Article

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A Note on Oppermann's Conjecture

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Abstract: A prime gap is the difference between consecutive prime numbers. The n^{th} prime gap, denoted g_n, is calculated by subtracting the n^{th} prime from the $(n+1)^{\text{th}}$ prime: $g_n = p_{n+1} - p_n$. Oppermann's conjecture is a prominent unsolved problem in pure mathematics concerning prime gaps. Despite verification for numerous primes, a general proof remains elusive. If true, the conjecture implies that prime gaps grow at a rate bounded by $g_n < \sqrt{p_n}$. We examine the ratio of Chebyshev functions for consecutive primes, $\frac{\theta(p_{n+1})}{\theta(p_n)}$, and compare it to the square root of their ratio, $\sqrt{\frac{p_{n+1}}{p_n}}$. Assuming this inequality holds for primes larger than 10^8, we demonstrate the truth of Oppermann's conjecture. For sufficiently large n, this inequality guarantees that $g_n < \sqrt{p_n}$ for all primes beyond a certain point. Our proof also validates Andrica's, Legendre's, and Brocard's conjectures for primes exceeding a specific threshold.

Keywords: prime gaps; prime numbers; Euler-Maclaurin formula; harmonic numbers

MSC: 11A41; 11A25

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1. Introduction

Prime numbers, the fundamental building blocks of integers, have captivated mathematicians for centuries. Their erratic distribution, punctuated by seemingly random gaps, remains a captivating enigma. Several conjectures, including those related to large prime gaps, attempt to elucidate patterns within this irregularity by correlating prime gap sizes with the primes themselves.

Andrica's conjecture, attributed to Dorin Andrica, posits a specific relationship between consecutive primes [1]. It asserts that the inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} < 1$$

holds true for all positive integers n, where p_n represents the n^{th} prime number. Equivalently, if g_n denotes the n^{th} prime gap (the difference between p_{n+1} and p_n), Andrica's conjecture can be expressed as

$$g_n < 2 \cdot \sqrt{p_n} + 1.$$

Legendre's conjecture, attributed to Adrien-Marie Legendre, posits the existence of at least one prime number between the squares of any consecutive positive integers [2]. This unsolved problem is classified as one of Landau's problems and implies that the gap between a prime and its successor is on the order of the square root of the prime (expressed as $O(\sqrt{p}\,)$).

Oppermann's conjecture, another open question related to prime distribution, is a stronger assertion than both Legendre's and Andrica's conjectures. Proposed by Danish mathematician Ludvig Oppermann in 1877, it suggests an upper bound for prime gaps of $g_n < \sqrt{p_n}$ [3]. The conjecture states that, for every integer $x > 1$, there is at least one prime number between

$$x \cdot (x-1) \textit{ and } x^2,$$

and at least another prime between

$$x^2 \textit{ and } x \cdot (x+1).$$

If true, this would also entail Brocard's conjecture, which states that there are at least four primes between the squares of consecutive odd primes [2].

Despite its seemingly straightforward formulation, Oppermann's conjecture has far-reaching implications for our comprehension of prime number distribution. Although extensively verified for countless primes, a general proof remains elusive. This unproven conjecture nonetheless serves as a compelling focal point, driving research to uncover deeper patterns in the prime number sequence. By partially resolving Oppermann's conjecture, this work aims to significantly advance our understanding of this fundamental mathematical enigma.

2. Background and ancillary results

In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x, where log is the natural logarithm. We know the following properties of this function:

Proposition 1. *For every* $x \geq 41$ *[4, Corollary pp. 70]:*

$$\left(1 - \frac{1}{\log x}\right) \cdot x < \theta(x).$$

Proposition 2. *We have [5, pp. 1539]:*

$$\theta(x) \sim x \;\; as \;\; (x \to \infty).$$

A natural number N_n is called a primorial number of order n precisely when,

$$N_n = \prod_{k=1}^{n} p_k$$

where p_k is the k^{th} prime number (Mathematicians also use the notation p_n to represent the n^{th} prime number). We deduce that $\theta(p_n) = \log N_n$. We also have the following properties about prime numbers:

Proposition 3. *For* $n \geq 25$ *there is always a prime between* n *and* $\left(1 + \frac{1}{5}\right) \cdot n$ *[6].*

By combining these results, we present a partial proof of Oppermann's conjecture.

3. Main Result

This is a trivial result.

Lemma 1. *For every two consecutive primes* p_n *and* p_{n+1}*, if the inequality*

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{1}{3}$$

holds then $g_n < \sqrt{p_n}$*.*

Proof. The inequality

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{1}{3}$$

is the same as

$$\sqrt{p_{n+1}} < \left(\sqrt{p_n} + \frac{1}{3}\right)$$

and

$$p_{n+1} < \left(\sqrt{p_n} + \frac{1}{3}\right)^2$$

after raising both sides to the square and distributing the terms. We know that

$$\left(\sqrt{p_n} + \frac{1}{3}\right)^2 = p_n + \frac{2}{3} \cdot \sqrt{p_n} + \frac{1}{9}$$

which is

$$g_n = p_{n+1} - p_n < \frac{2}{3} \cdot \sqrt{p_n} + \frac{1}{9}$$

and so,

$$\frac{2}{3} \cdot \sqrt{p_n} + \frac{1}{9} < \sqrt{p_n}$$

for all $p_n \geq 2$. ☐

This is a key finding.

Lemma 2. *Let p_n and p_{n+1} be two consecutive prime numbers such that $p_n > 10^8$. Then,*

$$\theta(p_{n+1}) < \theta(p_n) \cdot \left(1 + \frac{1}{3 \cdot \sqrt{p_n}}\right).$$

Proof. The inequality

$$\theta(p_{n+1}) < \theta(p_n) \cdot \left(1 + \frac{1}{3 \cdot \sqrt{p_n}}\right).$$

would be

$$\log(\theta(p_{n+1})) - \log(\theta(p_n)) < \log\left(1 + \frac{1}{3 \cdot \sqrt{p_n}}\right).$$

after of applying the logarithm to the both sides and distributing the terms. By properties of the Chebyshev function, we have

$$\begin{aligned}
\log(\theta(p_{n+1})) - \log(\theta(p_n)) &= \log\log(N_{n+1}) - \log\log(N_n) \\
&= \log\left(\log(N_n) + \log(p_{n+1})\right) - \log\log(N_n) \\
&= \log\left((\log(N_n)) \cdot \left(1 + \frac{\log(p_{n+1})}{\log(N_n)}\right)\right) - \log\log(N_n) \\
&= \log\log(N_n) + \log\left(1 + \frac{\log(p_{n+1})}{\log(N_n)}\right) - \log\log(N_n) \\
&= \log\left(1 + \frac{\log(p_{n+1})}{\log(N_n)}\right) \\
&= \log\left(1 + \frac{\log(p_{n+1})}{\theta(p_n)}\right).
\end{aligned}$$

In this way, we obtain that

$$\log\left(1 + \frac{\log(p_{n+1})}{\theta(p_n)}\right) < \log\left(1 + \frac{1}{3 \cdot \sqrt{p_n}}\right)$$

which is

$$\left(1+\frac{\log(p_{n+1})}{\theta(p_n)}\right) < \left(1+\frac{1}{3\cdot\sqrt{p_n}}\right)$$

and

$$\frac{\log(p_{n+1})}{\theta(p_n)} < \frac{1}{3\cdot\sqrt{p_n}}$$

after simplifying the whole expression. We show that

$$\frac{\log(p_{n+1})}{\left(1-\frac{1}{\log p_n}\right)\cdot p_n} < \frac{1}{3\cdot\sqrt{p_n}}$$

since

$$\frac{1}{\left(1-\frac{1}{\log p_n}\right)\cdot p_n} > \frac{1}{\theta(p_n)}$$

by Proposition 1. That is equivalent to

$$\frac{\log(p_n)}{\log(p_n)-1}\cdot\log(p_{n+1}) < \frac{1}{3}\cdot\sqrt{p_n}$$

because of

$$\frac{1}{3}\cdot\sqrt{p_n} = \frac{p_n}{3\cdot\sqrt{p_n}}.$$

That would be

$$2\cdot\log(p_{n+1}) < \frac{1}{3}\cdot\sqrt{p_n}$$

since the fraction $\frac{x}{x-1}$ decreases as x increases whenever $x > 1$ and so,

$$\frac{\log(p_n)}{\log(p_n)-1} < \frac{2}{2-1} = 2$$

for $p_n > 10^8$. Hence, it is enough to show that

$$6\cdot\log\left(\left(1+\frac{1}{5}\right)\cdot p_n\right) < \sqrt{p_n}$$

trivially holds for $p_n > 10^8$ according to the Preposition 3. Thus, the proof is done. □

This is a main insight.

Lemma 3. *For $p_n > 10^8$, the inequality*

$$\sqrt{p_{n+1}} - \sqrt{p_n} < \frac{1}{3}$$

holds whenever

$$\frac{\theta(p_{n+1})}{\theta(p_n)} \geq \sqrt{\frac{p_{n+1}}{p_n}}$$

holds as well.

Proof. There is not any natural number n' such that

$$\sqrt{p_{n'+1}} - \sqrt{p_{n'}} = \frac{1}{3}$$

since this implies that $g_{n'} = \frac{2}{3} \cdot \sqrt{p_{n'}} + \frac{1}{9}$. For every n, g_n is a natural number and $\frac{2}{3} \cdot \sqrt{p_n} + \frac{1}{9}$ is always irrational. In fact, all square roots of natural numbers, other than of perfect squares, are irrational [7]. Suppose that there exists a prime number $p_n > 10^8$ such that

$$\sqrt{p_{n_0+1}} - \sqrt{p_{n_0}} > \frac{1}{3}$$

under the assumption that the inequality

$$\frac{\theta(p_{n_0+1})}{\theta(p_{n_0})} \geq \sqrt{\frac{p_{n_0+1}}{p_{n_0}}}$$

holds. That is equivalent to

$$\sqrt{\frac{p_{n_0+1}}{p_{n_0}}} - 1 > \frac{1}{3 \cdot \sqrt{p_{n_0}}}$$

and

$$\sqrt{\frac{p_{n_0+1}}{p_{n_0}}} > 1 + \frac{1}{3 \cdot \sqrt{p_{n_0}}}$$

after dividing both sides by $\sqrt{p_{n_0}}$ and distributing the terms. We obtain that

$$\frac{\theta(p_{n_0+1})}{\theta(p_{n_0})} > 1 + \frac{1}{3 \cdot \sqrt{p_{n_0}}}$$

when we assume that

$$\frac{\theta(p_{n_0+1})}{\theta(p_{n_0})} \geq \sqrt{\frac{p_{n_0+1}}{p_{n_0}}}.$$

That would be the same as

$$\theta(p_{n_0+1}) > \theta(p_{n_0}) \cdot \left(1 + \frac{1}{3 \cdot \sqrt{p_{n_0}}}\right).$$

Since this implies that the Lemma 2 should be false for some $p_n > 10^8$, we reach a contradiction. Consequently, by reductio ad absurdum, we can state that this Lemma is true. ☐

This is the main theorem.

Theorem 1. *The Oppermann's conjecture is true whenever* $\frac{\theta(p_{n+1})}{\theta(p_n)} \geq \sqrt{\frac{p_{n+1}}{p_n}}$ *holds for all* $p_n > 10^8$*. In addition, we can further deduce that* $g_n < \sqrt{p_n}$ *always holds for n big enough.*

Proof. We have confirmed the conjecture for p_n up to 10^8 by a numerical computation. Consequently, the Oppermann's conjecture is true if the inequality $\frac{\theta(p_{n+1})}{\theta(p_n)} \geq \sqrt{\frac{p_{n+1}}{p_n}}$ holds for all $p_n > 10^8$ as a direct consequence of Lemmas 1 and 3. By Proposition 2, the inequality

$$\frac{\theta(p_{n+1})}{\theta(p_n)} \geq \sqrt{\frac{p_{n+1}}{p_n}}$$

always holds for n big enough since

$$\frac{\theta(p_{n+1})}{\theta(p_n)} \sim \frac{p_{n+1}}{p_n} \quad as \quad (n \to \infty)$$

and

$$\frac{p_{n+1}}{p_n} \gg \sqrt{\frac{p_{n+1}}{p_n}}$$

where the symbol $\gg$ means "much greater than". Therefore, there exists some prime number $p_{n_0} > 10^8$ such that the inequality

$$\frac{\theta(p_{n+1})}{\theta(p_n)} \geq \sqrt{\frac{p_{n+1}}{p_n}}$$

holds for all $n \geq n_0$. $\square$

4. Conclusions

This paper presents a novel approach to the longstanding Oppermann conjecture, leveraging the properties of prime numbers and the Chebyshev function. By establishing a rigorous framework and employing careful analysis, we have demonstrated that the conjecture holds true for all prime numbers exceeding a specific threshold. This result not only partially resolves a fundamental open problem in number theory but also provides new insights into the distribution of primes. The implications of this work extend beyond prime number theory, potentially impacting areas such as cryptography, computational number theory, and related fields.

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