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Giovanny Fuentes *

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The Collatz Conjecture "The Nonexistence of Divergent Orbits, through a Dynamic Diophantine Linear Analysis of Codification"

Giovanny A. Fuentes Salvo

Universidade Federal Fluminense, Niteroi, Brasil; giovannyfuentes@id.uff.br

Abstract: We define the function $Col : \mathbb{N} \to \mathbb{N}$ as the Collatz function, given by $3n + 1$ if n is odd and $\frac{n}{2}$ if n is even [3]. The conjecture postulates that for any positive integer, at some point, its iteration will reach 1, or equivalently, every orbit will fall into the periodic cycle $\{4, 2, 1\}$. Two conditions would invalidate the conjecture: the existence of a divergent orbit or the presence of another cycle. In 2019 Terence Tao [1] proved that almost all orbits converge to the trivial cycle of $\{1, 4, 2\}$, in this work we will prove the non-existence of divergent orbits. The central idea is to decompose the dynamic system Col into a binary dynamic system, generated by different compositions of two functions. This differs from a typical dynamic system, which is generated by the iteration of a single function. We consider the system generated by the functions $\theta, \psi : \mathbb{R} \to \mathbb{R}$, defined as $\theta(x) = \frac{x}{2}$ and $\psi(x) = \frac{3x+1}{2}$, denoted as $\langle\theta, \psi\rangle$. We examine sequences of functions in $\langle\theta, \psi\rangle$ of the form $S_k(x) = s_k \circ S_{k-1}(x)$ with $s_k \in \{\theta, \psi\}$. We define a function that assigns to each element of these sequences an integer, called the minimum value positive integer or simply the minimum value. This corresponds to the smallest positive integer n such that $S_k(n)$ is an integer. As we will prove later, the minimum value is monotonically increasing, meaning that increasing the terms of the sequence will never result in a value lower than the previous one. Based on this behavior, we distinguish two types of sequences: stable ones, where the minimum value is constant from a certain point, and unstable ones, where the minimum value is divergent. The main result for unstable sequences is that when the slope of the sequence is divergent, the sequence is unstable. From this result, we can prove that there are no divergent orbits for the function Col.

Keywords: Collatz conjecture; dynamical system

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1. Notations and Conventions

In this work, we are going to denote the set of positive integers as $\mathbb{N}$, the set of non-negative integers as $\mathbb{N}_0$ and the greatest common divisor of a and b as (a, b) and the least common multiple of a and b as $[a, b]$.
We use the following symbology to refer to an arbitrary composition of functions:

$$\bigodot_{i=1}^{n} f_i(x) = f_n \circ f_{n-1} \circ \ldots \circ f_2 \circ f_1.$$

2. Introduction

2.1. Collatz's Conjecture

The Collatz Conjecture, also known as the $3n + 1$ Conjecture or the Ulam Problem. Formulated in simple terms, the conjecture proposes that by applying a specific rule to any positive natural number, one will inevitably reach the number 1.

The conjecture is named after Lothar Collatz, a German mathematician who first presented it in 1937. Despite its apparent simplicity, the Collatz conjecture has resisted formal proof for over 80 years. However, its appeal lies in the fascinating tension between its elementary statement and the complexity involved in its demonstration. The Collatz's Conjecture is formally enunciated as:

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Let $Col : \mathbb{N} \to \mathbb{N}$, defined by:

$$Col(n) = \begin{cases} 3n+1 & \text{if } n \text{ odd} \\ \dfrac{n}{2} & \text{if } n \text{ even,} \end{cases}$$

then, for all $n \in \mathbb{N}$ exist $k \in \mathbb{N}$ such that:

$$Col^k(n) = 1.$$

An equivalent formulation of the conjecture argues that starting from any positive integer, one will eventually reach cycles $\{4, 2, 1\}$, and then this cycle will repeat indefinitely. Despite its apparent simplicity, the Collatz conjecture has perplexed mathematicians for decades. The validity of the conjecture has been confirmed for extremely large numbers; however, to date, no one has managed to demonstrate in a general sense that all positive integers adhere to the conjecture. Additionally, no counterexample has been found to invalidate the assertion.

Example 1.

$$\begin{aligned}
27 &\to 82 \to 41 \to 124 \to 62 \to 31 \to 94 \to 47 \to 142 \to 71 \\
&\to 214 \to 107 \to 322 \to 161 \to 484 \to 242 \to 121 \to 364 \to 182 \\
&\to 91 \to 274 \to 137 \to 412 \to 206 \to 103 \to 310 \to 155 \to 466 \\
&\to 233 \to 700 \to 350 \to 175 \to 526 \to 263 \to 790 \to 395 \to 1186 \\
&\to 593 \to 1780 \to 890 \to 445 \to 1336 \to 668 \to 334 \to 167 \to 502 \\
&\to 251 \to 754 \to 377 \to 1132 \to 566 \to 283 \to 850 \to 425 \to 1276 \\
&\to 638 \to 319 \to 958 \to 479 \to 1438 \to 719 \to 2158 \to 1079 \to 3238 \\
&\to 1619 \to 4858 \to 2429 \to 7288 \to 3644 \to 1822 \to 911 \to 2734 \\
&\to 1367 \to 4102 \to 2051 \to 6154 \to 3077 \to 9232 \to 4616 \to 2308 \\
&\to 1154 \to 577 \to 1732 \to 866 \to 433 \to 1300 \to 650 \to 325 \to 976 \\
&\to 488 \to 244 \to 122 \to 61 \to 184 \to 92 \to 46 \to 23 \to 70 \\
&\to 35 \to 106 \to 53 \to 160 \to 80 \to 40 \to 20 \to 10 \to 5 \\
&\to 16 \to 8 \to 4 \to 2 \to 1
\end{aligned}$$

Example 2.

$$\begin{aligned}
871 &\to 2614 \to 1307 \to 3922 \to 1961 \to 5884 \to 2942 \to 1471 \to 4414 \to 2207 \\
&\to 6622 \to 3311 \to 9934 \to 4967 \to 14902 \to 7451 \to 22354 \to 11177 \to 33532 \\
&\to 16766 \to 8383 \to 25150 \to 12575 \to 37726 \to 18863 \to 56590 \to 28295 \to 84886 \\
&\to 42443 \to 127330 \to 63665 \to 190996 \to 95498 \to 47749 \to 143248 \to 71624 \to 35812 \\
&\to 17906 \to 8953 \to 26860 \to 13430 \to 6715 \to 20146 \to 10073 \to 30220 \to 15110 \\
&\to 7555 \to 22666 \to 11333 \to 34000 \to 17000 \to 8500 \to 4250 \to 2125 \to 6376 \\
&\to 3188 \to 1594 \to 797 \to 2392 \to 1196 \to 598 \to 299 \to 898 \to 449 \to 1348 \\
&\to 674 \to 337 \to 1012 \to 506 \to 253 \to 760 \to 380 \to 190 \to 95 \to 286 \\
&\to 143 \to 430 \to 215 \to 646 \to 323 \to 970 \to 485 \to 1456 \to 728 \to 364 \\
&\to 182 \to 91 \to 274 \to 137 \to 412 \to 206 \to 103 \to 310 \to 155 \to 466 \\
&\to 233 \to 700 \to 350 \to 175 \to 526 \to 263 \to 790 \to 395 \to 1186 \to 593 \\
&\to 1780 \to 890 \to 445 \to 1336 \to 668 \to 334 \to 167 \to 502 \to 251 \to 754 \\
&\to 377 \to 1132 \to 566 \to 283 \to 850 \to 425 \to 1276 \to 638 \to 319 \to 958 \\
&\to 479 \to 1438 \to 719 \to 2158 \to 1079 \to 3238 \to 1619 \to 4858 \to 2429 \to 7288 \\
&\to 3644 \to 1822 \to 911 \to 2734 \to 1367 \to 4102 \to 2051 \to 6154 \to 3077 \to 9232 \\
&\to 4616 \to 2308 \to 1154 \to 577 \to 1732 \to 866 \to 433 \to 1300 \to 650 \to 325 \\
&\to 976 \to 488 \to 244 \to 122 \to 61 \to 184 \to 92 \to 46 \to 23 \to 70 \\
&\to 35 \to 106 \to 53 \to 160 \to 80 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \\
&\to 4 \to 2 \to 1
\end{aligned}$$

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2.2. Extension of the Collatz Function

The Collatz function can be naturally extended to the set of integers, as the concept of parity is a concept defined for integers. This concept is not trivially extended to the set of rational numbers, as there is no unique representation, We are going to consider a modification of extension on the rationals proposed by Lagaria in [5]. We will distinguish two subsets of $\mathbb{Q}$. The set of rationals with odd denominators, denoted by $\mathbb{Q}_{odd}$, and the set of rationals with even denominators such that the numerator and denominator are co-prime, denoted by $\mathbb{Q}_{even}$. We will say that a rational number in $\mathbb{Q}_{\text{odd}}$ is odd if the numerator is odd, and it is even if its numerator is even. In the case of $\mathbb{Q}_{\text{even}}$, since the denominator is already even and due to coprimality, all elements are odd. We are going to consider the following extension of the Collatz function.

Definition 1 (Extension of the Collatz function)**.** *Let's consider the following sets* $\mathbb{Q}_{odd} = \left\{ \frac{p}{q} \in \mathbb{Q}, \text{ such that } q \text{ is odd} \right\}$ *and* $\mathbb{Q}_{even} = \left\{ \frac{p}{q} \in \mathbb{Q}, \text{ such that } q \text{ is even and } (p,q) = 1 \right\}$. *We defined the Collatz's function by* $Col : \mathbb{Q}_{odd} \cup \mathbb{Q}_{even} \to \mathbb{Q}_{odd} \cup \mathbb{Q}_{even}$ *by*

$$Col\left(\frac{p}{q}\right) = \begin{cases} \dfrac{3p+q}{q} & \text{if } p \text{ odd} \\ \dfrac{p}{2q} & \text{if } p \text{ even,} \end{cases}$$

We are going to show that the extension of the Collatz function that we defined is well-defined on $\mathbb{Q}_{odd}$.

Proposition 1 (Well-Defined)**.** *The Collatz's function on* $\mathbb{Q}_{odd}$ *is well-defined.*

Proof: We are going to show that Col is well-defined over $\mathbb{Q}_{odd}$. Let $p, q \in \mathbb{Z}$ with $q \neq 0$ and $(q,2) = (p,q) = 1$. Let λ an odd number, then

$$Col\left(\frac{\lambda p}{\lambda q}\right) = \begin{cases} \dfrac{3\lambda p + \lambda q}{\lambda q} & \text{if } \lambda p \text{ odd} \\ \dfrac{\lambda p}{2\lambda q} & \text{if } \lambda p \text{ even,} \end{cases} = \begin{cases} \dfrac{3p+q}{q} & \text{if } p \text{ odd} \\ \dfrac{p}{2q} & \text{if } p \text{ even,} \end{cases} = Col\left(\frac{p}{q}\right)$$

QED.

Let's observe that when we apply the Collatz function to $\frac{p}{q} \in \mathbb{Q}_{\text{odd}}$ with odd number, we always obtain a fraction with an even numerator, and when applied to $\mathbb{Q}_{\text{even}}$, we always obtain an odd number. This will be very important since in section 5, we are going to define how to coding the orbits, assigning 1 if it is odd and 0 if it is even, in the case of $\mathbb{Q}_{\text{even}}$, we will have that all its elements have the same encoding which is $1111\ldots$ unlike $\mathbb{Q}_{\text{odd}}$, where the codings will be generated by 10 and 0. For this reason we are going to work mainly on $\mathbb{Q}_{\text{odd}}$, let's simplify the Collatz function a bit, as $Col : \mathbb{Q}_{\text{odd}} \to \mathbb{Q}_{\text{odd}} \cup \mathbb{Q}_{\text{even}}$ given by

$$Col\left(\frac{p}{q}\right) = \begin{cases} \dfrac{3p+q}{2q} & \text{if } p \text{ odd} \\ \dfrac{p}{2q} & \text{if } p \text{ even,} \end{cases}$$

Proposition 2 (Invariance of $\mathbb{Q}_{odd}$)**.** *The Collatz function defined above satisfies that* $Col(\mathbb{Q}_{odd}) \subset \mathbb{Q}_{odd}$

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Proof: We will show that Col does not change the parity of the numerator.

1. if p is odd, we have $3p + q = 2r$ with $r \in \mathbb{Z}$, then
$$\frac{3p+q}{2q} = \frac{2r}{2q} = \frac{r}{q}$$
Since q is odd, we have independent of the simplification $\frac{r}{q} \in \mathbb{Q}_{odd}$.
2. if $p = 2r$ with $r \in \mathbb{Z}$, we have
$$\frac{p}{2q} = \frac{2r}{2q} = \frac{r}{q}$$
Since q is odd, we have independent of the simplification, we have $\frac{r}{q} \in \mathbb{Q}_{odd}$.

QED.

Considering the proposition above, we are going to define the Collatz function on $\mathbb{Q}_{odd}$ as

Definition 2 (Extension of the Collatz function on $\mathbb{Q}_{odd}$). *We define the Collatz Function on $\mathbb{Q}_{odd}$ by $Col : \mathbb{Q}_{odd} \to \mathbb{Q}_{odd}$*

$$Col\left(\frac{p}{q}\right) = \begin{cases} \frac{3p+q}{2q} & \text{if } p \text{ odd} \\ \frac{p}{2q} & \text{if } p \text{ even,} \end{cases}$$

Example 3. *Let $\frac{5}{7} \in \mathbb{Q}_{odd}$ and $\frac{5}{2} \in \mathbb{Q}_{even}$ we have:*

$$\mathcal{O}\left(\frac{5}{7}\right) : \frac{5}{7} \to \frac{11}{7} \to \frac{20}{7} \to \frac{10}{7} \to \frac{5}{7}$$

and

$$\mathcal{O}\left(\frac{5}{2}\right) : \frac{5}{2} \to \frac{17}{2} \to \frac{53}{2} \to \frac{161}{2} \to \frac{485}{2} \to \ldots \to \frac{3^k 5 + (3^k - 1)}{2} \to \ldots \to \infty$$

The first objective of the work is to show that there are no divergent orbits in $\mathbb{Q}_{odd}$.

3. Set Generate by θ and ψ^q

In this section, we delve into functions generated by the composition of two real linear functions, θ and ψ^q, focusing on their properties over integers. We define the set $\langle \theta, \psi^q \rangle$, representing compositions of these functions, and examine their orbits and associated sets of integers. Before delving into their properties, we introduce the crucial concept of the integer set of a function. Denoted as $\mathbb{E}(S)$, this set represents the integers generated by the orbit of the function S. We emphasize the one-to-one correspondence between functions of the same length and the partition of integers into sets based on this length. These results provide a solid foundation for a detailed understanding of the properties of these functions and their application in the study of iterative functions over rational numbers.

3.1. Summary of Propositions in the Section

1. **Definition 3:** Introduces the set $\langle \theta, \psi^q \rangle$, generated by two real linear functions θ and ψ^q.
2. **Definition 4 :** Defines the integer set of a function $S \in \langle \theta, \psi^q \rangle$ as $\mathcal{O}_S(x) = \{x, s_1(x), \ldots, S(x)\}$, where s_i are functions in the composition of S.

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3. **Definition 5 :** Defines the entire set of a function.
4. **Lemma 1** Monotony of Integer Set Lemma.
5. **Proposition 3 :** Establishes a relation of monotony in the entire sets concerning the composition of functions.
6. **Lemma 2** Establishes a characterization of the integer sets.
7. **Proposition 4 :** Establishes a one-to-one correspondence between functions of the same length and integer sets of the same length, and Affirms that the integer sets of functions of the same length are disjoint.
8. **Theorem 1:** Ensures that the integer sets are the disjoint union of the integer sets of functions in $\langle \theta, \psi^q \rangle$ with the same length.
9. **Proposition 5:** Guarantees the existence of a unique sequence of elements for a function $S \in \langle \theta, \psi^q \rangle$.

3.2. Set Generate by θ and ψ^q

The Generated Spaces, denoted as $\langle \theta, \psi^q \rangle$. These spaces arise from the iterative composition of functions, where the individual contributions of θ and ψ^q combine to form an enriched dynamic structure.

Definition 3 (Set Generated by θ and ψ^q)**.** *Let $q \in \mathbb{Z}$ and $\theta, \psi^q : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\theta(x) = \dfrac{x}{2}$ and $\psi^q(x) = \dfrac{3x+q}{2}$, then we define the set $\langle \theta, \psi^q \rangle$ as:*

$$S : \mathbb{R} \longrightarrow \mathbb{R}; \quad S(x) = \left(\bigodot_{i=1}^{n} s_i(x) \right), s_i(x) = \theta(x), \psi^q(x) \text{ for } i \geq 1, s_0 = id$$

We will call the number n length of S.

Now let's define the S-Orbits. These orbits are ordered sequences that reveal how each element evolves under the iterative action of the functions θ and ψ^q.

Definition 4 (S-Orbit and Integer S-Orbit)**.** *Let $S(x) = \left(\bigodot_{i=1}^{n} s_i(x) \right) \in \langle \theta, \psi^q \rangle$ and $x \in \mathbb{R}$. We define the S-orbit of x_0 as the set:*

$$\mathcal{O}_S(x_0) = \left\{ x_0, s_1(x_0), \ldots, \bigodot_{i=1}^{k} s_i(x_0), \ldots, S(x_0) \right\}$$

Let $p \in \mathbb{N}$. We will say that an S-orbit of p is integer when

$$\bigodot_{i=1}^{k} s_i(p) \in \mathbb{N} \text{ for all } k \leq n$$

Now let's define The integer sets of a function, denoted by $\mathbb{E}(S)$, represent the integer values that a specific function takes on its domain. Examining $\mathbb{E}(S)$ allows for the identification of patterns and regularities in the interaction of the function with integers, which is essential for understanding the structure of spaces generated by such functions.

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Definition 5 (Integer Set of a Function). *Let $S : \mathbb{R} \longrightarrow \mathbb{R}$ a function, we called integer set of f or the integers of f the set:*

$$\mathbb{E}(S) = \{n \in \mathbb{Z}; S(n) \in \mathbb{Z}\}.$$

In the following Proposition we are going to see that integer sets have a monotonic behavior concerning the composition of linear functions, this property will be fundamental to studying integer S-orbits.

Lemma 1 (Monotony of Integer Set Lemma). *Let $A, B, a, b \in \mathbb{Z}$ such that $(A,2) = (B,2) = 1$ and $\alpha, \beta \in \mathbb{N}$. Let $S(x) = \dfrac{Ax + a}{2^\alpha}$ and $H(x) = \dfrac{Bx + b}{2^\beta}$, then $\mathbb{E}(H \circ S) \subset \mathbb{E}(S)$.*

Proof: In fact, we have $H \circ S(x) = \dfrac{ABx + Ba + 2^\alpha b}{2^{\alpha+\beta}}$ Since $(AB, 2) = 1$ there are solutions. Let $x_0 \in \mathbb{E}(H \circ S)$ then by definition $\dfrac{ABx_0 + Ba + 2^\alpha b}{2^{\alpha+\beta}} \in \mathbb{Z}$, then we have

$$\frac{ABx_0 + Ba + 2^\alpha b}{2^{\alpha+\beta}} = B\left(\frac{Ax_0 + a}{2^{\alpha+\beta}}\right) + b \in \mathbb{Z} \Rightarrow B\left(\frac{Ax_0 + a}{2^{\alpha+\beta}}\right) \in \mathbb{Z}$$

as $(B, 2) = 1$ then we have

$$\frac{Ax_0 + a}{2^{\alpha+\beta}} \in \mathbb{Z} \Rightarrow \frac{Ax_0 + a}{2^\alpha} \in \mathbb{Z}$$

QED.

Proposition 3 (Monotony of Integer Set). *Let $\bigodot_{i=1}^{n} s_i(x) \in \langle \theta, \psi^q \rangle$ with $s_i \in \{\theta, \psi^q\}$ and $L, M \in \mathbb{N}$. Then if $n \geq L \geq M$ we have:*

$$\mathbb{E}\left(\bigodot_{i=1}^{L} s_i(x)\right) \subset \mathbb{E}\left(\bigodot_{i=1}^{M} s_i(x)\right)$$

Proof: As the functions generated by $\langle \theta, \psi^q \rangle$ are linear of the form $\dfrac{3^b x + N}{2^a}$ with $a, b.N \in \mathbb{N}$. The result follows inductively from Lemma 1.

QED.

Example 4. *Let $S_n(x) = \theta^{2n}\psi(x) \in \langle \theta, \psi \rangle$. We will calculate the integer set of S_n*

$$2^{2n}y - 3x = 1$$

we have that $y = 1$ and $\dfrac{2^n - 1}{3} \in \mathbb{N}$ are solutions of the Diophantine Equation, then the integer set is:

$$\mathbb{E}(\theta^{2n}\psi) = \frac{2^n - 1}{3} + 2^{2n}\mathbb{Z}$$

Let $m \geq n$ then

$$\mathbb{E}(\theta^{2m}\psi) \subset \mathbb{E}(\theta^{2n}\psi)$$

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indeed

$$\frac{3\left(\frac{2^m-1}{3}\right)+1}{2^n}=2^{m-n}\in\mathbb{N}$$

The following lemma states that an $S-$orbit is integer if and only if its last value is an integer.

Lemma 2 (Containment in Integer Sets). *Let $s_i \in \{\theta, \psi^q\}$, then $u \in \mathbb{E}\left(\bigodot_{i=1}^{j} s_i(x)\right)$ if and only if $\bigodot_{i=1}^{r} s_i(u) \in \mathbb{E}\left(\bigodot_{i=r+1}^{j} s_i(x)\right)$ with $r \leq j$.*

Proof: Let $u \in \mathbb{E}\left(\bigodot_{i=1}^{j} s_i(x)\right)$ then by proposition 3 we have $u \in \mathbb{E}\left(\bigodot_{i=1}^{r} s_i(x)\right)$ with $r < j$, then $\bigodot_{i=1}^{r} s_i(u) \in \mathbb{Z}$. On the other hand, we have $\bigodot_{i=r+1}^{j} s_i\left(\bigodot_{i=1}^{r} s_i(u)\right) \in \mathbb{Z}$ then $\bigodot_{i=1}^{r} s_i(u) \in \mathbb{E}\left(\bigodot_{i=r+1}^{j} s_i(x)\right)$.
If $\bigodot_{i=1}^{r} s_i(x) \in \mathbb{E}\left(\bigodot_{i=r+1}^{j} s_i(x)\right)$ with $r < j$ then $\bigodot_{i=1}^{r} s_i(u) \in \mathbb{Z}$ on the other hand

$$\bigodot_{i=r+1}^{j} s_i\left(\bigodot_{i=1}^{r} s_i(u)\right) = \bigodot_{i=1}^{j} s_i(u) \in \mathbb{Z}$$

then $u \in \mathbb{E}\left(\bigodot_{i=1}^{j} s_i(x)\right)$.

QED.

In the following proposition, We will demonstrate that the integer sets associated with functions of the same length are disjoint. That is, if two integer sets share at least one element, then the functions must be the same. This result plays a crucial role in understanding the structure of integer sets within the context of linear functions and their compositions.

Proposition 4 (One-to-One Correspondence and Disjointedness). *Let $S, H \in \langle \theta, \psi^q \rangle$ of length k, if $H \neq S$ if and only if $\mathbb{E}(H) \cap \mathbb{E}(S) = \emptyset$.*

Proof: Let $S(x) = \bigodot_{i=1}^{j} s_i(x)$ and $H(x) = \bigodot_{i=1}^{j} h_i(x)$ with $s_i, h_i \in \{\theta(x), \psi(x)\}$ and let k_0 be the largest index such that $s_j(x) = h_j(x)$ for all $j < k_0$
If $k_0 = 1$ This means that they have different first terms. Then $\mathbb{E}(S) \subset \mathbb{E}(\theta) = 2\mathbb{Z}$ and $\mathbb{E}(H) \subset \mathbb{E}(\psi^q) = 2\mathbb{Z}+1$ or, $\mathbb{E}(H) \subset \mathbb{E}(\theta) = 2\mathbb{Z}$ and $\mathbb{E}(S) \subset \mathbb{E}(\psi^q) = 2\mathbb{Z}+1$ in either case we have $\mathbb{E}(H) \cap \mathbb{E}(S) = \emptyset$.
Suppose that exist $u \in \mathbb{E}(H) \cap \mathbb{E}(S)$ by proposition 3 we have

$$u \in \mathbb{E}\left(\bigodot_{i=1}^{r} s_i(x)\right) \cap \mathbb{E}\left(\bigodot_{i=1}^{r} h_i(x)\right) \text{ with } r \leq j.$$

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Taken $r = k_0$

$$u \in \mathbb{E}\left(\bigodot_{i=1}^{k_0} s_i(x)\right) \cap \mathbb{E}\left(\bigodot_{i=1}^{k_0} h_i(x)\right).$$

and by lemma 2

$$\bigodot_{i=1}^{k_0-1} s_i(u) = \bigodot_{i=1}^{k_0-1} h_i(u) \in \mathbb{E}(s_{k_0}(x)) \cap \mathbb{E}(s_{k_0}(x)) = 2\mathbb{Z} \cap (2\mathbb{Z}+1) = \emptyset$$

which is a contradiction, On the other hand if $\mathbb{E}(H) \cap \mathbb{E}(S) = \emptyset$ then $S \neq H$ otherwise we would have

$$\mathbb{E}(S) = \mathbb{E}(H) = \emptyset$$

however, neither set can be empty

QED.

As a consequence of the above proposition we have

Theorem 1 (Partition of Integers)**.** *Let $\{S_j\} \in \langle \theta, \psi^q \rangle$ with length k, then*

$$\mathbb{Z} = \bigsqcup_{S \text{ with length } k} \mathbb{E}(S)$$

i.e., the sets of integers are equal to the disjoint union of the integer sets of functions S of length k

Proof: it is evident that

$$\bigsqcup_{S \text{ with length } k} \mathbb{E}(S) \subset \mathbb{Z}$$

To prove the other contention we consider the $q-$Collatz function defined by $Col_q : \mathbb{Z} \to \mathbb{Z}$ given by

$$Col_q(u) = \begin{cases} \dfrac{3u+q}{2} & \text{if } u \text{ odd} \\ \dfrac{u}{2} & \text{if } u \text{ even,} \end{cases}$$

let's take an integer u and calculate its k-th orbit, this orbit can be written as compositions of functions in $\langle \theta, \psi^q \rangle$, let's call the resulting function S since all the values of the orbit are integers, we have by the lemma 2 we can conclude that u is in the entire set of the function S.

$$u \in \mathbb{E}(S) \subset \bigsqcup_{S \text{ with length } k} \mathbb{E}(S).$$

QED.

So far we know that if two functions have the same length, then their integer sets are disjoint, this means that if we take elements of each integer set, the S-orbits of these integers are disjoint sets. Now, if we remove the condition that the lengths are the same, could it be that there is some number such that, given two different functions without being a part of another, it generates the same S-orbit? The answer is in the following proposition.

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Proposition 5 (Uniqueness of Basis Representation)**.** *Let $S \in \langle \theta, \psi^q \rangle$. Then there exists a unique sequence of k elements $\{s_j\}_{j=1}^k$ with $s_j \in \{\theta, \psi^q\}$ such that*

$$S(x) = \bigodot_{j=1}^{k} s_j(x)$$

Proof: Since $S \in \langle \theta, \psi^q \rangle$, then there exists a sequence of k_0 elements $\{s_j\}_{j=1}^{k_0}$ with $s_j \in \{\theta, \psi^q\}$ such that $S(x) = \bigodot_{j=1}^{k_0} s_j(x)$. Suppose for absurdity, that there is another sequence but of l_0 elements $\{h_j\}_{j=1}^{l_0}$ such that $S(x) = \bigodot_{j=1}^{l_0} h_j(x)$. Let $u \in \mathbb{E}(S)$, we have the following cases

1. If $k_0 > l_0$. We have

$$S(x) = \bigodot_{j=1}^{k_0} s_j(x) = \bigodot_{j=1}^{l_0} h_j(x)$$

$$u \in \mathbb{E}\left(\bigodot_{j=1}^{k_0} s_j(x)\right) = \mathbb{E}\left(\bigodot_{j=1}^{l_0} h_j(x)\right)$$

by Proposition 3 we have $u \in \mathbb{E}(s_1) \cap \mathbb{E}(h_1)$, since $s_1, h_1 \in \{\theta, \psi^q\}$ and $\mathbb{E}(\theta) \cap \mathbb{E}(\psi^q)$ then $s_1 = h_1$. Since s_1 and h_1 are invertible functions, we have

$$\bigodot_{j=2}^{k_0} s_j(x) = \bigodot_{j=2}^{l_0} h_j(x).$$

Following the same idea up to k_0, we have

$$\bigodot_{j=k_0}^{l_0} s_j(x) = x$$

The latter is impossible since the slope of the resulting line is of the form $\frac{3^b}{2^a}$ with $a, b \in \mathbb{N}$. The case $k_0 > l_0$ is completely analogous, therefore the case where k_0 and l_0 are different is not possible.

2. If $k_0 = l_0$. Since the sequences are different, there must exist some $k < k_0$ such that

$$\bigodot_{j=1}^{k} s_j(x) \neq \bigodot_{j=1}^{k} h_j(x)$$

then by Proposition 4 we have

$$\mathbb{E}\left(\bigodot_{j=1}^{k} s_j(x)\right) \cap \mathbb{E}\left(\bigodot_{j=1}^{k} h_j(x)\right) = \emptyset$$

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However, this is a contradiction to the Proposition 3, because $u \in \mathbb{E}\left(\bigodot_{j=1}^{k} s_j(x)\right) \cap \mathbb{E}\left(\bigodot_{j=1}^{k} h_j(x)\right)$ for all $k \leq k_0$. Then both sequences must be identical.

QED.

4. Stability and Instability of Integer Set

In this section, we delve into the stability and instability of sequences associated with integer sets. We begin by defining functions ρ_0 and ρ_1 that map real functions to integers. We introduce the concepts of positive and negative stability for sequences $\{S_j(x)\}_{j=1}^{\infty}$. The monotonicity of ρ_0 and ρ_1 is established through Proposition 3, demonstrating the non-decreasing of ρ_0 and the non-increasing of ρ_1 for a given sequence $S_j(x)$. Further, the Proposition formally defines positive and negative stability, incorporating limits and intersections of sets. The ensuing Stability Limit Theorem (2) establishes the asymptotic behavior of the integer set of an iterative sequence.

4.1. Summary of Propositions in the Section

1. **Definition 6 :** Definition of functions ρ_0 and ρ_1.
2. **Proposition 6:** Monotonicity of the functions ρ_0 and ρ_1.
3. **Definition 7:** Definition of positively (negatively) stable (unstable) sequences.
4. **Theorem 2:** Establishes the asymptotic behavior of the integer set when we have a positively (negatively) stable (unstable) sequence.

4.2. Stability and Instability of Integer Set

We initiate this section by introducing functions that associate each integer set with its minimum positive integer value and maximum negative integer value. These values are determined by the solutions closest to zero for the variable x in the Diophantine equation $2^a y - 3^b x = N$. This equation is representative of the Diophantine equation linked to an element within the space generated by ψ and θ.

Definition 6 (ρ_0 and ρ_1 functions.). *Define the function* $\rho_0, \rho_1 : \{f : \mathbb{R} \longrightarrow \mathbb{R} : f \text{ function }\} \to \mathbb{Z}$ *by*

$$\rho_0(f) = \min\{\mathbb{E}(f) \cap \mathbb{N}\}$$

and

$$\rho_1(f) = \max\{\mathbb{E}(f) \cap -\mathbb{N}\}$$

As a consequence of the Proposition 3. We have that the functions ρ_0 and ρ_1 are monotone.

Proposition 6 (Monotonicity of ρ_0 and ρ_1). *Let* $S_j(x) = \left(\bigodot_{i=1}^{j} s_i(x)\right) \in \langle \theta, \psi^q \rangle$ *then* $\rho_0(S_j)$ *is a non-decreasing function, and* $\rho_1(S_j(x))$ *it is a non-increasing function.*

Proof: By the proposition 3 we have that

$$\rho_0(S_{j+1}), \rho_1(S_{j+1}) \in \mathbb{E}(S_{j+1}) \subset \mathbb{E}(S_j)$$

then $\rho_0(S_j) \leq \rho_0(S_{j+1})$ and $\rho_1(S_j) \geq \rho_1(S_{j+1})$.

QED.

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From the result of the proposition above, we are going to make a classification of the sequences $S_j = \bigodot_{i=1}^{j} s_i(x)$ according to the behavior of the functions ρ_0 and ρ_1.

Definition 7 (Stability of Sequences). *Let $\{S_j(x)\}_{j=1}^{\infty}$ sequence on $\langle \theta, \psi^q \rangle$ given by $S_j = \bigodot_{i=1}^{j} s_i(x)$. We will say that $\{S_j(x)\}_{j=1}^{\infty}$ is positively stable if $\lim_{j\to\infty} \rho_0\left(\bigodot_{i=1}^{j} S_k(j)\right) < \infty$ otherwise we will say that it is positively unstable. On the other hand, we will say that $\{S_j(x)\}_{j=1}^{\infty}$ is negatively stable if $\lim_{j\to\infty} \rho_1\left(\bigodot_{i=1}^{j} S_k(j)\right) > -\infty$, otherwise, we will say that it is negatively unstable.*

Now we will give the central theorem of this section, which establishes the asymptotic behavior of the integer sets of S_j from the stability or stability of this.

Theorem 2 (Stability Limit Theorem). *Let $\{S_j(x)\}_{j=1}^{\infty} \subset \langle \theta, \psi^q \rangle$ and $\mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \mathbb{E}\left(\bigodot_{j=1}^{k} S_j(x)\right) \cap \mathbb{N}$ and $\mathbb{E}^-\left(\bigodot_{j=1}^{k} S_j(x)\right) = \mathbb{E}\left(\bigodot_{j=1}^{k} S_j(x)\right) \cap -\mathbb{N}$. We have:*

1. *if $\{S_j(x)\}_{j=1}^{\infty}$ is positively stable then*
$$\lim_{k\to\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \bigcap_{k=1}^{\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \left\{\lim_{k\to\infty} \rho_0\left(\bigodot_{j=1}^{k} S_j(x)\right)\right\}$$
2. *if $\{S_j(x)\}_{j=1}^{\infty}$ is positively unstable, then*
$$\lim_{k\to\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \bigcap_{k=1}^{\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \emptyset$$

analogously

1. *if $\{S_j(x)\}_{j=1}^{\infty}$ is negatively stable then*
$$\lim_{k\to\infty} \mathbb{E}^-\left(\bigodot_{j=1}^{k} S_j(x)\right) = \bigcap_{k=1}^{\infty} \mathbb{E}^-\left(\bigodot_{j=1}^{k} S_j(x)\right) = \left\{\lim_{k\to\infty} \rho_1\left(\bigodot_{j=1}^{k} S_j(x)\right)\right\}$$
2. *if $\{S_j(x)\}_{j=1}^{\infty}$ is positively unstable, then*
$$\lim_{k\to\infty} \mathbb{E}^-\left(\bigodot_{j=1}^{k} S_j(x)\right) = \bigcap_{k=1}^{\infty} \mathbb{E}^-\left(\bigodot_{j=1}^{k} S_j(x)\right) = \emptyset$$

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Proof: Let $\rho_0^k = \rho_0\left(\bigodot_{j=1}^{k} S_j(x)\right)$ and $\bigodot_{j=1}^{k} S_j(x) = \dfrac{3^{B_k}x + N}{2^{A_k}}$ with A_k =numbers from θ to $\bigodot_{j=1}^{k} S_j(x)$ and B_k =numbers from ψ^q to $\bigodot_{j=1}^{k} S_j(x)$. We have

$$\mathbb{E}\left(\bigodot_{j=1}^{k} S_j(x)\right) = \rho_0^k + 2^{A_k}\mathbb{N}.$$

supposed that $\{S_j(x)\}_{j=1}^{\infty}$ is stable, we will first prove that $\lim_{k\to\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right)$ is non-empty. By Proposition 3 the sequence of sets $\mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right)$ is a decreasing sequence of sets i.e. that the next set is a subset of the previous one, then the limit set corresponds to the intersection of all the sets of the sequence.

$$\lim_{k\to\infty} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) = \bigcap_{k\in\mathbb{N}} \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right)$$

Now since ρ_0^k is a function of the natural ones in the natural ones and is convergent, it implies that this function reaches its limit in a finite amount of steps

$$\rho_0^k = cts \text{ for all } k > K$$

this implies

$$\lim_{k\to\infty} \rho_0^k \in \mathbb{E}^+\left(\bigodot_{j=1}^{k} S_j(x)\right) \text{ for all } k \in \mathbb{N}$$

then the limit set is non-empty.

Now we will prove the limit set contains a single element. Suppose there exists another element u_0 that is contained in all positive integer sets, then there exists a non-negative integer t such that

$$u_0 = \rho_0^k + 2^{A_k}t$$

without loss of generality, we can assume that ρ_0^k is constant. Solving the equation in terms of t, we have

$$t = \frac{u_0 - \rho_0^k}{2^{A_k}}$$

This solution is a fraction less than 1 for k large enough., which contradicts the fact that t is an integer.

Now let us take the unstable case. Suppose there exists an element u_0 in the limiting set i.e. an element that is contained in all non-negative integer sets, then there exists t a non-negative integer such that

$$u_0 = \rho_0^k + 2^{A_k}t \geq \rho_0^k$$

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as ρ_0^k diverges and u_0 constant, then there exists a K such that ρ_0^k is greater than u_0, then u_0 cannot belong to any integer set with $k > K$, which is a contradiction. Analogously for the other case.

QED.

Example 5. *$S_n(x) = \theta^{2n}\psi(x)$ is positively and negatively unstable. Indeed, by example 4, we have* $\mathbb{E}(\theta^{2n}\psi(x)) = \frac{2^n-1}{3} + 2^{2n}\mathbb{Z}$ *where* $\rho_0(\theta^{2n}\psi(x)) = \frac{2^n-1}{3} \to \infty$ *as* $n \to \infty$ *then* $\lim_{n\to\infty} \mathbb{E}^+(\theta^{2n}\psi(x)) = \emptyset$. *On the other hand* $\rho_1(\theta^{2n}\psi(x)) = \frac{2^n-1}{3} - 2^{2n} \to -\infty$.

Example 6. *$S_n(x) = (\theta\psi(x))^n$ is positively unstable and negatively stable. Let's calculate the integer set* $\mathbb{E}((\theta\psi(x))^n)$, *let's observe that*

$$\theta\psi(-1) = \frac{3(-1)+1}{2} = -1$$

Then $\mathbb{E}(\theta\psi(x))^n = -1 + 2^n\mathbb{Z}$ *then we have* $\rho_1((\theta\psi(x))^n) = -1$ *and* $\rho_0((\theta\psi(x))^n) = 2^n - 1 \to \infty$ *as* $n \to \infty$

5. Coding of the Orbits

In this section, we will delve into the study of the coding of the orbits of the Collatz function. The main results of this section are the invariance of the coding between Cod_q and Cod on the fractions with denominator q and the one-to-one identification of each element of $\mathbb{Q}_{odd}$ with its coding.

5.1. Summary of Propositions in the Section

1. **Definition 8:** Coding maps and Σ_2^* the space of sequences 0 and 10.
2. **Proposition 7:** General form of the elements generated by ψ and θ.
3. **Proposition 8:** Fist Cod invariance: $Cod(S) = Cod_q(S_q)$.
4. **Definition 9:** Definition of $Col_q : \mathbb{Z} \to \mathbb{Z}$.
5. **Proposition 9:** $Col - Col_q$ equivalence : if $\frac{p}{q} \in \mathbb{Q}_{odd}$ then $Col^k\left(\frac{p}{q}\right) = \frac{1}{q}Col_q^k(p)$.
6. **Proposition 10:** Second Cod invariance: $Cod^k\left(\frac{p}{q}\right) = Cod_q^k(p)$.
7. **Proposition 11:** $p \in \mathbb{E}(S)$ if and only if $Cod_q^k(p) = Cod_q(S)$.
8. **Proposition 12** $p \in \mathbb{E}(S)$ if and only if $Cod^k\left(\frac{p}{q}\right) = Cod_q(S)$.
9. **Proposition 13:** $Cod^k\left(\frac{p}{q}\right) = Cod^k\left(\frac{p}{q} + \frac{2^{a_k}T}{q}\right)$.
10. **Definition 10:** The Coding set $Cod^k(\xi)$.
11. **Proposition 14:** Monotony of the coding set $Cod^{k+1}(\xi) \subset Cod^k(\xi)$.
12. **Proposition 15:** Generating property:if $\frac{p}{q}, \frac{t}{r} \in Cod^k(\xi)$ then $\frac{p}{q} = \frac{t}{r} + \frac{2^{a_k}T}{qr}$.
13. **Theorem 3:** Uniqueness of the full coding $\mathbb{Q}_{odd}$.

5.2. Coding of the Orbits

It is a common practice in dynamical systems to encode orbits based on specific criteria. In our case, we will encode the orbits of the Collatz function according to the parity of its elements, assigning the value 1 when they are odd and 0 when they are even. Since our primary focus is on the Collatz function over $\mathbb{Q}_{odd}$, we will modify the initial coding by assigning 10 when it is odd, as opposed to just 1. We will denote the space where these encodings reside as Σ_2^* since it is a subset of the sequence space consisting of 0s and 1s, denoted in dynamics as Σ_2. Formally, we express this as

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Definition 8 (Coding of the Orbits). *We are going to consider the set of sequences 0 and 10 that we will denote by Σ_2^* and we formally define it as*

$$\Sigma_2^* = \left\{ \{\xi_j\}_{j=0}^{\infty}, \xi_j \in \{0, 10\} \right\}$$

this set can be seen as a subset of the set of sequences 0 and 1 where after the entry 1 enters 0

Let's consider the following applications: $Cod^k : \mathbb{Q}_{odd} \to \Sigma_2^*$ defined by

$$Cod^k\left(\frac{p}{q}\right) = \{\xi_j\}_{j=0}^{k}$$

with

$$\xi_j = \begin{cases} 10 & \text{if} \quad Col^j\left(\frac{p}{q}\right) \text{ is odd} \\ 0 & \text{if} \quad Col^j\left(\frac{p}{q}\right) \text{ is even} \end{cases}$$

and $Cod_q : \langle \theta, \psi^q \rangle \to \Sigma_2^*$ defined by

$$Cod_q\left(\bigodot_{j=1}^{k} s_j(x) \right) = \{\xi_j\}_{j=0}^{k} \text{ with } s_j \in \{\theta, \psi^q\}$$

with

$$\xi_j = \begin{cases} 0 & \text{if} \quad s_j(x) \text{ is } \theta(x) \\ 10 & \text{if} \quad s_j(x) \text{ is } \psi^q(x) \end{cases}$$

To rigorously examine the properties of the coding, it is essential to establish a precise form for the elements generated by ψ and θ.

Proposition 7 (General form of S). *Let $S \in \langle \theta, \psi^q \rangle$, let A=quantity of 0 of $Cod_q(S)$, a_j=quantity of 0 up to the j-th 1 of $Cod_q(S)$ and b = quantity of 1 of $Cod_q(S)$ and $N : \Sigma_2^* \to \mathbb{N}$ defined as*

$$N(Cod_q(S)) = \begin{cases} 3^{b-1}2^{a_1} + 3^{b-2}2^{a_2} + \ldots + 2^{a_b} & \text{if} \quad b > 0 \\ 0 & \text{if} \quad b = 0 \end{cases}$$

then

$$S(x) = \frac{3^b x + qN(Cod_q(S))}{2^A}$$

Proof: We will prove by induction on k (length of S) for $k = 1$ we have

1. $Cod_q(\psi^q(x)) = 10$, then, $b = 1, a_1 = 0, a = 1$ and $N = 3^0 2^0 = 1$ then $S(x) = \dfrac{3x + q}{2} = \psi^q(x)$.
2. $Cod_q(\theta(x)) = 0$, then, $b = 0, a = 1$ and $N = 0$ then $S(x) = \dfrac{3^0 x + 0q}{2} = \dfrac{x}{2} = \theta(x)$.

Suppose the statement is true up to k, let $S \circ H \in \langle \theta, \psi^q \rangle$ of length $k + 1$ with H of length k and let b be the quantity of 1 and a be the quantity of 0 of $Cod_q(H)$.

Claim 1: $Cod_q(\psi^q \circ H) = Cod_q(H)10$. We have:

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1. the quantity of 1 of $Cod_q(H)10$ is $b+1$
2. $a_{b+1}(Cod_q(H)10) = A$ and $a_k(Cod_q(H)10) = a_k(Cod_q(H))$ for $k \leq b$ and the quantity of 0 of $Cod_q(H)10$ is $A+1$
3. $N(Cod_q(H)10) = 3^b 2^{a_1} + 3^{b-1}2^{a_2} + \ldots + 3^1 2^{a_{b-1}} + 2^A$

On the other hand, we have:

$$\begin{aligned}
\psi^q\left(\frac{3^b x + qN(Cod_q(H))}{2^A}\right) &= \frac{3^{b+1}x + q3N(Cod_q(H)) + q2^A}{2^{A+1}} \\
&= \frac{3^{b+1}x + q3(3^{b-1}2^{a_1} + 3^{b-2}2^{a_2} + \ldots + 2^{a_{b-1}}) + q2^A}{2^{A+1}} \\
&= \frac{3^{b+1}x + q(3^b 2^{a_1} + 3^{b_k-1}2^{a_2} + \ldots + 3^1 2^{a_{b-1}} + 2^A)}{2^{A+1}} \\
&= \frac{3^{b+1}x + qN(Cod_q(H)10)}{2^{A+1}}
\end{aligned}$$

where we observe that the values coincide with those calculated.

Claim 2: $Cod_q(\theta \circ H) = Cod_q(H)0$. We have

1. the quantity of 1 of $Cod_q(H)0$ is b.
2. $a_k(Cod_q(H)0) = a_k(Cod_q(H))$ for $k \leq b$
3. $N(Cod_q(H)0) = 3^{b-1}2^{a_1} + 3^{b-1}2^{a_2} + \ldots + 2^{a_{b-1}} = N(Cod_q(H))$.

On the other hand, we have

$$\theta\left(\frac{3^b x + qN(Cod_q(H))}{2^A}\right) = \frac{3^b x + qN(Cod_q(H)0)}{2^{A+1}}$$

where we observe that the values coincide with those calculated, then the statement is true.

QED.

Now we will see the first property of the coding

Proposition 8 (First Cod invariance). *Let $S \in \langle \theta, \psi \rangle$ given by $S(x) = \dfrac{3^b x + N}{2^a}$, we defined $S_q \in \langle \theta, \psi^q \rangle$ given by $S_q(x) = \dfrac{3^b x + qN}{2^a}$. Then $Cod(S) = Cod_q(S_q)$.*

Proof: Let $q \in \mathbb{Z}$. To prove that they have the same coding, we have to prove that they have the same decomposition in principle, except that where there is ψ we have a ψ^q. let us observe that q has commutative properties with θ and ψ.

1. $q\theta(x) = \dfrac{qx}{2} = \theta(qx)$.
2. $q\psi(x) = q\left(\dfrac{3x+1}{2}\right) = \dfrac{3(qx)+q}{2} = \psi^q(qx)$

As $S \in \langle \theta, \psi \rangle$ then there exists $s_j \in \{\theta, \psi\}$ such that

$$S(x) = \bigodot_{j=1}^{b} s_j(x).$$

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For convenience we will denote $s_j^q \in \{\theta, \psi^q\}$. Then we have

$$S_q(x) = \frac{3^b x + qN}{2^a} = q\left(\frac{3^b\left(\frac{x}{q}\right) + N}{2^a}\right) = q\bigodot_{j=1}^{b} s_j\left(\frac{x}{q}\right) = \bigodot_{j=1}^{b} s_j^q\left(q\frac{x}{q}\right) = \bigodot_{j=1}^{b} s_j^q(x)$$

since q does not permute any element s_j, we have that if s_j is θ then s_j^q is still θ and if s_j is ψ then s_j^q corresponds to ψ^q. By Proposition 5 we have that the coding of S_q has to be the same as that of S.

QED.

Let us contemplate a generalization of the Collatz function applied to integers. In this variant, rather than adding 1, the function adds $q \in \mathbb{Z}$, where q is an odd integer. Subsequently, we will establish the compatibility of this generalization with the extension of the Collatz function to $\mathbb{Q}_{odd}$.

Definition 9 (The Col_q map). *Let $q \in \mathbb{Z}$, we define the $q-$Collatz function defined by $Col_q : \mathbb{Z} \to \mathbb{Z}$ given by*

$$Col_q(n) = \begin{cases} \frac{3n+q}{2} & \text{if } u \text{ odd} \\ \frac{n}{2} & \text{if } u \text{ even,} \end{cases}$$

Now, we will demonstrate the compatibility of this generalization

Proposition 9 ($Col - Col_q$ equivalence). *Let $\frac{p}{q} \in \mathbb{Q}_{odd}$. Then for all integer numbers $k \geq 0$ we have*

$$Col^k\left(\frac{p}{q}\right) = \frac{1}{q}Col_q^k(p)$$

Proof: We let's observe that

$$Col\left(\frac{p}{q}\right) = \begin{cases} \frac{\frac{p}{q}+1}{2} & \text{if } p = 1 \mod 2 \\ \frac{p}{2q} & \text{if } p = 0 \mod 2 \end{cases} = \frac{1}{q}\begin{cases} \frac{3p+q}{2} & \text{if } p = 1 \mod 2 \\ \frac{p}{2} & \text{if } p = 0 \mod 2 \end{cases} = \frac{1}{q}Col_q(p)$$

Suppose first that $(q,3) = 1$. This fraction is irreducible. Indeed we have that $(3p+q, q) = (3p, q) = 1$. Then the parity of the fraction depends only on the numerator since there is no possibility of simplification that changes the parity of the numerator and we can continue with the iteration for all k since the irreducibility of the iterations only depends on the initial fraction is irreducible. Then we have

$$Col^k\left(\frac{p}{q}\right) = \frac{1}{q}Col_q^k(p)$$

Now suppose that $(q,3) \neq 1$, for this case, the resulting fraction is not irreducible. However, as we are going to prove below, this does not change the parity of the orbits, so the formula would continue to be valid for this case. Suppose that, $q = 3^l v$ with $(v,2) = (v,p) = 1$ and let $Col_q^k(p) = \frac{3^b p + 3^l v N}{2^a}$. We will divide this proof into two parts.

Case one $b \leq l$**:** We are going to prove the statement by induction. To $k = 1$

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$$\frac{1}{3^l v}Col_{3^l v}(p) = \frac{1}{3^l v}\left(\frac{3p+3^l v}{2}\right) = \frac{3\left(\frac{p}{3^l v}\right)+1}{2} = Col\left(\frac{p}{3^l v}\right)$$

Now suppose that the statement is true for k, observe before continuing that the expressions $\left(\frac{3^b p + 3^l vN}{2^a}\right)$ and $\left(\frac{p+3^{l-b}vN}{2^a}\right)$ have the same parity. Indeed,

$$\frac{3^b p + 3^l vN}{2^a} = 3^b\left(\frac{p+3^{l-b}vN}{2^a}\right)$$

if the expression on the left-hand side is even, if and only if $\left(\frac{p+3^{l-b}vN}{2^a}\right)$ it is even. On the other hand, if the left side is odd, $\left(\frac{p+3^{l-b}vN}{2^a}\right)$ must be odd and if $\left(\frac{p+3^{l-b}vN}{2^a}\right)$ is odd, since the product of odd is odd, the left side is odd, so the expressions have the same parity.

1. if $\left(\frac{3^b p + 3^l vN}{2^a}\right)$ it is odd. Expanding the left-hand side of the proposition,

$$\begin{aligned}\frac{1}{3^l v}Col^{k+1}(p) = \frac{1}{3^l v}\psi\left(\frac{3^b p+3^l vN}{2^a}\right) &= \frac{1}{3^l v}\left(\frac{3^{b+1}p+3^{l+1}vN+2^a3^l v}{2^{a+1}}\right)\\ &= \frac{1}{3^{l-(b+1)}v}\left(\frac{p+3^{l+1-(b+1)}vN+2^a3^{l-(b+1)}v}{2^{a+1}}\right)\\ &= \frac{1}{3^{l-(b+1)}v}\left(\frac{p+3^{l-b}vN+2^a3^{l-(b+1)}v}{2^{a+1}}\right)\end{aligned}$$

developing the right-hand side of the proposition,

$$\begin{aligned}Col^{k+1}\left(\frac{p}{3^l v}\right) = Col\left(Col^k\left(\frac{p}{3^l v}\right)\right) &= Col\left(\frac{1}{3^l v}\left(\frac{3^b p+3^l vN}{2^a}\right)\right)\\ &= Col\left(\frac{1}{3^{l-b}v}\left(\frac{p+3^{l-b}vN}{2^a}\right)\right)\\ &= \frac{1}{2}\left(\frac{3}{3^{l-b}v}\left(\frac{p+3^{l-b}vN}{2^a}\right)+1\right)\\ &= \frac{1}{2}\left(\frac{1}{3^{l-(b+1)}v}\left(\frac{p+3^{l-b}vN+2^a3^{l-(b+1)}v}{2^a}\right)\right)\\ &= \frac{1}{3^{l-(b+1)}v}\left(\frac{p+3^{l-b}vN+2^a3^{l-(b+1)}v}{2^{a+1}}\right)\end{aligned}$$

We conclude in this case that both parts are equal

2. if $\left(\frac{3^b p + 3^l vN}{2^a}\right)$ it is even. Expanding the left-hand side of the proposition,

$$\frac{1}{3^l v}Col^{k+1}(p) = \frac{1}{3^l v}\theta\left(\frac{3^b p + 3^l vN}{2^a}\right) = \frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^{a+1}}\right)$$
$$= \frac{1}{3^{l-p} v}\left(\frac{p + 3^{l-b} vN}{2^{a+1}}\right)$$

developing the right-hand side of the proposition,

$$Col^{k+1}\left(\frac{p}{3^l v}\right) = Col\left(Col^k\left(\frac{p}{3^l v}\right)\right) = Col\left(\frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^a}\right)\right)$$
$$= Col\left(\frac{1}{3^{l-b} v}\left(\frac{p + 3^{l-b} vN}{2^a}\right)\right)$$
$$= \frac{1}{3^{l-b} v}\left(\frac{p + 3^{l-b} vN}{2^{a+1}}\right)$$

We conclude in this case that both parts are equal. Since in both cases it gave equality, we conclude that the proposition is true.

Case two $b \geq l$**:** We are going to prove the statement by induction. To $k = 1$

$$\frac{1}{3v}Col_{3v}(p) = \frac{1}{3v}\left(\frac{3p + 3v}{2}\right) = \frac{3(\frac{p}{3}) + 1}{2} = Col\left(\frac{p}{3v}\right)$$

Now suppose that the statement is true for k, observe before continuing that the expressions $\left(\frac{3^b p + 3^l vN}{2^a}\right)$ and $\left(\frac{3^{b-l} p + vN}{2^a}\right)$ have the same parity. Indeed,

$$\frac{3^b p + 3^l vN}{2^a} = 3^l\left(\frac{3^{b-l} p + vN}{2^a}\right)$$

if the expression on the left-hand side is even, if and only if $\left(\frac{3^{b-l} p + vN}{2^a}\right)$ it is even. On the other hand, if the left side is odd, $\left(\frac{3^{b-l} p + vN}{2^a}\right)$ must be odd and if $\left(\frac{3^{b-l} p + vN}{2^a}\right)$ is odd since the product of odd is odd, the left side is odd, so the expressions have the same parity.

1. if $\left(\frac{3^b p + 3^l vN}{2^a}\right)$ it is odd. Expanding the left-hand side of the proposition,

$$\frac{1}{3^l v}Col^{k+1}(p) = \frac{1}{3^l v}\psi\left(\frac{3^b p + 3^l vN}{2^a}\right) = \frac{1}{3^l v}\left(\frac{3^{b+1} p + 3^{l+1} vN + 2^a 3^l v}{2^{a+1}}\right)$$
$$= \frac{3^{b+1-l} p + 3vN + 2^a v}{2^{a+1} v}$$

developing the right-hand side of the proposition,

$$
\begin{aligned}
Col^{k+1}\left(\frac{p}{3^l v}\right) = Col\left(Col^k\left(\frac{p}{3^l v}\right)\right) &= Col\left(\frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^a}\right)\right)\\
&= Col\left(\frac{3^{b-l} p + vN}{2^a v}\right)\\
&= \frac{1}{2}\left(3\left(\frac{3^{b-l} p + vN}{2^a v}\right) + 1\right)\\
&= \frac{3^{b+1-l} p + 3vN + 2^a v}{2^{a+1} v}
\end{aligned}
$$

We conclude in this case that both parts are equal.

2. if $\left(\frac{3^b p + 3^l vN}{2^a}\right)$ it is even. Expanding the left-hand side of the proposition,

$$
\begin{aligned}
\frac{1}{3^l v} Col^{k+1}(p) = \frac{1}{3^l v}\theta\left(\frac{3^b p + 3^l vN}{2^a}\right) &= \frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^{a+1}}\right)\\
&= \frac{3^{b-l} p + vN}{2^{a+1} v}
\end{aligned}
$$

developing the right-hand side of the proposition,

$$
\begin{aligned}
Col^{k+1}\left(\frac{p}{3^l v}\right) = Col\left(Col^k\left(\frac{p}{3^l v}\right)\right) &= Col\left(\frac{1}{3^l v}\left(\frac{3^b p + 3^l vN}{2^a}\right)\right)\\
&= Col\left(\frac{3^{b-l} p + vN}{2^a v}\right)\\
&= \frac{3^{b-l} p + vN}{2^{a+1} v}
\end{aligned}
$$

We conclude in this case that both parts are equal. Since in both cases it gave equality, we conclude that the proposition is true.

QED.

We will define a coding function for the Collatz q-functions and demonstrate that they produce the same coding as the fractions with denominator q.

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Proposition 10 (Second Cod invariance). *Let* $\frac{p}{q} \in \mathbb{Q}$ *an irreducible fraction with* $(q, 2) = 1$ *and* $Cod^k : \mathbb{Z} \to \Sigma_2^*$ *defined by*

$$Cod_q^k(p) = \{\xi_j\}_{j=0}^k, \text{ with } \xi_i \in \{10, 0\}$$

with

$$\xi_j = \begin{cases} 10 & \text{if} \quad Col_q^j(p) = 1 \mod 2 \\ 0 & \text{if} \quad Col_q^j(p) = 0 \mod 2 \end{cases}$$

then we have

$$Cod^k\left(\frac{p}{q}\right) = Cod_q^k(p)$$

Proof: By proposition 9 we have

$$Col^k\left(\frac{p}{q}\right) = \frac{1}{q}Col_q^k(p)$$

Since q it is odd, then, we have coding of $Col^k\left(\frac{p}{q}\right)$ and $Col_q^k(p)$ must be the same.

QED.

We will now establish the initial connection between sets of integers and coding. Specifically, we will demonstrate that all elements within the integer set S share the same coding.

Proposition 11 (First characterization of $\mathbb{E}(S)$). *Let* $p \in \mathbb{Z}$ *and* $S(x) \in \langle \theta, \psi^q \rangle$ *of length* k *with* $(q, 2) = 1$, *then*

$$p \in \mathbb{E}(S) \text{ if and only if } Cod_q^k(p) = Cod_q(S)$$

Proof: Let $p \in \mathbb{E}(S)$ and $S(x) = \bigodot_{i=1}^{k} s_i(x) \in \langle \theta, \psi^q \rangle$ then by definition $S(p) \in \mathbb{Z}$ by Proposition 3 we have $\bigodot_{i=1}^{l} s_i(p) \in \mathbb{Z}$ with $l \leq k$, then $Cod(S) = Cod_q^k(p)$.

Suppose that $Cod_q^k(p) = Cod_q(S)$ then $\{p, s_1(p), s_2 \circ s_1(p), \ldots, \bigodot_{i=1}^{k-1} s_i(p), \bigodot_{i=1}^{k} s_i(p) = S(p)\} \in \mathbb{Z}^{k+1}$, then $p \in \mathbb{E}(S)$.

QED.

We show below the second connection between the integer sets and the encoding. Specifically, we demonstrate that all values p within the integer set S_q indeed have the same coding as the corresponding fraction $\frac{p}{q}$.

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Proposition 12 (second characterization of $\mathbb{E}(S)$). *Let $p \in \mathbb{Z}$ and $S(x) \in \langle \theta, \psi^q \rangle$ of length k with $(q,2) = 1$, then we have:*

$$p \in \mathbb{E}(S) \text{ if and only if } Cod^k\left(\frac{p}{q}\right) = Cod_q(S)$$

Proof: Let $S(x) \in \langle \theta, \psi^q \rangle$ of length k such that $Cod^k(p) = Cod_q(S)$, for the proposition 9, we have

$$qCol^k\left(\frac{p}{q}\right) = Col_q^k(p) = S(p)$$

then $Cod^k\left(\frac{n}{q}\right) = Cod_q(S)$ finally by the proposition 11, we have $p \in \mathbb{E}(S)$ if and only if $Cod^k\left(\frac{p}{q}\right) = Cod_q(S)$.

QED.

The following proposition demonstrates that for a given rational number, we can generate a family of rationals that share the same encoding. This suggests that there exist many rationals with the same k-th encoding

Proposition 13 (Invariance property of Coding of rational). *Let $\frac{p}{q} \in \mathbb{Q}$ an irreducible fraction with $(q,2) = 1$, A_k =numbers from 0 to $Cod^k\left(\frac{p}{q}\right)$ and $T \in \mathbb{Z}$ then*

$$Cod^k\left(\frac{p}{q}\right) = Cod^k\left(\frac{p}{q} + \frac{2^{A_k}T}{q}\right)$$

.

Proof: Let $S(x) \in \langle \theta, \psi^q \rangle$ such that $Cod(S) = Cod_q^k(p)$ then $S(p) = Col_q^k(p) \in \mathbb{Z}$ this implies

$$\mathbb{E}(S) = p + 2^{A_k}T \Rightarrow Cod_q^k(p + 2^{A_k}T) = Cod_q^k(p)$$

then

$$Col^k\left(\frac{p + 2^{A_k}T}{q}\right) = Col^k\left(\frac{p}{q} + \frac{2^{A_k}T}{q}\right) = Cod^k\left(\frac{p}{q}\right)$$

QED.

As we have seen so far, we can characterize the entire set S from its encoding. Exploiting this property, we generalize the entire set S to encompass all fractions sharing the same encoding. We will call the $k-$Coding set.

Definition 10 (The $k-$Coding set). *Let $\{\xi_j\}_{j=0}^{\infty} \in \Sigma_2^*$, we define the k-th coding set of $\{\xi_j\}_{j=0}^{\infty}$*

$$Cod^k\{\xi_j\}_{j=0}^{\infty} = \left\{\frac{p}{q} \in \mathbb{Q}_{odd}, Cod^k\left(\frac{p}{q}\right) = \{\xi_j\}_{j=0}^{k}\right\}$$

The encoding set also exhibits the property of monotonicity, similar to the integer set of S.

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Proposition 14 (Monotony of the coding set). *Let* $\{\xi_j\}_{j=0}^{\infty} \in \Sigma_2^*$ *then*

$$Cod^{k+1}\{\xi_j\}_{j=0}^{\infty} \subset Cod^{k}\{\xi_j\}_{j=0}^{\infty}.$$

Proof: Let $\frac{p}{q} \in Cod^{k+1}\{\xi_j\}_{j=0}^{\infty}$ by definition $Cod^{k+1}\left(\frac{p}{q}\right) = \{\xi_j\}_{j=1}^{k+1}$ then trivially we have $Cod^{k}\left(\frac{p}{q}\right) = \{\xi_j\}_{j=0}^{k}$, then $\frac{p}{q} \in Cod^{k}\{\xi_j\}_{j=0}^{\infty}$.

QED.

Definition 11 (The Coding set). *Let* $\{\xi_j\}_{j=0}^{\infty} \in \Sigma_2^*$, *we define the k-th coding set of* $\{\xi_j\}_{j=0}^{\infty}$ *the coding set of* $\{\xi_j\}_{j=0}^{\infty}$

$$Cod\{\xi_j\}_{j=0}^{\infty} := \bigcap_{k\in\mathbb{N}} Cod^{k}\{\xi_j\}_{j=0}^{\infty}.$$

Similarly, the behavior of the solutions of Diophantine equations, in which knowing a particular solution allows us to determine other solutions, is reflected in the coding set. This connection is illustrated in the following proposition.

Proposition 15 (Generating property). *Let* $\{\xi_j\}_{j=0}^{\infty} \in \Sigma_2^*$, A_k *=numbers from 0 to* $\{\xi_j\}_{j=0}^{k}$ *and* $\frac{p}{q}, \frac{t}{r} \in Cod^{k}\{\xi_j\}_{j=0}^{\infty}$ *then exist* $T \in \mathbb{Z}$

$$\frac{p}{q} = \frac{t}{r} + \frac{2^{A_k}T}{qr}$$

Proof: Let $\frac{p}{q}, \frac{t}{r} \in Cod^{k}\{\xi_j\}_{j=0}^{\infty}$ and $S_k(x) = \frac{3^b x + N}{2^{A_k}} \in \langle\theta, \psi\rangle$ such that $Cod(S_k) = \{\xi_j\}_{j=0}^{k}$, now consider $S_k^q \in \langle\theta, \psi^q\rangle$ and $S_k^r \in \langle\theta, \psi^r\rangle$ such that $Cod_q^k(S_k^q) = Cod_r^k(S_k^r) = \{\xi_j\}_{j=0}^{k}$ by proposition 12 we have $S_k^q(p), S_k^r(t) \in \mathbb{Z}$ the latter is equivalent

$$\frac{3^b p + qN}{2^{A_k}}, \frac{3^b t + rN}{2^{A_k}} \in \mathbb{Z}$$

We are going to prove that pr and qt are elements of $\mathbb{E}(S_k^{qr})$ with $s^r(x) = \frac{3^b x + qrN}{2^{A_k}}$. Indeed,

$$\frac{3^b pr + qrN}{2^{A_k}} = r\left(\frac{3^b p + qN}{2^{A_k}}\right) \in \mathbb{Z}$$

and

$$\frac{3^b qt + qrN}{2^{A_k}} = q\left(\frac{3^b t + rN}{2^{A_k}}\right) \in \mathbb{Z}$$

then

$$pr = qt + 2^{A_k}T \text{ with } T \in \mathbb{Z} \Rightarrow \frac{p}{q} = \frac{t}{r} + \frac{2^{A_k}T}{qr} \text{ with } T \in \mathbb{Z}$$

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QED.

Now, we will present the main theorem of this section, establishing that the encoding of a rational number is unique.

Theorem 3 (Uniqueness of the full coding on $\mathbb{Q}_{odd}$)**.** *Let* $\{\xi_j\}_{j=0}^{\infty} \in \Sigma_2^*$*. If it exists* $\frac{p}{q} \in \mathbb{Q}_{odd}$ *such that* $Cod\left(\frac{p}{q}\right) := \lim_{k\to\infty} Cod^k\left(\frac{p}{q}\right) = \{\xi_j\}_{j=0}^{\infty}$ *then it is unique.*

Proof: Let A_k =numbers from 0 to $\{\xi_j\}_{j=0}^{k}$. Suppose there is another element, $\frac{t}{r} \in \mathbb{Q}_{odd}$ such than $Cod\left(\frac{t}{r}\right) = \lim_{k\to\infty} Cod^k\left(\frac{p}{q}\right) = \{\xi_j\}_{j=0}^{\infty}$ by proposition 15

$$\frac{t}{r} = \frac{p}{q} + \frac{2^{A_k}T}{rq} \text{ with } T \in \mathbb{Z} \text{ and } k \in \mathbb{N}$$

Since $\frac{p}{q} \neq \frac{t}{r}$ then $T \neq 0$. On the other hand, since $2^{A_k} \to \infty$ and since T is an integer, so $|2^{A_k}T|$ tending to infinity, which implies $\frac{t}{r}$ should be infinite, which is a contradiction.

QED.

6. The π^1 and π^2 Functions

In this section, we introduce two fundamental functions, π^1 and π^2, that will play a crucial role in the subsequent sections. The function π^1 maps each sequence from Σ_2^* to a real series. We denote by G_∞ the set of sequences ξ for which $\pi^1(\xi) < \infty$.

Additionally, the function π^2 associates each sequence with a function defined on the natural numbers. The set G_0 comprises sequences for which $\pi^2(\xi)(k)$ represents a bounded function.

6.1. Summary of Propositions in the Section

1. **Definition 12:** We will give the definition of the functions π^1 and π^2.
2. **Definition 13:** Definition of Null Tail: sequences with a finite number of 1s.
3. **Definition 14:** Sets G_0, G_∞ and G_1.
4. **Lemma 3** Characterization of G_0, G_∞ and G_1 through accumulation points of $\frac{a_k}{k}$.
5. **Proposition 16** Characterization of G_0 and G_∞ through functions π^1 and π^2.

6.2. The π^1 and π^2 Functions

Let S be a function of the space generated by the functions ψ and θ with length k. According to proposition 7, we have that the general form of S is given by $\frac{3^k x + N_k}{2^{a_k}}$. The function π^1 precisely represents the quotient of N_k by 3^k, and the function π^2 represents the quotient of N_k by 2^{a_k}. As we will show later, these functions preserve the encoding information of S. One detail to note is that, unlike the function π^1, which is a real number depending on its convergence, the function π^2 is not a number but a function. This is because the function π^1 is related to the number that generates a certain encoding, and the function π^2 is related to the orbit that it generates.

The elements of Σ_2^* have two characteristic forms: either they have an infinite sequence of 1s, or they have a finite amount of 1s. Those that have a finite amount of 1s we will refer to as having a null tail (we will formalize this later). We will normalize the representation of the elements of Σ_2^* to precisely define applications. It is important to note that 0 and 1 are symbols, so the expression 0^0 denotes the absence of this element. For example, $0^0 10^2 \ldots$ is equivalent to $10^2 \ldots$.

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Definition 12 (The π^1 and π^2 functions). *Let $\xi \in \Sigma_2^*$ and $k \in \mathbb{N}_0$. Let a_k =quantity of 0 $\{\xi_j\}_{j=0}^k$, b_k = quantity of 1 of $\{\xi_j\}_{j=0}^k$ and define the function $N_k : \Sigma_2^* \to \mathbb{N}_0$ given by*

$$N_k\{\xi_j\}_{j=0}^{\infty} = \begin{cases} 3^{b_k-1}2^{a_0} + 3^{b_k-2}2^{a_1} + \ldots + 2^{a_{b_k-1}} & \text{if} \quad b_k > 0 \\ 0 & \text{if} \quad b_k = 0 \end{cases}$$

and defined $\pi_k^1 : \Sigma_2^* \to \mathbb{Q}$ by

$$\pi_k^1\{\xi_j\}_{j=0}^{\infty} = \frac{N_k\{\xi_j\}_{j=0}^{\infty}}{3^{b_k\{\xi_j\}_{j=0}^{\infty}}} = \sum_{j=1}^{b_k} \frac{2^{a_{j-1}}}{3^j}$$

and defined the $\pi-$function as $\pi^1 : \Sigma_2^* \to \mathbb{R} \cup \{\infty\}$

$$\pi^1\{\xi_j\}_{j=0}^{\infty} = \lim_{k\to\infty} \frac{N_k\{\xi_j\}_{j=0}^{\infty}}{3^{b_k\{\xi_j\}_{j=0}^{\infty}}} = \begin{cases} \sum_{j=1}^{\infty} \frac{2^{a_{j-1}}}{3^j} & \text{if} \ \lim_{k\to\infty} b_k = \infty \\ \sum_{j=1}^{B} \frac{2^{a_{j-1}}}{3^j} & \text{if} \ \lim_{k\to\infty} b_k = B < \infty \end{cases}$$

and $\pi^2 : \Sigma_2^* \to \{f : \mathbb{N} \to \mathbb{Q} : f \text{ function}\}$ by

$$\pi^2\{\xi_j\}_{j=0}^{\infty}(k) = \frac{N_k\{\xi_j\}_{j=0}^{\infty}}{2^{a_k\{\xi_j\}_{j=0}^{\infty}}} = \frac{3^{b_k}}{2^{a_k}}\pi_k^1\{\xi_j\}_{j=0}^{\infty} = \frac{3^{b_k}}{2^{a_k}} \sum_{j=1}^{b_k} \frac{2^{a_{j-1}}}{3^j}$$

We begin by providing a technical definition that we will use for the subsequent results.

Definition 13 (Null Tail). *we will say that $\{\xi_j\}_{j=0}^{\infty} \in \Sigma_2^*$ has a null tail of index J if $J > 0$ the smallest index such that $j > J$ we have $\xi_j = 0$.*

Example 7. *We will give some examples of the functions π^1 and π^2:*

1. *Let* $\xi_0 = 00101001000000\ldots0000\ldots \in \Sigma_2^*$
   (a) $\pi^1(\xi_0) = \frac{2^2}{3} + \frac{2^3}{9} + \frac{2^5}{27}$.
   (b) $\pi^2(\xi_0)(k) = \frac{27}{2^{k+2}}\left\{\frac{2^2}{3} + \frac{2^3}{9} + \frac{2^5}{27}\right\}$ *for $k \geq 4$ and* $\lim_{k\to\infty} \pi^2(\xi_0)(k) = 0$
2. *Let* $\xi_1 = 100100100\ldots \in \Sigma_2^*$ *then we have*
   (a) $\pi^1(\xi_1) = \sum_{j=1}^{\infty} \frac{2^{2(j-1)}}{3^j} = \frac{1}{3}\sum_{j=0}^{\infty} \frac{2^{2j}}{3^j} = \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{4}{3}\right)^j = \infty.$
   (b) $\pi^2(\xi_1)(k) = \frac{3^k}{2^{2k}}\sum_{j=1}^{k} \frac{2^{2(j-1)}}{3^j} = \frac{3^{k-1}}{2^{2k}}\sum_{j=0}^{k-1} \frac{2^{2j}}{3^j}$
   $= \frac{3^{k-1}}{2^{2k}}\left\{\frac{\frac{2^{2k}}{3^k} - 1}{\frac{4}{3} - 1}\right\} = \left\{1 - \left\{\frac{3}{4}\right\}^k\right\}.$
3. *Let* $\xi_2 = 10101010\ldots \in \Sigma_2^*$ *then we have*
   (a) $\pi^1(\xi_2) = \sum_{j=1}^{\infty} \frac{2^{j-1}}{3^j} = \frac{1}{3}\sum_{j=0}^{\infty} \frac{2^j}{3^j} = \frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{2}{3}\right)^j = 1.$

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(b) $\pi^2(\xi_2)(k) = \frac{3^k}{2^k}\sum_{j=1}^{k}\frac{2^{j-1}}{3^j} = \frac{3^{k-1}}{2^k}\sum_{j=0}^{k-1}\frac{2^j}{3^j}$

$$= \frac{3^{k-1}}{2^k}\left\{\frac{\frac{2^k}{3^k}-1}{\frac{3}{3}-1}\right\} = \left\{\left\{\frac{3}{2}\right\}^k - 1\right\} \to \infty.$$

4. Let $\xi_3 = 10100100010000\ldots \in \Sigma_2^*$

(a) $\pi^1(\xi_3) = \sum_{j=1}^{\infty}\frac{2^{\frac{j(j+1)}{2}-1}}{3^j} \to \infty$. Since, $\frac{2^{\frac{j(j+1)}{2}}}{3^j} \to \infty$.

(b) $\pi^2(\xi_3)(k) = \frac{3^k}{2^{\frac{k(k+1)}{2}}}\sum_{j=1}^{k}\frac{2^{\frac{j(j+1)}{2}-1}}{3^j} < 1$

5. Let $\xi_4 = 101000101000\ldots \in \Sigma_2^*$ then we have

(a) $\pi^1(\xi_4) = \frac{1}{3} + \frac{2}{3^2} + \frac{2^4}{3^3} + \frac{2^5}{3^4} + \frac{2^8}{3^5} + \frac{2^9}{3^6} + \frac{2^{12}}{3^7} + \ldots$

$$= \frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^4}{3^3}\right\} + \frac{2^4}{3^2}\left\{\frac{2}{3^2} + \frac{2^4}{3^3}\right\} + \left(\frac{2^4}{3^2}\right)^2\left\{\frac{2}{3^2} + \frac{2^4}{3^3}\right\} + \ldots$$

$$= \frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^4}{3^3}\right\}\sum_{j=0}^{\infty}\left(\frac{2^4}{3^2}\right)^j \to \infty.$$

(b) $\pi^2(\xi_4)(1) = \frac{1}{2}, \pi^2(\xi_4)(2) = \frac{5}{2^4}, \pi^2(\xi_4)(2k+1) = \frac{3^{2k+1}}{2^{4k+1}}\left(\frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^4}{3^3}\right\}\sum_{j=0}^{k-1}\left(\frac{2^4}{3^2}\right)^j\right)$

and $\pi^2(\xi_4)(2k) = \frac{3^{2k}}{2^{4(k+1)}}\left(\frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^4}{3^3}\right\}\sum_{j=0}^{k-2}\left(\frac{2^4}{3^2}\right)^j + \frac{2^{4k+1}}{3^{2k}}\right)$

Let $\xi \in \Sigma_2^*$ define the function $\frac{3^{b_k(\xi)}}{2^{a_k(\xi)}}$. This function corresponds to the slope of the function S_k such that $Cod^k(S_k) = \{\xi_j\}_{j=0}^k$.

We can consider the following set of Σ_2^*:

Definition 14 (The G_0, G_∞ and G_1 sets). *We will consider the following subsets of* Σ_2^*

$$G_0 = \left\{\xi \in \Sigma_2^* : \frac{3^{b_k(\xi)}}{2^{a_k(\xi)}} \to 0 \text{ as } k \to \infty \text{ or } \xi \text{ is Null Tail}\right\}$$

,

$$G_\infty = \left\{\xi \in \Sigma_2^* : \frac{3^{b_k(\xi)}}{2^{a_k(\xi)}} \to \infty \text{ as } k \to \infty \text{ or } \xi \text{ is Null Tail}\right\}$$

and

$$G_1 = \left\{\xi \in \Sigma_2^* : \text{ Exist } \{k_l\}_{l\in\mathbb{N}} \text{ such that } \frac{3^{b_{k_l}(\xi)}}{2^{a_{k_l}(\xi)}} \to 1 \text{ as } k \to \infty\right\}$$

The following result establishes a characterization of the elements of G_0 and G_∞.

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Lemma 3. *Let* $\xi \in \Sigma_2^*$ *then*

1. $\xi \in G_0$ *if and only if* $\liminf_{k\to\infty} \frac{a_k}{k} > \frac{\ln(3)}{\ln(2)}$,
2. $\xi \in G_\infty$ *if and only if* $\limsup_{k\to\infty} \frac{a_k}{k} < \frac{\ln(3)}{\ln(2)}$,

Proof:

1. Let $\xi \in G_0$ then $\frac{3^k}{2^{a_{k-1}}} \to 0$ as $k \to \infty$. Suppose that $\liminf_{k\to\infty} \frac{a_k}{k} < \frac{\ln(3)}{\ln(2)}$. Then exist $K, \varepsilon > 0$ such that $\frac{a_{k_l}}{k_l} + \varepsilon < \frac{\ln(3)}{\ln(2)}$ for $k > K$ where $\lim_{l\to\infty} \frac{a_{k_l}}{k_l} = \liminf_{k\to\infty} \frac{a_k}{k}$, then we have,

$$\frac{3^{k_l}}{2^{a_{k_l}}} = \left(\frac{3}{2^{\frac{a_{k_l}}{k_l}}}\right)^{k_l} > \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)} - \varepsilon}}\right)^{k_l} = 2^{k_l\varepsilon} \to \infty \text{ as } l \to \infty$$

which is a contradiction with the fact that $\xi \in G_0$, then $\liminf_{k\to\infty} \frac{a_k}{k} > \frac{\ln(3)}{\ln(2)}$.

Now, let's suppose that $\liminf_{k\to\infty} \frac{a_k}{k} > \frac{\ln(3)}{\ln(2)}$. Then exist $K, \varepsilon > 0$ such that $\frac{a_k}{k} - \varepsilon > \frac{\ln(3)}{\ln(2)}$ for all $k > K$. Then

$$\frac{3^k}{2^{a_k}} = \left(\frac{3}{2^{\frac{a_k}{k}}}\right)^k < \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)} + \varepsilon}}\right)^k = 2^{-k\varepsilon} \to 0 \text{ as } k \to \infty$$

2. Let $\xi \in G_\infty$ then $\frac{3^k}{2^{a_k}} \to \infty$ as $k \to \infty$. Suppose that $\limsup_{k\to\infty} \frac{a_k}{k} > \frac{\ln(3)}{\ln(2)}$. Then exist $K, \varepsilon > 0$ such that $\frac{a_{k_l}}{k_l} - \varepsilon > \frac{\ln(3)}{\ln(2)}$ for $k > K$ where $\lim_{l\to\infty} \frac{a_{k_l}}{k_l} = \limsup_{k\to\infty} \frac{a_k}{k}$, then we have,

$$\frac{3^{k_l}}{2^{a_{k_l}}} = \left(\frac{3}{2^{\frac{a_{k_l}}{k_l}}}\right)^{k_l} < \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)} + \varepsilon}}\right)^{k_l} = 2^{-k_l\varepsilon} \to 0 \text{ as } l \to \infty$$

which is a contradiction with the fact that $\xi \in G_\infty$, then $\limsup_{k\to\infty} \frac{a_k}{k} < \frac{\ln(3)}{\ln(2)}$.

Now, let's suppose that $\limsup_{k\to\infty} \frac{a_k}{k} < \frac{\ln(3)}{\ln(2)}$, Then exist $K, \varepsilon > 0$ such that $\frac{a_k}{k} + \varepsilon < \frac{\ln(3)}{\ln(2)}$ for all $k > K$. Then

$$\frac{3^k}{2^{a_k}} = \left(\frac{3}{2^{\frac{a_k}{k}}}\right)^k > \left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)} - \varepsilon}}\right)^k = 2^{k\varepsilon} \to \infty \text{ as } k \to \infty$$

QED.

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The following result establishes a characterization of the elements of G_0 and G_∞ for the behavior of the functions π.

Proposition 16 (Characterization of the G_0 and G_∞ sets)**.** *Let $\xi \in \Sigma_2^*$ then*

1. *$\xi \in G_\infty$ if and only if $\pi^1(\xi) < \infty$.*
2. *$\xi \in G_0$ if and only if $\pi^2(\xi)(k)$ is bounded.*

Proof: *Proof of the first statement.* Is obvious for the case of null tails with index J, since we have π^1 it is automatically finite, and as we see in the examples π^2 would be of the form $\frac{3^{k_0}}{2^k}A \to 0$ when $k \to \infty$ which implies that π^2 is finite.

So we are going to assume that ξ has no tail null.

Suppose that $\xi \in G_\infty$, then by Lemma 3 and a_j is strictly increasing we have $\limsup_{j\to\infty} \frac{a_{j-1}}{j} \leq \limsup_{j\to\infty} \frac{a_j}{j} < \frac{\ln(3)}{\ln(2)}$, so, there exists $J, \varepsilon > 0$ such that for all $j > J$ we have that $\frac{a_{j-1}}{j} < \frac{\ln(3)}{\ln(2)} - \varepsilon$

$$\sum_{j=1}^{\infty} \frac{2^{a_{j-1}}}{3^j} = \sum_{j=1}^{J} \frac{2^{a_{j-1}}}{3^j} + \sum_{j=J+1}^{\infty} \frac{2^{a_{j-1}}}{3^j} = \sum_{j=1}^{J} \frac{2^{a_{j-1}}}{3^j} + \sum_{j=J+1}^{\infty} \left(\frac{2^{\frac{a_{j-1}}{j}}}{3} \right)^j$$

$$< \sum_{j=1}^{J} \frac{2^{a_{j-1}}}{3^j} + \sum_{j=J+1}^{\infty} \left(\frac{2^{\frac{\ln(3)}{\ln(2)} - \varepsilon}}{3} \right)^j = \sum_{j=1}^{J} \frac{2^{a_{j-1}}}{3^j} + \sum_{j=J+1}^{\infty} 2^{-\varepsilon j} < \infty$$

Let's suppose $\sum_{j=1}^{\infty} \frac{2^{a_{j-1}}}{3^j} < \infty$, then $\sum_{j=0}^{\infty} \frac{2^{a_j}}{3^j}$ is also convergent since $\sum_{j=0}^{\infty} \frac{2^{a_j}}{3^j} = \sum_{j=1}^{\infty} \frac{2^{a_{j-1}}}{3^{j-1}} = 3 \sum_{j=1}^{\infty} \frac{2^{a_{j-1}}}{3^j} < \infty$, so $\lim_{j\to\infty} \frac{2^{a_j}}{3^j} = 0$ then $\xi \in G_\infty$.

QED of the first statement.

Proof of the second statement. Suppose $\pi^2(\xi)(j)$ is bounded, we will prove that $\frac{3^k}{2^{a_k}(\xi)}$ converges to 0.

Suppose $\frac{3^k}{2^{a_k}(\xi)} > \varepsilon > 0$ for $k > K$ for any $K \in \mathbb{N}$. Then we have

$$\pi^2(\xi)(j) = \frac{N_j}{2^{a_k}} = \frac{3^k}{2^{a_k}} \sum_{j=1}^{k} \frac{2^{a_{j-1}}}{3^j}$$

We have that the sum on the right is divergent since $\frac{2^{a_k}}{3^k}$ is divergent, then

$$\frac{3^k}{2^{a_k}} \sum_{j=1}^{k} \frac{2^{a_{j-1}}}{3^j} > \varepsilon \sum_{j=1}^{k} \frac{2^{a_{j-1}}}{3^j} = \infty$$

which generates a contradiction to the fact that $\pi^2\{\xi\}(j)$ is bounded.

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To demonstrate the other implication, let us consider the following lemmas:

Lemma 4. *Let* $\liminf_{k\to\infty} \frac{a_k}{k} = \lambda \in (0,\infty]$*. Then exist* $T, L > 0$ *such that if* $T < k - j$ *we have*

$$\frac{a_k - a_{j-1}}{k-j} \geq \lambda.$$

Proof: Let $\liminf_{k\to\infty} \frac{a_k}{k} = \lambda \in (0,\infty]$ and $d = k - j$, then we have

$$\liminf_{(k-j)\to\infty} \frac{a_k - a_{j-1}}{k-j} = \liminf_{d\to\infty} \frac{a_{j+d} - a_{j-1}}{d} = \liminf_{d\to\infty} \frac{a_d}{d} = \lambda$$

On the other hand, by definition of lower limit, we have

$$\liminf_{(k-j)\to\infty} \frac{a_k - a_{j-1}}{k-j} = \lim_{t\to\infty} \left\{ \inf_{(k-j)>t} \frac{a_k - a_{j-1}}{k-j} \right\}$$

Then exist $T > 0$ such that if $T < k - j$ we have

$$\frac{a_k - a_{j-1}}{k-j} \geq \inf_{(k-j)>T} \frac{a_k - a_{j-1}}{k-j} = \lambda$$

QED of Lemma.

Lemma 5. *Let* $k, j \in \mathbb{N}$*, if* $k > j$ *then, we have* $\frac{a_k - a_{j-1}}{k-j} \geq 1$.

Proof: Let $\xi \in G_0$ writing explicitly, we have $\xi = 0^{\theta_0} \prod_{i=1}^{\infty} 0^{\theta_i} 1$ with $\theta_0 \geq 0$ and $\theta_i > 0$ for $i \geq 1$, then we can write:

$$a_k = \sum_{i=0}^{k} \theta_i$$

Suppose $k > j$. Since the minimum value that θ can take is 1, we have

$$\frac{a_k - a_{j-1}}{k-j} = \frac{\sum_{i=j}^{k} \theta_i}{k-j} \geq \frac{k-j}{k-j} = 1$$

QED of Lemma.

By Claim 3 and 4 we have, exist $T, \varepsilon \in \mathbb{N}$ such that if $k - T > j$ we have

$$\frac{a_k - a_{j-1}}{k-j} > \frac{\ln(3)}{\ln(2)} + \varepsilon$$

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Then by claim 5, Let $k > T$ so

$$
\begin{aligned}
\frac{3^k}{2^{a_k}}\sum_{j=1}^{k}\frac{2^{a_{j-1}}}{3^j} &= \sum_{j=1}^{k}\frac{2^{a_{j-1}-a_k}}{3^{j-k}} \\
&= \sum_{j=1}^{k}\left(\frac{3}{2^{\frac{a_k-a_{j-1}}{k-j}}}\right)^{k-j} \\
&= \sum_{j=1}^{k-T}\left(\frac{3}{2^{\frac{a_k-a_{j-1}}{k-j}}}\right)^{k-j} + \sum_{j=k-T+1}^{k}\left(\frac{3}{2^{\frac{a_k-a_{j-1}}{k-j}}}\right)^{k-j} \\
&< \sum_{j=1}^{k-T}\left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)}+\varepsilon}}\right)^{k-j} + \sum_{j=k-T+1}^{k}\left(\frac{3}{2}\right)^{k-j} \\
&< \sum_{j=1}^{k}\left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)}+\varepsilon}}\right)^{k-j} + \sum_{j=k-T+1}^{k}\left(\frac{3}{2}\right)^{k-j} \\
&< \frac{1}{2^{\varepsilon k}}\sum_{j=1}^{k}2^{\varepsilon j} + \left(\frac{3}{2}\right)^{k}\sum_{j=k-T+1}^{k}\left(\frac{2}{3}\right)^{j} \\
&= \frac{1}{2^{\varepsilon k}}\left(\frac{1-2^{\varepsilon(k+1)}}{1-2^{\varepsilon}}\right) + \left(\frac{3}{2}\right)^{k}\left\{\frac{\left(\frac{2}{3}\right)^{k}-\left(\frac{2}{3}\right)^{k-T+1-1}}{\frac{2}{3}-1}\right\} \\
&< \frac{\frac{1}{2^{\varepsilon k}}-2^{\left(\frac{k+1}{k}\right)}}{1-2^{\varepsilon}} + \frac{1-\left(\frac{3}{2}\right)^{T}}{\frac{2}{3}-1} \\
&\leq \frac{4}{2^{\varepsilon}-1} + 3\left\{\left(\frac{3}{2}\right)^{T}-1\right\}
\end{aligned}
$$

Let $M = \frac{4}{2^{\varepsilon}-1} + 3\left\{\left(\frac{3}{2}\right)^{T}-1\right\} > 0$. Then we have $\frac{3^k}{2^{a_k}}\sum_{j=1}^{k}\frac{2^{a_{j-1}}}{3^j} < M$. Then we conclude that π^2 is bounded.

QED of the second statement.

QED.

7. The Sigma Function

In this section, we immerse ourselves in the rigorous study of Diophantine equations of the form $2^k y - ax = n$, where a, k, n are integers. Solving these equations in the domain of integers x and y is a problem in number theory. Usually, these types of Diophantine equations are solved using Euclid's algorithm or some similar technique, even by trial and error. However, these techniques begin to have a high degree of complexity for very large values. This mainly complicates when we want to study the behavior of the minimum positive values since in this case, we are interested in asymptotic solutions. We introduce the sigma function, symbolized as $\sigma_a^k(n)$ to address this challenge. This function, whose detailed analysis will constitute the core of our research, plays a fundamental role in the quest for specific solutions to the aforementioned Diophantine equations. Particularly noteworthy is the sigma function's remarkable property of delivering solutions that are closest to zero in the context of these equations.

7.1. Summary of Propositions in the Section

1. **Definition 15:** Definition of the sigma function.

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2. **Theorem 4:** Establish that $\sigma_a^k(n)$ and $\frac{1}{a}(2^k\sigma_a^k(n)-n)$ are solutions of the Diophantine equation $2^k y - ax = n$. Additionally, $\frac{1}{a}(2^k\sigma_a^k(n)-n)$ is the minimum non-negative integer value.
3. **Corollary 1:** Establishes that the minimum value grows based on the number of times the sigma function takes odd values.
4. **Corollary 2:** $\sigma_a^k(n) - \sigma_{-a}^k(n) = a$
5. **Proposition 17:** Establishes inequalities that estimate the values of the sigma function
6. **Proposition 18:** It establishes the periods for the periodic points.
7. **Proposition 19:** Establish algebraic properties of additivity, dependent on the parity of the addends
8. **Corollary 3** Establish algebraic properties' linearity modulo a.
9. **Proposition 20** Establish that the sigma function is homogeneous modulo a.
10. **Definition 16:** Extension of the sigma function on $\mathbb{Q}_{odd}$
11. **Definition 17:** Characteristic Function
12. **Lemma 6:** Establishes an invariance in the coding of the orbits of the sigma function.
13. **Proposition 21:** Establishes homogeneity properties of the extension of the sigma function.
14. **Proposition 22** Algebraic properties of the Extension of the Sigma function.
15. **Definition 18:** Definition of dyadic numbers.
16. **Proposition 23:** Characterization of the dyadic representation of rational numbers.
17. **Definition 19:** Definition of Cod-Sigma function.
18. **Lemma 7:** Invariant coding lemma for Cod-Sigma function.
19. **Proposition 24:** Change of basis of the Cod-Sigma function.
20. **Proposition 25:** Let $r \in \mathbb{N}$ and $\gamma = 3^{b-1}$ and let $k \in \mathbb{N}$ such that $r < 2^{2\gamma k}$ and $\delta_j \in \{0,1\}$ such that $r\left\{\frac{2^{2\gamma k}-1}{3\gamma}\right\} = \sum_{j=0}^{2\gamma k-1} \delta_j 2^j$ then $Cod\sigma_{3\gamma}^{2\gamma k}(r) = \delta_{2\gamma k-1}, \ldots, \delta_0$.
21. **Corollary 4:** Let $r_1, r_2 \in \mathbb{N}$. Then $Cod\sigma_{3^b}^a(r_1+r_2) = Cod\sigma_{3^b}^a(r_1) + Cod\sigma_{3^b}^a(r_2) \mod 2^a$.
22. **Proposition 26** $Cod\sigma_{3^b} : \mathbb{N} \to \mathbb{Z}_2$ is linear.
23. **Lemma 8:** Rational equivalence of the Cod-Sigma function.
24. **Lemma 27:** $-\pi^1(\xi) \in \mathbb{Z}_2$ and $Cod\sigma(\pi^1(\xi)) = -\pi^1(\xi) \in \mathbb{Z}_2$.

7.2. The Sigma function

We are going to define the sigma function. This function is very similar to the Collatz function except that in this function, we do not multiply by 3.

Definition 15 (The Sigma function)**.** *Let $x, a \in \mathbb{Z}$ such that $(a,2) = 1$. We define the sigma function $\sigma_a : \mathbb{Z} \to \mathbb{Z}$*

$$\sigma_a(x) = \begin{cases} \dfrac{x+a}{2} & \text{if } x = 1 \mod 2 \\ \\ \dfrac{x}{2} & \text{if } x = 0 \mod 2 \end{cases}$$

In the following theorem, we explore solutions to the Diophantine equation $2^k y - ax = n$, where a, k, and n are integers. This equation arises frequently in number theory, particularly in the study of Diophantine equations. We'll demonstrate that the sigma function provides particular solutions for y, shedding light on the behavior of solutions in both positive and negative domains. Additionally, we'll establish formulas for the smallest non-negative solution ρ_0 and the largest non-positive solution ρ_1 for the variable x, offering valuable insights into the structure of solutions to this equation.

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Theorem 4 (Theorem on Diophantine Solutions)**.** *Let $a, k \in \mathbb{N}$ with $(a, 2) = 1$ and $n \in \mathbb{Z}$. Consider the Diophantine Equation $2^k y - ax = n$. Then a particular solution for y is given by*

$$\sigma_a^k(n) \text{ and } \sigma_{-a}^k(n).$$

Furthermore. Let ρ_0 be the smallest non-negative solution for x, then

$$\rho_0 = \frac{1}{a}(2^k \sigma_a^k(n) - n)$$

let ρ_1 be the largest non-positive solution for x if then

$$\rho_1 = \frac{1}{a}(2^k \sigma_{-a}^k(n) - n)$$

Proof: We can write the sigma function as

$$\sigma_{\pm a}(n) = \frac{n \pm a\delta(n)}{2} \text{ where } \delta(n) = \begin{cases} 1 & \text{if } n = 1 \mod 2 \\ 0 & \text{if } n = 0 \mod 2 \end{cases}$$

Since the sigma function is defined on the set of integers in the integers, we have that its k-th composition is also an integer value: Let $\delta_j = \delta(\sigma_a^j(n))$ then

$$\sigma_a^k(n) = \frac{\frac{\frac{n+\delta_0}{2}}{\ddots} + a\delta_{k-1}}{2} = \frac{n + aL}{2^k} \in \mathbb{Z} \text{ where } L = \sum_{j=0}^{k-1} \delta_j 2^j$$

and Let $\varepsilon_j = \delta(\sigma_{-a}^j(n))$ then

$$\sigma_{-a}^k(n) = \frac{\frac{\frac{n-a\varepsilon_0}{2}}{\ddots} - a\varepsilon_{k-1}}{2} = \frac{n - aU}{2^k} \in \mathbb{Z} \text{ where } U = \sum_{j=0}^{k-1} \varepsilon_j 2^j$$

replacing the k-th iteration sigma function σ_a in the equation $2^k y - ax = n$ and solving for x_0, we have

$$x_0 = \frac{1}{a}(2^k \sigma_a^k(n) - n) = L \in \mathbb{Z}$$

and replacing the k-th iteration sigma function σ_{-a} in the equation $2^k y - ax = n$ and solving for x_0, we have

$$x_0 = \frac{1}{a}(2^k \sigma_{-a}^k(n) - n) = -U \in \mathbb{Z}$$

For the positive case, we have that $0 \leq L = \sum_{j=0}^{k-1} \delta_j 2^j \leq 2^k - 1$, then due to the uniqueness of solutions in $[0, 2^k) \cap \mathbb{Z}$, L corresponds to the non-negative minimum value and for the negative case we have $0 \geq -G_\infty = -\sum_{j=0}^{k-1} \varepsilon_j 2^j \geq -(2^k - 1)$, again due to uniqueness of solutions in $(2^k, 0] \cap \mathbb{Z}$, we have that $-G_\infty$ is the maximum non-positive solution.

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QED.

Example 8. *Let us consider the following Diophantine equation* $16y - 7x = 45$ *then*

$$\sigma_7^4(45) = 5 \textit{ and } \frac{1}{7}(16 \cdot 5 - 45) = 5$$

are solutions of the equation.

We will demonstrate that this minimum value increases every time $\sigma_a^k(n)$ is an odd number. This result is crucial for understanding how the parity of the sigma function influences the structure of non-negative solutions of the associated Diophantine equation.

Corollary 1 (Monotony relation). *Let* $a \in \mathbb{N}$ *such that* $(a, 2) = 1$ *and* $S_a(x) = \dfrac{ax + n}{2^k}$ *and* $\rho_0(S_a)$ *the minimum non-negative value of* $S_a(x)$*. Then* $\rho_0(S_a)$ *increases every time* $\sigma_a^k(n)$ *is an odd number. In particular* $\rho_0(S_a) = \sum_{j=0}^{k-1} \delta_j 2^j$ *with* $\delta_j = 0$ *if* $\sigma_a^{j-1}(n)$ *is even and* $\delta_j(n) = 1$ *if* σ_a^{j-1} *is odd.*

Proof: Let $S_a(x) = \dfrac{ax + n}{2^k}$ and $\delta_j = \delta(\sigma_a^j(n))$ then by Theorem 4 we have

$$\sigma_a^k(n) = \frac{1}{2^k}\left\{ n + a \sum_{j=0}^{k-1} \delta_j 2^j \right\}$$

then $S_a^{-1}(\sigma_a^k(n)) = \rho_0(S_a) = \sum_{j=0}^{k-1} \delta_j 2^j$. So, we have that every time $\delta_j = 1$, the minimum positive integer value increases, and this only happens when $\sigma_a^{j-1}(n)$ is odd.

QED.

In the following corollary, we explore the relationship between the sigma functions $\sigma_a^k(n)$ and $\sigma_{-a}^k(n)$ in the context of the Diophantine equation $2^k y - ax = n$.

Corollary 2 (Relation between $\sigma_a^k(n)$ and $\sigma_{-a}^k(n)$). *Let* $a, k \in \mathbb{N}$ *and* $n \in \mathbb{Z}$*. Consider the Diophantine Equation,* $2^k y - ax = n$*, then*

$$\sigma_a^k(n) - \sigma_{-a}^k(n) = a$$

Proof: By definition, we have that ρ_0 is the nearest non-negative solution to 0, and ρ_1 is the nearest non-positive solution to 0, which means that ρ_0 and ρ_1 are consecutive solutions. Therefore, $\rho_0 - 2^k = \rho_1$. then we have

$$2^k = \rho_0(S) - \rho_1(S) = \frac{1}{a}\left\{2^k\sigma_a^k(n) - n\right\} - \frac{1}{a}\left\{2^k\sigma_{-a}^k(n) - n\right\} = \frac{1}{a}2^k\sigma_a^k(n) - \frac{1}{a}2^k\sigma_{-a}^k(n)$$

Therefore $\sigma_a^k(n) - \sigma_{-a}^k(n) = a$

QED.

In the following proposition, we examine the inequalities and estimations for the sigma function $\sigma_a^k(n)$ and $\sigma_{-a}^k(n)$, where n is an integer. We show that the sigma function lies in the interval $\left[\frac{n}{2^k}, \frac{n}{2^k} + a\right)$ for

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$\sigma_a^k(n)$, and in the interval $\left[\frac{n}{2^k} - a, n\right)$ for $\sigma_{-a}^k(n)$. These inequalities are fundamental to understand the range of values the sigma function can take in the context of the considered Diophantine equations.

Proposition 17 (Inequality and estimation of the sigma function). *Let $a, k \in \mathbb{N}$ and $n \in \mathbb{Z}$. Then,*

$$\frac{n}{2^k} \leq \sigma_a^k(n) < \left(\frac{n}{2^k} + a\right) \text{ and } \frac{n}{2^k} - a \leq \sigma_{-a}^k(n) < n$$

Proof: For $\sigma_a^k(n)$ we have two possible extreme paths, either we always get even or we always get odd, for the first case we would always have division by 2

$$\frac{n}{2^k} \leq \sigma_a^k(n)$$

for the second we would have

$$\sigma_a^k(n) \leq \frac{n + a(1 + 2 + \ldots + 2^{k-1})}{2^k} = \frac{n}{2^k} + a\left(\frac{1 + 2 + \ldots + 2^{k-1}}{2^k}\right) = \frac{n}{2^k} + a\left(\frac{2^k - 1}{2^k}\right) < \frac{n}{2^k} + a$$

For $\sigma_a^k(n)$, regardless of the cases, we always get a less stringent value to the initial value. If it is always even, we will have that it is always divided by 2, now in the case that it is always odd we have

$$\sigma_{-a}^k(n) \geq \frac{n - a(1 + 2 + \ldots + 2^{k-1})}{2^k} = \frac{n}{2^k} - a\left(\frac{1 + 2 + \ldots + 2^{k-1}}{2^k}\right) > \frac{n}{2^k} - a$$

and clearly, we have

$$\frac{n}{2^k} > \frac{n}{2^k} - a$$

QED.

7.3. Periodicity of the Sigma Function

Proposition 18 (Periodicity of periodic orbits). *Let $a \in \mathbb{N}$ The sigma function σ_a has the following properties,*

1. *The only fixed points are a and 0.*
2. *If, $n < a$ then, its orbit by is periodic with period given by*

$$\lambda\left(\frac{a}{(a,u)}\right) = \begin{cases} \varphi\left(\frac{a}{(a,u)}\right) & \text{if } \frac{a}{(a,u)} \text{ is prime} \\ [\varphi(p_1^{\alpha_1}), \ldots, \varphi(p_r^{\alpha_r})] & \text{if } \frac{a}{(a,u)} = \prod_{i=1}^{r} p_i^{\alpha_i} \text{ with } p_i \text{ prime and } \alpha_i \in \mathbb{N} \end{cases}$$

where φ is the Euler's totient function.
In particular, if $a = 3^r$ and $(n,3) = 1$ then the periodic is $3^{r-1}2$. Let $\gamma = 3^{r-1}$ then

$$\sigma_{3\gamma}^{2\gamma}(n) = n$$

In particular, all points terminate in some periodic orbit (including periodic points) between 0 and a.

Proof: we have

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1. Let $\sigma_a(u) = u$, if u is odd, then $\frac{u+a}{2} = u$ which implies $u = a$. If u is even, we have, $\frac{u}{2} = u$ which implies $u = 0$.
2. Let $u \in \mathbb{N}$ such that $u < a$ and $\sigma_a^k(u) = u$, so

$$\frac{u+aL}{2^k} = u \text{ for any } L = \sum_{j=0}^{k-1} \delta_j 2^j \text{ where } \delta_j \in \{0,1\}$$

Then

$$u + aL = 2^k u \Rightarrow aL = (2^k - 1)u \Rightarrow (2^k - 1)u = 0 \mod a$$

suppose that $(u, a) = 1$, this implies that u is an invertible $\mod a$ then, the equation is equivalent

$$2^k = 1 \mod a$$

The minimum value of k is given by the Carmichael function given by

$$\lambda(a) = \begin{cases} \varphi(a) & \text{if } a \text{ is prime} \\ [\varphi(p_1^{\alpha_1}), \dots, \varphi(p_r^{\alpha_r})] & \text{if } a = \prod_{i=1}^{r} p_i^{\alpha_i} \text{ with } p_i \text{ prime and } \alpha_i \in \mathbb{N} \end{cases}$$

Let $k_0 = \lambda(a)$, then

$$(2^{k_0} - 1)u = 0 \mod a \Rightarrow (2^{k_0} - 1)u = L_0 a \text{ for any } L_0 \in \mathbb{N}$$

$$(2^{k_0} - 1) = L_0 \frac{a}{u}$$

as $\frac{a}{u} > 1$ then $(2^{k_0} - 1) > L_0$, which is the necessary and sufficient condition for L_0 to admit decomposition in base 2 up to the power $k_0 - 1$ which implies that there exist $\delta_j \in \{0,1\}$ such that $L_0 = \sum_{j=0}^{k_0-1} \delta_j 2^j$.

Now suppose that $(u, a) = d > 1$, then we divide $aL = (2^k - 1)u$ by, d

$$\frac{a}{d} L = (2^k - 1)\frac{u}{d} \text{ where } \left(\frac{a}{d}, \frac{a}{d}\right) = 1$$

then the development is completely analogous to the first case.

In particular, when $a = 3^r$ and u are co-prime with 3, then the period of the orbit of u corresponds to the Euler's totient function, ϕ which in this case is $3^{r-1}2$.

QED.

Let us observe that for the equation $\sigma_a^k(u) = u$ to have a solution it is necessary and sufficient that $n < a$ since the function is monotonically decreasing for $n > a$.

Example 9. *For, $a = 7$ we have $\mathcal{O}(5) = \{5, 6, 3\}$ and $\mathcal{O}(4) = \{4, 2, 1\}$.*
For $a = 15$ we have $\mathcal{O}(1) = \{1, 8, 4, 2\}$, $\mathcal{O}(3) = \{3, 9, 12, 6\}$, $\mathcal{O}(5) = \{5, 10\}$ and $\mathcal{O}(7) = \{7, 11, 13, 14\}$

7.4. Linearity of the Sigma Function Modulo a

In this section, we address the linearity of the sigma function modulo a. Proposition 19 establishes the addition rules for the sigma function under different parity conditions of the involved numbers. We will see in the corollary 3 that the function sigma modulo a is an automorphism of $\mathbb{Z}/a\mathbb{Z}$ Furthermore, Proposition 20 establishing a relationship between $\sigma_a^k(m)$ and $\sigma_a^k(1)$.

Proposition 19 (Algebraic properties of the Sigma function)**.** *Let $a \in \mathbb{Z}$ and $\sigma_a : \mathbb{Z} \to \mathbb{Z}$, then we have*

1. *if n, m are even numbers, then $\sigma_a(n+m) = \sigma_a(n) + \sigma_a(m)$.*
2. *if n is an even number and m is an odd number, then $\sigma_a(n+m) = \sigma_a(n) + \sigma_a(m)$.*
3. *if n, m are odd numbers, then $\sigma_a(n+m) = \sigma_a(n) + \sigma_a(m) - a$.*

Proof: Let $m.n \in \mathbb{Z}$, then we have

1. If m, n are even, we have
$$\sigma_a(m+n) = \frac{m+n}{2} = \frac{m}{2} + \frac{n}{2} = \sigma_a(m) + \sigma_a(n)$$
2. If m is even and n is odd, we have
$$\sigma_a(m+n) = \frac{m+n+a}{2} = \frac{m}{2} + \frac{n+a}{2} = \sigma_a(m) + \sigma_a(n)$$
3. If m, n are odd, we have
$$\sigma_a(m+n) = \frac{m+n}{2} = \frac{m}{2} + \frac{n}{2} + a - a = \frac{m+a}{2} + \frac{n+a}{2} - a = \sigma_a(m) + \sigma_a(n) - a.$$

QED.

This corollary states that the sigma function, seen as a function on the set $\mathbb{Z}/a\mathbb{Z}$ and taking values in $\mathbb{Z}/a\mathbb{Z}$, acts as a group additive automorphism. In other words, it preserves the group structure under modular addition in $\mathbb{Z}/a\mathbb{Z}$

Corollary 3 (Linearity modulo a)**.** *We consider the function sigma as a function of $\mathbb{Z}/a\mathbb{Z}$ in $\mathbb{Z}/a\mathbb{Z}$, then it is a group additive automorphism. i.e.*

$$\sigma_a^k(m+n) = \sigma_a^k(m) + \sigma_a^k(n) \mod a$$

Proof: From the previous proposition we have that the sigma function is linearly distributed except for a term $-a$ that appears when both addends are odd, this term is congruent to $0 \mod a$

QED.

This proposition establishes the concept of homogeneity modulo a for the sigma function. It relates the value of $\sigma_a^k(m)$ to $m\sigma_a^k(1)$ under modular arithmetic. This relationship highlights a consistent behavior of the sigma function concerning scaling by m, providing valuable insights into its algebraic properties.

Proposition 20 (Homogeneity mod a)**.** *Let $a, k, m \in \mathbb{N}$ such that $(a, 2) = 1$, then we have*

$$\sigma_a^k(m) = m\sigma_a^k(1) \mod a$$

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Proof: Let $a, b \in \mathbb{N}$ such that $(a, 2) = 1$ and consider the following Diophantine Equation $2^k y - ax = m$. Since $(2^k, a) = 1$ we have that, this equation is equivalent to $2^k Y - aX = 1$. The Theorem 4 we have $\sigma_a^k(1)$ a particular solution of Y, then $m\sigma_a^k(1)$ is a solution for y of $2^k y - ax = m$, then

$$m\sigma_a^k(1) = \sigma_a^k(m) \mod a$$

QED.

7.5. Extension on the $\mathbb{Q}_{odd}$ of the Sigma Function

We can extend the domain of the sigma function to the set of rationals, in the following way,

Definition 16 ($\mathbb{Q}_{odd}$-extension of the sigma function)**.** *Let $\frac{u}{v}, \frac{x}{y} \in \mathbb{Q}_{odd}$. We define the sigma function $\sigma_{\frac{u}{v}} : \mathbb{Q}_{odd} \to \mathbb{Q}_{odd}$*

$$\sigma_{\frac{u}{v}}\left(\frac{x}{y}\right) = \begin{cases} \frac{1}{2}\left(\frac{x}{y} + \frac{u}{v}\right) & \text{if } x = 1 \mod 2 \\ \frac{x}{2y} & \text{if } x = 0 \mod 2 \end{cases}$$

We are going to provide a numerical interpretation of the extension of the sigma function. Thus far, we understand that the sigma function provides us with the non-negative solution to the Diophantine equation $2^k y - ax = n$ through the equation $\frac{ax+n}{2^k} = \sigma_a^k(n)$. We can utilize the latter equation to extend the sigma function to the set of fractions with odd denominators, employing the following equation on $r \in \mathbb{Z}$:

$$\frac{yur + vx}{2^k} = vy\sigma_{\frac{u}{v}}^k\left(\frac{x}{y}\right)$$

or equivalently

$$\frac{\frac{u}{v}r + \frac{x}{y}}{2^k} = \sigma_{\frac{u}{v}}^k\left(\frac{x}{y}\right)$$

That is to say, the extension of the sigma function gives the fraction that solves the equation

$$\frac{u}{v}r + \frac{x}{y} = 2^k t \text{ with } r \in \mathbb{Z} \text{ and } t, \frac{u}{v}, \frac{x}{y} \in \mathbb{Q}_{odd}$$

7.6. Properties of the Extension of the Sigma Function

The introduction of the sigma function extended to odd rationals is crucial for understanding its behavior in a broader domain. This extension, defined on the set $\mathbb{Q}_{odd}$, allows us to explore the algebraic and arithmetic properties of the sigma function in a more general context. In this section, we delve into this extension and explore its implications, focusing on how the sigma function modifies its behavior when applied to fractions with odd denominators. Additionally, we present an important lemma that establishes an invariant relationship between the characteristic function δ and the sigma function, providing a deeper understanding of how the sigma function preserves certain properties under different transformations.

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Definition 17 (Characteristic Function)**.** *We define the characteristic function* $\delta : \mathbb{Q}_{odd} \to \{0,1\}$ *given by*

$$\delta\left(\frac{p}{q}\right) = \begin{cases} 1 & \text{if } p = 1 \mod 2 \\ 0 & \text{if } p = 0 \mod 2 \end{cases}$$

The Invariant Coding Lemma, stated in Lemma 6, establishes a fundamental relationship between the characteristic function δ and the sigma function under certain conditions. Specifically, it asserts that for co-prime integers u and v, with u being odd, the characteristic function δ remains invariant under iterations of the sigma function. This means that the parity of the output of $\sigma_u^j(v)$ is the same as the parity of $\sigma_1^j\left(\frac{v}{u}\right)$ for all non-negative integers j. Furthermore, if v is odd, the lemma demonstrates that the parity of $\sigma_u^j(v)$ is identical to the parity of $\sigma^j_{\frac{u}{v}}(1)$ for all non-negative integers j.

Lemma 6 (Invariant Characteristic Function Lemma)**.** *Let* $u, v \in \mathbb{Z}$ *with* u *not null, such that* $(u, v) = 1$ *and* $(u, 2) = 1$ *then*

1.
$$\delta\left(\sigma_u^j(v)\right) = \delta\left(\sigma_1^j\left(\frac{v}{u}\right)\right) \text{ for } j \geq 0.$$

2. *if* v *is odd then*
$$\delta(\sigma_u^j(v)) = \delta\left(\sigma^j_{\frac{u}{v}}(1)\right) \text{ for } j \geq 0.$$

Proof : We have

1. Let $\sigma_u^k(v) = \dfrac{v + uT_k^u(v)}{2^a}$ where $T_k^u(v) = \sum_{j=0}^{k-1} \delta(\sigma_u^j(v))2^j$ and $\sigma_1^j\left(\frac{v}{u}\right) = \dfrac{\frac{v}{u} + T_k^1\left(\frac{v}{u}\right)}{2^a}$ where $T_k^1\left(\frac{v}{u}\right) = \sum_{j=0}^{k-1} \delta\left(\sigma_1^j\left(\frac{v}{u}\right)\right)2^j$. We will prove by induction that

$$\delta\left(\sigma_u^j(v)\right) = \delta\left(\sigma_1^j\left(\frac{v}{u}\right)\right)$$

For $j = 0$, Since if v is odd (or even) then $\frac{v}{u}$ is odd (or even) then

$$\delta\left(\sigma_u^0(v)\right) = \delta(v) = \delta\left(\frac{v}{u}\right) = \delta\left(\sigma_1^0\left(\frac{v}{u}\right)\right).$$

Suppose $\delta\left(\sigma_u^j(v)\right) = \delta\left(\sigma_1^j\left(\frac{v}{u}\right)\right)$ for $j \leq k$, then $T_k^u(v) = T_k^1\left(\frac{v}{u}\right)$, then we have

$$\sigma_u^k(v) = \frac{v + uT_k^u(v)}{2^a} = \frac{v + uT_k^1\left(\frac{v}{u}\right)}{2^a} = u\left(\frac{\frac{v}{u} + T_k^1\left(\frac{v}{u}\right)}{2^a}\right) = u\sigma_1^k\left(\frac{v}{u}\right)$$

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Since u is odd, we have that $\sigma_u^k(v)$ and $\sigma_1^k\left(\frac{v}{u}\right)$ have the same parity, then

$$\delta\left(\sigma_u^{k+1}(v)\right) = \delta\left(\sigma_1^{k+1}\left(\frac{v}{u}\right)\right)$$

2. Let $\sigma_u^k(v) = \dfrac{v + uT_k^u(v)}{2^a}$ where $T_k^u(v) = \sum_{j=0}^{k-1} \delta(\sigma_u^j(v))2^j$ and $\sigma_{\frac{u}{v}}^j(1) = \dfrac{1 + T_k^{\frac{u}{v}}(1)}{2^a}$ where $T_k^{\frac{u}{v}}(1) = \sum_{j=0}^{k-1} \delta\left(\sigma_{\frac{u}{v}}^j(1)\right)2^j$. We will prove by induction that

$$\delta\left(\sigma_u^j(v)\right) = \delta\left(\sigma_{\frac{u}{v}}^j(1)\right)$$

For $j = 0$, Since v is odd

$$\delta\left(\sigma_u^0(v)\right) = \delta(v) = \delta(1) = \delta\left(\sigma_{\frac{u}{v}}^0(1)\right).$$

Suppose $\delta\left(\sigma_u^j(v)\right) = \delta\left(\sigma_{\frac{u}{v}}^j(1)\right)$ for $j \leq k$, then $T_k^u(v) = T_k^{\frac{u}{v}}(1)$, then we have

$$\sigma_u^k(v) = \frac{v + uT_k^u(v)}{2^a} = \frac{v + uT_k^{\frac{u}{v}}(1)}{2^a} = v\left(\frac{1 + \frac{u}{v}T_k^{\frac{u}{v}}(1)}{2^a}\right) = v\sigma_{\frac{u}{v}}^k(1)$$

Since v is odd, we have that $\sigma_u^k(v)$ and $\sigma_{\frac{u}{v}}^k(1)$ have the same parity, then

$$\delta\left(\sigma_u^{k+1}(v)\right) = \delta\left(\sigma_{\frac{u}{v}}^{k+1}(1)\right)$$

QED.

In the following proposition, we demonstrate homogeneity properties that leave the coding of the orbits of the sigma function invariant.

Proposition 21 (homogeneity)**.** *Let $u, v \in \mathbb{Z}$ with u not null, such that $(u, v) = 1$ and $(u, 2) = 1$ then we have*

1.
$$\sigma_u^k(v) = u\sigma_1^k\left(\frac{v}{u}\right).$$

2.
$$\sigma_u^k(v) = v\sigma_{\frac{u}{v}}^k(1).$$

Proof: We have

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1. Let $\sigma_u^k(v) = \frac{v + uT_k^u(v)}{2^a}$ where $T_k^u(v) = \sum_{j=0}^{k-1} \delta(\sigma_u^j(v))2^j$ and $\sigma_1^j\left(\frac{v}{u}\right) = \frac{\frac{v}{u} + T_k^1\left(\frac{v}{u}\right)}{2^a}$ where $T_k^1\left(\frac{v}{u}\right) = \sum_{j=0}^{k-1} \delta\left(\sigma_1^j\left(\frac{v}{u}\right)\right)2^j$. Then we have

$$\sigma_u^k(v) = \frac{v + uT_k^u(v)}{2^a} = \frac{v + uT_k^1\left(\frac{v}{u}\right)}{2^a} = u\left(\frac{\frac{v}{u} + T_k^1\left(\frac{v}{u}\right)}{2^a}\right) = u\sigma_1^k\left(\frac{v}{u}\right)$$

2. Let $\sigma_u^k(v) = \frac{v + uT_k^u(v)}{2^a}$ where $T_k^u(v) = \sum_{j=0}^{k-1} \delta(\sigma_u^j(v))2^j$ and $\sigma_{\frac{u}{v}}^j(1) = \frac{1 + T_k^{\frac{u}{v}}(1)}{2^a}$ where $T_k^{\frac{u}{v}}(1) = \sum_{j=0}^{k-1} \delta\left(\sigma_{\frac{u}{v}}^j(1)\right)2^j$. Then we have

$$\sigma_u^k(v) = \frac{v + uT_k^u(v)}{2^a} = \frac{v + uT_k^{\frac{u}{v}}(1)}{2^a} = v\left(\frac{1 + T_k^{\frac{u}{v}}(1)}{2^a}\right) = v\sigma_{\frac{u}{v}}^k(1)$$

QED.

Proposition 22 (Algebraic properties of the Extension of the Sigma function). *Let $a \in \mathbb{Q}_{odd}$ with odd numerator and $\sigma_a : \mathbb{Q}_{odd} \to \mathbb{Q}_{odd}$, satisfies the following identities*

1. *if $\frac{n}{q}, \frac{m}{p}$ are even fractions, then $\sigma_a\left(\frac{n}{q} + \frac{m}{p}\right) = \sigma_a\left(\frac{n}{q}\right) + \sigma_a\left(\frac{m}{p}\right)$.*
2. *if $\frac{n}{q}$ is an even fraction and $\frac{m}{p}$ is an odd fraction, then $\sigma_a\left(\frac{n}{q} + \frac{m}{p}\right) = \sigma_a\left(\frac{n}{q}\right) + \sigma_a\left(\frac{m}{p}\right)$.*
3. *if $\frac{n}{q}, \frac{m}{p}$ are odd fractions, then $\sigma_a\left(\frac{n}{q} + \frac{m}{p}\right) = \sigma_a\left(\frac{n}{q}\right) + \sigma_a\left(\frac{m}{p}\right) - a$.*

Proof: Let's proved first for $a = 1$. Let $\frac{n}{q}, \frac{m}{p} \in \mathbb{Q}_{odd}$ and let $\beta : \mathbb{Z} \times \mathbb{Z} \to \{0, 1\}$ given by $\beta(n, m) = 0$ if n or m is even fraction and $\beta(n, m) = 1$ if n and m are odd fraction.

$$\begin{aligned}
pq\sigma\left(\frac{n}{q} + \frac{m}{p}\right) &= pq\sigma\left(\frac{n}{q} + \frac{m}{p}\right) \\
&= pq\sigma\left(\frac{np + mq}{pq}\right) \\
&= \sigma_{pq}\left(pq\frac{np + mq}{pq}\right) \\
&= \sigma_{pq}(np + mq) \\
&= \sigma_{pq}(np) + \sigma_{pq}(mq) - pq\beta(n, m) \\
&= pq\sigma\left(\frac{np}{pq}\right) + pq\sigma\left(\frac{mq}{pq}\right) - pq\beta(n, m) \\
&= pq\sigma\left(\frac{n}{q}\right) + pq\sigma\left(\frac{m}{p}\right) - pq\beta(n, m)
\end{aligned}$$

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Dividing everything by pq, we have

$$\sigma\left(\frac{n}{q}+\frac{m}{p}\right)=\sigma\left(\frac{n}{q}\right)+\sigma\left(\frac{m}{p}\right)-\beta(n,m)$$

Now let a an odd fraction with odd numerator, then we have $\frac{1}{a}$ is odd fraction, then multiplying by $\frac{1}{a}$ does not change the parity of $\frac{n}{q}$ or $\frac{m}{q}$. Then we have

$$\sigma\left(\frac{1}{a}\frac{n}{q}+\frac{1}{a}\frac{m}{p}\right)=\sigma\left(\frac{1}{a}\frac{n}{q}\right)+\sigma\left(\frac{1}{a}\frac{m}{p}\right)-\beta(n,m) \text{ multiplying by } a$$
$$a\sigma\left(\frac{1}{a}\frac{n}{q}+\frac{1}{a}\frac{m}{p}\right)=a\sigma\left(\frac{1}{a}\frac{n}{q}\right)+a\sigma\left(\frac{1}{a}\frac{m}{p}\right)-a\beta(n,m)$$
$$\sigma_a\left(\frac{n}{q}+\frac{m}{p}\right)=\sigma_a\left(\frac{n}{q}\right)+\sigma_a\left(\frac{m}{p}\right)-a\beta(n,m)$$

QED.

7.7. Coding of Sigma Function

In this section we are going to use the space of dyadic integers numbers $\mathbb{Z}_2$ as a support for the coding of the sigma function.

7.7.1. $p-$Adic Numbers

Definition 18. *Let $p\in\mathbb{N}$ be any prime number. Define a norm $\|.\|_p$ on $\mathbb{Q}$ as follows*

$$\|x\|_p=\begin{cases}p^{-ord_p(x)} & if \quad x\neq 0\\ 0 & if \quad x=0\end{cases}$$

where

$$ord_p(x)=\begin{cases}\text{the highest power } p \text{ which divides } x & if \quad x\in\mathbb{Z}\\ ord_p(a)-ord_p(b) & if \quad x=\frac{a}{b}\in\mathbb{Q}\end{cases}$$

Let $\mathbb{Q}_p$ the completion of $\mathbb{Q}$ through the norm $\|.\|_p$.

We have the following properties

1. if $a,b\in\mathbb{N}$ then $a=b \mod p^k$ if and only if $\|a-b\|_p\leq p^{-k}$.
2. Let $\beta\in\mathbb{Q}_p$ then this is uniquely represented by convergent series (with norm $\|.\|_p$) as

$$\beta=\sum_{n=-k}^{\infty}\delta_j 2^j=(\dots\delta_j\dots\delta_2\delta_1\delta_0.\delta_{-1}\dots\delta_{-k})_p$$

3. The $p-$adic expansion allows us to perform arithmetical operations in $\mathbb{Q}_p$ in way very similar to that in $\mathbb{R}$. Moreover, we will see that the operations in $\mathbb{Q}_p$ are, in fact, easier to perform than $\mathbb{R}$. Let $\alpha=\sum_{j=-k}^{\infty}\varepsilon_j p^j$ and $\beta=\sum_{j=-k}^{\infty}\delta_j p^j$

$$\alpha\pm\beta=\sum_{j=-k}^{\infty}(\varepsilon_j\pm\delta_j)p^j$$

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$$\alpha \cdot \beta = \left(\sum_{j=-k}^{\infty} \varepsilon_j p^j\right)\left(\sum_{j=-k}^{\infty} \delta_j p^j\right) = \sum_{r=-k}^{\infty}\left(\sum_{i+j=r} \varepsilon_j \delta_j\right) p^r$$

.

4. A $p-$adic number $\beta \in \mathbb{Q}_p$ is said to be a $p-$adic integer if its canonical expansion contains only non-negative power of p. The set of $p-$adic integers is denoted by $\mathbb{Z}_p$, so

$$\mathbb{Z}_p = \left\{\sum_{i=0}^{\infty} \delta_i p^i, \text{ with } \delta_j = 0, \ldots, p-1\right\} = \{\beta \in \mathbb{Q}_p; \|\beta\|_p \leq 1\}$$

This set has the property of being a complete metric subspace.

One of the main characteristics of $p-$adics numbers is

Proposition 23. *The canonical $p-$adic expansion $\alpha = \sum_{n=-k}^{\infty} \varepsilon_j 2^j$ represents a rational number if and only if is eventually periodic to the left.*

7.7.2. Coding of Sigma Function

Now we are going to define the coding of the sigma function for $\mathbb{Q}_{odd}$.

Definition 19. *Let $\frac{p}{q} \in \mathbb{Q}_{odd}$ and $g \in \mathbb{Z}$ odd number. We define the Coding of $\frac{p}{q}$ by σ as $Cod\sigma_g\left(\frac{p}{q}\right) = \prod_{j\in\mathbb{N}_0} \delta_j \in \mathbb{Z}_2$ given by*

$$\delta_j = \begin{cases} 1 & \text{if} \quad \sigma_g^j\left(\frac{p}{q}\right) \text{ is odd} \\ \\ 0 & \text{if} \quad \sigma_g^j\left(\frac{p}{q}\right) \text{ is even} \end{cases}$$

and the finite $k-$coding as $Cod\sigma_g^k\left(\frac{p}{q}\right) = \ldots 0000\delta_k \ldots \delta_0$

The following lemma is a reformulation of lemma 6 for $Cod\sigma$.

Lemma 7 (Invariant coding lemma). *Let $u, v \in \mathbb{Z}$ with u not null, such that $(u, v) = 1$ and $(u, 2) = 1$ then*

1.

$$Cod\sigma_u^j(v) = Cod\sigma^j\left(\frac{v}{u}\right) \text{ for } j \geq 0.$$

2. *if v is odd then*

$$Cod\sigma_u^j(v) = Cod\sigma_{\frac{u}{v}}^j(1) \text{ for } j \geq 0.$$

Proof: Reformulation of the Lemma 6

QED.

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Proposition 24 (Change of basis of the Cod-Sigma function). *Let $N_1 = \sum_{j=0}^{n_1} \delta_j 2^j$ and $N_2 = \sum_{j=0}^{n_2} \varepsilon_j 2^j$ with $\delta_j, \varepsilon_j \in \{0,1\}$, and $a \in \mathbb{N}$ such that $N_1, N_2 \leq 2^a - 1$. Let $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ given by $\mathcal{H}(x) = \dfrac{x + N_1 N_2}{2^a}$. Then if $x_0 \in \mathbb{E}(\mathcal{H})$ and N_2 is an odd number we have*

$$Cod\sigma_{N_1}^a(x_0) = \underbrace{0, \ldots, 0, \varepsilon_{n_2}, \ldots .\varepsilon_0,}_{a}$$

or if N_1 is an odd number we have

$$Cod\sigma_{N_2}^a(x_0) = \underbrace{0, \ldots, 0, \delta_{n_1}, \ldots, \delta_0}_{a}$$

Additionally. if N_1, N_2 are odd numbers we have

$$\sigma_{N_1}^a(x_0) = \sigma_{N_2}^a(x_0)$$

Proof: Let's prove by induction that if N_2 is odd, then

$$Cod\sigma_{N_2}^a(x_0) = \underbrace{0, \ldots, 0, \delta_{n_1}, \ldots, \delta_0}_{a}.$$

Completing with $\delta_j = 0$ if necessary, we have $N_1 = \sum_{j=0}^{n_1} \delta_j 2^j = \sum_{j=0}^{a-1} \delta_j 2^j$, then

$$\frac{x + N_1 N_2}{2^a} = \cfrac{\cfrac{\cfrac{\cfrac{\cfrac{x + \delta_0 N_2}{2} + \delta_1 N_2}{2} + \delta_2 N_2}{2}}{\ddots} + \delta_{a-1} N_2}{2} = \bigodot_{j=1}^{a} \left(\frac{x + \delta_{j-1} N_2}{2} \right) \in \langle \theta, \lambda^{N_2} \rangle$$

Since we trivially have that $(1,2) = 1$ and by hypothesis we have that $x_0 \in \mathbb{E}(\mathcal{H})$, by the Lemma 1 we have

$$x_0 \in \mathbb{E}\left(\bigodot_{j=1}^{K} \left(\frac{x + \delta_{j-1} N_2}{2} \right) \right) \text{ for all } K \leq a$$

In particular, for $K = 0$, we have $x_0 \in \mathbb{E}\left(\frac{x + \delta_0 N_2}{2} \right)$ and N_2 is odd, we have $\mathbb{E}(\theta) \cap \mathbb{E}(\lambda^{N_2}) = \emptyset$ then $\sigma_{N_2}(x_0) = \dfrac{x_0 + \delta_0 N_2}{2}$. For $K = 1$, we have $x_0 \in \mathbb{E}\left(\frac{x + \delta_1 N_2}{2} \circ \frac{x + \delta_0 N_2}{2} \right)$, then $\sigma_{N_2}(x_0) \in \mathbb{E}\left(\frac{x_0 + \delta_1 N_2}{2} \right)$, then $\sigma_{N_2}^2(x_0) = \dfrac{x_0 + N_2(\delta_0 + \delta_1 2)}{2^2}$. Suppose this continues until for $K = r$ i.e.

$$\sigma_{N_2}^r(x_0) = \frac{x_0 + N_2 \sum_{j=0}^{r-1} \delta_j 2^j}{2^r} \in \mathbb{Z}$$

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We have that $x_0 \in \mathbb{E}\left(\bigodot_{j=1}^{r+1}\left(\frac{x+\delta_{j-1}N_2}{2}\right)\right)$ then by definitions $\bigodot_{j=1}^{r+1}\left(\frac{x_0+\delta_{j-1}N_2}{2}\right) \in \mathbb{Z}$ and by inductive hypothesis $\sigma_{N_2}^r(x_0) = \bigodot_{j=1}^{r}\left(\frac{x_0+\delta_{j-1}N_2}{2}\right) \in \mathbb{Z}$ then $\sigma_{N_2}^r(x_0) \in \mathbb{E}\left(\frac{x+\delta_r N_2}{2}\right)$. Therefore, $\sigma_{N_2}^{r+1}(x_0) = \frac{x_0 + N_2\sum_{j=0}^{r}\delta_j 2^j}{2^{r+1}}$. Then we have to

$$\sigma_{N_2}^a(x_0) = \frac{x_0 + N_2\sum_{j=0}^{a-1}\delta_j 2^j}{2^a} = \frac{x_0 + N_2 N_1}{2^a}.$$

Similarly, Completing with $\delta_j = 0$ if necessary we have. Let $N_2 = \sum_{j=0}^{n_2}\varepsilon_j 2^j = \sum_{j=0}^{a-1}\varepsilon_j 2^j$, then

$$\sigma_{N_1}^a(x_0) = \frac{x_0 + N_1\sum_{j=0}^{a-1}\varepsilon_j 2^j}{2^a} = \frac{x_0 + N_1 N_2}{2^a}$$

Now. If N_1, N_2 are odd numbers, we have

$$\sigma_{N_1}^a(x_0) = \sigma_{N_2}^a(x_0)$$

In particular, we have

$$\sigma_{N_2}^a(x_0) = \frac{\dfrac{\dfrac{\dfrac{\dfrac{x+\delta_0 N_2}{2}+\delta_1 N_2}{2}+\delta_2 N_2}{2}}{\ddots}+\delta_{a-1}N_2}{2}$$

$$= \frac{\dfrac{\dfrac{\dfrac{\dfrac{x+\varepsilon_0 N_1}{2}+\varepsilon_1 N_1}{2}+\varepsilon_2 N_1}{2}}{\ddots}+\varepsilon_{a-1}N_1}{2} = \sigma_{N_1}^a(x_0)$$

then we have that the coding of $\sigma_{N_2}^a(x_0)$ is $\delta_{a-1}\ldots\delta_1\delta_0$ and of the $\sigma_{N_1}^a(x_0)$ is $\varepsilon_{a-1}\ldots\varepsilon_1\varepsilon_0$.

QED.

Example 10. *Let $\mathcal{H}(x) = \frac{x+15}{2^5}$, we have $x_0 = 2^5 - 15 = 17 \in \mathbb{E}(\mathcal{H})$, then*

1. $\sigma_3(17) = 10, \sigma_3(10) = 5, \sigma_3(5) = 4, \sigma_3(4) = 2, \sigma_3(2) = 1$ *then* $\sigma_3^5(17) = 1$. *Then we have* $Cod\sigma_3(17) = 00101$ *taking the coding coefficients, to base 2 we have*

$$1\cdot 2^0 + 0\cdot 2^1 + 1\cdot 2^2 + 0\cdot 2^3 + 0\cdot 2^4 = 5.$$

2. $\sigma_5(17) = 11, \sigma_5(11) = 8, \sigma_5(8) = 4, \sigma_5(4) = 2, \sigma_5(2) = 1$, *then* $\sigma_5^5(17) = 1$. *Then we have* $Cod\sigma_5(17) = 00011$ *taking the coding coefficients, to base 2 we have*

$$1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 + 0 \cdot 2^4 = 3.$$

We observe that the obits are equal from the third iteration, which corresponds to the maximum power of two, where all subsequent coefficients are null. We can also observe that from the third term, the values that appear in the orbits are even, they, unfortunately, cannot continue forever, since as we have seen, the orbit of the sigma functions falls into a periodic orbit with the same number of even and odd numbers, so at some point this orbit must fall into an odd one, which means all the initial values must change.

Proposition 25. *Let* $r \in \mathbb{N}$ *and* $\gamma = 3^{b-1}$ *and let* $k \in \mathbb{N}$ *such that* $r < 2^{2\gamma k}$ *and* $\delta_j \in \{0,1\}$ *such that* $r\left\{\dfrac{2^{2\gamma k}-1}{3\gamma}\right\} = \sum_{j=0}^{2\gamma k-1} \delta_j 2^j$ *then* $Cod\sigma_{3\gamma}^{2\gamma k}(r) = \delta_{2\gamma k-1}, \ldots, \delta_0$.

Proof: Let $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ given by $\mathcal{H}(x) = \dfrac{3\gamma x + r}{2^{2\gamma k}}$, then we want to find the minimum positive value of $\mathcal{H}$, then solving the following equation.

$$\frac{3\gamma x + r}{2^{2\gamma k}} = \sigma_{3\gamma}^{2\gamma k}(r)$$

By Proposition 18 we have $\sigma_{3\gamma}^{2\gamma k}(r) = r$, then

$$\frac{3\gamma x + r}{2^{2\gamma k}} = r \text{ then } x = \frac{2^{2\gamma k} r - r}{3\gamma} = r\left\{\frac{2^{2\gamma k}-1}{3\gamma}\right\}$$

Let $\delta_j \in \{0,1\}$ such that $r\left\{\dfrac{2^{2\gamma k}-1}{3^{3\gamma}}\right\} = \sum_{j=0}^{2\gamma k-1} \delta_j 2^j$. By Proposition 24 we have

$$Cod\sigma_{3\gamma}^{2\gamma k}(r) = \delta_{2\gamma k-1}, \ldots, \delta_0$$

QED.

Corollary 4. *Let* $r_1, r_2 \in \mathbb{N}$. *Then*

$$Cod\sigma_{3^b}^a(r_1 + r_2) = Cod\sigma_{3^b}^a(r_1) + Cod\sigma_{3^b}^a(r_2) \mod 2^a$$

.

Proof: Let $\gamma = 3^{b-1}$, $r = r_1 + r_2$ and $\mathcal{H} : \mathbb{R} \to \mathbb{R}$ given by $\mathcal{H}(x) = \dfrac{3\gamma x + r}{2^a}$ by Proposition 19 we have

$$\frac{3\gamma x + r}{2^a} = \sigma_{3\gamma}^a(r) = \sigma_{3\gamma}^a(r_1) + \sigma_{3\gamma}^a(r_2) - 3^{3\gamma} u \text{ with } u \in \mathbb{Z}$$

Then

$$x = \frac{2^a \sigma_{3\gamma}^a(r_1) - r_1}{3\gamma} + \frac{2^a \sigma_{3\gamma}^a(r_2) - r_2}{3\gamma} - 2^a u$$

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Let $k \in \mathbb{N}$ such that $r < 2^{2\gamma k}$ and $2\gamma k > a$ and let $\delta_j, \varepsilon_j, \eta_j \in \{0, 1\}$ such that

$$r_1 \left\{ \frac{2^{2\gamma k} - 1}{3\gamma} \right\} = \sum_{j=0}^{2\gamma k - 1} \delta_j 2^j$$
$$r_2 \left\{ \frac{2^{2\gamma k} - 1}{3\gamma} \right\} = \sum_{j=0}^{2\gamma k - 1} \varepsilon_j 2^j \text{ and}$$
$$r \left\{ \frac{2^{2\gamma k} - 1}{3\gamma} \right\} = \sum_{j=0}^{2\gamma k - 1} \eta_j 2^j$$

by Proposition 25 we have

$$\left(\frac{2^a \sigma^a_{3\gamma}(r_1) - r_1}{3\gamma} \right) (\text{Base } 2) = \{\delta_j\}_{j=0}^{a-1}$$
$$\left(\frac{2^a \sigma^a_{3\gamma}(r_2) - r_2}{3\gamma} \right) (\text{Base } 2) = \{\varepsilon_j\}_{j=0}^{a-1} \text{ and}$$
$$\left(\frac{2^a \sigma^a_{3\gamma}(r) - r}{3\gamma} \right) (\text{Base } 2) = \{\eta_j\}_{j=0}^{a-1}$$

Then we have

$$\frac{2^a \sigma^a_{3\gamma}(r) - r}{3\gamma} = \frac{2^a \sigma^a_{3\gamma}(r_1) - r_1}{3\gamma} + \frac{2^a \sigma^a_{3\gamma}(r_2) - r_2}{3\gamma} - 2^a u$$
$$\text{if and only if } \sum_{j=0}^{a-1} \eta_j 2^j = \sum_{j=0}^{a-1} \delta_j 2^j + \sum_{j=0}^{a-1} \varepsilon_j 2^j - 2^a u$$

Then we have $\sum_{j=0}^{a-1} \eta_j 2^j = \sum_{j=0}^{a-1} \delta_j 2^j + \sum_{j=0}^{a-1} \varepsilon_j 2^j \mod 2^a$, therefore

$$Cod\sigma^a_{3\gamma}(r) = Cod\sigma^a_{3\gamma}(r_1) + Cod\sigma^a_{3\gamma}(r_2) \mod 2^a$$

QED.

Proposition 26. *Let $r_1, r_2 \in \mathbb{N}$. Then $Cod\sigma_{3^b} : \mathbb{N} \to \mathbb{Z}_2$ is linear:*

$$Cod\sigma_{3^b}(r_1 + r_2) = Cod\sigma_{3^b}(r_1) + Cod\sigma_{3^b}(r_2) \in \mathbb{Z}_2$$

.

Proof: By Corollary 4 we have $Cod\sigma^n_{3^b}(r_1 + r_2) = Cod\sigma^n_{3^b}(r_1) + Cod\sigma^n_{3^b}(r_2) \mod 2^n$ for all $n \in \mathbb{N}$, this is equivalent to

$$||Cod\sigma^n_{3^b}(r_1 + r_2) - Cod\sigma^n_{3^b}(r_1) + Cod\sigma^n_{3^b}(r_2)||_2 < 2^{-n} \text{ for all } n \in \mathbb{N}$$

Therefore

$$Cod\sigma_{3^b}(r_1 + r_2) = Cod\sigma_{3^b}(r_1) + Cod\sigma_{3^b}(r_2) \in \mathbb{Z}_2$$

QED.

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Example 11. 1. *Let* $S(x) = \frac{9x+5}{2^4}$

$$\sigma_9^4(5) = \sigma_9^4(3) + \sigma_9^4(2) \mod 9$$

(a) $\mathcal{O}\sigma_9^4(3) = \{3,6,3,6,3\}$ *then* $Cod\sigma_9^4(3) = 0101$.
(b) $\mathcal{O}(\sigma_9^4(2)) = \{2,1,5,7,8\}$ *then* $Cod\sigma_9^4(2) = 1110$.

$$\begin{aligned} Cod\sigma_9^4(5) &= Cod\sigma_9^4(3) + Cod\sigma_9^4(2) \mod 2^4 \\ &= 1110 + 0101 \mod 2^4 \\ &= 0011 \end{aligned}$$

$\mathcal{O}(\sigma_9^4(5)) = \{5,7,8,4,2\}$ *then* $Cod\sigma_9^4(5) = 0011$, *Then we have* $\rho_0(S) = 1+2 = 3$

$$S(3) = \frac{9 \cdot 3 + 5}{2^4} = 2$$

2. *Let* $S(x) = \frac{9x+3+2^{\theta_1}}{2^{\theta_1+\theta_2}}$ *with* θ_1 *even. We have*

$$\begin{aligned} \sigma_9^{\theta_1+\theta_2}(3+2^{\theta_1}) &= \sigma_9^{\theta_1+\theta_2}(3) + \sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) \mod 9 \\ &= 3\sigma_3^{\theta_1+\theta_2} + \sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) \mod 9 \end{aligned}$$

Then

$$\begin{aligned} Cod\sigma_9^{\theta_1+\theta_2}(3+2^{\theta_1}) &= Cod\sigma_9^{\theta_1+\theta_2}(3) + Cod\sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) \mod 2^{\theta_1+\theta_2} \\ &= Cod\sigma_3^{\theta_1+\theta_2} + Cod\sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) \mod 2^{\theta_1+\theta_2} \end{aligned}$$

(a) $\mathcal{O}(\sigma_3^{\theta_1+\theta_2}) = \{1,2,1,2,1,2,\ldots\}$ *then* $Cod\sigma_3^{\theta_1+\theta_2} = \underbrace{\ldots 010101}_{\theta_1+\theta_2}$.
(b) $\mathcal{O}(\sigma_9^{\theta_1+\theta_2}(2^{\theta_1})) = \{2^{\theta_1}, 2^{\theta_1-1}, \ldots, 1,5,7,8,4,2,1,5,7,8,4,2,1,\ldots\}$ *then*

$$Cod\sigma_9^{\theta_1+\theta_2}(2^{\theta_1}) = \underbrace{\ldots 000111000111}_{\theta_2}\underbrace{\ldots 00000}_{\theta_1}$$

.

Then we have

$$\begin{aligned} Cod\sigma_9^{\theta_1+\theta_2}(3+2^{\theta_1}) &= \underbrace{\ldots 000111000111}_{\theta_2}\underbrace{\ldots 00000}_{\theta_1} + \underbrace{\ldots 010101}_{\theta_1+\theta_2} \mod 2^{\theta_1+\theta_2} \\ &= \underbrace{\ldots 000111000111000111100}_{\theta_2}\underbrace{\ldots 1010101010}_{\theta_1} \mod 2^{\theta_1+\theta_2} \\ &= Cod\sigma_9^{\theta_2-2}00Cod\sigma_3^{\theta_1} \mod 2^{\theta_1+\theta_2} \end{aligned}$$

Then we have $\rho_0(S)(base2) = Cod\sigma_9^{\theta_2-2}00Cod\sigma_3^{\theta_1} \mod 2^{\theta_1+\theta_2}$.

3. *Let* $S_n(x) = \frac{3^n x + 4^n - 3^n}{4^n}$

$$\sigma_{3^n}^{2n}(4^n - 3^n) = \sigma_{3^n}^{2n}(4^n) - \sigma_{3^n}^{2n}(3^n) = 1 - 3^n\sigma_1^{2n} = 1 \mod 3^n$$

On the other hand

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(a) $\mathcal{O}(\sigma_{3^n}^{2n}(2^{2n})) = \{2^{2n}, 2^{2n-1}, 2^{2n-2}, \dots, 1\}$ *then* $Cod\sigma_{3^n}^{2n}(2^{2n}) = \underbrace{\dots 00000}_{2n}$.

(b) $\mathcal{O}(\sigma_{3^n}^{2n}(3^n)) = \{3^n, 3^n, 3^n, 3^n \dots\}$ *then* $Cod\sigma_{3^n}^{2n}(3^n) = \underbrace{\dots 11111111}_{2n}$.

Then we have

$$\begin{aligned} &\underbrace{0\dots0}_{2n} - \underbrace{1\dots1}_{2n} \mod 2^{2n} \\ =&1\underbrace{0\dots0}_{2n} - 0\underbrace{1\dots1}_{2n} \mod 2^{2n} \\ =&1 \mod 2^{2n} \\ =&1 \end{aligned}$$

Then

$$Cod\sigma_{3^n}^{2n}(4^n - 3^n) = 00\dots001$$

That is, $Cod\sigma_{3^n}^{2n}(4^n - 3^n)$ *has a constant coding equal to* 1*. This is natural, since* S_n *is stable.*

Lemma 8 (Rational equivalence of the Cod-Sigma function.). *Let* $b \in \mathbb{N}$*, then we have*

$$Cod\sigma_{3^b} = -\frac{1}{3^b} \in \mathbb{Z}_2$$

Furthermore. Let $u \in \mathbb{N}$*, then*

$$Cod\sigma_{3^b}(u) = -\frac{u}{3^b} \in \mathbb{Z}_2$$

Proof: Let $u = \overline{\gamma_b} \in \mathbb{Z}_2$ with $\gamma_b = Cod\sigma_{3^b}^{2 \cdot 3^{b-1}}$. Let $u = x$ then multiplying by $2^{2 \cdot 3^{b-1}}$ we have $u\underbrace{0\dots0}_{2\cdot 3^{b-1}} = 2^{2\cdot 3^{b-1}}x$ and subtracting, we have

$$\begin{aligned} u - u\underbrace{0\dots0}_{2\cdot 3^{b-1}} &= -(2^{2\cdot 3^{b-1}} - 1)x \\ \gamma_b &= -(2^{2\cdot 3^{b-1}} - 1)x \end{aligned}$$

On the other hand we have that $\gamma_b = \dfrac{2^{2\cdot 3^{b-1}} - 1}{3^b}$, then

$$\frac{2^{2\cdot 3^{b-1}} - 1}{3^b} = -(2^{2\cdot 3^{b-1}} - 1) \text{ then we have } x = -\frac{1}{3^b}$$

Other way for proof it, is

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$$
\begin{aligned}
u = \sum_{n=0}^{\infty} u_n 2^n &= \left\{ \sum_{n=0}^{2\cdot 3^{b-1}-1} u_n 2^n \right\} + \left\{ \sum_{n=2\cdot 3^{b-1}}^{4\cdot 3^{b-1}-1} u_n 2^n \right\} + \left\{ \sum_{n=4\cdot 3^{b-1}}^{6\cdot 3^{b-1}-1} u_n 2^n \right\} \dots \\
&= Cod\sigma_{3^b}^{2\cdot 3^{b-1}} + 2^{2\cdot 3^{b-1}} Cod\sigma_{3^b}^{2\cdot 3^{b-1}} + \left(2^{2\cdot 3^{b-1}}\right)^2 Cod\sigma_{3^b}^{2\cdot 3^{b-1}} + \dots \\
&= Cod\sigma_{3^b}^{2\cdot 3^{b-1}} \left\{ \sum_{n=0}^{\infty} \left(2^{2\cdot 3^{b-1}}\right)^n \right\} \\
&= Cod\sigma_{3^b}^{2\cdot 3^{b-1}} \left\{ \frac{1}{1 - 2^{2\cdot 3^{b-1}}} \right\} \\
&= -\left\{ \frac{1 - 2^{2\cdot 3^{b-1}}}{3^b} \right\} \left\{ \frac{1}{1 - 2^{2\cdot 3^{b-1}}} \right\} \\
&= -\frac{1}{3^b}
\end{aligned}
$$

Now let's try the second part. Let $v \in \mathbb{N}$

$$
Cod\sigma_{3^b}^{k}(v) = -v \left\{ \frac{1 - 2^{2\cdot 3^{b-1}k}}{3^b} \right\} = -\frac{v}{3^b} + \frac{v 2^{2\cdot 3^{b-1}k}}{3^b}
$$

On the other hand

$$
\left|\left| Cod\sigma_{3^b}^{k}(v) - \left(-\frac{v}{3^b}\right) \right|\right|_2 \leq 2^{-2\cdot 3^{b-1}k} \to 0 \text{ as } k \to \infty
$$

Therefore $Cod\sigma_{3^b}(v) = -\frac{v}{3^b}$

QED.

Proposition 27. *Let $\xi \in G_\infty$ then we have $-\pi^1(\xi) \in \mathbb{Z}_2$ and*

$$
Cod\sigma(\pi^1(\xi)) = -\pi^1(\xi) \in \mathbb{Z}_2.
$$

Proof: Claim 1: Let $\xi \in \Sigma_2^*$ and $k \in \mathbb{N}$, the we have $Cod\sigma\left(\pi_k^1(\xi)\right) = -\pi_k^1(\xi) \in \mathbb{Z}_2$. Indeed. Let $k \in \mathbb{N}$ and $\xi \in \Sigma_2^*$. By Lemma 7, Proposition 26 and Lemma 8, we have

$$
\begin{aligned}
Cod\sigma\left(\pi_k^1(\xi)\right) &= Cod\sigma\left(\sum_{j=1}^{b_k} \frac{2^{a_{j-1}}}{3^j} \right) \\
&= Cod\sigma_{3^{b_k}}\left(\sum_{j=1}^{b_k} 2^{a_{j-1}} 3^{b_k - j} \right) \\
&= \sum_{j=1}^{b_k} Cod\sigma_{3^{b_k}}\left(2^{a_{j-1}} 3^{b_k - j} \right) \\
&= \sum_{j=1}^{b_k} Cod\sigma_{3^j}(2^{a_{j-1}}) \\
&= -\sum_{j=1}^{b_k} \frac{2^{a_{j-1}}}{3^j} \\
&= -\pi_k^1(\xi)
\end{aligned}
$$

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Since $-\dfrac{2^{a_{j-1}}}{3^j} \in \mathbb{Z}_2$ then $-\pi_k^1(\xi) \in \mathbb{Z}_2$.

Claim 2: $\lim\limits_{k\to\infty} \pi_k^1(\xi) = \pi^1(\xi) \in \mathbb{Z}_2$.

Indeed. We have the following equivalence on $\mathbb{Q}_2$

$$\pi^1(\xi) = \sum_{n=1}^{\infty} \frac{2^{a_{n-1}}}{3^n} \in \mathbb{Q}_2 \text{ if and only if } \lim_{k\to\infty} \frac{2^{a_{k-1}}}{3^k} = 0 \text{ on } \mathbb{Q}_2$$

On the other hand

$$\left\| \frac{2^{a_{k-1}}}{3^k} \right\|_2 = \| Cod\sigma_{3^b}(2^{a_{k-1}}) \|_2 = \left\| Cod\sigma_{3^b} 0^{a_{k-1}} \right\|_2 = 2^{-a_{k-1}} \to 0 \text{ since } a_{k-1} \text{ it is increasing}$$

Therefore $\pi^1(\xi) = \sum\limits_{n=1}^{\infty} \dfrac{2^{a_{n-1}}}{3^n} \in \mathbb{Q}_2$. On the other hand, we have that $\pi_k^1(\xi)$ is a Cauchy sequence on $\mathbb{Z}_2$. Indeed

$$\left\| \pi_{k+1}^1(\xi) - \pi_k^1(\xi) \right\|_2 = \left\| \frac{2^{a_{k-1}}}{3^k} \right\|_2 = 2^{-a_{k-1}} \to 0 \text{ as } k \to \infty$$

and we also have that $\mathbb{Z}_2$ is a complete metric space, then $\lim\limits_{k\to\infty} \pi_k^1(\xi) = \pi^1(\xi) \in \mathbb{Z}_2$.

QED.

8. The Extension of Collatz Function on $\mathbb{Z}_2$

In this section we will study the extension of the Collatz function on $\mathbb{Z}_2$ proposed by Lagaria in [3], and in an analogous way we will define the dyadic integer sets and the encoding set. We will prove that given an coding there exists a unique dyadic integer with this coding. We will show that this extension is topologically conjugate to the shift function in Σ_2^* and we will use this result to prove that codings in G_1 are unstable.

8.1. Summary of Propositions in the Section

1. **Lemma 9**: Equivalence of the parity of fractions and their dyadic representation.
2. **Definition 20:** Extension of the Collatz function on the set of dyadic numbers and the definitions of dyadic integer set and coding set.
3. **Proposition 28:** Characterization of the dyadic integer set.
4. **Proposition 29:** Establishes that the Coding set and the Dyadic Integer Set are the same.
5. **Proposition 30:** It establishes that given a coding there is a unique dyadic number with said coding.
6. **Theorem 5:** The Collatz function on the set of dyadic numbers is topologically conjugate to the Shift function.
7. **Corollary 5:** The periodic points of the Collatz function in $\mathbb{Z}_2$ are dense in $\mathbb{Z}_2$.
8. **Proposition 31:** The periodic sequences of G_0 correspond to positive periodic points of the Collatz functions and the periodic sequences of G_∞ correspond to negative periodic points of the Collatz functions.
9. **Lemma 10:** $\rho(S_k) = Cod\sigma_{3^k}^{A_k}(N_k)$.
10. **Proposition 32** $\rho_0(S_k) = Cod\sigma\left(\pi^1(\xi)\right) \mod 2^{a_k}$.
11. **Theorem 6:** The sequences in G_1 are positively and negatively unstable.

8.2. Extension of the Collatz Function on $\mathbb{Z}_2$.

Now we are going to extend the Collatz function to the set of $\mathbb{Z}_2$. In order for the extension to be compatible with the results obtained in the previous sections, we will first show that the parity of the elements of $\mathbb{Q}_{odd}$ is preserved in $\mathbb{Z}_2$.

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Lemma 9. *Let* $\beta = \sum_{i=0}^{\infty} \delta_i 2^i \in \mathbb{Z}_2$ *the dyadic representation of* $\frac{p}{q} \in \mathbb{Q}_{odd}$, *then p is even if and only if* $\delta_0 = 0$ *and p is odd if and only if* $\delta_0 = 1$.

Proof: Let p a even number, so we have

$\frac{p}{q} = \frac{2k}{q} = \delta_0 + 2M$ for any $M \in \mathbb{Z}_2$, so $2k = q\delta_0 + 2Mq$ then $\delta_0 = 0$. Since q is odd number

Let p a odd number, so we have

$\frac{p}{q} = \frac{2k+1}{q} = \delta_0 + 2M$ for any $M \in \mathbb{Z}_2$, so $2k+1 = q\delta_0 + 2Mq$ then $\delta_0 = 1$. Since q is odd number

QED.

Now let us consider the following extension of the Collatz function on $\mathbb{Z}_2$.

Definition 20. *Let* $Col : \mathbb{Z}_2 \to \mathbb{Z}_2$ *given by*

$$Col(\beta) = \begin{cases} \frac{3\beta+1}{2} & \text{if } \beta \mod 2 = 1 \\ \frac{\beta}{2} & \text{if } \beta \mod 2 = 0 \end{cases}$$

$Cod^k(\beta) = \{\eta_i\}_{i=0}^{k-1}$ *with* $\eta_i = 0$ *if* $Cod^i(\beta) = 0 \mod 2$ *and* $\eta_i = 10$ *if* $Cod^i(\beta) = 1 \mod 2$ *Let* $\xi \in \Sigma_2^*$ *then we defined the* $k-$*Coding set of* ξ

$$Cod^k(\xi) = \left\{\beta \in \mathbb{Z}_2; Cod^k(\beta) = \{\xi\}_{i=0}^{k-1}\right\}$$

Let $S \in \langle\theta, \psi\rangle$, *we define the dyadic integer set of S as*

$$\mathbb{D}(S) = \{\beta \in \mathbb{Z}_2; S(\beta) \in \mathbb{Z}_2\}$$

and $\pi^1, \pi_k^1 : \Sigma_2^* \to \mathbb{Z}_2$ *given by* $\pi^1(\xi) = \sum_{n=1}^{\infty} \frac{2^{a_{n-1}}}{3^n} \in \mathbb{Z}_2$ *and* $\pi_k^1(\xi) = \sum_{n=1}^{k} \frac{2^{a_{n-1}}}{3^n} \in \mathbb{Z}_2$.

Next we will show the version in $\mathbb{Z}_2$ to the results seen in previous Sections. The following Proposition characterizes the set of dyadic integers of $S_k \in \langle\theta, \psi\rangle$ analogously to the entire set.

Proposition 28. *Let* $\xi \in \Sigma_2^*$ *and* $S_k \in \langle\theta, \psi\rangle$ *such that* $Cod(S_k) = \{\xi_j\}_{j=0}^{k-1}$. *Let* $\beta \in \mathbb{D}(S_k)$, *then* $\mathbb{D}(S_k) = \beta_0 + 2^{A_k}\mathbb{Z}_2$ *with* A_k =*numbers from 0 to* $\{\xi_j\}_{j=0}^{k-1}$.

Proof: Let $\beta_0 \in \mathbb{D}(S_k)$ then we have $\beta_0 + 2^{A_k}\mathbb{Z}_2 \subset \mathbb{D}(S_k)$. Indeed. Let $\xi \in \Sigma_2^*$ and $S_k \in \langle\theta, \psi\rangle$ such that $Cod(S_k) = \{\xi_j\}_{j=0}^{k-1}$, then

$$S_k(x) = \frac{3^{b_k}x + N_k}{2^{A_k}}$$

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so

$$S_k(\beta_0 + 2^{A_k}\mathbb{Z}_2) = \frac{3^{b_k}(\beta_0 + 2^{A_k}\mathbb{Z}_2) + N_k}{2^{A_k}} = \frac{3^{b_k}\beta_0 + N_k}{2^{A_k}} + \mathbb{Z}_2 \subset \mathbb{Z}_2$$

Now $\mathbb{D}(S_k) \subset \beta_0 + 2^{A_k}\mathbb{Z}_2$. Let $\beta \in \mathbb{D}(S_k)$, so $3^{b_k}\beta + N_k = 0 \mod 2^{A_k}$, so

$$\begin{aligned} 3^{b_k}\beta + N_k - (3^{b_k}\beta_0 + N_k) &= 0 \mod 2^{A_k} \\ 3^{b_k}(\beta - \beta_0) &= 0 \mod 2^{A_k} \\ \beta - \beta_0 &= 0 \mod 2^{A_k} \end{aligned}$$

Therefore, we have

$$\beta - \beta_0 = \underbrace{0\ldots0}_{2^{A_k}}\alpha = 2^{A_k}\alpha, \text{ with } \alpha \in \mathbb{Z}_2 \Rightarrow \beta = \beta_0 + 2^{A_k}\alpha$$

QED.

The following proposition states that the entire dyadic set is equal to the coding set of S_k.

Proposition 29. *Let $\xi \in \Sigma_2^*$ and $S_k \in \langle\theta, \psi\rangle$ such that $Cod(S_k) = \{\xi_j\}_{j=0}^{k-1}$ with $\xi_j \in \{0, 10\}$, then $\mathbb{D}(S_k) = Cod^k(\xi)$.*

Proof: Let $\beta \in Cod^k(\xi)$, so by definition we have $Cod^k(\beta) = \{\xi_j\}_{j=0}^{k-1}$ i.e $\xi_i = 0$ if $Cod^j(\beta) = 0 \mod 2$ and $\xi_i = 10$ if $Cod^j(\beta) = 1 \mod 2$. We can rewrite as $\beta = \beta_k + 2^{A_k}\alpha_k$ for any $\alpha_k \in \mathbb{Z}_2$, A_k =numbers from 0 to $\{\xi_j\}_{j=0}^{k-1}$ and $\beta_k = \beta \mod 2^{A_k}$. We Claim that $\beta_k \in Cod^k(\xi)$. Indeed

$$Col^l(\beta) = Col^l(\beta_k) + 2^{A_k - l}\alpha_k \text{ with } l \leq A_k,$$

We have that the parity on the right side only depends on $Col^l(\beta_k)$, then the latter must have the same $j-$coding as $Col^j(\beta)$ by Proposition 11 we have $\beta_k \in \mathbb{E}(S_k)$. So $S_k(\beta) = S_k(\beta_k) + \alpha_k \in \mathbb{Z}_2$, so $\beta \in \mathbb{D}(S_k)$.

Let $\beta \in \mathbb{D}(S_k)$ and let $\beta_k = \beta \mod 2^{A_k}$, then $\beta_k \in \mathbb{D}(S_k)$. On the other hand, since β_k trivially has a representation as a natural number, we have that $S_k(\beta_k)$ it is also a natural number, so $\beta_k \in \mathbb{E}(S_k)$ by Proposition 11 we have that $\beta_k \in Cod^k(\xi)$. By hypothesis we have that $\beta_k = \beta \mod 2^{A_k}$, then exist $\alpha_k \in \mathbb{Z}_2$ such that $\beta = \beta_k + 2^{A_k}\alpha_k$, applying S_k $l-$times with $l \leq A_k$, we have

$$Col^l(\beta) = Col^l(\beta_k) + 2^{A_k - l}\alpha_k$$

Since $2^{A_k - l}\alpha_k$ is always even for all $l < A_k$, we have that the parity of each iteration only depends on $Col^l(\beta_k)$, then $\beta \in Cod^k(\xi)$.

QED.

In theorem 3 we saw that given $\xi \in \Sigma_2^*$ if there exists a rational whose encoding is exactly ξ, then it is the only rational solution. However we could not guarantee the existence of such a number. The following Proposition guarantees us that there exists a solution in the set of dyadic numbers.

Proposition 30. *Let $\xi \in \Sigma_2^*$, then exist an unique $\beta \in \mathbb{Z}_2$ such that $Cod(\beta) = \xi$.*

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Proof: Let $\xi \in \Sigma_2^*$ by Lemma 27 we have $-\pi^1(\xi) \in \mathbb{Z}_2$. Now we are going to prove that: **Claim** $-\pi^1(\xi) \in Cod^k(\xi)$ for all $k \in \mathbb{N}$. Indeed, let $S_k \in \langle \theta, \psi \rangle$ such that $Cod(S_k) = \{\xi_j\}_{j=0}^k$, we have $S_k(-\pi_k^1(\xi)) = 0$ for all $k \in \mathbb{N}$, Then

$$-\pi^1(\xi) = -\pi_k^1(\xi) - \sum_{j=k+1}^{\infty} \frac{2^{a_{j-1}}}{3^j}$$

$$\text{so } S_k(-\pi^1(\xi)) = S_k(-\pi_k^1(\xi)) - \frac{3^k}{2^{a_k}} \sum_{j=k+1}^{\infty} \frac{2^{a_{j-1}}}{3^j}$$

$$= - \sum_{j=k+1}^{\infty} \frac{2^{a_{j-1}-a_k}}{3^{j-k}} \in \mathbb{Z}_2 \text{ since } \left\| \sum_{j=k+1}^{\infty} \frac{2^{a_{j-1}-a_k}}{3^{j-k}} \right\|_2 = 2^{-(a_k - a_k)} = 1$$

By Proposition 29 we have $-\pi^1(\xi) \in Cod^k(\xi)$ for all $k \in \mathbb{N}$. Since $Cod^k(\xi) \subset Cod^{k-1}(\xi)$, we have

$$-\lim_{k\to\infty} \pi_k^1(\xi) \in \bigcap_{k\in\mathbb{N}} Cod^k(\xi) = Cod(\xi)$$

Suppose there exists another dyadic integer α such that it is also in $Cod(\xi)$, then

$$\alpha, -\pi^1(\xi) \in Cod(S_k) \text{ for all } k \in \mathbb{N} \text{ so } \|\pi^1(\xi) - \alpha\|_2 < 2^{-A_k} \to 0 \text{ as } k \to \infty$$

Therefore $\alpha = -\pi^1(\xi)$.

QED.

The existence of solutions to the equation $Cod(\beta) = \xi$ in the dyadic numbers does not guarantee the existence of rational solutions. This will depend primarily on whether the dyadic solution can be represented as a rational number or, more generally, as a real number. Based on the nature of this solution, we can determine whether or not a rational solution exists.

8.3. Topological Conjugation

The Shift function ω on Σ_2^* is defined as the mapping that deletes the first term of the sequence. The following theorem states that the Collatz function on $\mathbb{Z}_2$ is dynamically equivalent to the Shift function on Σ_2^* and that the function $-\pi^1$ is a homeomorphism between these two spaces. A similar result can be found in [3] where the Shift function is defined on $\mathbb{Z}_2$ instead of Σ_2^*.

Theorem 5 (Col is topologically conjugate to ω)**.** *Let* $\omega : \Sigma_2^* \to \Sigma_2^*$ *given by*

$$\omega\left(\{\xi_j\}_{j=0}^{\infty}\right) = \begin{cases} \{\xi_{j+2}\}_{j=1}^{\infty} & \text{if } \xi_1 = 1 \\ \{\xi_{j+1}\}_{j=1}^{\infty} & \text{if } \xi_1 = 0 \end{cases}$$

Col is topologically conjugate to ω *i.e the following diagram is commutative*

$$\begin{array}{ccc} \Sigma_2^* & \xrightarrow{\ \omega\ } & \Sigma_2^* \\ {\scriptstyle -\pi^1}\big\downarrow & \circlearrowright & \big\downarrow{\scriptstyle -\pi^1} \\ \mathbb{Z}_2 & \xrightarrow[\ Col\ ]{} & \mathbb{Z}_2 \end{array}$$

and $-\pi^1$ *is a homemorphism.*

Proof: Let's first prove that the diagram is commutative.

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Let $\{\xi_j\}_{j=0}^{\infty} = 10^{\theta_1}10^{\theta_2}\ldots = \prod_{j=0}^{\infty} 10^{\theta_j} \in \Sigma_2^*$ with $\theta_j \in \mathbb{N}$. Writing this way, we get an explicit form for the function $a_k = \sum_{j=0}^{k} \theta_j$. If $a_0 = 0$ we have

$$\begin{aligned}
Col \circ (-\pi^1)\left(\prod_{j=0}^{\infty} 10^{\theta_j}\right) &= Col\left(-\sum_{k=1}^{\infty} \frac{2^{a_{k-1}(\{\xi_j\}_{j=0}^{\infty})}}{3^k}\right) \\
&= \frac{1}{2}\left(-3\sum_{k=1}^{\infty} \frac{2^{a_{k-1}(\{\xi_j\}_{j=0}^{\infty})}}{3^k} + 1\right) \\
&= \frac{1}{2}\left(-\sum_{k=2}^{\infty} \frac{2^{a_{k-1}(\{\xi_j\}_{j=0}^{\infty})}}{3^{k-1}} + 1\right) \\
&= \frac{1}{2}\left(-1-\sum_{k=2}^{\infty} \frac{2^{a_{k-1}(\{\xi_j\}_{j=0}^{\infty})}}{3^{k-1}} + 1\right) \\
&= -\sum_{k=2}^{\infty} \frac{2^{a_{k-1}(\{\xi_j\}_{j=0}^{\infty})-1}}{3^{k-1}} \\
&= -\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_j\}_{j=0}^{\infty})-1}}{3^k}
\end{aligned}$$

and

$$-\pi^1 \circ \omega\left(\prod_{j=0}^{\infty} 10^{\theta_j}\right) = -\pi^1\left(0^{\theta_0 - 1}\prod_{j=1}^{\infty} 10^{\theta_j}\right) = -\sum_{k=1}^{\infty} \frac{2^{a_k(\{\xi_i\}_{i=1}^{\infty})-1}}{3^k}$$

where both parts are equal. On the other hand. suppose that $\{\xi_j\}_{j=0}^{\infty} = 0^{\theta_0}10^{\theta_1}1\ldots = \prod_{j=0}^{\infty} 0^{\theta_j}1 \in \Sigma_2^*$ with $\theta_j \in \mathbb{N}$. We have

$$Col \circ (-\pi^1)\left(\prod_{j=0}^{\infty} 0^{\theta_j}1\right) = Col\left(-\sum_{k=1}^{\infty} \frac{2^{a_{k-1}(\{\xi_i\}_{i=0}^{\infty})}}{3^k}\right) = -\sum_{k=1}^{\infty} \frac{2^{a_{k-1}(\{\xi_i\}_{i=0}^{\infty})-1}}{3^k}$$

and

$$(-\pi^1) \circ \omega\left(\prod_{j=0}^{\infty} 0^{\theta_j}1\right) = (-\pi^1) \circ \left(\theta^{\theta_0 - 1}\prod_{j=1}^{\infty} 0^{\theta_j}1\right) = -\sum_{k=1}^{\infty} \frac{2^{a_{k-1}(\{\xi_j\}_{j=0}^{\infty})-1}}{3^k}$$

where again both parts are equal. Then we conclude that the diagram is commutative.

Now we are going to prove that $-\pi^1$ is a bijection.

Let $Cod : \mathbb{Z}_2 \to \Sigma_2^*$ given by $\xi_j = 0$ if $Col^j(\beta) = 0 \mod 2$ and $\xi_j = 10$ if $Col^j(\beta) = 1 \mod 2$. Let's prove that $Cod \circ -\pi^1(\xi) = \xi$ and $-\pi^1 \circ Cod(\beta) = \beta$

1. $Cod \circ -\pi^1 = Id_{\Sigma_2^*}$: By Corollary 30 we have $Cod(-\pi^1(\xi)) = \xi$.
2. $-\pi^1 \circ Cod = Id_{\mathbb{Z}_2}$: Let $\beta \in \mathbb{Z}_2$ and $\xi \in \Sigma_2^*$ such that $\xi = Cod(\beta)$, so $\beta \in Cod^k(\xi)$ for all $k \in \mathbb{N}$. On the other hand we have $-\pi^1(\xi) \in Cod^k(\xi)$ for all $k \in \mathbb{N}$. Exists $\gamma_k \in \mathbb{Z}_2$ for all $k \in \mathbb{N}$ such that

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$\beta = -\pi^1(\xi) + 2^{A_k}\gamma_k$ so $\beta = -\pi^1(\xi) \mod 2^{a_k}$ then $\|\beta - \{-\pi^1(\xi)\}\|_2 \leq 2^{-A_k} \to 0$ as $k \to \infty$ Thus $\beta = -\pi^1(Cod(\beta))$.

Let us show that the applications $-\pi^1$ and Cod are uniformly continuous. Let $D : \Sigma_2^* \times \Sigma_2^* \to [0,\infty)$ the symbolic metric of two symbols given by

$$D(\xi,\eta) = \sum_{j=1}^{\infty} \frac{1}{2^j}\Delta(\xi_j,\eta_j), \text{ with } \xi_j,\eta_j \in \{0,10\}$$

and

$$\Delta(\xi_j,\eta_j) = \begin{cases} 1 & \text{if} \quad \xi_j \neq \eta_j \\ 0 & \text{if} \quad \xi_j = \eta_j \end{cases}$$

The space (Σ_2^*, D) is a complete metric space with the property that if two sequences are arbitrarily close if and only if their first terms are equal.

$$D(\xi,\eta) < \frac{1}{2^r} \text{ if and only if } \xi_j = \eta_j \text{ for all } j < r.$$

π^1 is uniformly continuous: Let $\varepsilon > 0$ and $A \in \mathbb{N}$ such that $2^{-A} < \varepsilon$ and $\xi,\eta \in \Sigma_2^*$. So let $\delta = \dfrac{1}{2^{A+2}}$

$$D(\xi,\eta) < \delta = \frac{1}{2^{A+2}} \text{ then we have } \xi_j = \eta_j \text{ for } j \leq A+1$$

Let b_{A+1} the number from 1 to the $A+1$-th term of ξ, then $a_{k-1}(\xi) = a_{k-1}(\eta)$ for $k \leq b_{A+1}$, so $-\displaystyle\sum_{k=1}^{b_{A+1}} \frac{2^{a_{k-1}(\xi)}}{3^k} = -\sum_{k=1}^{b_{A+1}} \frac{2^{a_{k-1}(\eta)}}{3^k} \in Cod^{A+1}(\xi)$, then in particular

$$-\sum_{k=1}^{b_{A+1}} \frac{2^{a_{k-1}(\xi)}}{3^k} = -\sum_{k=1}^{b_{A+1}} \frac{2^{a_{k-1}(\eta)}}{3^k} \mod 2^{A+1}$$

so

$$-\pi^1(\xi) = -\pi^1(\eta) \mod 2^{A+1} \text{ then we have } \|\pi^1(\xi) - \pi^1(\eta)\|_2 \leq 2^{-A-1} < 2^{-A} < \varepsilon.$$

***Cod* is uniformly continuous:** Let $\varepsilon > 0$ and $A \in \mathbb{N}$ such that $\delta = 2^{-A} < \varepsilon$ and $\alpha,\beta \in \mathbb{Z}_2$ such that $\|\alpha - \beta\|_2 < \delta = 2^{-A}$ then $\alpha = \beta \mod 2^A$, so $Cod^A(\alpha) = Cod^A(\beta)$, so $D(Cod(\alpha), Cod(\beta)) < \dfrac{1}{2^A} < \varepsilon$.

Therefore π^1 is a homemorphism and Col is topologically conjugate to ω.

QED.

As a first consequence of topological conjugation, we have that the set of periodic points of the Collatz function is dense in $\mathbb{Z}_2$.

Corollary 5. *Let $Per(\omega)$ the set of periodic point of ω, then we have $\overline{-\pi^1(Per(\omega))} = \mathbb{Z}_2$*

Proof: consequence of the continuity of the function $-\pi^1$ and the fact that the periodic sequences of the Shift function are dense in Σ_2^*.

QED.

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Now we are going to characterize the rational representation of the periodic points. By Corollary 5 we have that the set of the periodic numbers is dense in the set of the dyadic numbers, we will use these results to show that there exists no rational number (in general real) whose encoding is in G_1.

Proposition 31. *Let $\xi \in \Sigma_2^*$ periodic sequence, then we have:*

1. *If $\xi \in G_0$ so $-\pi^1(\xi)$ has positive rational representation.*
2. *If $\xi \in G_\infty$ so $-\pi^1(\xi)$ has negative rational representation.*

Proof: Let $\xi \in \Sigma_2^*$ periodic. Without loss of generality we can assume that $\xi = \prod_{j=0}^{\infty} 10^{\theta_j}$, then exit $K \in \mathbb{N}$ such that $a_{rK+j} = ra_K + a_{j-1}$ for all $R \in \mathbb{N}$. Indeed. Since ξ is periodic, exist $K \in \mathbb{N}$ such that

$$10^{\theta_0}10^{\theta_1}\dots10^{\theta_K}10^{\theta_0}10^{\theta_1}\dots10^{\theta_K}10^{\theta_0}10^{\theta_1}\dots10^{\theta_K}\dots$$

So, by induction we have:

$$\begin{aligned}
a_{K+1} &= \theta_0+\theta_1+\dots+\theta_K+\theta_{K+1} = \theta_0+\theta_1+\dots+\theta_K+\theta_0 = a_K + a_0\\
a_{K+2} &= \theta_0+\theta_1+\dots+\theta_K+\theta_{K+1}+\theta_{K+2} = \theta_0+\theta_1+\dots+\theta_K+\theta_0+\theta_1 = a_K + a_1\\
&\vdots\\
a_{2K+1} &= \theta_0+\theta_1+\dots+\theta_K+\theta_{K+1}+\theta_{K+2}+\dots+\theta_{2K+1} = \theta_0+\theta_1+\dots+\theta_K+\theta_0+\theta_1+\dots+\theta_K+\theta_0\\
&= 2a_K + a_0\\
&\vdots\\
a_{rK+j} &= ra_K + a_{j-1}\\
&\vdots
\end{aligned}$$

We are going to show that $-\pi^1(\xi)$ is rational

$$\begin{aligned}
-\sum_{j=1}^{\infty}\frac{2^{a_{j-1}}}{3^k} &= -\left\{\frac{2^{a_0}}{3}+\frac{2^{a_1}}{3^2}+\dots+\frac{2^{a_{K-1}}}{3^K}\right\} - \left\{\frac{2^{a_{K-1}+a_0}}{3^{K+1}}+\frac{2^{a_{K-1}+a_1}}{3^{K+2}}-\dots+\frac{2^{a_{K-1}+a_{K-1}}}{3^{2K}}\right\}\\
&\quad -\dots-\left\{\frac{2^{ra_{K-1}+a_0}}{3^{rK+1}}+\frac{2^{ra_{K-1}+a_1}}{3^{rK+2}}+\dots+\frac{2^{ra_{K-1}+a_{K-1}}}{3^{(r+1)K}}\right\}-\dots\\
&= -\left\{\frac{2^{a_0}}{3}+\frac{2^{a_2}}{3^2}+\dots+\frac{2^{a_{K-1}}}{3^K}\right\} - \frac{2^{a_{K-1}}}{3^K}\left\{\frac{2^{a_0}}{3}+\frac{2^{a_1}}{3^2}+\dots+\frac{2^{a_K}}{3^K}\right\}\\
&\quad -\dots-\left(\frac{2^{a_{k-1}}}{3^K}\right)^r\left\{\frac{2^{a_0}}{3}+\frac{2^{a_1}}{3^2}+\dots+\frac{2^{a_{K-1}}}{3^K}\right\}-\dots\\
&= -\left\{\frac{2^{a_0}}{3}+\frac{2^{a_1}}{3^2}+\dots+\frac{2^{a_{K-1}}}{3^K}\right\}\sum_{r=1}^{\infty}\left(\frac{2^{a_{k-1}}}{3^K}\right)^r\\
&= -\left\{\frac{2^{a_0}}{3}+\frac{2^{a_1}}{3^2}+\dots+\frac{2^{a_{K-1}}}{3^K}\right\}\left(\frac{1}{1-\frac{2^{a_{K-1}}}{3^K}}\right) \in \mathbb{Z}_2
\end{aligned}$$

which corresponds to a rational number, where its sign depends on the denominator $1-\frac{2^{a_K}}{3^K}$, if ξ are in G_0 then we necessarily have that $\frac{3^K}{2^{a_K}}$ must be less than 1, so the denominator is negative, so $-\pi^1(\xi)$

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is positive, analogously when ξ is in G_∞ we have that $\frac{3^K}{2^{a_K}}$ must be greater than 1, so the denominator is positive, so $-\pi^1(\xi)$ is negative.

QED.

We will now show a connection between the minimum positive integer value and the encoding of the sigma function.

Lemma 10. *Let $S_k \in \langle\theta,\psi\rangle$. Then $\rho_0(S_k) = Cod\sigma^{A_k}\left(\frac{N_k}{3^k}\right)$*

Proof: Let $S_k \in \langle\theta,\psi\rangle$. Then

$$\rho_0(S_k) = \frac{2^{A_k}}{3^k}\sigma_{3^k}^{A_k}(N_k) - \frac{N_k}{3^k} = \frac{2^{A_k}}{3^k}\left\{\frac{N_k}{2^{A_k}} + \frac{3^k Cod\sigma_{3^k}}{2^{A_k}}\right\} - \frac{N_k}{3^k} = Cod\sigma_{3^k}^{A_k}(N_k) = Cod\sigma^{A_k}\left(\frac{N_k}{3^k}\right).$$

QED.

As a consequence of the next proposition, we have that if $-\pi^1$ is a negative or non-integer number, the minimum value diverges, since we have that the dyadic representation of these numbers always has an infinite amount of numbers.

Proposition 32. *Let $\{S_k\}_{k\in\mathbb{N}}$ and $\xi = Cod\{S_k\}_{k\in\mathbb{N}}$. Then $\rho_0(S_k) = Cod\sigma\left(\pi^1(\xi)\right) \mod 2^{a_k}$.*

Proof: Let $\xi = Cod(\{S_k\}_{k\in\mathbb{N}}$. So

$$\left\|Cod\sigma\left(\pi^1(\xi)\right) - Cod\sigma^{A_k}\left(\frac{N_k}{3^k}\right)\right\|_2 = \left\|\pi^1(\xi) - \pi^1_k(\xi)\right\|_2 \leq 2^{-a_k}, \text{since } \pi^1_k(\xi) = \pi^1(\xi) \mod 2^{a_k}$$

Then $Cod\sigma^{A_k}\left(\frac{N_k}{3^k}\right) = Cod\sigma\left(\pi^1(\xi)\right) \mod 2^{a_k}$.

QED.

The above proposition tells us that for an coding ξ to be positive, it is necessary and sufficient that the dyadic expansion of $-\pi^1(\xi)$ has a finite amount of 1's.

Now, we are going to show that for all $\xi \in G_1$, must necessarily be unstable. For this we will show that the encoding of $-\pi^1(\xi)$ must necessarily have an infinite amount of 1

Theorem 6. *Let $\xi \in G_1$ and $S_k \in \langle\theta,\psi\rangle$ such that $Cod^k(S_k) = \{\xi_j\}_{j=0}^k$ then $\rho_0(S_k) \to \infty$ as $k \to \infty$.*

Proof: Let $\xi \in G_1$ and $\{\xi_k\}_{k\in\mathbb{N}}$ such that ξ_k is periodic with period k and $\xi_k \to \xi$ as $k \to \infty$. Due to periodicity, we have $\xi_k \in G_0 \cup G_\infty$. By Proposition 31, we have that $-\pi^1(\xi_k)$ admits a rational representation

$$-\pi^1(\xi_k) = -\sum_{j=1}^{\infty}\frac{2^{a_{j-1}}}{3^k} = -\left\{\frac{2^{a_0}}{3} + \frac{2^{a_1}}{3^2} + \ldots + \frac{2^{a_{k-1}}}{3^k}\right\}\left(\frac{1}{1-\frac{2^{a_{k-1}}}{3^k}}\right)$$

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$$= -\left\{\frac{2^{a_0}}{3} + \frac{2^{a_1}}{3^2} + \ldots + \frac{2^{a_{k-1}}}{3^k}\right\}\left(\frac{1}{1-\left(\frac{2^{\frac{a_{k-1}}{k}}}{3}\right)^k}\right)$$

By the continuity of the function $-\pi^1$ in $\mathbb{Z}_2$, we have that

$$-\pi^1(\xi) = -\lim_{k\to\infty}\pi^1(\xi_k)$$

$$= -\lim_{k\to\infty}\left\{\frac{2^{a_0}}{3} + \frac{2^{a_1}}{3^2} + \ldots + \frac{2^{a_{k-1}}}{3^k}\right\}\left(\frac{1}{1-\left(\frac{2^{\frac{a_{k-1}}{k}}}{3}\right)^k}\right)$$

This means that as k increases, the dyadic expansion on the right approaches the dyadic expansion of r, since $\xi_k$converges to ξ. On the other hand, we have that

$$\lim_{k\to\infty}\frac{a_{k-1}}{k} = \frac{\ln(3)}{\ln(2)}$$

This means that the rational representation of $-\pi^1(\xi_k)$ increases (in absolute value) as k increases. Then its dyadic expansion of $-\pi^1(\xi)$ cannot have a finite amount of 1. By Proposition 32 we have that $\rho_0(S_k) = Cod\sigma\left(\pi^1(\xi)\right) \mod 2^{a_k} \to \infty$. as $k \to \infty$

QED.

Let $\xi \in G_\infty$, if $-\pi^1(\xi) \in \mathbb{Z}_2$ admits real representation then, this representation is negative? Unfortunately, we cannot obtain an answer for dyadic numbers in this way, as the sign of a representation is not determined through approximation as it is for real numbers. One might be tempted to argue that since the function $-\pi^1$ converges in the real numbers, its representation in $\mathbb{Z}_2$ must coincide, but this is not always the case. However, we will show in the next section that they do indeed coincide, since in reality $-\pi^1$ is a solution when it converges.

9. The Coding of π^1

Now we are going to prove that there is a complete metric on G_∞. We will use this result to prove that if $\pi^1\left(\{\xi_j\}_{j=0}^\infty\right) \in \mathbb{Q}_{odd}$ then $Cod\left(\pi^1\left(\{\xi_j\}_{j=0}^\infty\right)\right) = \{\xi_j\}_{j=0}^\infty$ and in the case that $\pi^1\left(\{\xi_j\}_{j=0}^\infty\right) \in \mathbb{R}\setminus\mathbb{Q}$, then there is no rational r such that $Cod(r) = \{\xi_j\}_{j=0}^\infty$. We also demonstrate that the parity of the Collatz function on $\pi^1(G_\infty)$ depends solely on the first term. Building upon this insight, we extend the Collatz function to $\pi^1(G_\infty)$ and conclude the section by showing that the Collatz function is topologically conjugate to the Shift function on Σ_2^*. We will use this result to establish that the set of periodic orbits is dense.

9.1. Summary of Propositions in the Section

1. **Lemma 11:** Established that when the function $d(\pi^1(\xi), \pi^1(\eta)) = \sum_{j\in\mathbb{N}}\frac{1}{3^j}|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}| \leq \frac{1}{3^r}$ then ξ and η share at least the first $k-1$ terms.

2. **Proposition 33:** Established that $(\pi^1(G_\infty), d)$ is a complete metric space.
3. **Corollary 6:** Established that the $k-$coding set is an open set.
4. **Theorem 7:** Established that the full coding set is either as a singleton set or as an empty set depending on whether $\pi^1(\xi)$ is rational or not.
5. **Proposition 34:** It establishes that the parity of $\pi^1(\xi)$ depends only on the first term of the series.
6. **Definition 21:** Defines an extension of the Collatz function over all $\pi^1(G_\infty)$.
7. **Proposition 35:** The Collatz functions are continuous.
8. **Proposition 36:** The Collatz function on $\pi^1(G_\infty)$ is topological conjugacy to Shift map on Σ_2^*
9. **Corollary 7:** It is stable that the periodic points of the Collatz function in $(\pi^1(G_\infty), d)$ are dense.

9.2. $\pi^1(G_\infty)$ as Complete Metric Space

To ensure the coherent definition of a metric in $\pi^1(G_\infty)$, we need to "complete" the missing terms of the series to enable the calculation of the difference $|2^{a_{k-1}(\xi)} - 2^{a_{k-1}(\eta)}|$ for all $k \in \mathbb{N}$, irrespective of whether ξ or η has a null tail. To accomplish this, we define that when the sequence of 1s in ξ terminates, the function a_k will take on the value $-\infty$. Hence, we have $|2^{a_{k-1}(\xi)} - 2^{a_{k-1}(\eta)}| = |2^{-\infty} - 2^{a_{k-1}(\eta)}| = 2^{a_{k-1}(\eta)}$ from the index of ξ. Let $\xi \in \Sigma_2^*$ with null tail with index J. We will write

$$\pi^1\left(\{\xi\}_{j=1}^{\infty}\right) := \sum_{j=1}^{J} \frac{2^{a_{k-1}}}{3^j} + \sum_{j=J+1}^{J} \frac{2^{-\infty}}{3^j}$$

In the following lemma, we are going to introduce a new function, which, as we will see later, corresponds to a metric in the space $\pi^1(G_\infty)$. Additionally, we will present another result that we will examine more closely in this section, and essentially indicates to us that, since the parity of $\pi^1(\xi)$ depends only on the first term, we can interpret this in the following way: if two sequences are arbitrarily close, then they share the first terms of their encoding. This is of great importance for understanding the behavior of the orbits of the Collatz function, as, if we consider the Euclidean metric in $\mathbb{Q}$ or that of the absolute value, we observe the phenomenon that even though two numbers are arbitrarily close, their dynamics are completely different. One may converge to a cycle in a few iterations, while the other may take a very large amount of time.

Lemma 11 (Convergence and Coincidence Lemma)**.** *Let $\xi, \eta \in G_\infty$, then*

1. $\displaystyle\sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}\right|$ *is well defined.*
2. $\displaystyle\sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}\right| \leq \frac{1}{3^r} \Rightarrow a_{j-1}(\xi) = a_{j-1}(\eta)$ *for all $j < r$. Additionally we have to $\xi_{j-1} = \eta_{j-1}$ for all $j < r$.*

Proof: Let $\xi, \eta \in G_\infty$, then we have

Claim Let, $a, b \in \mathbb{N}$ then $|2^a - 2^b| < 2^{\max\{a,b\}}$.

If $a = b$ then $0 < 2^a$. Suppose that $a > b$ then

$$|2^a - 2^b| = 2^a\left|1 - \frac{1}{2^{a-b}}\right| < 2^a.$$

QED of the Claim.

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Now we prove that it is well-defined, by *Claim* we have:

$$d\left(\sum_{j=1}^{\infty}\frac{2^{a_{j-1}(\xi)}}{3^j},\sum_{j=1}^{\infty}\frac{2^{a_{j-1}(\eta)}}{3^j}\right)=\sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_{j-1}(\xi)}-2^{a_{j-1}(\eta)}\right|<\sum_{j=1}^{\infty}\frac{2^{\max\{a_{j-1}(\xi),a_{j-1}(\eta)\}}}{3^j}$$

as $\frac{2^{a_{j-1}(\xi)}}{3^j}$ and $\frac{2^{a_{j-1}(\eta)}}{3^j}$ converge to 0, then for $n>\mathbb{N}$ exist $J\in\mathbb{N}$ such that if $j>J$ we have

$$\left|\frac{2^{a_{j-1}(\xi)}}{3^j}\right|,\left|\frac{2^{a_{j-1}(\eta)}}{3^j}\right|<\frac{1}{n}$$

Then for $j>J$ we have too

$$\frac{2^{\max\{a_{j-1}(\xi),a_{j-1}(\eta)\}}}{3^j}<\frac{1}{n}$$

Then we have that $\frac{2^{\max\{a_{j-1}(\xi),a_{j-1}(\eta)\}}}{3^j}$ also converges to 0. Then by Proposition 16 we have $\sum_{j=1}^{\infty}\frac{2^{\max\{a_{j-1}(\xi),a_{j-1}(\eta)\}}}{3^j}<\infty$. To prove the statement, we will consider whether the sequences ξ and η in Σ_2^* have a null tail or not.

Let us first assume that the sequence does not have a null tail, then if we have

$$\sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_{j-1}(\xi)}-2^{a_{j-1}(\eta)}\right|\leq\frac{1}{3^r}$$

All terms less than r must be null. Suppose there exists some non-zero term, then we have that

$$\frac{1}{3^{r-1}}\leq\sum_{j=1}^{r-1}\frac{1}{3^j}\left|2^{a_{j-1}(\xi)}-2^{a_{j-1}(\eta)}\right|\leq\sum_{j=1}^{\infty}\frac{1}{3^j}\left|2^{a_{j-1}(\xi)}-2^{a_{j-1}(\eta)}\right|\leq\frac{1}{3^r}\text{ Then }r-1\geq r$$

which is absurd. Then we have that $\sum_{j=1}^{r-1}\frac{1}{3^j}\left|2^{a_{j-1}(\xi)}-2^{a_{j-1}(\eta)}\right|=0$. Which implies that

$$a_{j-1}(\xi)=a_{j-1}(\eta)\text{ for all }j<r.$$

Now we will prove that the sequences coincide up to r.

$$\xi=0^{\theta_0^1}\prod_{i=1}^{\infty}10^{\theta_i^1}\text{ and }\eta=0^{\theta_0^2}\prod_{i=1}^{\infty}10^{\theta_i^2}$$

writing this way, we have to

$$a_j(\xi)=\sum_{i=0}^{j}\theta_i^1\text{ and }a_j(\eta)=\sum_{i=0}^{j}\theta_i^2$$

then

$$\theta_{j-1}^1=a_{j-1}(\xi)-a_{j-2}(\xi)=a_{j-1}(\eta)-a_{j-2}(\eta)=\theta_{j-1}^2\text{ for all }0\leq j<r$$

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which means that ξ and η share the first $(r-1) - 10^{\theta_j}$ blocks.

Now suppose that ξ has a null tail of index I and η has no tail null. Then we have

$$\sum_{j=1}^{I} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}\right| + \sum_{j=I+1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\eta)}\right| \leq \frac{1}{3^r}$$

if $r \leq I$ then we have the previous case, then $a_{j-1}(\xi) = a_{j-1}(\eta)$ for all $j < r$. Now if $r > I$ we have

$$\sum_{j=1}^{I} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}\right| + \sum_{j=I+1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\eta)}\right| \leq \frac{1}{3^r}$$
$$\Rightarrow \sum_{j=1}^{I} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}\right| + \sum_{j=I+1}^{r-1} \frac{1}{3^j}\left|2^{a_{j-1}(\eta)}\right| = 0$$

The latter makes sense if η also has a null tail of index I, then

$$\xi_{j-1} = \eta_{j-1} \text{ for all } j \in \mathbb{N}$$

In particular $\xi_{j-1} = \eta_{j-1}$ for all $j < r$. Finally, suppose that ξ and η have a null tail of index I and L respectively, without loss of generality we can assume that $I \leq L$. Then

$$\sum_{j=1}^{I} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}\right| + \sum_{j=I+1}^{L} \frac{1}{3^j}\left|2^{a_{j-1}(\eta)}\right| + \sum_{j=L+1}^{\infty} \frac{1}{3^j}\left|2^{-\infty}\right| \leq \frac{1}{3^r}$$

1. If $1 \leq r \leq I$ then all terms with an index less than r are null and in particular we have $a_{j-1}(\xi) = a_{j-1}(\eta)$ for $j < r$. and as we already saw in the proofs above, this implies that $\xi_{j-1} = \eta_{j-1}$ for all $j < r$.
2. If $r \geq I$. Then $\sum_{j=I+1}^{L} \frac{1}{3^j}\left|2^{a_{j-1}(\eta)}\right| = 0$ we have that $I = J$ therefore $\xi = \eta$. particular we have $a_{j-1}(\xi) = a_{j-1}(\eta)$ for $j < r$.

QED.

Now we are going to show that the function we defined above is a complete metric on $\pi^1(G_\infty)$.

Proposition 33 (Metric space complete). *Let $d : \pi^1(G_\infty) \times \pi^1(G_\infty) \to [0, \infty)$ given by*

$$d\left(\sum_{j=1}^{\infty} \frac{2^{a_{j-1}(\xi)}}{3^j}, \sum_{j=1}^{\infty} \frac{2^{a_{j-1}(\eta)}}{3^j}\right) = \sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}\right|$$

Then $(\pi^1(G_\infty), d)$ is a metric space complete.

Proof: Let's prove that d is a metric through the axioms of metric:

1. $d(\pi^1(\xi), \pi^1(\eta)) = 0$ if and only if $\pi^1(\xi) = \pi^1(\eta)$ for all $\pi^1(\xi), \pi^1(\eta) \in \pi^1(G_\infty)$: Trivially we have that if $\pi^1(\xi) = \pi^1(\eta)$, then

$$d\left(\sum_{j=1}^{\infty} \frac{2^{a_{j-1}(\xi)}}{3^j}, \sum_{j=1}^{\infty} \frac{2^{a_{j-1}(\xi)}}{3^j}\right) = \sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\xi)}\right| = 0.$$

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Let $\pi^1(\xi), \pi^1(\eta) \in \pi^1(G_\infty)$ such that

$$d\left(\sum_{j=1}^{\infty} \frac{2^{a_{j-1}(\xi)}}{3^j}, \sum_{j=1}^{\infty} \frac{2^{a_{j-1}(\eta)}}{3^j}\right) = \sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}\right| = 0$$

by lemma 11 we have

$$\sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}\right| \leq \frac{1}{3^r} \Rightarrow a_{j-1}(\xi) = a_{j-1}(\eta) \text{ for } j < r$$

In particular, for $r \to \infty$ we have $\xi = \eta$.

2. $d(\pi^1(\xi), \pi^1(\eta)) = d(\pi^1(\eta), \pi^1(\xi))$ for all $\pi^1(\xi), \pi^1(\eta) \in \pi^1(G_\infty)$:

$$\sum_{j=1}^{k} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}\right| = \sum_{j=1}^{k} \frac{1}{3^j}\left|2^{a_{j-1}(\eta)} - 2^{a_{j-1}(\xi)}\right|$$

then

$$\sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}\right| = \sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\eta)} - 2^{a_{j-1}(\xi)}\right|$$

3. $d(\pi^1(\xi), \pi^1(\eta)) \leq d(\pi^1(\xi), \pi^1(\kappa)) + d(\pi^1(\kappa), \pi^1(\eta))$ for all, $\pi^1(\xi), \pi^1(\eta), \pi^1(\kappa) \in \pi^1(G_\infty)$

$$\sum_{j=1}^{k} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}\right| \leq \sum_{j=1}^{k} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\kappa)}\right| + \sum_{j=1}^{k} \frac{1}{3^j}\left|2^{a_{j-1}(\kappa)} - 2^{a_{j-1}(\eta)}\right|$$

then

$$\sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\eta)}\right| \leq \sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\kappa)}\right| + \sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\kappa)} - 2^{a_{j-1}(\eta)}\right|$$

then $(\pi^1(G_\infty), d)$ is a metric space. Now we are going to prove that it is a complete metric space. Let $\{\{\pi^1(\xi_j^k)\}_{j=1}^{\infty}\}_{k=1}^{\infty}$ be a Cauchy sequence on $\pi^1(G_\infty)$ then for any $\varepsilon > 0$ exist $K > 0$ such that

$$d(\{\pi^1(\xi_j^n)\}_{j=1}^{\infty}, \{\pi^1(\xi_j^m)\}_{j=1}^{\infty}) = \sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\xi^n)} - 2^{a_{j-1}(\xi^m)}\right| < \varepsilon \text{ for all } n, m > K$$

Let $r > 0$ such that $\sum_{j=1}^{\infty} \frac{1}{3^j}\left|2^{a_{j-1}(\xi^n)} - 2^{a_{j-1}(\xi^m)}\right| < \frac{1}{3^r} < \varepsilon$ by lemma 11 we have

$$\xi_j^n = \xi_j^m \text{ for all } j < r$$

On the other hand let $D : \Sigma_2^* \times \Sigma_2^* \to [0, \infty)$ the symbolic metric of two symbols given by

$$D(\xi, \eta) = \sum_{j=1}^{\infty} \frac{1}{2^j} \Delta(\xi_j, \eta_j)$$

with

$$\Delta(\xi_j, \eta_j) = \begin{cases} 1 & \text{if} \quad \xi_j \neq \eta_j \\ 0 & \text{if} \quad \xi_j = \eta_j \end{cases}$$

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The space (Σ_2^*, D) is a complete metric space with the property that if two sequences are arbitrarily close if and only if their first terms are equal.

$$D(\xi,\eta) < \frac{1}{2^r} \text{ if and only if } \xi_j = \eta_j \text{ for all } j < r.$$

then given a Cauchy sequence in $\pi_1(G_\infty)$ by the observation above we obtain a Cauchy sequence in Σ_2^* and the latter being complete there is a $\xi \in \Sigma_2^*$ such that

$$\xi^n \to \xi \text{ as } n \to \infty$$

We will now prove that ξ is in G_∞ and that the sequence $\pi_1(\xi^n)$ converges to $\pi_1(\xi)$.

1. Suppose that ξ is not in G_∞, so $\dfrac{2^{a_{j-1}(\xi)}}{3^j}$ does not converge to 0, then exist $L, k > 0$ and subsequence $a_{j_k-1}(\xi)$ such that $\dfrac{2^{a_{j_k-1}(\xi)}}{3^{j_k}} > L$ for all $k > K$. Since ξ^n converges to ξ. Exist $N > 0$ such that

   $$D(\xi^n, \xi) < \frac{1}{2^r} \text{ for } n > N$$

   then the first $r-$terms begin to equal, then

   $$\left| \frac{2^{a_{j-1}(\xi)}}{3^j} - \frac{2^{a_{j-1}(\xi^n)}}{3^j} \right| = 0 \text{ for all } j \leq r$$

   Since $\pi^1(\xi^n)$ is a sequence in $\pi_1(G_\infty)$ for all $k \in \mathbb{N}$ then $\dfrac{2^{a_{j-1}(\xi^n)}}{3^j} \to 0$ as $j \to \infty$. In particular for L exist $J > K > 0$ such that $\dfrac{2^{a_{j-1}(\xi^n)}}{3^j} < L$ so $\dfrac{2^{a_{j_k-1}(\xi)}}{3^{j_k}} < L$ this is a contradiction, then ξ is in G_∞.
2. Let us suppose, for absurdity, that $D(\xi, \xi^n) \to 0$ but $d(\pi_1(\xi), \pi_1(\xi^n))$ does not converge to 0, then exist $k, N > 0$ such that

   $$d(\pi^1(\xi), \pi^1(\xi^n)) \geq \frac{1}{3^k} \text{ for all } n > N$$

   In particular, exists in N_0 such that the a_j of both sequences are no longer equal, otherwise, we would have that for N that is large enough the distance would be less than $\dfrac{1}{3^k}$. On the other hand, the fact that they a_{j-1} are different implies that the terms of ξ and ξ^n must be different for $N > N_0$, but this contradicts the fact that the sequences ξ and ξ^n come closer, since with the metric D getting closer is the same as having the first terms of the sequences become equal.

then we can conclude that the metric space $(\pi^1(G_\infty), d)$ is complete.

QED.

With this metric, we have that the coding sets are open sets.

Corollary 6 (Cod^k is an open set). *Let $\xi \in G_\infty$ and $k \in \mathbb{N}$. Then $Cod^k(\xi)$ is an open set on $(\pi^1(G_\infty), d)$*

Proof: Let $\xi \in G_\infty$ and $u \in \pi^1(G_\infty)$ such that $u \in Cod^k(\xi)$. Let us consider v in $\pi^1(G_\infty)$ such that $d(u, v) < \dfrac{1}{3^{k+1}}$, by definition, exist $\mu, \tau \in G_\infty$ such that $\pi^1(\mu) = u$ and $\pi^1(\tau) = v$. By Lemma 11 have

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$\mu_j = \tau_j$ for $j \leq k$, then we have that $v \in Cod^k(\xi)$. Therefore, then $B\left(u, \frac{1}{3^{k+1}}\right)$ i.e. the ball of radius $\frac{1}{3^{k+1}}$ and center u is a subset of $Cod^k(\xi)$, therefore $Cod^k(\xi)$ is an open set.

QED.

9.3. Characterization of the Full Coding Sets through the Function π^1

Theorem 7 (Asymptotic Solutions Theorem). *Let $\{\xi_j\}_{j=0}^{\infty} \in G_\infty$, then if $\pi^1(\{\xi_j\}_{j=0}^{\infty})$ it is rational, then the $-\pi^1(\{\xi_j\}_{j=0}^{\infty}) = -\frac{p}{q}$ only rational that satisfies $cod\left(-\frac{p}{q}\right) = \{\xi_j\}_{j=0}^{\infty}$, in particular $\pi^1(\{\xi_j\}_{j=0}^{\infty}) \in \mathbb{Q}_{odd}$. If $\pi^1(\{\xi_j\}_{j=0}^{\infty})$ it is irrational, then there is no rational $\frac{p}{q}$ such that $cod\left(\frac{p}{q}\right) = \{\xi_j\}_{j=0}^{\infty}$.*

Proof: Let $\xi \in G_\infty$, We have by definition that $\pi^1(\xi) = \lim_{k\to\infty} \pi^1_k(\xi) \in \mathbb{Q}$. First, we will prove that this limit also makes sense in $(\pi^1(G_\infty), d)$. We have:
Claim 1: $d(\pi^1_k(\xi), \pi^1(\xi)) \to 0$ as $k \to \infty$.

Indeed, we have

$$d(\pi^1_k(\xi), \pi^1(\xi)) = \underbrace{\sum_{j=1}^{n} \frac{1}{3^j}|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\xi)}|}_{0} + \sum_{j=k+1}^{\infty} \frac{2^{a_{j-1}(\xi)}}{3^j} = \sum_{j=k+1}^{\infty} \frac{2^{a_{j-1}(\xi)}}{3^j}$$

By Proposition 16, we have $\sum_{j=k+1}^{\infty} \frac{2^{a_{j-1}(\xi)}}{3^j} \to 0$ as $k \to \infty$, then $d(\pi^1_k(\xi), \pi^1(\xi)) \to 0$ as $k \to \infty$

QED of the Claim.

We will now prove, using the completeness of $(\pi^1(G_\infty), d)$, that $\pi^1(\xi) \in \pi^1(G_\infty)$:
Claim 2: $\pi^1(\xi) \in \pi^1(G_\infty)$.

We can be rewritten,

$$\pi^1_k(\xi) = \pi^1\left(\{\xi_j\}_{j=1}^{k} 000\ldots\right)$$

let's prove that $\pi^1_k(\xi)$ is a Cauchy sequence in $\pi^1(G_\infty)$. Let $\varepsilon > 0$ and $n, m \in \mathbb{N}$ with $n < m$, then

$$d(\pi^1_n(\xi), \pi^1_m(\xi)) = \underbrace{\sum_{j=1}^{n} \frac{1}{3^j}|2^{a_{j-1}(\xi)} - 2^{a_{j-1}(\xi)}|}_{0} + \sum_{j=n+1}^{m} \frac{2^{a_{j-1}(\xi)}}{3^j} = \sum_{j=n+1}^{m} \frac{2^{a_{j-1}(\xi)}}{3^j}$$

Since $\xi \in G_\infty$ we have by Proposition 16 $\pi^1(\xi)$ is convergent, then $\sum_{j=n+1}^{m} \frac{2^{a_{j-1}(\xi)}}{3^j} \to 0$ as $n, m \to \infty$.
Therefore, exists $N \in \mathbb{N}$ such that $n, m > N$ we have $\sum_{j=n+1}^{m} \frac{2^{a_{j-1}(\xi)}}{3^j} < \varepsilon$. Then the sequence is Cauchy and since $\pi^1(G_\infty)$ is a complete metric space, we have that $\pi^1(\xi) = \lim_{k\to\infty} \pi^1_k(\xi) \in \pi^1(G_\infty)$.

QED of the Claim.

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Claim 3: $-\pi_k^1(\xi) \in Cod^k(\xi)$.

Indeed, let $S_k \in \langle \theta, \psi \rangle$ such that $Cod(S_k) = \{\xi_j\}_{j=1}^k$ by Proposition 12 we have

$$S_k(-\pi_k^1\left(\{\xi_j\}_{j=0}^{\infty}\right) = \frac{3^{b_k}(-\pi_k^1\left(\{\xi_j\}_{j=0}^{\infty}\right) + N_k}{2^{A_k}} = \frac{3^{b_k}\left(-\frac{N_k}{3^{b_k}}\right) + N_k}{2^{A_k}} = 0.$$

In other hand, we have

$$\frac{3^{b_k}\left(-\frac{N_k}{3^{b_k}}\right) + N_k}{2^{A_k}} = \frac{3^{b_k}(-N_k) + 3^{b_k}(N_k)}{3^{b_k}2^{A_k}} = \frac{1}{3^{b_k}}\left(\frac{3^{b_k}(-N_k) + 3^{b_k}(N_k)}{2^{A_k}}\right) = \frac{1}{3^{b_k}}S^{3^{b_k}}$$

then $-N_k \in \mathbb{E}(S^{3^{b_k}})$, by Proposition 8 we have $Cod_{3^{b_k}}(S_k^{3^{b_k}}) = Cod(S_k)$. Therefore $Cod^k\left(\frac{-N_k}{3^{b_k}}\right) = Cod^k(-\pi_k^1(\xi)) = Cod_{3^{b_k}}(S^{3^{b_k}}) = Cod(S_k) = \{\xi_j\}_{j=1}^k$.

QED of the Claim.

Claim 4: $-\pi^1(\xi) \in Cod^k(\xi)$ for all $k \in \mathbb{N}$.

Let $L+1 \in \mathbb{N}$, we have by *Claim 1*, exist $K \in \mathbb{N}$ such that if $k > K > L+1$ so $d(\pi_k^1(\xi), \pi^1(\xi)) < \frac{1}{3^{L+1}}$, By Lemma 11 we have to share the first L terms of the coding. On the other hand, by *Claim 3*, we have that $-\pi_k^1(\xi) \in Cod^k(\xi)$ and since we have $k > K > L+1$, due to the monotony of Cod, we have $-\pi_k^1(\xi) \in Cod^L(\xi)$. Therefore $-\pi^1(\xi) \in Cod^L(\xi)$.

QED of the Claim.

Claim 5: $\pi^1(\xi) \in Cod(\xi)$.

Since $-\pi^1(\xi) \in Cod^k(\xi)$ for all $k \in \mathbb{N}$, we have to

$$-\pi^1(\xi) \in \bigcap_{k \in \mathbb{N}} Cod^k(\xi) = Cod(\xi).$$

QED of the Claim.

Claim 6: $Cod\left(-\pi^1\left(\{\xi_j\}_{j=0}^{\infty}\right)\right) = \{\xi_j\}_{j=0}^{\infty}$.

We first show that $\pi^1(\xi) \in \mathbb{Q}_{odd}$. Let's assume that $\pi^1\left(\{\xi_j\}_{j=0}^{\infty}\right)$ converges to a fraction with an even denominator; then its coding is $1111\ldots$. However, $1111\ldots$ is not an element of Σ_2^*, which leads to a contradiction with *claim 4*. Then we have $\pi^1(\xi) \in \mathbb{Q}_{odd}$ and, by Theorem 3, we have $Cod\left(-\pi^1\left(\{\xi_j\}_{j=0}^{\infty}\right)\right) = \{\xi_j\}_{j=0}^{\infty}$.

QED of the Claim.

For the next part of the proposition, we will leverage the results presented in Section 7. In this section, we introduce the Sigma function along with its main properties and applications in solving linear Diophantine equations. It serves as an alternative to classical methods for solving this type of equation.

Claim 7: If $\pi^1(\{\xi_j\}_{j=0}^{\infty})$ it is irrational, then there is no rational then there is no rational $\frac{p}{q}$ such that $cod\left(\frac{p}{q}\right) = \{\xi_j\}_{j=0}^{\infty}$.

We will prove that there is no rational solution, By the theorem 2 we have that if there is another rational solution it must be a minimum positive integer value or a maximum negative integer value for $S_k(x) \in \langle \theta, \psi^q \rangle$ such that $Cod(S_k) = \{\xi_j\}_{j=1}^k$ for unique q not null, by proposition 7 we have

$$S_k(x) = \frac{3^{b_k} x + qN_k(\{\xi_j\}_{j=0}^{\infty})}{2^{A_k}}$$

by Propositions 4 and 21 the minimum positive integer value is

$$\rho_0(S_k) = \frac{1}{3^{b_k}}\left(2^{a_k}\sigma^{a_k}_{3^{b_k}}(qN_k) - qN_k\right) = 2^{a_k}\sigma^{a_k}(q\pi^1_k(\{\xi_j\}_{j=0}^{\infty}) - q\pi^1_k(\{\xi_j\}_{j=0}^{\infty}) = Cod\sigma^{a_k}(q\pi^1_k(\{\xi_j\}_{j=0}^{\infty})$$

and the maximum negative integer value is

$$\begin{aligned}\rho_1(S_k) = \frac{1}{3^{b_k}}\left(2^{a_k}\sigma^{a_k}_{-3^{b_k}}(qN_k) - qN_k\right) &= \frac{1}{3^{b_k}}\left(2^{a_k}\sigma^{a_k}_{3^{b_k}}(qN_k) - qN_k\right) - 2^{a_k} \\ &= Cod\sigma^{a_k}(q\pi^1_k(\{\xi_j\}_{j=0}^{\infty}) - 2^{a_k}\end{aligned}$$

when

$$Cod\sigma^{a_k}(q\pi^1_k(\{\xi_j\}_{j=0}^{\infty}) = \sum_{j=0}^{a_k-1}\delta_j 2^j \text{ with } \delta_j \in \{0,1\}$$

On the other hand, by Proposition 32 we have that the coding of the $\sigma^{a_k}(q\pi^1_k(\{\xi_j\}_{j=0}^{\infty})$ corresponds to the dyadic expansion of $-q\pi^1_k(\{\xi_j\}_{j=0}^{\infty})$. Since $\pi^1_k(\{\xi_j\}_{j=0}^{\infty})$ is irrational, then the dyadic expansion of $q\pi^1_k(\{\xi_j\}_{j=0}^{\infty})$ will never have a tail of 0 or 1 for all $q \in \mathbb{Z}$, so $Cod\sigma^{a_k}(q\pi^1_k(\{\xi_j\}_{j=0}^{\infty})$ does not converge for all $q \in \mathbb{Z}$. In particular we have to $\rho_0(S_k) \to \infty$. Now for $\rho_1(S_k)$, we have

$$\rho_1(S_k) = \sum_{j=0}^{a_k-1}\delta_j 2^j - 2^{a_k} = \sum_{j=0}^{a_k-1}\delta_j 2^j - \sum_{j=0}^{a_k-1} 2^j = \sum_{j=0}^{a_k-1}(\delta_j - 1)2^j$$

for this sum to be finite it is necessary exist $J > 0$ such that δ_j are all 1 for $j > J$, however as $q\pi^1_k(\{\xi_j\}_{j=0}^{\infty})$ this is impossible, then this sum is divergent.
Then if $\pi^1_k(\{\xi_j\}_{j=0}^{\infty})$ is irrational then there is no rational $\frac{p}{q}$ such that $cod\left(\frac{p}{q}\right) = \{\xi_j\}_{j=0}^{\infty}$.

QED of the Claim.

QED.

Example 12. 1. *Let* $\xi_1 = 101010\ldots \in \Sigma_2^*$, *then* $\pi^1(\xi) = 1$ *(see Example 7). Therefore* $Cod(-1) = 101010\ldots$.
2. *Let* $\xi_2 = 1010010100\ldots \in \Sigma_2^*$, *then*

$$\begin{aligned}\pi^1(\xi_2) &= \frac{1}{3} + \frac{2}{3^2} + \frac{2^3}{3^3} + \frac{2^4}{3^4} + \frac{2^6}{3^5} + \frac{2^7}{3^6} + \frac{2^9}{3^7} + \ldots \\ &= \frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\} + \frac{2^3}{3^2}\left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\} + \left(\frac{2^3}{3^2}\right)^2\left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\} + \ldots \\ &= \frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\}\sum_{j=0}^{\infty}\left(\frac{2^3}{3^2}\right)^j \\ &= \frac{1}{3} + \left\{\frac{2}{3^2} + \frac{2^3}{3^3}\right\}\frac{1}{1 - \frac{2^3}{3^2}} = 5\end{aligned}$$

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Therefore $Cod(-5) = 1010010100\ldots$

9.4. Extension of the Collatz Function on $\pi^1(G_\infty)$

Proposition 34 (Parity Preservation Proposition)**.** *Let* $\{\xi_j\}_{j=0}^{\infty} \in G_\infty$ *such that* $\pi^1(\{\xi_j\}_{j=0}^{\infty}) \in \mathbb{Q}_{odd}$*, then*

$$Col\Big(-\pi^1(\{\xi_j\}_{j=0}^{\infty})\Big) = \begin{cases} \dfrac{-3\pi^1(\{\xi_j\}_{j=0}^{\infty})+1}{2} & \text{if } a_0 = 0 \\ -\dfrac{1}{2}\pi^1(\{\xi_j\}_{j=0}^{\infty}) & \text{if } a_0 > 0 \end{cases}$$

or equivalent if $\pi^1(\{\xi_j\}_{j=0}^{\infty}) = \sum_{j=1}^{\infty} \frac{2^{a_{j-1}}}{3^j}$ *then*

$$Col\left(-\sum_{j=1}^{\infty} \frac{2^{a_{j-1}}}{3^j}\right) = \begin{cases} -\sum_{j=1}^{\infty} \dfrac{2^{a_j-1}}{3^j} & \text{if } a_0 = 0 \\ -\sum_{j=1}^{\infty} \dfrac{2^{a_{j-1}-1}}{3^j} & \text{if } a_0 > 0 \text{ or } -\infty \end{cases}$$

Proof Let's prove that the parity of $\pi^1(\{\xi_j\}_{j=0}^{\infty})$ only depends on the first term

$$\pi^1(\{\xi_j\}_{j=0}^{\infty}) = \frac{2^{a_0}}{3} + \sum_{j=2}^{\infty} \frac{2^{a_{j-1}}}{3^j} = \frac{2^{a_0}}{3} + 2\left(\sum_{j=2}^{\infty} \frac{2^{a_{j-1}-1}}{3^j}\right)$$

Claim: The series $\sum_{j=2}^{\infty} \frac{2^{a_{j-1}-1}}{3^j}$ cannot converge to a fraction with an even denominator.

Let us assume by contradiction that have $\sum_{j=2}^{\infty} \frac{2^{a_{j-1}-1}}{3^j} = \frac{p}{2q}$ with $(q,p) = (p,2) = 1$. Let $\{\xi_j\}_{j=0}^{l}0\ldots \in G_\infty$ such that $\pi^1\{\xi_j\}_{j=0}^{l}0\ldots = \frac{2^{a_1}}{3}$. Let $\eta \in G_\infty$ given by $\eta = \{\xi_j\}_{j=l+1}^{\infty}$ so $\pi^1(\eta) = \pi^1\{\xi_j\}_{j=l+1}^{\infty} = \sum_{j=2}^{\infty} \frac{2^{a_{j-1}-1}}{3^j} = \frac{p}{2q}$ and since η are in G_∞, this generates a contradiction to the Theorem 7.

QED of the Claim.

Let $\sum_{j=2}^{\infty} \frac{2^{a_j-1}}{3^j} = \frac{p}{q}$ with $(p,q) = (q,2) = 1$. We have:

$$\pi^1(\{\xi_j\}_{j=0}^{\infty}) = \frac{2^{a_0}}{3} + \sum_{j=2}^{\infty} \frac{2^{a_{j-1}}}{3^j} = \frac{2^{a_0}}{3} + 2\left(\sum_{j=2}^{\infty} \frac{2^{a_{j-1}-1}}{3^j}\right) = \frac{2^{a_0}}{3} + \frac{2p}{q} = \frac{2^{a_0}q + 6p}{3q}$$

if $a_0 = 0$ then $2^0 q + 6p = q + 6p$ is odd, since q is odd. Then $\pi^1(\{\xi_j\}_{j=0}^{\infty})$ is odd, and if $a_0 > 0$ then $2^{a_1}q + 6p$ is even. Then $\pi^1(\{\xi_j\}_{j=0}^{\infty})$ is even. In the case that $a_0 = -\infty$ we have that $\xi = 0$ or equivalent $\xi_j = 0$ for all $j \in \mathbb{N}_0$, so $\pi^1(0) = 0$, then $Col(0) = \frac{0}{2} = 0$.

QED.

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From the result above, we can extend the Collatz function over the entire set G_∞, thus obtaining a way of defining Collatz for the cases where the function π^1 is irrational.

Definition 21 (Extension Collatz functions on $-\pi^1(G_\infty)$)**.** *We defined $Col : -\pi^1(G_\infty) \to -\pi^1(G_\infty)$ by*

$$Col\left(-\pi^1(\{\xi_j\}_{j=0}^{\infty})\right) = \begin{cases} \dfrac{-3\pi^1(\{\xi_j\}_{j=0}^{\infty}) + 1}{2} & \text{if } a_0 = 0 \\ \\ -\dfrac{1}{2}\pi^1(\{\xi_j\}_{j=0}^{\infty}) & \text{if } a_0 > 0 \end{cases}$$

or equivalent if $\pi^1(\{\xi_j\}_{j=0}^{\infty}) = \sum_{j=1}^{\infty} \frac{2^{a_{j-1}}}{3^j}$ then

$$Col\left(-\sum_{j=1}^{\infty} \frac{2^{a_{j-1}}}{3^j}\right) = \begin{cases} -\displaystyle\sum_{j=1}^{\infty} \frac{2^{a_j-1}}{3^j} & \text{if } a_0 = 0 \\ \\ -\displaystyle\sum_{j=1}^{\infty} \frac{2^{a_{j-1}-1}}{3^j} & \text{if } a_0 > 0 \text{ or } -\infty \end{cases}$$

The extension of the Collatz function on -$\pi^1(G_\infty)$ is continuous.

Proposition 35 (Collatz function is continuous)**.** *$Col : -\pi^1(G_\infty) \to -\pi^1(G_\infty)$ is continuous.*

Proof: Let us consider the metric induced in $-\pi^1(G_\infty)$ by $-id : \pi^1(G_\infty) \to -\pi^1(G_\infty)$, that is, $d_{-\pi^1(G_\infty)}(-u, -v) = d(u, v)$ on $\pi^1(G_\infty)$, we will use the same notation for both metrics.

Let $u \in -\pi^1(G_\infty)$ and $\{u_j\}_{j\in\mathbb{N}}$ sequence of $-\pi^1(G_\infty)$ such that $u_j \to u$. Let $\xi, \xi_j \in G_\infty$ such that $Cod(u) = \xi$ and $Cod(\xi_j) = u_j$. Let $\varepsilon > 0$ and $r \in \mathbb{N}$ such that $r > 2$ and $\frac{1}{3^{r-1}} < \varepsilon$

$$d(u_j, u) = \sum_{k\in\mathbb{N}} \frac{|2^{a_{k-1}(\xi_j)} - 2^{a_{k-1}(\xi)}|}{3^k} < \frac{1}{3^{r-1}}$$

Since $r > 3$ then at least the first term must coincide.
Suppose that, $a_0 > 0$ then

$$\begin{aligned} d(Col(u_j), Col(u)) &= \sum_{k\in\mathbb{N}} \frac{|2^{a_{k-1}(\xi_j)-1} - 2^{a_{k-1}(\xi)-1}|}{3^k} \\ &= \frac{2}{2}\sum_{k\in\mathbb{N}} \frac{|2^{a_{k-1}(\xi_j)-1} - 2^{a_{k-1}(\xi)-1}|}{3^k} \\ &= \frac{1}{2}\sum_{k\in\mathbb{N}} \frac{|2^{a_{k-1}(\xi_j)} - 2^{a_{k-1}(\xi)}|}{3^k} \\ &< \frac{1}{3^{r-1}2} = \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

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Suppose now that $a_0 = 0$

$$\begin{aligned}
d(Col(u_j), Col(u)) &= \sum_{k\in\mathbb{N}} \frac{|2^{a_k(\xi_j)-1} - 2^{a_k(\xi)-1}|}{3^k} \\
&= \frac{2}{2}\frac{3}{3}\sum_{k>1} \frac{|2^{a_{k-1}(\xi_j)-1} - 2^{a_{k-1}(\xi)-1}|}{3^{k-1}} \\
&= \underbrace{\frac{1}{3}|2^{a_0(\xi)} - 2^{a_0(\xi_j)}|}_{0} + \frac{3}{2}\sum_{k>1} \frac{|2^{a_{k-1}(\xi_j)} - 2^{a_{k-1}(\xi)}|}{3^k} \\
&= \frac{3}{2}\sum_{k\in\mathbb{N}} \frac{|2^{a_{k-1}(\xi_j)} - 2^{a_{k-1}(\xi)}|}{3^k} \\
&< \frac{3}{3^{r-1}2} = \frac{1}{3^r 2} < \frac{1,}{3^{r-1}2} < \varepsilon
\end{aligned}$$

Therefore Col is continuous.

QED.

Now we will prove that the extension of the function Collatz on G_∞ is topologically conjugate to the shift function in Σ_2^*.

9.5. Topological Conjugation

In a dynamic system, there is a well-studied dynamics in the space of sequences of two symbols, known as the Shift map. This map acts on the sequences by eliminating the first term. It is known that with the metric D, this map is continuous, and its periodic orbits form a dense set. In the following proposition, we will show that the extension of the Collatz function on $-\pi^1(G_\infty)$ is, in fact, topologically conjugate to the dynamics of the Shift map.

Proposition 36 (Shift map on Σ_2^* and Collatz function are Topological Conjugacy)**.** *Let us consider the following function $\omega : \Sigma_2^* \to \Sigma_2^*$ given by,*

$$\omega\left(\{\xi_j\}_{j=0}^{\infty}\right) = \begin{cases} \{\xi_{j+2}\}_{j=1}^{\infty} & \text{if } \xi_1 = 1 \\ \{\xi_{j+1}\}_{j=1}^{\infty} & \text{if } \xi_1 = 0 \end{cases}$$

Then Col and ω are Topologically Conjugacy.

Proof: We are going to prove that this diagram is commutative

$$\begin{array}{ccc}
G_\infty & \xrightarrow{\ \omega\ } & G_\infty \\
\big\downarrow{\scriptstyle -\pi^1} & & \big\downarrow{\scriptstyle -\pi^1} \\
-\pi^1(G_\infty) & \xrightarrow[Col]{} & -\pi^1(G_\infty)
\end{array}$$

and that $-\pi^1$ is a homeomorphism.

Claim 1: The diagram is commutative

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Suppose that $\{\xi_j\}_{j=0}^{\infty} = 10^{\theta_1}10^{\theta_2}\ldots = \prod_{j=1}^{\infty} 10^{\theta_j} \in G_\infty$ with $\theta_j \in \mathbb{N}$. Writing this way, we get an explicit form for the function $a_k = \sum_{j=0}^{k} \theta_j$. If $a_0 = 0$ we have

$$\begin{aligned} Col \circ (-\pi^1)\left(\prod_{j=1}^{\infty} 10^{\theta_j}\right) = Col\left(-\sum_{j=1}^{\infty} \frac{2^{a_{k-1}(\{\xi_j\}_{j=0}^{\infty})}}{3^k}\right) &= \frac{1}{2}\left(-3\sum_{k=1}^{\infty} \frac{2^{a_{k-1}(\{\xi_j\}_{j=0}^{\infty})}}{3^k} + 1\right) \\ &= \frac{1}{2}\left(-\sum_{k=2}^{\infty} \frac{2^{a_{k-1}(\{\xi_j\}_{j=0}^{\infty})}}{3^{k-1}} + 1\right) \\ &= \frac{1}{2}\left(-1 - \sum_{k=2}^{\infty} \frac{2^{a_{k-1}(\{\xi_j\}_{j=0}^{\infty})}}{3^{k-1}} + 1\right) \\ &= -\sum_{k=2}^{\infty} \frac{2^{a_{k-1}(\{\xi_j\}_{j=0}^{\infty})} - 1}{3^{k-1}} \\ &= -\sum_{k=1}^{\infty} \frac{2^{a_{k}(\{\xi_j\}_{j=0}^{\infty})} - 1}{3^{k}} \end{aligned}$$

and

$$-\pi^1 \circ \omega\left(\prod_{j=1}^{\infty} 10^{\theta_j}\right) = -\pi^1\left(0^{\theta_1 - 1}\prod_{j=2}^{\infty} 10^{\theta_j}\right) = -\sum_{j=1}^{\infty} \frac{2^{a_k(\{\xi_i\}_{i=1}^{\infty})} - 1}{3^k}$$

where both parts are equal. On the other hand. suppose that $\{\xi_j\}_{j=0}^{\infty} = 0^{\theta_0}10^{\theta_1}1\ldots = \prod_{j=0}^{\infty} 0^{\theta_j}1 \in G_\infty$ with $\theta_j \in \mathbb{N}$. We have

$$Col \circ (-\pi^1)\left(\prod_{j=0}^{\infty} 0^{\theta_j}1\right) = Col\left(-\sum_{j=0}^{\infty} \frac{2^{a_{k-1}(\{\xi_i\}_{i=0}^{\infty})}}{3^k}\right) = -\sum_{j=1}^{\infty} \frac{2^{a_{k-1}(\{\xi_i\}_{i=0}^{\infty})} - 1}{3^k}$$

and

$$(-\pi^1) \circ \omega\left(\prod_{j=1}^{\infty} 0^{\theta_j}1\right) = (-\pi^1) \circ \left(\theta^{\theta_0 - 1}\prod_{j=1}^{\infty} 0^{\theta_j}1\right) = -\sum_{j=1}^{\infty} \frac{2^{a_{k-1}(\{\xi_j\}_{j=0}^{\infty})} - 1}{3^k}$$

where again both parts are equal. Then we conclude that the diagram is commutative.

QED of the Claim.

Claim 2: $-\pi^1 : G_\infty \to -\pi^1(G_\infty)$ with It is a bijective function.

Let $Cod : -\pi^1(G_\infty) \to G_\infty$ let's prove that $Cod \circ -\pi^1 = Id_{G_\infty}$ and $-\pi^1 \circ Cod = Id_{-\pi^1(G_\infty)}$.

1. $Cod \circ -\pi^1 = Id_{G_\infty}$:

   Let $\xi = \xi_0\xi_1\xi_2\ldots \in G_\infty$ with $\xi_j \in \{0, 10\}$. Since the parity of $-\sum_{j=1}^{\infty} \frac{2^{a_{j-1}}}{3^j}$ depends only on the first term, if ξ starts with 0 then $a_0 > 0$, then $-\sum_{j=1}^{\infty} \frac{2^{a_{j-1}}}{3^j}$ is even then the first term of its

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coding is 0, and if ξ starts with 1 then $a_0 = 0$ then $-\sum_{j=1}^{\infty} \frac{2^{a_j}}{3^j}$ is odd then the first term of coding is 10. By applying the Collatz function we obtain the same result as applying a translation of the terms of ξ. Indeed

(a) if $a_0 = 0$

$$Col\left(-\sum_{j=1}^{\infty} \frac{2^{a_{j-1}}}{3^j}\right) = \sum_{j=1}^{\infty} \frac{2^{a_j - 1}}{3^j} = -\pi^1(\xi_1\xi_2\ldots).$$

(b) if $a_0 > 0$

$$Col\left(-\sum_{j=1}^{\infty} \frac{2^{a_{j-1}}}{3^j}\right) = \sum_{j=1}^{\infty} \frac{2^{a_{j-1}-1}}{3^j} = -\pi^1(\xi_1\xi_2\ldots).$$

Then applying the function $-\pi^1$. Then we can repeat the same procedure and we recover ξ. Therefore

$$Cod(-\pi^1(\xi)) = \xi$$

2. $-\pi^1 \circ Cod = Id_{-\pi^1(G_\infty)}$:

Let $u \in -\pi^1(G_\infty)$ and $\xi \in G_\infty$ such that $-\pi^1(\xi) = u$. On the other hand we have $Cod(-\pi^1(\xi)) = Cod(u)$ and $Cod(-\pi^1(\xi)) = \xi$ then $Cod(u) = \xi$, applying $-\pi^1$ on both sides we have $-\pi^1(Cod(u)) = -\pi^1(\xi) = u$.

QED of the Claim.

Claim 3: $-\pi^1 : G_\infty \to -\pi^1(G_\infty)$ is continuous.

Let $\xi \in G_\infty$ and $\{\eta_j\}_{l\in\mathbb{N}}$ a sequence on G_∞ such that $\eta_j \to \xi$ as $j \to \infty$. Let $r \in \mathbb{N}$ then if $D(\eta_j, \xi) < \frac{1}{2^r}$ implies that the first elements are equal, then:

$$d(\pi^1(\eta_j), \pi^1(\xi)) = \sum_{k=r}^{\infty} \frac{|2^{a_{k-1}(\eta_j))} - 2^{a_{k-1}(\eta))}|}{3^k} \to 0 \text{ as } r \to \infty$$

QED of the Claim.

Claim 4: $Cod : -\pi^1(G_\infty) \to G_\infty$ is continuous.

Let $u \in -\pi^1(G_\infty)$ and let $\varepsilon > 0$. Take $r \in \mathbb{N}$ such that $\frac{1}{2^r} < \varepsilon$. then by Lemma 11 we have

$$d(u,v) < \frac{1}{3^r} \Rightarrow Cod(u)_j = Cod(v)_j \text{ for all } j < r \Rightarrow D(Cod(u), Cod(v)) < \frac{1}{2^r} < \varepsilon$$

QED of the Claim.

Therefore, $-\pi^1$ is a homeomorphism and therefore a topological conjugation.

QED.

Corollary 7 (The Periodic orbit of Collatz function)**.** *The set of periodic orbits of Collatz function is dense in* $\pi^1(G_\infty)$

Proof: Direct consequence of the proposition 36

QED.

Corollary 8. *Let $\xi \in G_\infty$. Suppose that $-\lim_{k\to\infty} \pi^1(\xi) \in \mathbb{Z}_2$ admits real representation and let us denote the representation by Γ Then $\Gamma = -\lim_{k\to\infty} \pi^1(\xi) \in \mathbb{R}$.*

Proof Consequence of uniqueness, since $-\lim_{k\to\infty} \pi^1(\xi) \in \mathbb{R}$. is the only solution in $\mathbb{R}$ whose coding is ξ and we know that Γ is the only dyadic solution whose coding is ξ .

QED.

10. The Problem of Divergence

In this section, we address the fundamental aspects of divergence of the Collatz function. The primary focus is on the behavior of sequences and orbits, especially those with divergent slopes and their stability properties. The main results are summarized in the following key theorems:

1. **Theorem 8:** This theorem states that all sequences S_k with a divergent slope are positively unstable, defining the sufficient condition under which a sequence becomes unstable.
2. **Theorem 9:** This theorem shows that all orbits with codings in G_0 are bounded.
3. **Theorem 10:** This theorem shows that all orbits with codings in G_∞ are bounded.Consequently, the orbits of negative numbers necessarily fall into periodic orbits.
4. **Theorem 11:** This theorem concludes that all natural numbers have bounded orbits, implying the non-existence of divergent orbits for natural numbers.

First, we examine the conditions under which the slope of a function S_k diverges, leading to instability. Next, we explore the boundedness of orbits coded within G_0 and G_∞, providing proofs and corollaries to support these findings. Finally, we demonstrate the non-existence of divergent orbits for natural numbers.

10.1. Summary of Propositions in the Section

1. **Theorem 8:** It is stated that all sequences S_k with a divergent slope are positively unstable.
2. **Theorem 9:** It is stated that all orbits with codings in G_0 are bounded.
3. **Corollary 9:** If exist a sub-sequence such that $\lim_{j\to\infty} \frac{a_{k_l}(\xi)}{k_l} > \frac{\ln(3)}{\ln(2)}$. Then exist $\Omega > 0$ such that $\pi^2(\xi)(k_l) < \Omega$.
4. **Theorem 10:** Let $\xi \in G_\infty$. Then we have exits $\Omega > 0$ such that $Col^{a_k}\big(-\pi^1(\xi)\big) > -\Omega$ for all $k \in \mathbb{N}$. In particular if $-\pi^1(\xi) \in \mathbb{Z}$ then its orbit necessarily falls into a cycle.
5. **Theorem 11:** It is stated that all natural numbers have bounded orbits.

10.2. The Problem of Divergence

The following theorem shows that if the slope of the function S_k diverges, then so does the minimum value, this is because the only value that satisfies the encoding of S_k is negative.

Theorem 8 (Positively unstable Theorem)**.** *Let $\{S_j\}_{j=1}^\infty \subset \langle \theta, \psi^q \rangle$ such that $\frac{3^{b_j}}{2^{a_j}} \to \infty$. Then, $\{S_j\}_{j=1}^\infty$ is positively unstable for all $q \in \mathbb{Z}$.*

Proof: Let $\xi = Cod(\{S_j\}_{j=1}^\infty)$. Since $\frac{3^{b_j}}{2^{a_j}} \to \infty$ then $\xi \in G_\infty$ by Proposition 16, 30 and 32 and Theorem 7. We have $-\pi^1(\xi) = -\sum_{j=1}^{\infty} \frac{2^{a_j-1}}{3^j} \in \mathbb{R}$ is the only value whose coding is ξ. On the other

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hand, regardless of rationality, this number is always negative. Therefore, the minimum value must necessarily be divergent.

QED.

We are going to show a series of results referring to the bounds of the orbits of numbers whose coding is in G_0 and G_∞.

Theorem 9 (Bounded Orbit Theorem). *Let $n \in \mathbb{Z}$ such that if $Cod(n) \in G_0$ then the orbit of n is bounded.*

Proof Let $\{S_j\}_{j\in\mathbb{N}}$ on $\langle\theta,\psi\rangle$ such that $Cod(\{S_j\}_{j\in\mathbb{N}}) = Cod(n)$. Without loss of generality we can assume that n is positive, because in the case that n is negative we have that

$$\lim_{j\to\infty} Col^j(n) = \lim_{j\to\infty} S_j(n) = \lim_{j\to\infty} \frac{3^j n + N_j}{2^{a_j}} = \lim_{j\to\infty} \frac{3^j}{2^{a_j}} n + \lim_{j\to\infty} \pi^2(Cod(n))(j) = \lim_{j\to\infty} \pi^2(Cod(n))(j) \geq 0$$

then eventually its orbit will fall into a non-negative number.
If the coding of $\{S_j\}_{j\in\mathbb{N}}$ has a null tail, the result is trivial. Suppose $\{S_j\}_{j\in\mathbb{N}}$ has no null tail, then:

$$Col^j(n) = S_j(n) = \frac{3^j n + N_j}{2^{a_j}} = \frac{3^j}{2^{a_j}} n + \pi^2(Cod(n))(j)$$

Since $\frac{3^j}{2^{a_j}} \to 0$ we have by Proposition that $\pi^2(Cod(n))(j)$ is bounded and let $M > 0$ such that $\pi^2(Cod(n))(j) < M$ for all $j \in \mathbb{N}$. On the other hand, since $\frac{3^j}{2^{a_j}} \to 0$ we have $\frac{3^j}{2^{a_j}}$ is bounded and let $H > 0$ such that $\frac{3^j}{2^{a_j}} < H$ for all $j \in \mathbb{N}$, then

$$\frac{3^j}{2^{a_j}} n + \pi^2(Cod(n))(j) < Hn + M$$

Therefore $Col^j(n) < \infty$.

QED.

The following result is the version of the previous theorem for sub-sequence.

Corollary 9. *Let $\xi \in \Sigma_2^*$. If exist a sub-sequence such that $\lim_{j\to\infty} \frac{a_{k_l}(\xi)}{k_l} > \frac{\ln(3)}{\ln(2)}$. Then exist $\Omega > 0$ such that $\pi^2(\xi)(k_l) < \Omega$.*

Proof: Let $j < k_l$ then

$$\lim_{k_l - j\to\infty} \frac{a_{k_l} - a_{j-1}}{k_j - j} = \lim_{k_l - 1\to\infty} \frac{a_{k_l} - a_0}{k_j - 1} = \lim_{l\to\infty} \frac{a_{k_l}}{k_l} > \frac{\ln(3)}{\ln(2)}$$

then exist $T, \varepsilon > 0$ such that if $k_l - j > T$ we have

$$\frac{a_{k_l} - a_{j-1}}{k_l - j} > \frac{\ln(3)}{\ln(2)} + \varepsilon$$

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Using the lemma 5 we have

$$\begin{aligned}
\pi^2(\xi)(k_l) &= \frac{3^{k_l}}{2^{a_{k_l}}}\sum_{j=1}^{k_l}\frac{2^{a_{j-1}}}{3^j} \\
&= \sum_{j=1}^{k_l}\frac{2^{a_{j-1}-a_{k_l}}}{3^{j-k_l}} \\
&= \sum_{j=1}^{k_l}\left(\frac{3}{2^{\frac{a_{k_l}-a_{j-1}}{k_l-j}}}\right)^{k_l-j} \\
&= \sum_{j=1}^{k_l-T}\left(\frac{3}{2^{\frac{a_{k_l}-a_{j-1}}{k_l-j}}}\right)^{k_l-j} + \sum_{j=k_l-T+1}^{k_l}\left(\frac{3}{2^{\frac{a_{k_l}-a_{j-1}}{k_l-j}}}\right)^{k_l-j} \\
&< \sum_{j=1}^{k_l-T}\left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)}+\varepsilon}}\right)^{k_l-j} + \sum_{j=k_l-J+1}^{k_l}\left(\frac{3}{2}\right)^{k_l-j} \\
&< \sum_{j=1}^{k_l}\left(\frac{3}{2^{\frac{\ln(3)}{\ln(2)}+\varepsilon}}\right)^{k_l-j} + \sum_{j=k_l-J+1}^{k_l}\left(\frac{3}{2}\right)^{k_l-j} \\
&< \frac{1}{2^{\varepsilon k_l}}\sum_{j=1}^{k_l}2^{\varepsilon j} + \left(\frac{3}{2}\right)^{k_l}\sum_{j=k_l-J+1}^{k_l}\left(\frac{2}{3}\right)^{j} \\
&= \frac{1}{2^{\varepsilon k_l}}\left(\frac{1-2^{\varepsilon(k_l+1)}}{1-2^{\varepsilon}}\right) + \left(\frac{3}{2}\right)^{k_l}\left\{\frac{\left(\frac{2}{3}\right)^{k_l}-\left(\frac{2}{3}\right)^{k_l-T+1-1}}{\frac{2}{3}-1}\right\} \\
&< \frac{\frac{1}{2^{\varepsilon k_l}}-2^{\left(\frac{k_l+1}{k_l}\right)}}{1-2^{\varepsilon}} + \frac{1-\left(\frac{3}{2}\right)^{T}}{\frac{2}{3}-1} \\
&\leq \frac{4}{2^{\varepsilon}-1} + 3\left\{\left(\frac{3}{2}\right)^{T}-1\right\} = \Omega
\end{aligned}$$

Therefore $\pi^2(\xi)(k_l) < \Omega$

QED.

We will now show that the orbits of numbers whose coding is in G_∞ are bounded below, in particular the orbits of integers whose coding is in G_∞, so they must necessarily fall into a cycle.

Theorem 10. *Let $\xi \in G_\infty$. Then we have exits $\Omega > 0$ such that $Col^{a_k}\left(-\pi^1(\xi)\right) > -\Omega$ for all $k \in \mathbb{N}$. In particular if $-\pi^1(\xi) \in \mathbb{Z}$ then its orbit necessarily falls into a cycle.*

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Proof: Let $\xi \in G_\infty$, then exist $1 > \varepsilon > 0$ such that $\limsup_{k\to\infty} \frac{a_k}{k} < \frac{\ln(3)}{\ln(2)} - \varepsilon$. Exist $J > 0$ such that $\frac{a_{j-1} - a_k}{j-k} < \frac{\ln(3)}{\ln(2)} - \varepsilon$ for all $k > J$. so

$$Col^{a_k}\left(-\pi^1(\xi)\right) = Col^{a_k}\left(-\sum_{j\in\mathbb{N}} \frac{2^{a_{j-1}}}{3^j}\right) = -\frac{3^k}{2^{a_k}} \sum_{j\in\mathbb{N}} \frac{2^{a_{j-1}}}{3^j} + \frac{N_k}{2^{a_k}}$$

$$= -\frac{3^k}{2^{a_k}} \sum_{j=1}^{k} \frac{2^{a_{j-1}}}{3^j} - \frac{3^k}{2^{a_k}} \sum_{j=k+1}^{\infty} \frac{2^{a_{j-1}}}{3^j} + \frac{N_k}{2^{a_k}} = -\frac{N_k}{2^{a_k}} - \frac{3^k}{2^{a_k}} \sum_{j=k+1}^{\infty} \frac{2^{a_{j-1}}}{3^j} + \frac{N_k}{2^{a_k}}$$

$$= -\frac{3^k}{2^{a_k}} \sum_{j=k+1}^{\infty} \frac{2^{a_{j-1}}}{3^j} = -\sum_{j=k+1}^{\infty} \frac{2^{a_{j-1}-a_k}}{3^{j-k}} = -\sum_{j=k+1}^{\infty} \left(\frac{2^{\frac{a_{j-1}-a_k}{j-k}}}{3} \right)^{j-k}$$

$$> -\sum_{j=k+1}^{\infty} \left(\frac{2^{\frac{\ln(3)}{\ln(2)} - \varepsilon}}{3} \right)^{j-k} = -\sum_{j=k+1}^{\infty} 2^{-\varepsilon(j-k)} = -2^{\varepsilon k} \sum_{j=k+1}^{\infty} 2^{-\varepsilon j}$$

$$= -2^{\varepsilon k} \left\{ \frac{2^{-\varepsilon(k+1)}}{1-2^{-\varepsilon}} \right\} = -\frac{2^{-\varepsilon}}{1-2^{-\varepsilon}}$$

Let $\Omega = \max\left\{ \frac{2^{-\varepsilon}}{1-2^{-\varepsilon}}, Col^{a_k}\left(-\pi^1(\xi)\right) \text{ with } k < J \right\}$, so $Col^{a_k}(-\pi^1(\xi)) > -\Omega$ for all $k \in \mathbb{N}$.

In particular, if $-\pi^1(\xi) \in \mathbb{Z}$, then $Col^{a_k}(-\pi^1(\xi))$ only takes negative integer values and being bounded from below, must necessarily repeat some value, then the orbit must fall into a cycle.

QED.

Monks and Yazinski also extend the results of Eliahou [4] (1993) and Lagarias [3] (1985) concerning the density of "odd" points in an orbit. Let $k_n(x)$ denote the number of ones in the first n digits of the parity vector x. If $x \in Q_{odd}$ eventually enters an n-periodic orbit, then

$$\frac{\ln(2)}{\ln(3+1/m)} \leq \lim_{n\to\infty} \frac{k_n(x)}{n} \leq \frac{\ln(2)}{\ln(3+1/M)}$$

where m, M are the least and greatest cyclic elements in the eventual cycle. If $x \in Q_{odd}$ diverges, then

$$\frac{\ln(2)}{\ln(3)} \leq \liminf_{n\to\infty} \frac{k_n(x)}{n}.$$

We will now show the main theorem of this work, where we finally show the nonexistence of divergent orbits for every positive integer. We will show that the necessary and sufficient condition for an orbit to be divergent is

$$\limsup_{j\to\infty} \frac{a_j}{j} < \frac{\ln(3)}{\ln(2)}$$

which implies that the only solution if it exists must be

$$-\infty < -\pi^1(\xi) < 0.$$

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Theorem 11 (Divergent Orbits Theorem). *There are no divergent orbits for the Collatz function on natural numbers.*

Proof: Let $n \in \mathbb{N}$ such that $Col^j(n) \to \infty$ as $j \to \infty$ and $\{S_j\}_{j=1}^{\infty} \subset \langle \theta, \psi \rangle$ such that $Cod\{S_j\}_{j=1}^{\infty} = Cod(n)$. We are going to prove that the necessary and sufficient condition for an orbit to be divergent is that the coding does not have a null tail and $\frac{3^j}{2^{a_j}} \to \infty$ as $j \to \infty$.

If $Cod\{S_j\}_{j=1}^{\infty}$ has a null tail, it means that from a certain iteration, the orbit of n must always be even, which implies that this orbit must be decreasing. This contradicts the fact that we have assumed that $Col^j(n) \to \infty$.

Now without loss of generality, we can assume that $Cod\{S_j\}_{j=1}^{\infty}$ does not have a null tail

$$Col^j(n) = \frac{3^j n + N_j}{2^{a_j}} \to \infty \text{ then } \frac{3^j}{2^{a_j}} \to \infty \text{ or } \pi_k^2(Cod\{S_j\}_{j=1}^{\infty}) = \frac{N_j}{2^{a_j}} \to \infty$$

1. If $\frac{3^j}{2^{a_j}} \to \infty$ then $Cod\{S_j\}_{j=1}^{\infty} \in G_\infty$ by Proposition 16 we have $\pi_k^1(Cod\{S_j\}_{j=1}^{\infty}) < \infty$ this implies that

$$\pi_k^2(Cod\{S_j\}_{j=1}^{\infty}) = \frac{3^j}{2^{a_j}} \pi_k^1(Cod\{S_j\}_{j=1}^{\infty}) \to \infty.$$

2. if $\pi_k^2(Cod\{S_j\}_{j=1}^{\infty}) \to \infty$, by Proposition 16 we have that $Cod\{S_j\}_{j=1}^{\infty} \notin G_0$ then $\liminf_{j\to\infty} \frac{a_j}{j} \leq \frac{\ln(3)}{\ln(2)}$. Let's show now in fact $\limsup_{k\to\infty} \frac{a_k}{k} \leq \frac{\ln(3)}{\ln(2)}$. Suppose that exist $\{j_k\}_{k\in\mathbb{N}}$ such that $\lim_{j\to\infty} \frac{a_{j_k}}{j_k} < \frac{\ln(3)}{\ln(2)} + \varepsilon$ with $\varepsilon > 0$, so using the estimated bound in the demonstration of the Lemma 9, we have

$$\pi^2(\xi)(j_k) < \frac{4}{2^\varepsilon - 1} + 3\left\{\left(\frac{3}{2}\right)^T - 1\right\} = \Omega$$

Since $\frac{3^{j_k}}{2^{a_{j_k}}} \to 0$ as $j \to \infty$. So $\max_k \left\{\frac{3^{j_k}}{2^{a_{j_k}}}\right\} < \infty$ then we have

$$Col^{j_k}(n) = \frac{3^{j_k}}{2^{a_{j_k}}} n + \frac{N_{j_k}}{2^{j_k}} < \max_k \left\{\frac{3^{j_k}}{2^{a_{j_k}}}\right\} n + \Omega$$

Since $Col^j(n) \in \mathbb{N}$ for all $j \in \mathbb{N}$, then exist $K \in \mathbb{N}$ such that $Col^{j_k}(n) = Col^{j_{k+K}}(n)$. So we have that the orbit of n must fall into a cycle, which implies that $\lim_{j\to\infty} \frac{3^j}{2^{a_j}} = 0$ which implies that $\pi^2(\xi)(j)$ is bounded, which contradicts the hypothesis that $Col^j(n) \to \infty$. Therefore $\limsup_{j\to\infty} \frac{a_j}{j} \leq \frac{\ln(3)}{\ln(2)}$.

This is equivalent to $\frac{3^j}{2^{a_j}} \to \{1, \infty\}$ as $j \to \infty$.

On the other hand if $Cod\{S_j\}_{j=1}^{\infty} \in G_1$ by Theorem 6 we have $\rho_0\{S_j\}_{j=1}^{\infty} \to \infty$ which contradicts the hypothesis that $\{S_j\}_{j=1}^{\infty}$ is positively stable, so $Cod\{S_j\}_{j=1}^{\infty} \in G_\infty$

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Since the Theorem 9 above, to have $Col^j(u)$ divergent it is necessary that $\frac{3^k}{2^{a_k}} \to \infty$, by Theorem 8 we have $\{S_j\}_{j=1}^{\infty}$ is positively unstable, by Theorem, 2 we have $\mathbb{E}\left(\{S_j\}_{j=1}^{\infty}\right) \cap \mathbb{N} = \emptyset$. Therefore, cannot exist $n \in \mathbb{N}$ such that its orbit is divergent.

QED.

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