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Posted Date: 15 August 2024

doi: 10.20944/preprints202408.1150.v1

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Article

Sum of the Squares of the Extended (k,t)–Fibonacci Numbers

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Abstract: In two previous articles, the extended (k,t)–Fibonacci numbers were presented and some of their properties were studied, both in general and particularizing them for the Leonardo numbers. In a second article we study the sum of the extended (k,t)–Fibonacci numbers as well as the sums of these odd and even numbers. In this article we study the sum of the squares of these numbers with their properties as well as the generating function, the recurrence relation and the Binet formula. Finally we particularize these results for the generalized t–Leonardo numbers.

Keywords: k–Fibonacci numbers; binet identity; recurrence relation; generating function; leonardo numbers

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1. Introduction

Leonardo of Pisa or Fibonacci (12th-13th centuries) was a mathematician from the city-state of Pisa who made great contributions to Mathematics, the most important of which was perhaps the introduction of decimal numbering in Europe from a practical point of view.

As a mathematician, he is most famous for the so-called Fibonacci numbers $\{0,1,1,2,3,5,8,\ldots\}$ in which each number is the sum of the two previous ones $F_n = F_{n-1} + F_{n-2}$.

It is believed that he also introduced the so-called Leonardo numbers created with the same previous relationship but adding a unit to the indicated sum and the first two being 1 and 1: $Le_n = Le_{n-1} + Le_{n-2} + 1$.

Leonardo numbers gained importance in the mid-20th century because they were used by computer scientist Edsger W. Dijkstra in his algorithm for finding the shortest path between two vertices of a graph.

There are several generalizations of the Fibonacci numbers, one of which leads to the so-called k–Fibonacci numbers. This section begins with a reminder of these numbers as well as some of their properties. Later, some of the properties of the extended (k,t)–Fibonacci numbers demonstrated in the previous articles are shown. And these results are particularized to the case of the Leonardo numbers.

The concept of Fibonacci number has been generalized in different ways [11,12]. One of them has given rise to the concept of k–Fibonacci number [8,9,14,16].

1.1. k–Fibonacci Numbers

We remember this concept as well as some of its properties.

Definition 1. *For any integer number $k \geq 1$, the k–Fibonacci numbers is defined as $F_{k,n+1} = k\,F_{k,n} + F_{k,n-1}$ with initial conditions $F_{k,0} = 0$ and $F_{k,1} = 1$*

Some of the properties that the k–Fibonacci numbers verify and that we will need later are summarized in the previous cites. In particular, and since we will use it throughout this article, we indicate the Binet identity,

$$F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2} \qquad (1)$$

 doi:10.20944/preprints202408.1150.v1

where $\sigma_1 = \dfrac{k+\sqrt{k^2+4}}{2}$ and $\sigma_2 = \dfrac{k-\sqrt{k^2+4}}{2}$ are the characteristic roots of the relation of the definition. If $k=1$, the root $\sigma_1 = \dfrac{1+\sqrt{5}}{2}$ is the golden ratio ϕ, while $\sigma_2 = \dfrac{1-\sqrt{5}}{2}$ is ψ.

Among other properties, these roots verify $\sigma_1+\sigma_2 = k$, $\sigma_1 \cdot \sigma_2 = -1$, $\sigma^2 = k\,\sigma + 1$, $\sigma_1 - \sigma_2 = \sqrt{k^2+4}$.

The sum of the first k–Fibonacci numbers is $\sum_{j=0}^{n} F_{k,j} = \dfrac{F_{k,n}+F_{k,n+1}-1}{k}$.

The generating function of the k–Fibonacci numbers is $f(k,x) = \dfrac{x}{1-k\,x-x^2}$ and the negative k–Fibonacci numbers are defined as $F_{k,-n} = (-1)^{n+1}F_{k,n}$.

1.2. Leonardo Numbers

A Leonardo number [1,2] is defined by mean of the recurrence relation $Le_n = Le_{n-1} + Le_{n-2} + 1$ with initial conditions $Le_0 = 1$ and $Le_1 = 1$. The first few terms of the sequence of Leonardo are $\{1,1,3,5,9,15,25,41,\ldots\}$, A001595 in the OEIS [13].

The Leonardo numbers are related to the Fibonacci numbers as $Le_n = 2F_{n+1} - 1$

In [6], we have found a generalization of the Leonardo numbers defined as $Le_n(t) = Le_{n-1}(t) + Le_{n-2}(t) + t$ with initial conditions $Le_0(t) = 1$ and $Le_1(t) = 1$. The first few terms of the sequence obtained are $\{Le_n\} = \{1,1,k+(1+t),k^2+(t+1)k+(t+1),k^3+(1+t)k^2+(2+t)k+(1+2t),\ldots\}$

1.3. Extended (k,t)–Fibonacci Numbers

Let t be a positive integer number. It defines the extended (k,t)–Fibonacci numbers [6,7] by mean of the linear non–homogeneus recurrence

$$T(k,t,n) = k\,T(k,t,n-1) + T(k,t,n-2) + t \tag{2}$$

with initial conditions $T(k,t,0) = 1$ and $T(k,t,1) = 1$.

According to that definition, this sequence is

$$\{T(k,t,n)\} = \{1,1,k+(t+1),k^2+(t+1)k+(t+1),k^3+(t+1)k^2+(t+2)k+(2t+1),\ldots\}$$

If $k=1$, this sequence takes the form $\{1,1,2+t,3+2t,5+4t,8+7t,13+12t,21+20t,\ldots\}$ which can be considered a generalization of Leonardo sequence and it is indicated as $\left\{Le_n(t)\right\}$. Later, if $t=1$, the classical Leonardo sequence appears.

Extended (k,t)–Fibonacci numbers $T(k,t,n) = T_n$ are related to the k–Fibonacci numbers by the formula

$$T_n = \frac{(k+t)(F_{k,n}+F_{k,n-1})-t}{k} \tag{3}$$

T_n verify the homogeneous linear recurrence $T_n = (k+1)T_{n-1} - (k-1)T_{n-2} - T_{n-3}$ and its generating function is $f(k,t,x) = \dfrac{1-k\,x-(1-k-t)x^2}{(1-x)(1-k\,x-x^2)}$.

On the other hand, the sum of the extended (k,t)–Fibonacci numbers is given by the formula $\sum_{j=0}^{n} T_j = \dfrac{1}{k}\left(\dfrac{k+t}{k}(F_{k,n+1}+2F_{k,n}+F_{k,n-1}-2)+k-n\,t\right)$

For $k=1$, the sum of the generalized Leonardo numbers $Le_n(t)$ is

$$\sum_{j=0}^{n} Le_j(t) = (t+1)(F_{n+3}-2) - n\,t + 1$$

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Moreover, $s(k,t,x) = \dfrac{1-k\,x-(1-k-t)x^2}{(1-x)^2(1-k\,x-x^2)}$ is the generating function of the sequence of partial sums $\left\{\sum_{j=0}^{n} T(k,t,j)\right\}$ and its Binet formula is

$$\sum_{j=0}^{n} T(k,t,j) = \frac{k+t}{k^2}\frac{1}{\sqrt{k^2+4}}\left(\sigma_1^{n-1}(\sigma_1+1)^2 - \sigma_2^{n-1}(\sigma_2+1)^2\right) - \frac{2(k+t)}{k^2} - \frac{n\,t}{k} + 1$$

2. Sum of the Squares of the Extended (k,t)–Fibonacci Numbers

In this paper we will study the squares of the extended (k,t)–Fiboacci numbers and their sum to later find their generating function, the recurrence relation verified by these addends and their Binet formula.

We will use Formula (3) to find the sum of the squares of the extended (k,t)–Fibonacci numbers $\sum_{j=0}^{n} T^2(k,t,j)$.

As $T(k,t,n) = \dfrac{(k+t)(F_{k,n}+F_{k,n-1})-t}{k}$ then

$$T^2(k,t,n) = \frac{1}{k^2}\left((k+t)^2(F_{k,n}+F_{k,n-1})^2 - 2t(k+t)(F_{k,n}+F_{k,n-1}) + t^2\right)$$

So

$$\begin{aligned}\sum_{j=0}^{n} T^2(k,t,j) = {} & \frac{(k+t)^2}{k^2}\left(\sum_{j=0}^{n} F_{k,j}^2 + 2\sum_{j=0}^{n} F_{k,j}F_{k,j-1} + \sum_{j=0}^{n} F_{k,j-1}^2\right) \\ & - \frac{2t(k+t)}{k^2}\left(\sum_{j=0}^{n} F_{k,j} + \sum_{j=0}^{n} F_{k,j-1}\right) + \frac{t^2}{k^2}\sum_{j=0}^{n} 1\end{aligned}$$

We will find the sum of these numbers taking into account the following formulas that have been proven in [3,5,8,9]:

$$\begin{aligned}\sum_{j=0}^{n} F_{k,j}^2 &= \frac{1}{k}F_{k,n}F_{k,n+1} \\ \sum_{j=0}^{n} F_{k,j-1}^2 &= \frac{1}{k}(F_{k,n-1}F_{k,n}+k) \\ \sum_{j=0}^{n} F_{k,j-1}F_{k,j} &= \frac{1}{k}\left(F_{k,n}^2 + \frac{(-1)^n-1}{2}\right) \\ \sum_{j=0}^{n} F_{k,j} &= \frac{1}{k}(F_{k,n}+F_{k,n+1}-1) \\ \sum_{j=0}^{n} F_{k,j-1} &= \frac{1}{k}(F_{k,n-1}+F_{k,n}+k-1) \\ \sum_{j=0}^{n} 1 &= n+1\end{aligned}$$

Consequently, if

$$\begin{aligned} m_1[k,n) &= F_{k,n}F_{k,n+1} + F_{k,n-1}F_{k,n} + k + 2\,F_{k,n}^2 + (-1)^n - 1 \\ m_2(k,n) &= 2F_{k,n} + F_{k,n+1} + F_{k,n-1} + k - 2 \\ m_3(n) &= n+1\end{aligned}$$

then

$$\sum_{j=0}^{n} T^2(k,t,j) = \frac{(k+t)^2}{k^3} m_1(k,n) - \frac{2t(k+t)}{k^3} m_2(k,n) + \frac{t^2}{k^2} m_3(n)$$

Below we will particularize this result to the case of Leonardo numbers.

2.1. *Sum of the Squares of the Leonardo Numbers*

If in the previous formula it is $k = 1$, the sum of generalized t–Leonardo numbers is $\sum_{j=0}^{n} Le_j^2(t) = (1+t)^2(F_n F_{n+3} + (-1)^n) - 2t\,(1+t)(F_{n+3} - 1) + t^2(n+1)$

Taking into account that $F_{n+3} = F_{n+1} + F_{n+2}$, this formula can be simplified as $\sum_{j=0}^{n} Le_j^2(t) = ((t+1)F_{n+1} - 2t)((t+1)F_{n+2} - 2t) + t^2(n-1) + 2t$

The first few terms of this sequence are

$$\left\{\sum_{j=0}^{n} Le_j^2(t)\right\} = \{1, 2, 6 + 4\,t + {'}t^2, 15 + 16\,t + 5t^2, 40 + 56\,t + 21t^2, \ldots\} \tag{4}$$

If, moreover, $k = 1$, for the classical Leonardo numbers this last formula becomes $\sum_{j=0}^{n} Le_j^2 = 4(F_{n+1} - 1)(F_{n+2} - 1) + n + 1$.

The sequence generated is $\left\{\sum_{j=0}^{n} Le_j^2\right\} = \{1, 2, 11, 36, 117, 342, 967, 2648, 7137, \ldots\}$ that is not indexed in the OEIS.

2.2. *Generating Function*

We will find the generating function of the sum of the squares of the extended (k,t)–Fibonacci numbers from the formula

$$\sum_{j=0}^{n} T^2(k,t,j) = \frac{(k+t)^2}{k^2}\left(\sum_{j=0}^{n} F_{k,j}^2 + 2\sum_{j=0}^{n} F_{k,j}F_{k,j-1} + \sum_{j=0}^{n} F_{k,j-1}^2\right) - \frac{2t(k+t)}{k^2}\left(\sum_{j=0}^{n} F_{k,j} + \sum_{j=0}^{n} F_{k,j-1}\right) + \frac{t^2}{k^2}\sum_{j=0}^{n} 1$$

and taking into account the generating function of each addend as indicated in the following relation:

Addend	Generating function
$\sum_{j=0}^{n} F_{k,j}^2$	$f_1(k,x) = \dfrac{x}{1-(k^2+1)(x+x^2)+x^3}$
$\sum_{j=0}^{n} F_{k,j-1}F_{k,j}$	$f_2(k,x) = \dfrac{k\,x^2}{(1-x)(1-(k^2+1)(x+x^2)+x^3)}$
$\sum_{j=0}^{n} F_{k,j-1}^2$	$f_3(k,x) = \dfrac{1-(k^2+1)x - k^2\,x^2}{(1-x)(1-(k^2+1)(x+x^2)+x^3)}$
$\sum_{j=0}^{n} F_{k,j}$	$f_4(k,x) = \dfrac{x}{(1-x)(1-k\,x-x^2)}$
$\sum_{j=0}^{n} F_{k,j-1}$	$f_5(k,x) = \dfrac{1-k\,x}{(1-x)(1-k\,x-x^2)}$
$\sum_{j=0}^{n} 1$	$f_6(x) = \dfrac{1}{(1-x)^2}$

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Proof. The three first generating functions are proven in [3,4].

H.S.Wilf demonstrates in [15] that if $f(x)$ is the generating function of the sequence $\{a_n\}$, then $\frac{1}{1-x} f(x)$ is the generating function of the sequence of partial sums $\sum_{j=0}^{n} a_j$. As the generating function of the sequence $\{F_{k,n}\}$ is $\frac{x}{1-k\,x-x^2}$ [9], the generating functions 4 and 5 are deduced.

Finally, the generating function of the constant sequence $\{1\}$ is $\frac{1}{1-x}$, from where $f_6(x)$. □

So, the generating function of the sequence $\sum_{j=0}^{n} T^2(k,t,j)$ is

$$\begin{aligned} f[k,t,x] &= \frac{(k+t)^2}{k^2}(f_1(k,x)+2\,f_2(k,x)+f_3(k,x)) \\ &\quad -\frac{2t(k+t)}{k^2}(f_4(k,x)+f_5(k,x))+\frac{t^2}{k^2}f_6(x) \end{aligned}$$

The expansion of the denominator $(1-(k^2+1)(x+x^2)+x^3)(1-x)^2(1-k\,x-x^2)$ of this generating function is a polynomial of the seventh degree, so the recurrence relation, although easy to calculate, lacks practical use.

As for the Binet formula, its calculation is much more complex and its practical application lacks the greatest interest due to this complexity.

For these reasons, we will dedicate the next sections to Leonardo numbers.

2.3. Generating Function of the Sum of the Squares of the Generalized t-Leonardo Numbers

We will find the generating function of the generalized t–Leonardo numbers from the equation

$$\sum_{j=0}^{n} Le_j^2(t) = (1+t)^2 \sum_{j=0}^{n} F_{j+1}^2 - 2t(1+t)\sum_{j=0}^{n} F_{j+1} + \sum_{j=0}^{n} t^2$$

To do this we will take into account the previous table so the generating function of

- $\sum_{j=0}^{n} F_{j+1}^2$ is $f_1(x) = \frac{1}{1-2(x+x^2)+x^3}$
- $\sum_{j=0}^{n} F_{j+1}$ is $f_2(x) = \frac{1}{(1-x)(1-x-x^2)} = \frac{1}{1-2\,x+x^3}$
- $\sum_{j=0}^{n} 1$ is $f_3(x) = \frac{1}{(1-x)^2}$

So, the generating function of the sequence of partial sums $\left\{\sum_{j=0}^{n} Le_j^2(t)\right\}$ is

$$f(t,x) = \frac{(1+t)^2}{1-2(x+x^2)+x^3} - \frac{2t(1+t)}{(1-x)(1-x-x^2)} + \frac{t^2}{(1-x)^2} \quad (5)$$

And for the sequence of the sums of the squares of the classical Leonardo numbers ($t=1$)
$f(x) = \frac{1-3x+7x^2-3x^3+x^4-x^5}{(1-x)^2(1-x-x^2)(1-2(x+x^2)+x^3)}$

2.4. Recurrence Relation for the Sum of the Squares of the Generalized t-Leonardo Numbers

The denominator of the generating function, Equation (5), is

$$d(x) = (1-2(x+x^2)+x^3)(1-x-x^2)(1-x)^2 = 1-5x+6x^2+4x^3-10x^4+2x^5+3x^6-x^7$$

 doi:10.20944/preprints202408.1150.v1

Then, if $\{q_n(t)\} = \left\{ \sum_{j=0}^{n} Le_j^2(t) \right\}$, it is verified the recurrence relation $q_n(t) = 5\,q_{n-1}(t) - 6\,q_{n-2}(t) - 4\,q_{n-3}(t) + 10\,q_{n-4}(t) - 2\,q_{n-5}(t) - 3\,q_{n-6}(t) + q_{n-7}(t)$ with the initial conditions the seven first terms of Equation (4).

And for the classical Leonardo numbers it is the same recurrence relation, $q_n = 5\,q_{n-1} - 6\,q_{n-2} - 4\,q_{n-3} + 10\,q_{n-4} - 2\,q_{n-5} - 3\,q_{n-6} + q_{n-7}$ with the initial conditions $\{1, 2, 11, 36, 117, 342, 967\}$

3. Sum of the Alternated Squares Extended (k, t)–Fibonacci Numbers

The aim of this section is to study the sum $\sum_{j=0}^{n} (-1)^j T_j^2$.

If $T_n = T(k, t, n)$, the square of Equation (3) is

$$T_n^2 = \frac{1}{k^2}\Big((k+t)^2(F_{k,n}^2 + 2\,F_{k,n}F_{k,n-1} + F_{k,n-1}^2)) - 2t(k+t)(F_{k,n} + F_{k,n-1}) + t^2\Big)$$

So,

$$\begin{aligned}\sum_{j=0}^{n}(-1)^j T_j^2 \;&=\; \frac{(k+t)^2}{k^2}\left(\sum_{j=0}^{n}(-1)^j F_{k,j}^2 + 2\sum_{j=0}^{n}(-1)^j F_{k,j}F_{k,j-1} + \sum_{j=0}^{n}(-1)^j F_{k,j-1}^2\right)\\ &\quad - \frac{2t(k+t)}{k^2}\left(\sum_{j=0}^{n}(-1)^j F_{k,j} + \sum_{j=0}^{n}(-1)^j F_{k,j-1}\right) + \frac{t^2}{k^2}\sum_{j=0}^{n}(-1)^j\end{aligned}$$

To find its value we will take into account the following equalities:

$$\begin{aligned}S_1 = \sum_{j=0}^{n}(-1)^j F_{k,j}^2 \;&=\; \frac{1}{k^2+4}((-1)^n F_{k,2n+1} - 2n - 1)\\ S_2 = \sum_{j=0}^{n}(-1)^j F_{k,j}F_{k,j-1} \;&=\; \frac{1}{k^2+4}((-1)^n F_{k,2n} + k\,n)\\ S_3 = \sum_{j=0}^{n}(-1)^j F_{k,j-1}^2 \;&=\; \frac{1}{k^2+4}((-1)^n\,F_{k,2n-1} + k^2 + 2n + 3)\\ S_4 = \sum_{j=0}^{n}(-1)^j F_{k,j} \;&=\; \frac{1}{k}((-1)^n(F_{k,n+1} - F_{k,n}) - 1)\\ S_5 = \sum_{j=0}^{n}(-1)^j F_{k,j-1} \;&=\; \frac{1}{k}((-1)^n(F_{k,n} - F_{k,n-1}) + 1) + 1\\ S_6 = \sum_{j=0}^{n}(-1)^j(-1)^j \;&=\; \frac{(-1)^n + 1}{2}\end{aligned}$$

Proof. The first three formulas are demonstrated in a similar way, so we will only prove the first one.

The fourth and fifth are proven in [10].

The last one is obvious.

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For the first formula, applying Binet Identity,

$$\begin{aligned}
\sum_{j=0}^{n}(-1)^j F_{k,j}^2 &= \frac{1}{k^2+4}\sum_{j=0}^{n}(-1)^j\left(\sigma_1^{2j}+\sigma_2^{2j}-2(-1)^j\right)\\
&= \frac{1}{k^2+4}\left(\sum_{j=0}^{n}(-1)^j\left(\sigma_1^{2j}+\sigma_2^{2j}\right)-\sum_{j=0}^{n}2\right)\\
&= \frac{1}{k^2+4}\left(\frac{(-1)^n\sigma_1^{2n+2}+1}{\sigma_1^2+1}+\frac{(-1)^n\sigma_2^{2n+2}+1}{\sigma_2^2+1}-2(n+1)\right)\\
&= \frac{1}{(k^2+4)^2}\Big((-1)^n\sigma_1^{2n}+\sigma_2^2+(-1)^n\sigma_1^{2n+2}+1\\
&\qquad +(-1)^n\sigma_2^{2n}+\sigma_1^2+(-1)^n\sigma_2^{2n+2}+1\Big)-\frac{2(n+1)}{k^2+4}\\
&= \frac{1}{k^2+4}((-1)^n F_{k,2n+1}+1-2n-2)\\
&= \frac{1}{k^2+4}((-1)^n F_{k,2n+1}-2n-1)
\end{aligned}$$

Then

$S_1+2S_2+S_3=\frac{2}{k^2+4}((-1)^n F_{k,2n}+k\,n)+(-1)^n F_{k,n}^2+1$

$S_4+S_5=(-1)^n F_{k,n}+1$

$S_6=\frac{(-1)^n+1}{2}$

And finally

$$\begin{aligned}
\sum_{j=0}^{n}(-1)^j T_j^2 &= \frac{(k+t)^2}{k^2}\left(\frac{2}{k^2+4}((-1)^n F_{k,2n}+k\,n)+(-1)^n F_{k,n}^2+1\right)\\
&\quad -\frac{2t\,(k+t)}{k^2}((-1)^n F_{k,n}+1)+\frac{t^2}{k^2}\frac{(-1)^n+1}{2}
\end{aligned} \tag{6}$$

□

For the generalized t–Leonardo numbers the sequence generated is

$$\{1,0,4+4t+t^2,-5-8t-3t^2,20+32t+13t^2,-44-80t-36t^2,\ldots\}$$

In this case

$$\begin{aligned}
\sum_{j=0}^{n}(-1)^j Le_j^2(t) &= (1+t)^2\left(\frac{2}{5}((-1)^n F_{2n}+n)+(-1)^n F_n^2+1\right)\\
&\quad -2t\,(1+t)((-1)^n F_n+1)+t^2\,\frac{(-1)^n+1}{2}
\end{aligned}$$

3.1. Generating Function

With the help of Mathematica we ave found the recurrence relation into this sequence, $u_n = -3\,u_{n-1}+2\,u_{n-2}+8\,u_{n-3}-2\,u_{n-4}-6\,u_{n-5}+u_{n-6}+u_{n-7}$ with initial conditions the first six terms of the previous sequence.

And for the classical Leonardo numbers ($t=1$)
$\sum_{j=0}^{n}(-1)^j Le_j^2 = 4\,(-1)^n\left(\frac{2}{5}F_{2n}+F_n^2-F_n\right)+\frac{8}{5}n+\frac{(-1)^n+1}{2}$, generating the sequence $\{1, 0, 9, -16, 65, -160, 465, -1216, 3273, -8608, 22721, \ldots\}$ with the same recurrence relation that in the case of the generalized t–Leonardo numbers but initial conditions $\{1,0,9,-16,65,-160,465\}$

 doi:10.20944/preprints202408.1150.v1

As for the Binet formula, just substitute the Binet identity for the k–Fibonacci numbers $F_{k,n} = \frac{\sigma_1^n - \sigma_2^n}{\sigma_1 - \sigma_2}$ in the preceding formulas.

4. Conclusions

We have studied the sum of the squares of the extended (k,t)–Fibonacci numbers and then the corresponding alternate sum.

Subsequently, the Binet formula, the generating function and the recurrence relation that verify the elements of these sequences have been found.

In both cases, the results have been particularized to Leonardo numbers, both the generalized and the classical ones.

The path is now open to continue research on extended Fibonacci numbers. Among them we can study the behavior of powers greater than 2, the results that can be obtained by applying binomial transformations to them or the study of their prime numbers for determined or arbitrary values of t.

Funding: "This research received no external funding".

Use of Artificial Intelligence Author hereby declares that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc) and text generators have been used during writing of this manuscript.

Conflicts of Interest: Author has declared that no competing interests exist.

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