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Liuyi Chen and [Qianqian Xia](#)*

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Article

Some Aspects of Differential Topology of Subcartesian Spaces

Liuyi Chen and Qianqian Xia *

Nanjing University of Information Science & Technology, Nanjing, 210044, China

* Correspondence: inertialtec@sina.com

Abstract: In this paper we investigate the differential topological properties of a subclass of singular space-subcartesian spaces. First, we get further result on partition of unity for differential spaces. Second, we establish the tubular neighborhood theorem for subcartesian spaces with constant structural dimension. Third, we generalize the concept of Morse functions on smooth manifolds to differential spaces. For subcartesian space with constant structural dimension, we provide examples of Morse functions; With the assumption that the subcartesian space can be embedded as a bounded subset of an Euclidean space, we prove that smooth bounded functions on this space can be approximated by Morse functions; We study the infinitesimal stability of Morse functions on subcartesian spaces. Classical results on Morse functions on smooth manifolds can be treated directly as corollaries of our results here.

Keywords: Singular space; Subcartesian space; Partition of unity; Tubular neighborhood; Morse function; Differential topology

MSC: 58A40, 57R70, 57R12

1. Introduction

There has long been perceived the need for an extension of the framework of smooth manifolds in differential geometry, which is too restrictive and does not admit certain basic geometric intuitions. There have been several different definitions which attempt to describe singular spaces, for example, Spalleks differentiable spaces [1], real algebraic varieties [2],[3], orbifolds [4], diffeology [5] et al.. Among them Sikorskis [6] theory of differential spaces studies the differential geometry of a large class of singular spaces which both contains the theory of manifolds and allows the investigation of singularities. Analogous to algebraic geometry, which is the investigation of geometry in terms of polynomials, the theory of differential space is the investigation of geometry in terms of differentiable functions.

It follows that smooth manifolds can be characterized as differential spaces such that every point has a neighborhood diffeomorphic to an open subset of $\mathbb{R}^n$, where n is the dimension of the manifold, the differential structures on U and V are generated by restrictions of smooth functions of S and $\mathbb{R}^n$, respectively, and diffeomorphism is in the sense of differential space. This definition can be weakened by not requiring V is open in $\mathbb{R}^n$ and allowing n to be an arbitrary non-negative integer. And it follows the concept of subcartesian space.

The theory of subcartesian spaces has been developed by Śniatycki et.al. in recent few years. There has been a lot of results on this topic [7–16]. See [17] for a systematic treatment on this topic.

In this paper, we investigate the properties of subcartesian spaces from the perspective of differential topology. We will show that some differential topological properties of smooth manifolds can be extended to subcartesian spaces. The first property is partition of unity, which plays a central role in analysis of smooth manifolds. As has been shown in [17], any Hausdorff, locally compact and second countable differential space possesses partition of unity. Here we get further results on partition of unity for differential spaces.

The second property we will investigate for subcartesian space is the tubular neighborhood property. Assume that the subcartesian space is with constant structural dimension. Let $i : S \rightarrow \mathbb{R}^m$ be an embedding which is ensured in [14] and let N be the normal bundle of S in $\mathbb{R}^m$ which can be considered as a subcartesian space after being endowed with differential structure. We first prove that there exists a local diffeomorphism between an open neighborhood of zero section of the normal bundle N of S in $\mathbb{R}^m$ and a subset containing S of $\mathbb{R}^m$ with constant structural dimension m , where the open neighborhood of zero section and the subset containing S of $\mathbb{R}^m$ are considered as differential subspace of N and $\mathbb{R}^m$ respectively. Further, by taking advantage of the partition of unity, we get a global tubular neighborhood: there exists a diffeomorphism between an open neighborhood of zero section of the normal bundle N of S in $\mathbb{R}^m$ defined by $\Delta_\epsilon = \{\xi \in N \mid |\xi| < \epsilon(\tau(\xi))\}$, and a subset containing S of $\mathbb{R}^m$ with constant structural dimension m . We finally get a global result : there exists a diffeomorphism between the normal bundle N of S in $\mathbb{R}^m$ and a subset containing S of $\mathbb{R}^m$. Our results generalize the tubular neighborhood theorem in smooth manifolds to more general cases—the subcartesian space with constant structural dimension.

The third property we will investigate for subcartesian space is the Morse theory. In classical Morse theory[18], Morse functions in smooth manifolds are defined as smooth functions whose critical points are nondegenerate, i.e., the Hessian matrix of second derivatives has nonvanishing determinant. In this paper, by taking advantage of the definition of derivation on differential space, we extend the definition of Morse functions on smooth manifolds to differential spaces. We then study some basic properties of Morse functions on subcartesian spaces. Precisely, assume that the subcartesian space S has constant structural dimension. We prove the following results:

1. Morse functions on S are plentiful. Let S be embedded in an Euclidean space $\mathbb{R}^n$, then for almost all $p \in S$, the function L_p in S defined by $L_p(q) = \|p - q\|^2$ is a Morse function on S ;
2. Assume that S can be embedded as a bounded subset of an Euclidean space. Then any smooth bounded function on S can be approximated by Morse functions;
3. The set of critical points of a Morse function is discrete in S ;
4. If S is compact, then the Morse functions are infinitesimal stable.

It follows immediately that the corresponding results on smooth manifolds[18] can be treated as corollaries of our results here.

To the best of our knowledge, our work is the first attempt to initiate a systematic study of the differential topological properties for differential spaces.

The paper is organized as follows. In Section 2, we recall some basic definitions and results on differential and subcartesian spaces which will be used in our paper. We then define Morse functions on differential spaces. In Section 3, we present existing results and prove further results on partition of unity for differential spaces. In Section 4, we investigate tubular neighborhood property for subcartesian space with constant structural dimension. In Section 5, we first provide examples of Morse functions on subcartesian spaces with constant structural dimension, by which we then prove the approximation theorem for subcartesian spaces which can be embedded as a bounded subset of $\mathbb{R}^m$. In Section 6, we study stability of Morse functions on compact subcartesian spaces with constant structural dimension. We make conclusions in Section 7.

2. Subcartesian Space

Definition 1. [17] A differential structure on a topological space S is a family $C^\infty(S)$ of real-valued functions on S satisfying the following conditions:

1. The family

$$\{f^{-1}(I) \mid f \in C^\infty(S) \text{ and } I \text{ is an open interval in } \mathbb{R}\}$$

is a subbasis for the topology of S .

2. If $f_1, \dots, f_n \in C^\infty(S)$ and $F \in C^\infty(\mathbb{R}^n)$, then $F(f_1, \dots, f_n) \in C^\infty(S)$.

3. If $f : S \rightarrow \mathbb{R}$ is a function such that, for every $x \in S$, there exist an open neighborhood U of x , and a function $f_x \in C^\infty(S)$ satisfying

$$f_x|_U = f|_U,$$

then $f \in C^\infty(S)$. Here, the subscript vertical bar $|$ denotes a restriction. $(S, C^\infty(S))$ is said to be a differential space. Functions $f \in C^\infty(S)$ are called smooth functions on S .

Definition 2. [17] Let S_1 and S_2 be two differential space. A map $\phi : S_1 \rightarrow S_2$ is C^∞ if $\phi^*f = f \circ \phi \in C^\infty(S_1)$ for every $f \in C^\infty(S_2)$. A C^∞ map ϕ between differential spaces is a diffeomorphism if it is invertible and its inverse is C^∞ .

An alternative way of constructing a differential structure on a set S , goes as follows. Let $\mathcal{F}$ be a family of real-valued functions on S . Endow S with the topology generated by a subbasis

$$\{f^{-1}(I) | f \in \mathcal{F} \text{ and } I \text{ is an open interval in } \mathbb{R}\}.$$

Define $C^\infty(S)$ by the requirement that $h \in C^\infty(S)$ if, for each $x \in S$, there exist an open subset U of S , functions $f_1, \dots, f_n \in \mathcal{F}$, and $F \in C^\infty(\mathbb{R}^n)$ such that

$$h|_U = F(f_1, \dots, f_n)|_U.$$

Clearly, $\mathcal{F} \in C^\infty(S)$. It is proved in [17] that $C^\infty(S)$ defined here is a differential structure on S . We refer to it as the differential structure on S generated by $\mathcal{F}$.

Let S be a differential space with a differential structure $C^\infty(S)$, and let T be an arbitrary subset of S endowed with the subspace topology (open sets in T are of the form $T \cap U$, where U is an open subset of S). Let

$$S(T) = \{f|_T | f \in C^\infty(S)\}.$$

Definition 3. [17] The space $S(T)$ of restrictions to $T \subseteq S$ of smooth functions on S generates a differential structure $C^\infty(T)$ on T such that the differential-space topology of S coincides with its subspace topology. In this differential structure, the inclusion map $i : T \rightarrow S$ is smooth.

In other words, $S(T)$ is the space of restrictions to T of smooth functions on S .

Definition 4. [17] A differential space S is said to be subcartesian if every point of S has a neighbourhood U diffeomorphic to a subset of some Cartesian space $\mathbb{R}^n$. $(U, \Phi, \mathbb{R}^n)$ is said to be a local chart of p , where $\Phi : U \rightarrow \Phi(U) \subseteq \mathbb{R}^n$ is the diffeomorphism.

Definition 5. [17] A derivation of $C^\infty(S)$ is a linear map

$$X : C^\infty(S) \rightarrow C^\infty(S) : f \rightarrow X(f)$$

satisfying Leibniz rule

$$X(f_1 f_2) = X(f_1) f_2 + f_1 X(f_2)$$

for every $f_1, f_2 \in C^\infty(S)$.

We denote by $Der C^\infty(S)$ the space of derivations of $C^\infty(S)$. This has the structure of a Lie algebra, with the Lie bracket $[X_1, X_2]$ defined by

$$[X_1, X_2](f) = X_1(X_2(f)) - X_2(X_1(f)),$$

for every $X_1, X_2 \in Der C^\infty(S)$ and $f \in C^\infty(S)$.

Definition 6. [17] A derivation of $C^\infty(S)$ at $x \in S$ is a linear map $v : C^\infty(S) \rightarrow \mathbb{R}$ such that

$$v(f_1 f_2) = v(f_1) f_2(x) + f_1(x) v(f_2)$$

for every $f_1, f_2 \in C^\infty(S)$.

We denote by $\text{Der}_x(C^\infty(S))$ the space of derivations of $C^\infty(S)$ at $x \in S$.

We interpret derivations of $C^\infty(S)$ at $x \in S$ as tangent vectors to S at x . The set of all derivations of $C^\infty(S)$ at x is denoted by $T_x S$ and is called the tangent space to S at x .

If X is a derivation of $C^\infty(S)$, then for every $x \in S$ we have a derivation $X(x)$ of $C^\infty(S)$ at x given by

$$X(x) : C^\infty(S) \rightarrow \mathbb{R} : f \rightarrow X(x)f = (Xf)(x). \quad (1)$$

The derivation (1) is called the value of X at x . Clearly, the derivation X is uniquely determined by the collection $\{X(x) | x \in S\}$ of its values at all points in S .

Let S be a differential space of $\mathbb{R}^n$. Let $N(S)$ denote the ideal of functions in $C^\infty(\mathbb{R}^n)$ that vanish identically on S :

$$N(S) = \{F \in C^\infty(\mathbb{R}^n) | F|_S = 0\}.$$

Proposition 1. [17] A smooth vector field Y on $\mathbb{R}^n$ restricts to a derivation of $C^\infty(S)$ if $Y(F) \in N(S)$, for every $F \in N(S)$.

Definition 7. [17] A point $x \in S$ is called a critical point of $f \in C^\infty(S)$ if $X(f) = 0$, for each $X \in \text{Der}_x(C^\infty(S))$.

If x is a critical point of $f \in C^\infty(S)$, then consider the smooth distribution on S defined by $\text{TS} = \text{span}_{\mathbb{R}}\{X | X \in \text{Der}(C^\infty(S))\}$. We can define a bilinear symmetric functional f_x^{**} on $\text{TS}(x)$, called the Hessian of f at x as follows. Let $v, w \in \text{TS}(x)$. Then there exists $V, W \in \text{Der}(C^\infty(S))$, such that $V(x) = v, W(x) = w$. We define

$$\begin{aligned} f_x^{**} : \text{TS}(x) \times \text{TS}(x) &\rightarrow \mathbb{R} \\ f_x^{**}(v, w) &= VW(f)(x). \end{aligned} \quad (2)$$

f_x^{**} is well-defined. Let $\tilde{V}, \tilde{W} \in \text{Der}(C^\infty(S))$ such that $\tilde{V}(x) = v, \tilde{W}(x) = w$. We have

$$\begin{aligned} \tilde{V}\tilde{W}(f)(x) &= \tilde{V}(x)\tilde{W}f = V(x)\tilde{W}(f) = V\tilde{W}(f)(x) = [V, \tilde{W}](f)(x) + \tilde{W}V(f)(x) \\ &= 0 + W(x)V(f) = WV(f)(x) = [W, V]f(x) + VW(f)(x) \\ &= VW(f)(x), \end{aligned} \quad (3)$$

where $[W, V]f(x) = 0$ because $[W, V] \in \text{Der}(C^\infty(S))$ and x is a critical point of f . From (3) we also have that f_x^{**} is symmetric and bilinear.

Definition 8. A point $x \in S$ is called a nondegenerate critical point of $f \in C^\infty(S)$ if x is a critical point of S such that f_x^{**} is nondegenerate.

Definition 9. A smooth function $f \in C^\infty(S)$ is said to be a Morse function if each critical point $x \in S$ of f is nondegenerate.

Definition 10. [17] Let S be a subcartesian space. The structural dimension of S at a point $x \in S$ is the smallest integer n such that for some open neighbourhood $U \subseteq S$ of x , there is a diffeomorphism of U onto a subset $V \subseteq \mathbb{R}^n$. The structural dimension of S is the smallest integer n such that for every point $x \in S$, the structural dimension n_x of S at x satisfies that $n_x \leq n$.

It follows from the definition of subcartesian space that a subcartesian space is locally compact. We make the following assumption: throughout the paper, the subcartesian spaces are second countable.

Lemma 1. *Let S be a subcartesian space with constant structural dimension n and $\Phi : S \rightarrow \mathbb{R}^m$ be a smooth map. Let $\mathfrak{D}$ be an open cover of S . Then there exist locally finite open covers $(U_j)_{j \in \mathbb{Z}_{>0}}, (V_j)_{j \in \mathbb{Z}_{>0}}, (W_j)_{j \in \mathbb{Z}_{>0}}$ such that $cl(U_j) \subseteq V_j, cl(V_j) \subseteq W_j, cl(W_j)$ is compact, for each $j > 0$, where $(W_j, \mathbb{R}^n, \phi_j)$ is a local chart of S and $\mathfrak{W} = \{W_j\} \prec \mathfrak{D}$. Besides, there exists a smooth extension $\tilde{\Phi}$ of Φ on U_j , that is, $\tilde{\Phi} \circ \phi_j|_{U_j} = \Phi|_{U_j}$.*

Proof. The proof follows by replacing $(2)p \in W \subseteq (G_{h+1}/cl(G_{h-2})) \cap \mathcal{V}$ in proof of Lemma 3.3 in [14] with $(2)p \in W \subseteq (G_{h+1}/cl(G_{h-2})) \cap \mathcal{V} \cap Q$, where $Q \in \mathfrak{D}$ is an open subset containing p , and by replacing f, Φ and n_j in the proof of Lemma 3.3 in [14] with $\Phi, \tilde{\Phi}$ and n . $\square$

We have the embedding theorem for subcartesian space.

Theorem 1. [14] *Let S be a subcartesian space with structural dimension n . Then there exists a proper embedding map $\Psi : S \rightarrow \mathbb{R}^m$, where $m \geq 2n + 1$.*

Theorem 2. [17] *For a subcartesian space S , the structural dimension at x is equal to $\dim T_x S$.*

The subcartesian space is said to be with constant structural dimension if the structural dimension of each $x \in S$ is the same.

Throughout the paper, we will focus on the subcartesian space with constant structural dimension. The remaining part of this section we will show that S is a metric space.

Definition 11. *A smooth Riemannian metric on a subcartesian space is a symmetric positive definite bilinear form $g(x)$ in $T_x S$, for each $x \in S$, such that for each smooth section σ of TS , the function $g(x)(\sigma(x), \sigma(x)) \in C^\infty(S)$.*

Theorem 3. [19] *Let S be a subcartesian space with structural dimension n . Then there exists a smooth Riemannian metric on S .*

Definition 12. *Let S be a subcartesian space with constant structural dimension. Given two points $p, q \in S$, the distance $d(p, q)$ is defined by $d(p, q) = \infimum$ of the lengths of all curves $\gamma_{p,q}$, where $\gamma_{p,q}$ is a piecewise differentiable curve joining p to q .*

Proposition 2. *With the distance d , S is a metric space.*

- (1) $d(p, x) \leq d(p, q) + d(q, x)$, for $p, q, x \in S$;
- (2) $d(p, q) = d(q, p)$;
- (3) $d(p, q) \geq 0$, and $d(p, q) = 0$ if and only if $p = q$.

Proof. We only need to show that if $d(p, q) = 0$ then $p = q$. Assume that p, q are two distinct points. It follows that there is a normal ball $B_r(p)$ (which is diffeomorphic to a subset V of $T_p S$ with $g(p)(v, v) < r^2$, for $v \in V$) that does not contain q . Since $d(p, q) = 0$, there exists a curve c joining p and q of length less than r . Hence the segment of c must contain in $B_r(p)$ and hence c can not join p and q . This makes a contradiction.

The remaining item follow directly from the definition of $d(p, q)$. $\square$

3. Partition of Unity

Definition 13. *A countable partition of unity on a differential space S is a countable family of functions $\{f_i\} \in C^\infty(S)$ such that:*

- (a) *The collection of their supports is locally finite.*

- (b) $f_i(x) \geq 0$ for each i and each $x \in S$.
 (c) $\sum_{i=1}^{\infty} f_i(x) = 1$ for each $x \in S$.

The following theorem in [17] establishes the existence of a partition of unity for locally compact, second countable Hausdorff differential spaces.

Theorem 4. [17] *Let S be a differential space with differential structure $C^\infty(S)$, and let $\{U_\alpha\}$ be an open cover of S . If S is Hausdorff, locally compact and second countable, then there exists a countable partition of unity $\{f_i\} \in C^\infty(S)$, subordinate to $\{U_\alpha\}$ and such that the support of each f_i is compact.*

Here we establish further results on partition of unity for differential spaces.

Lemma 2. *Let S be a Hausdorff, locally compact and second countable differential space with differential structure $C^\infty(S)$. Let $F \subseteq S$ be a non-empty closed subset and $G \subseteq S$ be an open subset such that $F \subseteq G$. Then there exists a smooth function $g \in C^\infty(S)$, such that $F \subseteq \{p \in S | g(p) = 1\} \subseteq \text{supp } g \subseteq G$.*

Proof. Let $H = S \setminus F$. Then $\{S \setminus F, G\}$ is an open cover of S . It follows from Theorem 4 that there exists a countable partition of unity $\{f_i\} \in C^\infty(S)$, subordinate to $\{S \setminus F, G\}$ such that the support of each f_i is compact.

Define $g = \sum_{\text{supp } f_i \subseteq G} f_i$. Since the collection of the supports of $\{f_i\}$ is locally finite, it follows from condition 3 in Definition 1 that $g \in C^\infty(S)$. And we have

$$F = S \setminus H \subseteq \{p \in S | g(p) = 1\} \subseteq \text{supp } g \subseteq G.$$

Then the result follows immediately. $\square$

Corollary 1. *Let S be a Hausdorff, locally compact and second countable differential space with differential structure $C^\infty(S)$. Let $\{G_i\}$ be a family of locally finite open subsets. Let $K_i \subseteq G_i$ be compact such that $\cup_i K_i = S$. Then there exists a family of smooth functions $\{v_i\}$ such that*

- (1) $0 \leq v_i \leq 1, \sum_i v_i = 1$;
 (2) $K_i \subseteq \text{supp } v_i \subseteq G_i$.

Proof. It follows from Lemma 2 that there exists $\mu_i \in C^\infty(S)$ such that $K_i \subseteq \{p \in S | \mu_i(p) = 1\} \subseteq \text{supp } \mu_i \subseteq G_i$.

Since $\{G_i\}$ is locally finitely, it follows that $\sum_j \mu_j < +\infty$. Further, $\sum_j \mu_j \in C^\infty(S)$. On the other hand, since $\cup_i K_i = S$, it follows that $\sum_j \mu_j \geq 1$.

Define $v_i = \mu_i / \sum_j \mu_j$. $v_i \in C^\infty(S)$. It follows immediately that $\{v_i\}$ satisfies conditions (1) and (2). $\square$

4. Tubular Neighborhoods

Let S be a subcartesian space with constant structural dimension n . It follows from Theorem 1 that there exists a proper embedding map $i : S \rightarrow \mathbb{R}^m$, where $m \geq 2n + 1$.

Define $N \subseteq S \times \mathbb{R}^m$ by

$$N = \{(q, v) | q \in S, v \text{ perpendicular to } i_*(T_q S) \text{ at } q\}. \quad (4)$$

Denote by $\tau : N \rightarrow S$ the projection $\pi(q, v) = q$. The differential structural $C^\infty(N)$ of N is generated by the family of functions $\{f \circ \tau, df | f \in C^\infty(\mathbb{R}^m)\}$.

Since S is a subcartesian space with constant structural dimension n , it follows that $\dim T_q S = n$, for each $q \in N$. Hence the dimension of the linear space $Q = \{v | v \text{ perpendicular to } i_*(T_q S) \text{ at } q\}$ is $m - n$ for each $q \in S$. $(N, \tau, S, \mathbb{R}^{m-n})$ is a vector bundle on S , where $\tau : N \rightarrow S$ is a smooth map and

$\tau^{-1}(U)$ is diffeomorphic to $U \times \mathbb{R}^{m-n}$ where U is an open subset of S . Hence N is a subcartesian space with constant structural dimension $n + (m - n) = m$.

Lemma 3. [20] Let X, Y be metric spaces. X is locally compact and second countable. Let A be a closed subset of X . Assume that the continuous map $\psi : X \rightarrow Y$ satisfies that

- (1) $\psi : X \rightarrow Y$ is a local homeomorphism;
- (2) $\psi|_A$ is an injection.

Then there exists an open neighborhood G of A in X and an open neighborhood $H = \psi(G)$ of $B = \psi(A)$ in Y , such that $\psi|_G$ is a homeomorphism from G to H .

Let $\psi : N \rightarrow \mathbb{R}^m$ be defined by $\psi((q, v)) = i(q) + v$. We have

Lemma 4. There exists an open neighborhood G of zero section Z of N such that $\psi|_G : G \rightarrow \psi(G)$ is a diffeomorphism between the subcartesian space G and $\psi(G)$.

Proof. For any $q \in S$ and $0_q \in Z, \psi(0_q) = q \in S$. Consider $(d\psi)_{0_q} : T_{0_q}N \rightarrow \mathbb{R}^m = i_*(T_qS) \oplus (i_*(T_qS))^\perp$. Due to the local product property of $(N, \tau, S, \mathbb{R}^{m-n})$, we have $T_{0_q}N = T_{0_q}Z \oplus T_{0_q}N_q$. Since $\psi|_Z : Z \rightarrow N$ is a diffeomorphism, we have $(d\psi)_{0_q}|_{(T_{0_q}Z)} : T_{0_q}Z \rightarrow T_qS$ is a linear isomorphism. Besides, $(d\psi)_{0_q}(T_{0_q}N_q) : T_{0_q}N_q \rightarrow (T_qS)^\perp$ is a linear isomorphism. Hence $(d\psi)_{0_q}(T_{0_q}N) : T_{0_q}N \rightarrow (T_qS)^\perp \oplus T_qS = \mathbb{R}^m$ is a linear isomorphism. Since N is a subcartesian space with constant structural dimension m , let $(U, \phi, \mathbb{R}^m)$ be a local chart of N , then ψ can be locally extended to be a smooth map $\tilde{\psi}$ from an open subset of $\mathbb{R}^m$ to $\mathbb{R}^m$. Since $(d\psi)_{0_q} : T_{0_q}N \rightarrow \mathbb{R}^m$ is a linear isomorphism, it follows that $(d\tilde{\psi})_0$ is a linear isomorphism. Hence $\tilde{\psi}$ is a local diffeomorphism around 0, which yields that ψ is a local diffeomorphism around 0_q . Since q is arbitrary, we get that there exists an open neighborhood X of zero section Z of N such that $\psi|_X : X \rightarrow \psi(X)$ is a local diffeomorphism. On the other hand, $\psi : Z \rightarrow S$ is a diffeomorphism.

Since N is a subcartesian space with constant structural dimension, it follows from Proposition 2 that N is a metric space and hence X is a metric space as an open subset of N . Then it follows from Lemma 3 that there exists an open neighborhood $G \subseteq X$ of Z and an open neighborhood $H = \psi(G)$ of S in $\psi(X)$, such that $\psi|_G : G \rightarrow H$ is a homeomorphism. Since $\psi|_X : X \rightarrow \psi(X)$ is a local diffeomorphism, it follows immediately that $\psi|_G$ is a diffeomorphism. This completes the proof of the lemma. $\square$

Consider the vector bundle $(N, \tau, S, \mathbb{R}^{m-n})$ on S . Due to the local trivial property of the vector bundle together with partition of unity on S , there exists a smooth Riemannian metric on $(N, \tau, S, \mathbb{R}^{m-n})$.

Lemma 5. Let β be a smooth Riemannian metric on $(N, \tau, S, \mathbb{R}^{m-n})$. Let Z be a zero section of N and G be an open neighborhood of Z . Then there exist a smooth function $\epsilon > 0$ on S such that

$$\Delta_\epsilon = \{\xi \in N \mid |\xi| < \epsilon(\tau(\xi))\} \subseteq G,$$

where $|\cdot|$ is the norm determined by the Riemannian metric β .

Proof. We first claim that for any $q \in S$, there exist an open neighborhood Q of q in S and $\delta > 0$ such that

$$\{\xi \in \tau^{-1}(Q) \mid |\xi| < \delta\} \subseteq G.$$

Consider the local trivial neighborhood U of q . Then there exists a diffeomorphism $h : \tau^{-1}(U) \rightarrow U \times \mathbb{R}^{m-n}$. Since $h(\tau^{-1}(U) \cap G)$ is an open neighborhood of Z in $U \times \mathbb{R}^{m-n}$ and since S is locally

compact, it follows that there exist an open neighborhood $Q \subseteq U$ of q where $Q \subseteq cl(Q) \subseteq U$ and $cl(Q)$ is compact, and $\gamma > 0$, such that

$$\{(x, v) \in cl(Q) \times \mathbb{R}^{m-n} \mid \|v\| < \gamma\} \subseteq h(\tau^{-1}(U) \cap G).$$

Denote by $\lambda(x, v) = |h^{-1}(x, v)|$, for all $(x, v) \in cl(Q) \times \mathbb{R}^{m-n}$. Since $cl(Q) \times S^{m-n-1}$ is compact, there exists $\mu > 0$ such that $\lambda(x, v) \geq \mu$, for any $(x, v) \in cl(Q) \times S^{m-n-1}$. Hence $\lambda(x, v) \geq \mu\|v\|$, for any $(x, v) \in cl(Q) \times \mathbb{R}^{m-n}$. Let $\delta = \mu\gamma$. We have $\lambda(x, v) = |h^{-1}(x, v)| < \delta$ yields that $\|v\| < \gamma$. Hence we have

$$h(\{\xi \in \tau^{-1}(Q) \mid \|\xi\| < \delta\}) \subseteq \{(x, v) \in cl(Q) \times \mathbb{R}^{m-n} \mid \|v\| < \gamma\} \subseteq h(\tau^{-1}(U) \cap G).$$

It follows immediately that $\{\xi \in \tau^{-1}(Q) \mid \|\xi\| < \delta\} \subseteq G$.

Hence there exists an open cover $\mathfrak{D} = \{Q\}$ of S and a family $\{\delta_Q\}$ such that $\{\xi \in \tau^{-1}(Q) \mid \|\xi\| < \delta_Q\} \subseteq G$.

Then it follows from Lemma 1 that there exist locally finite open covers $(U_j)_{j \in \mathbb{Z}_{>0}}, (V_j)_{j \in \mathbb{Z}_{>0}}, (W_j)_{j \in \mathbb{Z}_{>0}}$ such that $cl(U_j) \subseteq V_j, cl(V_j) \subseteq W_j$, and $cl(W_j)$ is compact, for each $j > 0$, where $\mathfrak{W} = \{W_j\} \prec \mathfrak{D}$. And for each W_j , there exists $\delta_j > 0$ such that $\{\xi \in \tau^{-1}(W_j) \mid \|\xi\| < \delta_j\} \subseteq G$.

We claim that there exists a smooth function $\epsilon > 0$ on S such that $\epsilon(x) < \delta_k$, for any $x \in U_k, k \in \mathbb{N}$. It follows from Corollary 1 that there exists partition of unity $\{\lambda_i\} \subseteq C^\infty(S)$ such that $U_k \subseteq \{x \in S \mid \lambda_k(x) > 0\} \subseteq \text{supp } \lambda_k \subseteq V_k, k = 1, 2, \dots$.

Given $j \in \mathbb{N}$, denote by $\epsilon_j = \min\{\delta_k \mid cl(V_k) \cap cl(V_j) \neq \emptyset, k \in \mathbb{N}\}, j = 1, 2, \dots$. Let $\epsilon(x) = \sum_{j=1}^{\infty} \epsilon_j \lambda_j(x)$. For $x \in U_k$, we have $\epsilon(x) < \sum_{cl(V_j) \cap cl(V_k) \neq \emptyset} \epsilon_j \lambda_j(x) \leq \delta_k \sum_{j=1}^{\infty} \lambda_j(x) = \delta_k$.

Hence we have proved that

$$\Delta_\epsilon = \{\xi \in N \mid \|\xi\| < \epsilon(\tau(\xi))\} \subseteq G.$$

□

We have the following tubular neighborhood theorem for subcartesian space.

Theorem 5. Let S be a subcartesian space with constant structural dimension. Let $i : N \rightarrow \mathbb{R}^m$ be an embedding. Let N be defined by (4) and $\tau : N \rightarrow S$ be the projection. Let $\psi : N \rightarrow \mathbb{R}^m$ be defined by $\psi((q, v)) = i(q) + v$. There exists a smooth function $\epsilon > 0$ on S , such that $\psi|_{\Delta_\epsilon} : \Delta_\epsilon \rightarrow \psi(\Delta_\epsilon)$ is a diffeomorphism between the subcartesian space Δ_ϵ and $\psi(\Delta_\epsilon)$, where $\Delta_\epsilon = \{\xi \in N \mid \|\xi\| < \epsilon(\tau(\xi))\}$ and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^m$. Further, define $\rho = \tau \circ \psi^{-1}$. Then $\rho : \psi(\Delta_\epsilon) \rightarrow S$ is a smooth map satisfying that $\rho(x) = x$, for any $x \in S$. Besides, $\rho \circ \psi|_{\Delta_\epsilon} = \tau|_{\Delta_\epsilon}$. $\psi(\Delta_\epsilon)$ is said to be the tubular neighborhood of S in $\mathbb{R}^m$ and $\rho : \psi(\Delta_\epsilon) \rightarrow S$ is said to be the contraction map of the tubular neighborhood.

The above result can be extended to the following global result.

Theorem 6. Let S be a subcartesian space with constant structural dimension. Let $i : N \rightarrow \mathbb{R}^m$ be an embedding. Let N be defined by (4) and $\tau : N \rightarrow S$ be the projection. There exists a diffeomorphism $\omega : N \rightarrow \omega(N) \subseteq \mathbb{R}^m$ such that $\omega(0_x) = x$, for any $x \in S$. Further, there exists a smooth contraction map $\rho : \omega(N) \rightarrow S$, such that $\rho \circ \omega = \tau$.

Proof. It follows from Theorem 5 that there exists a smooth function $\epsilon > 0$ on S such that $\psi : \Delta_\epsilon \rightarrow \psi(\Delta_\epsilon) \subseteq \mathbb{R}^m$ is a diffeomorphism. Besides, there exists a contraction map $\rho : \psi(\Delta_\epsilon) \rightarrow S$ such that $\rho \circ \psi|_{\Delta_\epsilon} = \tau|_{\Delta_\epsilon}$.

Define a smooth map $\theta : N \rightarrow \Delta_\epsilon$ by $\theta(\xi) = \frac{\epsilon(\tau(\xi))}{\sqrt{1+\|\xi\|^2}} \xi$. We claim that θ is a diffeomorphism. Consider the smooth map $\gamma : \Delta_\epsilon \rightarrow N$ by $\gamma(\eta) = \frac{\eta}{\sqrt{\epsilon(\tau(\eta))^2 - \|\eta\|^2}}$. It follows that $\gamma \circ \theta = Id : N \rightarrow N$.

and $\theta \circ \gamma = Id : \Delta_\epsilon \rightarrow \Delta_\epsilon$. Hence θ is a bijection. Since both θ and γ are smooth, it follows immediately that θ is a diffeomorphism.

Let $\omega = \phi \circ \theta : N \rightarrow \phi(\Delta_\epsilon)$. Then ω is a diffeomorphism which satisfies that $\omega(0_x) = \phi \circ \theta(0_x) = \phi(0_x) = x$, for all $x \in S$. Besides, $\rho \circ \omega = \rho \circ \phi \circ \theta = \tau \circ \theta = \tau$. This completes the proof of the theorem. $\square$

5. Approximating Bounded Smooth Functions by Morse Functions on Subcartesian Spaces

Let S be embedded in $\mathbb{R}^m$ with constant structural dimension n , i.e. $i : S \rightarrow \mathbb{R}^m$ be an embedding. Let $p \in \mathbb{R}^m$. Define the function $L_p : S \rightarrow \mathbb{R}$ by

$$L_p(q) = \|p - q\|^2. \quad (5)$$

It will be proved that for almost all p , the function L_p is a Morse function in S .

From the above section we know that N defined by (4) is a subcartesian space with constant structural dimension m .

Consider $\psi : N \rightarrow \mathbb{R}^m$ be $\psi(q, v) = q + v$.

Definition 14. $e \in \mathbb{R}^m$ is a focal point of (S, q) if $e = q + v$ where $(q, v) \in N$ and $\ker d\psi \neq 0$. The point e is called a focal point of S if e is a focal point of (S, q) for some $q \in S$.

Theorem 7. Let S be a subcartesian space with constant structural dimension n and $\Phi : S \rightarrow \mathbb{R}^n$ is smooth, the image of the set of the points where $d\Phi$ is singular has measure 0 in $\mathbb{R}^n$.

Proof. It follows from Lemma 1 that there exist open cover $(U_j)_{j \in \mathbb{Z}_{>0}}$, and a local chart $(U_j, \mathbb{R}^k, \phi_j)$ for each j , such that there exists a smooth extension $\tilde{\Phi}$ of Φ on U_j , that is, $\tilde{\Phi} \circ \phi_j|_{U_j} = \Phi|_{U_j}$.

Since the structural dimension of S is n , it follows that the set of points on U_j where $d\Phi$ is singular is the same as the set of points on U_j where $d(\tilde{\Phi} \circ \phi_j)$ is singular. From Sard's Theorem we know that the image of the set of points where $d\tilde{\Phi}$ is singular has measure 0 in $\mathbb{R}^n$. It follows that the image of the set of points on U_j where $d\Phi$ is singular has measure 0 in $\mathbb{R}^n$. Then the image of the set of the points where $d\Phi$ is singular is a union of countable sets where each set has measure 0 in $\mathbb{R}^n$. Hence the image of the set of the points where $d\Phi$ is singular has measure 0 in $\mathbb{R}^n$. $\square$

Corollary 2. For almost all $x \in \mathbb{R}^m$, the point x is not a focal point of S .

Proof. The point x is a focal point of S if and only if x is in the image of the set of points where $d\psi$ is singular. The result follows from Theorem 7. $\square$

Let $q \in S$ with $(u_1, \dots, u_n)$ being local coordinates for q . Then the inclusion $i : S \rightarrow \mathbb{R}^m$ can be locally extended to be a smooth map $x = (x_1(u_1, \dots, u_n), \dots, x_m(u_1, \dots, u_n))$.

Define the matrices associated with the coordinate system by

$$(g_{ij}) = \left(\left(\frac{\partial x}{\partial u_i} \right)^T \frac{\partial x}{\partial u_j} \right).$$

Consider the vector $\frac{\partial^2 x}{\partial u_i \partial u_j}$. Let v be a unit vector which is perpendicular to $i_*(T_q S)$. Define the vector l_{ij} to be the normal component of $\frac{\partial^2 x}{\partial u_i \partial u_j}$. Given any unit vector v which is normal to S at q , we have the matrix

$$(v^T \frac{\partial^2 x}{\partial u_i \partial u_j}) = (v^T l_{ij}).$$

The coordinates $(u_1, \dots, u_n)$ can be chosen such that (g_{ij}) evaluated at q is the identity matrix. Then the eigenvalues of the matrix $(v^T l_{ij})$ are called the principal curvature $K_1, \dots, K_n$ of S at q in

the normal direction v . $K_1^{-1}, \dots, K_n^{-1}$ are called principle radii of curvature. If the matrix $(v^T l_{ij})$ is singular, then one or more of the K_i will be zero; and hence the corresponding K_i^{-1} will not be defined.

Now consider the normal line $l = q + tv$. We have

Lemma 6. *The focal points of (S, q) along l are precisely the points $q + K_i^{-1}v$, where $1 \leq i \leq n, K_i \neq 0$. Thus there are at most n focal points of (S, q) along l , each being counted with its proper multiplicity.*

Proof. Choose $m - n$ vector fields $w_i(u_1, u_2, \dots, u_n)$ which are unit vectors orthogonal to each other and to $i_*(TS)$. We can introduce local coordinates $(u_1, \dots, u_n, t_1, \dots, t_{m-n})$ for N , which corresponds to the point

$$(x(u_1, \dots, u_n), \sum_{i=1}^{m-n} t^i w_i(u_1, \dots, u_n)) \in N.$$

Then the map $\psi : N \rightarrow \mathbb{R}^n$ has the local coordinate expression

$$(u_1, \dots, u_n, t_1, \dots, t_{m-n}) \rightarrow x(u_1, \dots, u_n) + \sum_{i=1}^{m-n} t^i w_i(u_1, \dots, u_n).$$

Since S has constant structural dimension n , we have $\{\frac{\partial}{\partial u_i}, \frac{\partial}{\partial t_i}\}$ span TN around $\tau^{-1}(q)$. Hence we have

$$\begin{aligned} d\psi\left(\frac{\partial}{\partial u_i}\right) &= \frac{\partial x}{\partial u_i} + \sum_{j=1}^{m-n} t_j \frac{\partial w_j}{\partial u_i}, \\ d\psi\left(\frac{\partial}{\partial t_i}\right) &= w_i. \end{aligned} \quad (6)$$

Taking the inner products of these vectors with the basis vector $\frac{\partial x}{\partial u_i}, w_j$, we then get a matrix

$$\begin{pmatrix} ((\frac{\partial x}{\partial u_i})^T \frac{\partial x}{\partial u_j} + \sum_{l=1}^{m-n} t_l (\frac{\partial w_l}{\partial u_i})^T \frac{\partial x}{\partial u_j}) & (\sum_{l=1}^{m-n} t_l (\frac{\partial w_l}{\partial u_i})^T w_j) \\ 0 & \text{identity matrix} \end{pmatrix},$$

since $\frac{\partial x}{\partial u_i}, w_j$ are orthogonal.

Since

$$0 = \frac{\partial}{\partial u_i} (w_l^T \frac{\partial x}{\partial u_j}) = (\frac{\partial w_l}{\partial u_i})^T \frac{\partial x}{\partial u_j} + w_l^T \frac{\partial^2 x}{\partial u_i \partial u_j},$$

we have $((\frac{\partial x}{\partial u_i})^T \frac{\partial x}{\partial u_j} + \sum_{l=1}^{m-n} t_l (\frac{\partial w_l}{\partial u_i})^T \frac{\partial x}{\partial u_j}) = (g_{ij} - \sum_{l=1}^{m-n} t_l w_l^T l_{ij}) = (g_{ij} - tv^T l_{ij})$.

Since (g_{ij}) evaluated at q is the identity matrix, it follows that the above matrix is singular at (q, tv) if and only if $t = K_i^{-1}$, where K_i the principal curvature of S at q in the normal direction v .

Since $q + tv$ is the focal point of (S, q) if and only if $d\psi$ is singular at (q, tv) if and only if the above matrix is singular at (q, tv) , the result follows immediately. $\square$

Now fix $p \in \mathbb{R}^m$, let's study the function L_p defined above.

$$L_p(x(u_1, \dots, u_k)) = \|x(u_1, \dots, u_k) - p\|^2 = x^T x - 2x^T p + p^T p.$$

We have

$$(L_p)_* \frac{\partial}{\partial u_i} = 2\left(\frac{\partial x}{\partial u_i}\right)^T (x - p).$$

Hence q is a critical point of L_p if and only if $q - p$ is normal to $i_*(T_q S)$ at q .

Since $\frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} L_p = 2((\frac{\partial x}{\partial u_i})^T \frac{\partial x}{\partial u_j} + (\frac{\partial^2 x}{\partial u_i \partial u_j})^T (x - p))$. It follows from the proof of Lemma 6 that $(\frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} L_p)$ is singular at q if and only if $p = q + tv$ where v is unit vector normal to $i_*(T_q S)$ at q and

$t = K_i$ where K_i is the principal curvature of S at q in the normal direction v . Hence we have proved that

Lemma 7. *The point $q \in S$ is a degenerate critical point of L_p if and only if p is a focal point of (S, q) .*

Theorem 8. *For almost all $p \in \mathbb{R}^m$ the function $L_p : S \rightarrow \mathbb{R}$ has no degenerate critical point.*

Proof. The result follows from Lemma 7 and Corollary 2. $\square$

Theorem 9. *Assume that S can be embedded as a bounded subset of $\mathbb{R}^m$. Let $f \in C^\infty(S)$ be bounded. Then for any $\epsilon > 0$, there exists a Morse function $g \in C^\infty(S)$, such that*

$$|g(y) - f(y)| < \epsilon,$$

for any $y \in S$.

Proof. Let $h : S \rightarrow \mathbb{R}^m$ be the bounded embedding, with the first coordinate h_1 is precisely the given smooth function f . Let c be a large number. Choose a point

$$p = (-c + \epsilon_1, \epsilon_2, \dots, \epsilon_m)$$

close to $(-c, 0, \dots, 0) \in \mathbb{R}^m$ such that the function $L_p : S \rightarrow \mathbb{R}$ is a Morse function and let

$$g(x) = \frac{L_p(x) - c^2}{2c}.$$

g is a Morse function and by computation we have

$$g(x) = f(x) + \sum_{i=1}^m h_i(x)^2/2c - \sum_{i=1}^m \epsilon_i h_i(x)/c + \sum_{i=1}^m \epsilon_i^2/2c - \epsilon_1.$$

Since $h_i, i = 1, \dots, m$ is bounded, choose c sufficiently large and ϵ_i sufficiently small, then

$$|g(y) - f(y)| < \epsilon,$$

for any $y \in S$. This completes the proof. $\square$

6. Infinitesimal Stability of Morse Functions on Subcartesian Spaces

In this section, we study stability of Morse functions on subcartesian spaces. see [21] for a systematic treatment on stability theory of Morse functions on smooth manifolds.

Lemma 8. [21] *Let f be a smooth function in $\mathbb{R}^n$ with $f(0) = 0$. Then*

$$f(x_1, \dots, x_n) = \sum_{i=1}^n x_i g_i(x_1, \dots, x_n),$$

where g_i are smooth functions on $\mathbb{R}^n$ such that $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

Lemma 9. [21] *Let p be a non-degenerate critical point for $f \in C^\infty(\mathbb{R}^n)$. Then there is a local coordinate system $(y^1, \dots, y^n)$ in a neighborhood U of p with $y^i(p) = 0$ for all i and such that the identity*

$$f(y^1, \dots, y^n) = f(p) - (y^1)^2 - \dots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \dots + (y^n)^2$$

holds throughout U .

Lemma 10. Let $f \in C^\infty(S)$. Let $x \in S$ be a nondegenerate critical point of f . Then there is a local coordinate system $(\phi, U, \mathbb{R}^n)$ of x with local coordinate system $(y^1, \dots, y^n)$ in $\mathbb{R}^n$ such that f has a smooth extension $\tilde{f}$ on $\mathbb{R}^n$

$$\tilde{f}(y^1, \dots, y^n) = f(p) - (y^1)^2 - \dots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \dots + (y^n)^2. \quad (7)$$

Proof. Let $(\phi, U, \mathbb{R}^n)$ be a local coordinate system of x such that $\phi(x) = 0$. Let $\tilde{f}$ be a smooth extension of f . Hence $\tilde{f}(0) = f(p)$. Since S has constant structural dimension and x is a nondegenerate critical point of f , it follows that 0 is a nondegenerate critical point of $\tilde{f}$. Then the result follows immediately from Lemma 9. $\square$

Corollary 3. The set of critical points of a Morse function on S is discrete.

Proof. The result follows from Lemma 10 directly. $\square$

Definition 15. Let S_1, S_2 be two subcartesian spaces. Let $\Phi : S_1 \rightarrow S_2$ be smooth.

(a) Let $\pi_{S_2} : TS_2 \rightarrow S_2$ be the canonical projection, and let $w : S_1 \rightarrow TS_2$ be smooth. Then w is a derivation along Φ if $\pi_{S_2} \circ w = \Phi$. Let $C_\Phi^\infty(S_1, S_2)$ denote the set of derivation along Φ .

(b) Φ is infinitesimally stable if for every w , a derivation along Φ , there is a derivation s on S_1 and a derivation t on S_2 such that

$$w = (d\Phi)s + t \circ \Phi.$$

Theorem 10. Let S be compact. Let $f \in C^\infty(S)$ be a Morse function all of whose critical values are distinct, i.e., if p and q are distinct critical points of f in S , then $f(p) \neq f(q)$. Then f is infinitesimal stable.

Proof. Let $w : S \rightarrow \mathbb{R} \times \mathbb{R}$ be a derivation along f . Then $w(x) = (f(x), \bar{w}(x))$ for every $x \in S$, where $\bar{w} \in C^\infty(S)$. Let s be a derivation on S . Then $df(s)(x) = (f(x), s(f)(x))$. Let t be a vector field on $\mathbb{R}$. Then $t \circ f(x) = (f(x), \bar{t}(f(x)))$, where $\bar{t} \in C^\infty(\mathbb{R})$. The condition of infinitesimal stability reduces in this case to the following: for every $w \in C^\infty(S)$, there exists a derivation s of S and a function $t \in C^\infty(\mathbb{R})$ such that

$$w = df(s) + t \circ f. \quad (8)$$

We now show how to solve the above equation. Since S is compact, it follows that there is only a finite number of critical points of f . Since all the critical values of f are distinct, we choose $t \in C^\infty(\mathbb{R})$ such that $t(f(x)) = w(x)$ for every critical point x of f . To solve (8), it is sufficient to solve

$$w = df(s), \quad (9)$$

where $w \in C^\infty(S)$ satisfies that $w(x) = 0$ for x being critical point of f . We now construct s .

Around each point p in S , choose an open neighborhood U_p with local coordinates $(U_p, \Phi_p, \mathbb{R}^n)$, such that both f and w have smooth extensions $\tilde{f}$ and $\tilde{w}$ on $\mathbb{R}^n$ with $\tilde{f} \circ \Phi = f$ and $\tilde{w} \circ \Phi = w$. Besides,

(a) If p is a regular point, choose U_p so small that $(df)_q \neq 0$ for every $q \in U_p$. Choose a derivation s^p on U_p such that $(df)(s^p) \neq 0$ on U_p .

(b) If p is a critical point, then $\tilde{f} = c + \epsilon_1 x_1^2 + \dots + \epsilon_n x_n^2$ where $\epsilon_1, \dots, \epsilon_n = \pm 1$. $w(p_i) = 0$ and since $\tilde{w}(0) = 0$, it follows from Lemma 8 that $\tilde{w} = \sum_{i=1}^n h_i(x) x_i$, where $h_i, i = 1, \dots, n$ are smooth functions on $\mathbb{R}^n$.

The collection $\{U_p\}_{p \in S}$ forms an open covering of S . Since S is compact, there exists a finite subcovering $U_1, \dots, U_m$ corresponding to $p_1, \dots, p_m$. Let $\rho_1, \dots, \rho_m$ be a partition of unity subordinate to this covering. Choose derivations s_i on S ($1 \leq i \leq m$) as follows:

(a) if p_i is a regular point, then let

$$s_i(x) = \begin{cases} \frac{w(x)\rho_i(x)s^{p_i}(x)}{df(s^{p_i})(x)}, & x \in U_{p_i} \\ 0, & x \in S \setminus U_{p_i}. \end{cases}$$

(b) If p_i is a critical point, let $\tilde{s}_i = \sum_{j=1}^n \frac{\epsilon_j h_j}{2} \frac{\partial}{\partial x_j}$ on $\mathbb{R}^n$. Since S has constant structural dimension n , it follows from Proposition 1 that $\tilde{s}_i$ defines a derivation $\hat{s}_i$ on U_i , since for any $f \in N(S)$, $\frac{\partial}{\partial x_i}(f) \in N(S)$, otherwise $f^{-1}(0)$ has dimension less than n . Let $s_i = \rho_i \hat{s}_i$.

If p_i is a regular point, then

$$s_i(f) = \begin{cases} \frac{w\rho_i s^{p_i}(f)}{df(s^{p_i})} = w\rho_i, & x \in U_i \\ 0, & x \in S \setminus U_i. \end{cases}$$

If p_i is a singular point, then

$$s_i(f) = \begin{cases} \rho_i \sum_{j=1}^n \frac{\epsilon_j h_j}{2} \frac{\partial}{\partial x_j} (c + \epsilon_1 x_1^2 + \cdots + \epsilon_n x_n^2) = \rho_i \sum_{j=1}^n h_j x_j = \rho_i w, & \text{on } U_i \\ 0, & \text{on } S \setminus U_i. \end{cases}$$

Let $s = s_1 + \cdots + s_n$. It follows that $s(f) = \rho_1 w + \cdots + \rho_n w = \sum_{i=1}^n \rho_i w = w$. Hence (9) is solved. The result follows immediately. $\square$

7. Conclusions

In this paper we have initiated a study of the differential topological properties for a subclass of singular space—the subcartesian space. We first get further results on partition of unity for differential spaces. We then study the tubular neighborhood and get the tubular neighborhood theorem for subcartesian space with constant structural dimension, both locally and globally. By taking advantage of the definition of derivations we define Morse functions on differential spaces. For subcartesian space S with constant structural dimension, we provide examples of Morse functions, which show that Morse functions are plentiful. Further, assume that S admits a bounded embedding in some Euclidean space, we show that bounded smooth functions on S can be approximated by Morse functions. We prove that the set of critical points of any Morse function is discrete in S . Further, if S is compact, we prove that the Morse functions are infinitesimal stable. In future we would like to discover more properties on the differential topological aspects of subcartesian spaces.

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