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K. MAHESH KRISHNA *

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Noncommutative Donoho-Elad-Gribonval-Nielsen-Fuchs Sparsity Theorem

K. Mahesh Krishna

School of Mathematics and Natural Sciences, Chanakya University Global Campus, NH-648, Haraluru Village, Devanahalli Taluk, Bengaluru Rural District, Karnataka State, 562 110, India, kmaheshak@gmail.com

Abstract: Breakthrough Sparsity Theorem, derived independently by Donoho and Elad *[Proc. Natl. Acad. Sci. USA, 2003]*, Gribonval and Nielsen *[IEEE Trans. Inform. Theory, 2003]* and Fuchs *[IEEE Trans. Inform. Theory, 2004]* says that unique sparse solution to NP-Hard ℓ_0-minimization problem can be obtained using unique solution of P-Type ℓ_1-minimization problem. In this paper, we derive noncommutative version of their result using frames for Hilbert C*-modules.

Keywords: Sparse solution; Frame; Hilbert C*-module.

MSC: 42C15; 46L08

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1. Introduction

Let $\mathcal{H}$ be a finite dimensional Hilbert space over $\mathbb{K}$ ($\mathbb{C}$ or $\mathbb{R}$). A finite collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{H}$ is said to be a **frame** (also known as **dictionary**) [1,2] for $\mathcal{H}$ if there are $a, b > 0$ such that

$$a\|h\|^2 \leq \sum_{j=1}^{n} |\langle h, \tau_j\rangle|^2 \leq b\|h\|^2, \quad \forall h \in \mathcal{H}.$$

A frame $\{\tau_j\}_{j=1}^n$ for $\mathcal{H}$ is said to be **normalized** if $\|\tau_j\| = 1$ for all $1 \leq j \leq n$. Note that any frame can be normalized by dividing each element by its norm. Given a frame $\{\tau_j\}_{j=1}^n$ for $\mathcal{H}$, we define the analysis operator

$$\theta_\tau : \mathcal{H} \ni h \mapsto \theta_\tau h := (\langle h, \tau_j\rangle)_{j=1}^n \in \mathbb{K}^n.$$

Adjoint of the analysis operator is known as the synthesis operator whose expression is

$$\theta_\tau^* : \mathbb{K}^n \ni (a_j)_{j=1}^n \mapsto \theta_\tau^*(a_j)_{j=1}^n := \sum_{j=1}^{n} a_j\tau_j \in \mathcal{H}.$$

Given $d \in \mathbb{K}^n$, let $\|d\|_0$ be the number of nonzero entries in d. Following ℓ_0-minimization problem appears in many of electronic devices.

Problem 1. *Let $\{\tau_j\}_{j=1}^n$ be a normalized frame for $\mathcal{H}$. Given $h \in \mathcal{H}$, solve*

$$\textit{minimize } \{\|d\|_0 : d \in \mathbb{K}^n\} \quad \textit{subject to} \quad \theta_\tau^* d = h.$$

Recall that $c \in \mathbb{K}^n$ is said to be a unique solution to Problem 1 if it satisfies following two conditions.

(i) $\theta_\tau^* c = h$.
(ii) If $d \in \mathbb{K}^n$ satisfies $\theta_\tau^* d = h$, then

$$\|d\|_0 > \|c\|_0.$$

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In 1995, Natarajan showed that Problem 1 is NP-Hard [3,4]. As the operator θ_τ^* is surjective, for a given $h \in \mathcal{H}$, there is a $d \in \mathbb{K}^n$ such that $\theta_\tau^* d = h$. Thus the central problem is to say when the solution to Problem 1 is unique. It is well-known that [5–7] following problem is the closest convex relaxation problem to Problem 1.

Problem 2. *Let $\{\tau_j\}_{j=1}^n$ be a normalized frame for $\mathcal{H}$. Given $h \in \mathcal{H}$, solve*

$$\textit{minimize } \{\|d\|_1 : d \in \mathbb{K}^n\} \quad \textit{subject to} \quad \theta_\tau^* d = h.$$

There are several linear programmings available to obtain solution of Problem 2 and it is a P-problem [8–10].
Most important result which shows that by solving Problem 2 we also get a solution to Problem 1 is obtained independently by Donoho and Elad [11], Gribonval and Nielsen [12] and Fuchs [13,14] is the following.

Theorem 3. *[11–16] (**Donoho-Elad-Gribonval-Nielsen-Fuchs Sparsity Theorem**) Let $\{\tau_j\}_{j=1}^n$ be a normalized frame for $\mathcal{H}$. If $h \in \mathcal{H}$ can be written as $h = \theta_\tau^* c$ for some $c \in \mathbb{K}^n$ satisfying*

$$\|c\|_0 < \frac{1}{2}\left(1 + \frac{1}{\max_{1\leq j,k\leq n, j\neq k} |\langle \tau_j, \tau_k\rangle|}\right),$$

then c is the unique solution to Problem 2 and Problem 1.

Our fundamental motivation comes from the following question: What is the noncommutative analogue of Theorem 3? This is then naturally connected with the notion of Hilbert C*-modules which are first introduced by Kaplansky [17] for modules over commutative C*-algebras and later developed for modules over arbitrary C*-algebras by Paschke [18] and Rieffel [19]. We end the introduction by recalling the definition of Hilbert C*-modules.

Definition 4. *[17–19] Let $\mathcal{A}$ be a unital C*-algebra. A left module $\mathcal{E}$ over $\mathcal{A}$ is said to be a (left) Hilbert C*-module if there exists a map $\langle \cdot, \cdot \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ such that the following hold.*

(i) $\langle x, x\rangle \geq 0, \forall x \in \mathcal{E}$. *If* $x \in \mathcal{E}$ *satisfies* $\langle x, x\rangle = 0$, *then* $x = 0$.
(ii) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle, \forall x, y, z \in \mathcal{E}$.
(iii) $\langle ax, y\rangle = a\langle x, y\rangle, \forall x, y \in \mathcal{E}, \forall a \in \mathcal{A}$.
(iv) $\langle x, y\rangle = \langle y, x\rangle^*, \forall x, y \in \mathcal{E}$.
(v) $\mathcal{E}$ *is complete w.r.t. the norm* $\|x\| := \sqrt{\|\langle x, x\rangle\|}, \forall x \in \mathcal{E}$.

2. Noncommutative Donoho-Elad-Gribonval-Nielsen-Fuchs Sparsity Theorem

Observe that the notion of frames is needed for Theorem 3. Thus we want noncommutative frames. These are introduced in 2002 by Frank and Larson in their seminal paper [20]. We begin by recalling the definition of noncommutative frames for Hilbert C*-modules. This notion is already well-developed in parallel with Hilbert space frame theory [21–23]. In the paper, we consider only finite rank modules.

Definition 5. *[20] Let $\mathcal{E}$ be a Hilbert C*-module over a unital C*-algebra $\mathcal{A}$. A collection $\{\tau_j\}_{j=1}^n$ in $\mathcal{E}$ is said to be a (modular) **frame** for $\mathcal{E}$ if there are real $a, b > 0$ such that*

$$a\langle x, x\rangle \leq \sum_{j=1}^n \langle x, \tau_j\rangle\langle \tau_j, x\rangle \leq b\langle x, x\rangle, \quad \forall x \in \mathcal{E}.$$

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A collection $\{\tau_j\}_{j=1}^n$ in a Hilbert C*-module $\mathcal{E}$ over unital C*-algebra $\mathcal{A}$ with identity 1 is said to have **unit inner product** if

$$\langle \tau_j, \tau_j \rangle = 1, \quad \forall 1 \leq j \leq n.$$

Let $\mathcal{A}$ be a unital C*-algebra. For $n \in \mathbb{N}$, let $\mathcal{A}^n$ be the standard left Hilbert C*-module over $\mathcal{A}$ with inner product

$$\langle (a_j)_{j=1}^n, (b_j)_{j=1}^n \rangle := \sum_{j=1}^n a_j b_j^*, \quad \forall (a_j)_{j=1}^n, (b_j)_{j=1}^n \in \mathcal{A}^n.$$

Hence norm on $\mathcal{A}^n$ is

$$\|(a_j)_{j=1}^n\|_2 := \left\| \sum_{j=1}^n a_j a_j^* \right\|^{\frac{1}{2}}, \quad \forall (a_j)_{j=1}^n \in \mathcal{A}^n.$$

We define

$$\|(a_j)_{j=1}^n\|_1 := \sum_{j=1}^n \|a_j\|, \quad \forall (a_j)_{j=1}^n \in \mathcal{A}^n.$$

A frame $\{\tau_j\}_{j=1}^n$ for $\mathcal{E}$ gives the modular analysis morphism

$$\theta_\tau : \mathcal{E} \ni x \mapsto \theta_\tau x := (\langle x, \tau_j \rangle)_{j=1}^n \in \mathcal{A}^n$$

and the modular synthesis morphism

$$\theta_\tau^* : \mathcal{A}^n \ni (a_j)_{j=1}^n \mapsto \theta_\tau^*(a_j)_{j=1}^n := \sum_{j=1}^n a_j \tau_j \in \mathcal{E}.$$

With these notions, we generalize Problems 1 and 2. In the entire paper, $\mathcal{E}$ denotes a finite rank Hilbert C*-module over a unital C*-algebra $\mathcal{A}$.

Problem 6. *Let $\{\tau_j\}_{j=1}^n$ be a unit inner product frame for $\mathcal{E}$. Given $x \in \mathcal{E}$, solve*

$$\text{minimize } \{\|d\|_0 : d \in \mathcal{A}^n\} \quad \text{subject to} \quad \theta_\tau^* d = x.$$

Problem 7. *Let $\{\tau_j\}_{j=1}^n$ be a unit inner product frame for $\mathcal{E}$. Given $x \in \mathcal{E}$, solve*

$$\text{minimize } \{\|d\|_1 : d \in \mathcal{A}^n\} \quad \text{subject to} \quad \theta_\tau^* d = x.$$

A very powerful property used to show Theorem 3 is the notion of null space property (see [15,24]). We now define the same property for Hilbert C*-modules. We use following notations. Let $\{e_j\}_{j=1}^n$ be the canonical basis for $\mathcal{A}^n$. Given $M \subseteq \{1, \ldots, n\}$ and $d = (d_j)_{j=1}^n \in \mathcal{A}^n$, define

$$d_M := \sum_{j \in M} d_j e_j.$$

Definition 8. *A unit inner product frame $\{\tau_j\}_{j=1}^n$ for $\mathcal{E}$ is said to have the (modular)* ***null space property*** *(we write NSP) of order $k \in \{1, \ldots, n\}$ if for every $M \subseteq \{1, \ldots, n\}$ with $o(M) \leq k$, we have*

$$\|d_M\|_1 < \frac{1}{2}\|d\|_1, \quad \forall d \in \ker(\theta_\tau^*), d \neq 0.$$

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We first relate NSP with Problem 7.

Theorem 9. *Let $\{\tau_j\}_{j=1}^n$ be a unit inner product frame for $\mathcal{E}$ and let $1 \leq k \leq n$. The following are equivalent.*

(i) If $x \in \mathcal{E}$ can be written as $x = \theta_\tau^* c$ for some $c \in \mathcal{A}^n$ satisfying $\|c\|_0 \leq k$, then c is the unique solution to Problem 7.
(ii) $\{\tau_j\}_{j=1}^n$ satisfies the NSP of order k.

Proof. (i) $\implies$ (ii) Let $M \subseteq \{1, \dots, n\}$ with $o(M) \leq k$ and let $d \in \ker(\theta_\tau^*), d \neq 0$. Then we have

$$0 = \theta_\tau^* d = \theta_\tau^*(d_M + d_{M^c}) = \theta_\tau^*(d_M) + \theta_\tau^*(d_{M^c})$$

which gives

$$\theta_\tau^*(d_M) = \theta_\tau^*(-d_{M^c}).$$

Define $c := d_M \in \mathcal{A}^n$ and $x := \theta_\tau^*(d_M)$. Then we have $\|c\|_0 \leq o(M) \leq k$ and

$$x = \theta_\tau^* c = \theta_\tau^*(-d_{M^c}).$$

By assumption (i), we then have

$$\|c\|_1 = \|d_M\|_1 < \|-d_{M^c}\|_1 = \|d_{M^c}\|_1.$$

Rewriting previous inequality gives

$$\|d_M\|_1 < \|d\|_1 - \|d_M\|_1 \implies \|d_M\|_1 < \frac{1}{2}\|d\|_1.$$

Hence $\{\tau_j\}_{j=1}^n$ satisfies the NSP of order k.

1. $\implies$ (i) Let $x \in \mathcal{E}$ can be written as $x = \theta_\tau^* c$ for some $c \in \mathcal{A}^n$ satisfying $\|c\|_0 \leq k$. Define $M := \operatorname{supp}(c)$. Then $o(M) = \|c\|_0 \leq k$. By assumption (ii), we then have

$$\|d_M\|_1 < \frac{1}{2}\|d\|_1, \quad \forall d \in \ker(\theta_\tau^*), d \neq 0. \tag{1}$$

Let $b \in \mathcal{A}^n$ be such that $x = \theta_\tau^* b$ and $b \neq c$. Define $a := b - c \in \mathcal{A}^n$. Then $\theta_\tau^* a = \theta_\tau^* b - \theta_\tau^* c = x - x = 0$ and hence $a \in \ker(\theta_\tau^*), a \neq 0$. Using Inequality (1), we get

$$\begin{aligned}\|a_M\|_1 < \frac{1}{2}\|a\|_1 &\implies \|a_M\|_1 < \frac{1}{2}(\|a_M\|_1 + \|a_{M^c}\|_1) \\ &\implies \|a_M\|_1 < \|a_{M^c}\|_1. \end{aligned} \tag{2}$$

Using Inequality (2) and the information that c is supported on M, we get

$$\begin{aligned}\|b\|_1 - \|c\|_1 &= \|b_M\|_1 + \|b_{M^c}\|_1 - \|c_M\|_1 - \|c_{M^c}\|_1 \\ &= \|b_M\|_1 + \|b_{M^c}\|_1 - \|c_M\|_1 = \|b_M\|_1 + \|(b-c)_{M^c}\|_1 - \|c_M\|_1 \\ &= \|b_M\|_1 + \|a_{M^c}\|_1 - \|c_M\|_1 > \|b_M\|_1 + \|a_M\|_1 - \|c_M\|_1 \\ &\geq \|b_M\|_1 + \|(b-c)_M\|_1 - \|c_M\|_1 \geq \|b_M\|_1 - \|b_M\|_1 + \|c_M\|_1 - \|c_M\|_1 = 0.\end{aligned}$$

Hence c is the unique solution to Problem 7.
□

Using Theorem 9 we obtain modular version of Theorem 3.

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Theorem 10. *Let $\{\tau_j\}_{j=1}^n$ be a unit inner product frame for $\mathcal{E}$. If $x \in \mathcal{E}$ can be written as $x = \theta_\tau^* c$ for some $c \in \mathcal{A}^n$ satisfying*

$$\|c\|_0 < \frac{1}{2}\left(1 + \frac{1}{\max\limits_{1\leq j,k\leq n, j\neq k} \|\langle \tau_j, \tau_k\rangle\|}\right), \tag{3}$$

then c is the unique solution to Problem 7.

Proof. We show that $\{\tau_j\}_{j=1}^n$ satisfies the NSP of order $k := \|c\|_0$. Then Theorem 9 says that c is the unique solution to Problem 7. Let $x \in \mathcal{E}$ can be written as $x = \theta_\tau^* c$ for some $c \in \mathcal{A}^n$ satisfying $\|c\|_0 \leq k$. Let $M \subseteq \{1, \dots, n\}$ with $o(M) \leq k$ and let $d = (d_j)_{j=1}^n \in \ker(\theta_\tau^*), d \neq 0$. Then we have

$$\theta_\tau \theta_\tau^* d = 0.$$

For each fixed $1 \leq k \leq n$, above equation gives

$$\begin{aligned} 0 &= \langle \theta_\tau \theta_\tau^* (d_j)_{j=1}^n, e_k\rangle = \langle \theta_\tau^* (d_j)_{j=1}^n, \theta_\tau^* e_k\rangle \\ &= \langle \theta_\tau^* (d_j)_{j=1}^n, \tau_k\rangle = \sum_{j=1}^n d_j \langle \tau_j, \tau_k\rangle \\ &= d_k\langle \tau_k, \tau_k\rangle + \sum_{j=1, j\neq k}^n d_j\langle \tau_j, \tau_k\rangle = d_k + \sum_{j=1, j\neq k}^n d_j\langle \tau_j, \tau_k\rangle. \end{aligned}$$

Therefore

$$d_k = -\sum_{j=1, j\neq k}^n d_j\langle \tau_j, \tau_k\rangle, \quad \forall 1 \leq k \leq n.$$

By taking norm,

$$\begin{aligned} \|d_k\| &= \left\|\sum_{j=1, j\neq k}^n d_j\langle \tau_j, \tau_k\rangle\right\| \leq \sum_{j=1, j\neq k}^n \|d_j\langle \tau_j, \tau_k\rangle\| \\ &\leq \sum_{j=1, j\neq k}^n \|d_j\|\|\langle \tau_j, \tau_k\rangle\| \leq \left(\max_{1\leq j,k\leq n, j\neq k} \|\langle \tau_j, \tau_k\rangle\|\right) \sum_{j=1, j\neq k}^n \|d_j\| \\ &= \left(\max_{1\leq j,k\leq n, j\neq k} \|\langle \tau_j, \tau_k\rangle\|\right)\left(\sum_{j=1}^n \|d_j\| - \|d_k\|\right) \\ &= \left(\max_{1\leq j,k\leq n, j\neq k} \|\langle \tau_j, \tau_k\rangle\|\right)(\|d\|_1 - \|d_k\|), \qquad \forall 1 \leq k \leq n. \end{aligned}$$

By rewriting above inequality we get

$$\left(1 + \frac{1}{\max\limits_{1\leq j,k\leq n, j\neq k} \|\langle \tau_j, \tau_k\rangle\|}\right)\|d_k\| \leq \|d\|_1, \quad \forall 1 \leq k \leq n. \tag{4}$$

Summing Inequality (4) over M leads to

$$\left(1+\frac{1}{\max\limits_{1\leq j,k\leq n, j\neq k}\|\langle\tau_j,\tau_k\rangle\|}\right)\|d_M\|_1 = \left(1+\frac{1}{\max\limits_{1\leq j,k\leq n, j\neq k}\|\langle\tau_j,\tau_k\rangle\|}\right)\sum_{k\in M}\|d_k\|$$
$$\leq \|d\|_1\sum_{k\in M}1 = \|d\|_1 o(M).$$

Finally using Inequality (3)

$$\begin{aligned}\|d_M\|_1 &\leq \left(1+\frac{1}{\max\limits_{1\leq j,k\leq n, j\neq k}\|\langle\tau_j,\tau_k\rangle\|}\right)^{-1}\|d\|_1 o(M)\\ &\leq \left(1+\frac{1}{\max\limits_{1\leq j,k\leq n, j\neq k}\|\langle\tau_j,\tau_k\rangle\|}\right)^{-1}\|d\|_1 k\\ &= \left(1+\frac{1}{\max\limits_{1\leq j,k\leq n, j\neq k}\|\langle\tau_j,\tau_k\rangle\|}\right)^{-1}\|d\|_1\|c\|_0\\ &< \frac{1}{2}\|d\|_1.\end{aligned}$$

Hence $\{\tau_j\}_{j=1}^n$ satisfies the NSP of order k. □

Theorem 11. *(Noncommutative Donoho-Elad-Gribonval-Nielsen-Fuchs Sparsity Theorem) Let* $\{\tau_j\}_{j=1}^n$ *be a unit inner product frame for* $\mathcal{E}$. *If* $x\in\mathcal{E}$ *can be written as* $x=\theta_\tau^* c$ *for some* $c\in\mathcal{A}^n$ *satisfying*

$$\|c\|_0 < \frac{1}{2}\left(1+\frac{1}{\max\limits_{1\leq j,k\leq n, j\neq k}\|\langle\tau_j,\tau_k\rangle\|}\right),$$

then c *is the unique solution to Problem 6.*

Proof. Theorem 10 says that c is the unique solution to Problem 7. Let $d\in\mathcal{A}^n$ be such that $x=\theta_\tau^* d$. We claim that $\|d\|_0 > \|c\|_0$. If this fails, we must have $\|d\|_0 \leq \|c\|_0$. We then have

$$\|d\|_0 < \frac{1}{2}\left(1+\frac{1}{\max\limits_{1\leq j,k\leq n, j\neq k}\|\langle\tau_j,\tau_k\rangle\|}\right).$$

Theorem 10 again says that d is also the unique solution to Problem 7. Therefore we must have $\|c\|_1 < \|d\|_1$ and $\|c\|_1 > \|d\|_1$ which is a contradiction. So claim holds and we have $\|d\|_0 > \|c\|_0$. □

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