \bar{c} + \tau_{32} \bar{c}^2) \bar{c} \|\dot{u}^{(3)}\| + \tau_{32} c^2 \bar{c} \|\dot{u}^{(2)}\|}{c(c^2 \tau_{32} - c \tau_{32} \bar{c})^2} \right] ds + \int_0^t \frac{e_3^2 \bar{c} \|\ddot{u}^{(3)}\|}{c^3} ds \cong \\
&\cong \beta \int_0^t \left[|e_3 e_1| \frac{\rho_3}{c^2 \tau_{31}^3} + |e_2 e_3| \frac{(1+\beta)}{(1-\beta)^3} \left(\frac{1}{c \tau_{32}^2} + \frac{2 \|\dot{u}^{(2)}\|}{c^2 \tau_{32}} \right) + \frac{e_3^2 \|\ddot{u}^{(3)}\|}{c^2} \right] ds.
\end{aligned}$$

In the 3D-Kepler formulation the first particle P_1 is put at the origin $O(0,0,0)$, that is, P_1 $(x_1^{(1)}(t)=0, x_2^{(1)}(t)=0, x_3^{(1)}(t)=0)$, $t \in [0, \infty)$. Then $E_{kin}^{(1)}(t)=0$.

We pass to the spherical coordinates, that is, the particle $P_n (n=2,3)$ is located at the point

$$x_1^{(n)} = \rho_n \cos \varphi_n \cos \lambda_n; x_2^{(n)} = \rho_n \sin \varphi_n \cos \lambda_n; x_3^{(n)} = \rho_n \sin \lambda_n$$

where $\rho_n(\theta) \geq 0$; $\varphi_n(\theta) \geq 0$; $\lambda_n(\theta) \in \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right]$, $0 < \delta < \frac{\pi}{2}$ and then the velocities are:

$$\begin{aligned}
u_1^{(n)} &= \dot{\rho}_n \cos \varphi_n \cos \lambda_n - \rho_n \dot{\varphi}_n \sin \varphi_n \cos \lambda_n - \rho_n \dot{\lambda}_n \cos \varphi_n \sin \lambda_n \\
u_2^{(n)} &= \dot{\rho}_n \sin \varphi_n \cos \lambda_n + \rho_n \dot{\varphi}_n \cos \varphi_n \cos \lambda_n - \rho_n \dot{\lambda}_n \sin \varphi_n \sin \lambda_n \\
u_3^{(n)} &= \dot{\rho}_n \sin \lambda_n + \rho_n \dot{\lambda}_n \cos \lambda_n
\end{aligned}$$

where $\dot{\rho}^{(2)} = d\rho^{(2)} / dt$. For the accelerations we have (cf. [1,2]):

$$\begin{aligned}
\ddot{u}_2^{(n)} &\approx \ddot{\rho}_n \sin \varphi_n \cos \lambda_n + \rho_n \ddot{\varphi}_n \cos \varphi_n \cos \lambda_n - \rho_n \ddot{\lambda}_n \sin \varphi_n \sin \lambda_n \\
\ddot{u}_3^{(n)} &\approx \ddot{\rho}_n \sin \lambda_n + \rho_n \ddot{\lambda}_n \cos \lambda_n.
\end{aligned}$$

In the same manner we obtain the expressions for the second derivatives ($n=2,3$):

$$\begin{aligned}
\ddot{u}_1^{(n)} &\approx \ddot{\rho}_n \cos \varphi_n \cos \lambda_n - \ddot{\varphi}_n \rho_n \sin \varphi_n \cos \lambda_n - \ddot{\lambda}_n \rho_n \cos \varphi_n \sin \lambda_n \\
\ddot{u}_2^{(n)} &\approx \ddot{\rho}_n \sin \varphi_n \cos \lambda_n + \ddot{\varphi}_n \rho_n \cos \varphi_n \cos \lambda_n - \ddot{\lambda}_n \rho_n \sin \varphi_n \sin \lambda_n \\
\ddot{u}_3^{(n)} &\approx \ddot{\rho}_n \sin \lambda_n + \ddot{\lambda}_n \rho_n \cos \lambda_n.
\end{aligned}$$

Then $\rho_2 = \rho_2(t)$; $\varphi_2 = \varphi_2(t)$; $\rho_3 = \rho_3(t - \tau_{23})$; $\varphi_3 = \varphi_3(t - \tau_{23})$;

$$\tau_{21}(t) = \frac{1}{c} \sqrt{\sum_{\alpha=1}^3 [x_\alpha^{(2)}(t) - x_\alpha^{(1)}(t - \tau_{21}(t))]^2} = \frac{\rho_2(t)}{c}; \tau_{31}(t) = \frac{1}{c} \sqrt{\sum_{\alpha=1}^3 [x_\alpha^{(3)}(t) - x_\alpha^{(1)}(t - \tau_{31}(t))]^2} = \frac{\rho_3(t)}{c};$$

$$\rho_2(t) = \rho_{20} + \int_0^t r_2(s) ds \geq \rho_{20} - R_2 \frac{e^{\mu T} - 1}{\mu} \approx \rho_{20}; \rho_3(t) = \rho_{30} + \int_0^t r_3(s) ds \geq \rho_{30} - R_3 \frac{e^{\mu T} - 1}{\mu} \approx \rho_{30};$$

$$\vec{\xi}^{(23)} = (0, \rho_2 \cos \varphi_2 - \rho_3 \cos \varphi_3, \rho_2 \sin \varphi_2 - \rho_3 \sin \varphi_3);$$

$$\tau_{23}(t) = \frac{1}{c} \sqrt{\langle \vec{\xi}^{(23)}, \vec{\xi}^{(23)} \rangle} = \frac{1}{c} \sqrt{\sum_{\alpha=1}^3 [x_\alpha^{(2)}(t) - x_\alpha^{(3)}(t - \tau_{23}(t))]^2} = \frac{1}{c} \sqrt{\rho_2^2 + \rho_3^2 - 2\rho_2 \rho_3 \cos(\varphi_2 - \varphi_3)} \geq \frac{|\rho_2(t) - \rho_3(t - \tau_{23})|}{c};$$

$$\begin{aligned}
|\rho_2(t) - \rho_3(t - \tau_{23})| &= \left| \rho_{20} + \int_0^t r_2(s) ds - \rho_{30} - \int_0^{t-\tau_{23}} r_3(s) ds \right| = \left| \rho_{20} + \int_0^t r_2(s) ds - \rho_{30} - \int_0^t r_3(s) ds + \int_{t-\tau_{23}}^t r_3(s) ds \right| \geq \\
&\geq |\rho_2(t) - \rho_3(t)| - R_3 \frac{e^{\mu t} - e^{\mu(t-\tau_{23})}}{\mu} = |\rho_2(t) - \rho_3(t)| - R_3 e^{\mu t} \frac{1 - e^{-\mu \tau_{23}}}{\mu} \geq |\rho_2(t) - \rho_3(t)| - (R_3 e^{\mu T}) / \mu \approx |\rho_3(t) - \rho_2(t)|;
\end{aligned}$$

$$|\rho_2(t) - \rho_3(t)| \geq |\rho_{20} - \rho_{30}| - (R_2 + R_3) e^{\mu T} / \mu > 0 \approx |\rho_{20} - \rho_{30}|; \frac{1}{\tau_{23}(t)} \leq \frac{c}{|\rho_{20} - \rho_{30}|};$$

$$\begin{aligned}\tau_{32}(t) &= \frac{1}{c} \sqrt{\langle \tilde{\xi}^{(32)}, \tilde{\xi}^{(32)} \rangle} = \frac{1}{c} \sqrt{\rho_3^2 + \rho_2^2 - 2\rho_2\rho_3 \cos(\varphi_3 - \varphi_2)} \geq \frac{|\rho_3(t) - \rho_2(t - \tau_{32})|}{c}; \\ |\rho_3(t) - \rho_2(t - \tau_{32})| &= \left| \rho_{30} + \int_0^t r_3(s) ds - \rho_{20} - \int_0^{t-\tau_{32}} r_2(s) ds \right| = \left| \rho_{30} + \int_0^t r_3(s) ds - \rho_{20} - \int_0^t r_2(s) ds + \int_{t-\tau_{32}}^t r_2(s) ds \right| \geq \\ &\geq |\rho_3(t) - \rho_2(t)| - R_2 \frac{e^{\mu t} - e^{\mu(t-\tau_{32})}}{\mu} = |\rho_3(t) - \rho_2(t)| - R_2 e^{\mu t} \frac{1 - e^{-\mu \tau_{32}}}{\mu} \geq |\rho_3(t) - \rho_2(t)| - (R_2 e^{\mu T}) / \mu \approx |\rho_3(t) - \rho_2(t)|; \\ |\rho_3(t) - \rho_2(t)| &\geq |\rho_{30} - \rho_{20}| - (R_3 + R_2) e^{\mu T} / \mu > 0 \approx |\rho_{30} - \rho_{20}|; \frac{1}{\tau_{32}(t)} \leq \frac{c}{|\rho_{30} - \rho_{20}|}.\end{aligned}$$

Recall that $\sqrt{\langle \ddot{u}^{(n)}, \ddot{u}^{(n)} \rangle} \leq \bar{c}$ implies

$$\begin{aligned}\sqrt{\langle \ddot{u}^{(n)}, \ddot{u}^{(n)} \rangle} &= \sqrt{r_n^2 + \rho_n^2 \phi_n^2 + \rho_n^2 \eta_n^2} \leq e^{\mu T} \sqrt{R_n^2 + \rho_n^2 \Phi_n^2 + \rho_n^2 Y_n^2} \leq \bar{c} \quad \text{and} \\ \sqrt{\langle \dot{\ddot{u}}^{(n)}, \dot{\ddot{u}}^{(n)} \rangle} &= \sqrt{\ddot{\rho}_n^2 + \rho_n^2 \ddot{\phi}_n^2 \cos^2 \lambda_n + \rho_n^2 \ddot{\lambda}_n^2} \leq e^{\mu T} \omega_n \sqrt{R_n^2 + \rho_n^2 \Phi_n^2 + \rho_n^2 Y_n^2} \leq \omega_n \bar{c}; \\ \sqrt{\langle \ddot{\ddot{u}}^{(n)}, \ddot{\ddot{u}}^{(n)} \rangle} &= \sqrt{\ddot{\rho}_n^2 + \rho_n^2 \ddot{\phi}_n^2 \cos^2 \lambda_n + \rho_n^2 \ddot{\lambda}_n^2} \leq e^{\mu T} \omega_n^2 \sqrt{R_n^2 + \rho_n^2 \Phi_n^2 + \rho_n^2 Y_n^2} \leq \omega_n^2 \bar{c}.\end{aligned}$$

Since $\frac{(1+\beta)}{(1-\beta)^3} = 1.03$, then

$$\begin{aligned}|E_2(t)| &\leq \beta \int_0^t \left[|e_2 e_1| \frac{\rho_2}{c^2 \tau_{21}^3} + |e_2 e_3| \frac{(1+\beta)}{(1-\beta)^3} \left(\frac{1}{\tau_{23}^2 c} + \frac{2 \|\dot{\ddot{u}}^{(2)}\|}{c^2 \tau_{23}} \right) + \frac{e_2^2 \|\ddot{\ddot{u}}^{(2)}\|}{c^2} \right] ds \leq \\ &\leq \frac{|e_2 e_1|}{137} \int_0^t \left[\frac{c}{\rho_{20}^2} + 1.03 \left(\frac{c^2}{|\rho_2(t) - \rho_3(t)|^2} \frac{1}{c} + \frac{2c \|\dot{\ddot{u}}^{(3)}\|}{c^2 |\rho_2(t) - \rho_3(t)|} \right) + \frac{\|\ddot{\ddot{u}}^{(2)}\|}{c^2} \right] ds \leq \\ &\leq \frac{|e_2 e_1|}{137} \int_0^t \left[\frac{c}{\rho_{20}^2} + 1.03 \left(\frac{c}{|\rho_{20} - \rho_{30}|^2} + \frac{2\omega_3 \bar{c}}{c |\rho_{20} - \rho_{30}|} \right) + \frac{\omega_2^2 \bar{c}}{c^2} \right] ds \leq \\ &\leq \frac{|e_2 e_1|}{137} \left[\frac{c}{\rho_{20}^2} + 1.03 \left(\frac{c}{|\rho_{20} - \rho_{30}|^2} + \frac{2\omega_3 \bar{c}}{c |\rho_{20} - \rho_{30}|} \right) + \frac{\omega_2^2 \bar{c}}{c^2} \right] \int_0^t e^{\mu s} ds \leq \\ &\leq \frac{|e_2 e_1|}{137} \left[\frac{c}{\rho_{20}^2} + 1.03 \left(\frac{c}{|\rho_{20} - \rho_{30}|^2} + \frac{2\omega_3 \bar{c}}{c |\rho_{20} - \rho_{30}|} \right) + \frac{\omega_2^2 \bar{c}}{c^2} \right] \frac{e^{\mu T} - 1}{\mu}\end{aligned}$$

and

$$\begin{aligned}|E_3(t)| &\leq \beta \int_0^t \left[|e_3 e_1| \frac{\rho_3}{c^2 \tau_{31}^3} + |e_2 e_3| \frac{(1+\beta)}{(1-\beta)^3} \left(\frac{1}{\tau_{32}^2 c} + \frac{2 \|\dot{\ddot{u}}^{(2)}\|}{c^2 \tau_{32}} \right) + \frac{e_3^2 \|\ddot{\ddot{u}}^{(3)}\|}{c^2} \right] ds \leq \\ &\leq \frac{|e_2 e_1|}{137} \left[\frac{c}{\rho_{30}^2} + 1.03 \left(\frac{c}{|\rho_{30} - \rho_{20}|^2} + \frac{2\omega_2 \bar{c}}{c |\rho_{30} - \rho_{20}|} \right) + \frac{\omega_3^2 \bar{c}}{c^2} \right] \int_0^t e^{\mu s} ds \leq \\ &\leq \frac{|e_2 e_1|}{137} \left[\frac{c}{\rho_{30}^2} + 1.03 \left(\frac{c}{|\rho_{30} - \rho_{20}|^2} + \frac{2\omega_2 \bar{c}}{c |\rho_{30} - \rho_{20}|} \right) + \frac{\omega_3^2 \bar{c}}{c^2} \right] \frac{e^{\mu T} - 1}{\mu}.\end{aligned}$$

6. Conclusion - Numerical Verification of the Above Formulas

For the He-atom both electrons are on the first stationary state, that is, one can assume

$\omega_2 = \omega_3 = \omega = 4.1 \times 10^{16}$, $T = 2\pi / \omega = 1.53 \times 10^{-16}$. Let us choose $\mu = 4.12 \times 10^{16} > \omega = 4.1 \times 10^{16}$ such that

$$\mu T = 4.12 \times 10^{16} \times 1.53 \times 10^{-16} \approx 6.3 \Rightarrow e^{\mu T} = e^{6.3} \approx 545 \quad \text{and}$$

$$\Phi_2 = \Phi_3 = \Phi \leq \omega e^{-\mu T} = 4.1 \times 10^{16} \times 0.0018 = 7.38 \times 10^{13} \quad \text{and}$$

$$R_2 = R_3 \leq \bar{c} e^{-\mu T} = \frac{1}{137} \times 3 \times 10^8 \times 1.8 \times 10^{-3} \approx 0.039 \times 10^5 = 3.9 \times 10^3. \quad \text{It is known that}$$

$$e_2 = e_3 \approx 1.6 \times 10^{-19} \text{ C}, \quad \beta = 1/137, \quad c \approx 3 \times 10^8 \text{ m/sec}, \quad \frac{(1+\beta)}{(1-\beta)^3} = \frac{138 \times 137^2}{136^3} = 1.03. \quad \text{Consequently}$$

for $n = 2$:

$$\begin{aligned} |E_2(t)| &\leq \frac{|e_2 e_1|}{137} \left[\frac{c}{\rho_{20}^2} + 1.03 \left(\frac{c}{|\rho_{20} - \rho_{30}|^2} + \frac{2\omega\bar{c}}{c|\rho_{20} - \rho_{30}|} \right) + \frac{\omega^2\bar{c}}{c^2} \right] \frac{e^{\mu T} - 1}{\mu} = \\ &= \frac{|e_2 e_1|}{137} \left(\frac{c}{\rho_{20}^2} + \frac{1.03 \times c}{|\rho_{20} - \rho_{30}|^2} + \left(\frac{2.06}{|\rho_{20} - \rho_{30}|} + \frac{\omega}{c} \right) \omega\beta \right) \frac{e^{\mu T} - 1}{\mu} = \\ &= 10^{-46} \left(\frac{7.41}{\rho_{20}^2} + \frac{7.657}{|\rho_{20} - \rho_{30}|^2} + \frac{1.54 \times 10^7}{|\rho_{20} - \rho_{30}|} \right) + 10^{-46} \times 10^{15} \approx 10^{-46} \left(\frac{7.41}{\rho_{20}^2} + \frac{7.66}{|\rho_{20} - \rho_{30}|^2} + \frac{1.54 \times 10^7}{|\rho_{20} - \rho_{30}|} \right). \end{aligned}$$

Since $\rho_{20} = 5.3 \times 10^{-11}$ then

$$\begin{aligned} 10^{-46} \left(\frac{7.41}{\rho_{20}^2} + \frac{7.66}{|\rho_{20} - \rho_{30}|^2} + \frac{1.54 \times 10^7}{|\rho_{20} - \rho_{30}|} \right) &= 10^{-46} \left(\frac{7.41 \times 10^{22}}{5.3^2} + \frac{7.66}{|\rho_{20} - \rho_{30}|^2} + \frac{1.54 \times 10^7}{|\rho_{20} - \rho_{30}|} \right) \approx \\ &\approx 10^{-46} \left(\frac{7.66}{|\rho_{20} - \rho_{30}|^2} + \frac{1.54 \times 10^7}{|\rho_{20} - \rho_{30}|} \right) = \bar{E}_2. \end{aligned}$$

Let us compare the obtained energy with the energy of the first stationary state from quantum mechanics (cf. [4]). Since $W_p = -\frac{1}{\varepsilon_0^2} \cdot \frac{e_2^4 m_2}{8\hbar^2} \frac{1}{p^2}$ ($p = 1, 2, \dots$) for $p = 1$ one obtains

$$W_1 = -\frac{1}{\varepsilon_0^2} \cdot \frac{e_2^4 m_2}{8\hbar^2} = -\frac{1}{8.86^2 \times 10^{-24}} \cdot \frac{(1.6 \times 10^{-19})^4 \times 9.1 \times 10^{-31}}{8.662^2 \times 10^{-68}} = -2.2 \times 10^{-18}.$$

It is easy to see that one can choose $|\rho_{02} - \rho_{03}|$ in such a way that $\bar{E}_2 = 2.2 \times 10^{-22}$. Indeed, for $|\rho_{02} - \rho_{03}| = \alpha \times 10^{-12}$ we obtain

$$\begin{aligned} \bar{E}_2 &= 10^{-46} \left(\frac{7.66}{|\rho_{20} - \rho_{30}|^2} + \frac{1.54 \times 10^7}{|\rho_{20} - \rho_{30}|} \right) = 10^{-46} \left(\frac{7.66}{\alpha^2} \times 10^{24} + \frac{1.54 \times 10^{19}}{\alpha} \right) = \\ &= 10^{-46} \frac{7.66}{\alpha^2} \times 10^{24} + 10^{-46} \left(\frac{1.54 \times 10^{19}}{\alpha} \right) \approx 10^{-22} \frac{7.66}{\alpha^2} = 2.2 \times 10^{-22} \end{aligned}$$

for $\alpha = 1.8659$.

We note that similar results can be obtained if $\omega_2 \neq \omega_3$ but $\omega_2 \approx \omega_3$.

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