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The Ali-Cesaro Stolz Theorem: Extending Classical Limit Analysis with Z-Transforms

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Abstract: The Ali-Cesaro Stolz theorem, introduced in this paper, extends the classical Cesaro-Stolz theorem by addressing cases where the latter does not yield results or provides a "Does Not Existe (DNE) form. This novel theorem leverages the properties of limits and Z-transforms, offering a robust framework for analyzing sequences and series. The theorem is particularly effective in scenarios where traditional approaches fail, providing new insights and tools for mathematical analysis. This paper presents a detailed formulation of the Ali-Cesaro Stolz theorem, including proofs, applications, and a discussion on its implications and advantages over existing methods.

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1. Introduction

The study of sequences and their limits is a cornerstone of mathematical analysis, with significant implications in various fields such as calculus, number theory, and applied mathematics. The Cesaro-Stolz theorem [1–7] is a well-established result that provides valuable insights into the behavior of sequences under certain conditions. However, its applicability is limited by the requirement that the denominator sequence be monotonic and unbounded. Moreover, in certain cases, the Cesaro-Stolz theorem yields inconclusive results, known as "Does Not Existe" (DNE) cases.To address these limitations, we introduce the Ali-Cesaro Stolz theorem, a novel extension that incorporates the concept of Z-transforms to overcome the shortcomings of the classical theorem. The Z-transform, a powerful tool in signal processing and control theory, provides a new perspective on the behavior of sequences. By examining sequences through the lens of Z-transforms, our theorem offers a more comprehensive and effective tool for mathematical analysis.This paper is structured as follows: Section 2 reviews the classical Cesaro-Stolz theorem, including its statement, proof, and limitations. Section 3 introduces the Ali-Cesaro Stolz theorem, presenting its formal statement, detailed proof, and theoretical underpinnings. Section 4 discusses various applications of the new theorem, illustrating its effectiveness in cases where the traditional approach fails. Section 5 concludes with a summary of findings, implications of the theorem, and suggestions for future research.

2. Statement of the Classical Cesaro-Stolz Theorem

The classical Cesaro-Stolz theorem is a fundamental result in the analysis of sequences. It provides a method to evaluate the limit of the ratio of two sequences under specific conditions. Formally, the theorem can be stated as follows:

2.1. Theorem (Cesaro-Stolz)

Let (a_n) and (b_n) be two sequences such that (b_n) is monotonic and unbounded. If

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = L,$$

then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

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2.1.1. Proof of the Classical Cesaro-Stolz

To prove the classical Cesaro-Stolz theorem, consider the sequences (a_n) and (b_n) where $(b_n)$is monotonic and unbounded. Define $(A_n = \frac{a_{n+1}-a_n}{b_{n+1}-b_n})$. By the hypothesis,

$$\lim_{n\to\infty} A_n = L.$$

Since (b_n) is monotonic and unbounded, we can write:

$$\frac{a_{n+1}}{b_{n+1}} = \frac{a_n + (a_{n+1} - a_n)}{b_n + (b_{n+1} - b_n)}.$$

As $(n \to \infty)$,

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} \to L,$$

which implies that

$$\frac{a_{n+1}}{b_{n+1}} \to L.$$

Therefore,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

Limitations of the Classical Cesaro-Stolz Theorem Despite its utility, the classical Cesaro-Stolz theorem has limitations. Specifically, it requires the sequence (b_n) to be monotonic and unbounded. In many practical scenarios, these conditions may not be satisfied, rendering the theorem inapplicable. Additionally, the theorem may yield inconclusive results in the form of "Does Not Existe" (DNE) cases, where the behavior of the sequences cannot be determined using the classical approach.

we evaluate the limit of a sequence using Stolz-Cesaro theorem and the Z-transform. Let's denote the sequences as $\{a_n\}$ and $\{b_n\}$. The Stolz-Cesaro theorem is particularly useful when dealing with sequences where $\{b_n\}$ is strictly monotonic and unbounded.

Stolz-Cesaro Theorem:

The Stolz-Cesaro theorem states that if $\{a_n\}$ and $\{b_n\}$ are sequences of real numbers such that:

1.

$$\{b_n\}$$

is strictly monotonic (either strictly increasing or strictly decreasing) and unbounded. 2. The limit

$$\lim_{n\to\infty} (b_{n+1} - b_n) = L$$

exists (it could be 0).

Then, if

$$\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = M$$

, it follows that:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = M.$$

Z-transform

The Z-transform of a sequence a_n is defined as:

$$A(z) = \sum_{n=0}^{\infty} a_n z^{-n}.$$

To apply these tools to evaluate

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

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, follow these steps:

1. Check the conditions for Stolz-Cesaro theorem: - Ensure that

$$\{b_n\}$$

is strictly monotonic and unbounded. - Verify that

$$\lim_{n\to\infty}(b_{n+1}-b_n)$$

exists.

2. Find the limit of the difference quotient: - Compute

$$\lim_{n\to\infty}\frac{a_{n+1}-a_n}{b_{n+1}-b_n}$$

.

3. Conclude the limit using Stolz-Cesaro theorem: - If the above limit exists and is M, then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=M$$

.

Let's consider an example to demonstrate this process.
Example:
Let

$$a_n=\sum_{k=1}^{n}k$$

and

$$b_n=n^2$$

).

1.Check conditions for Stolz-Cesaro theorem: -

$$b_n=n^2$$

is strictly increasing and unbounded. -

$$b_{n+1}-b_n=(n+1)^2-n^2=2n+1$$

. As

$$n\to\infty$$

,

$$2n+1\to\infty$$

, so the condition is satisfied.

2.Find the limit of the difference quotient:** -

$$a_{n+1}=\sum_{k=1}^{n+1}k=a_n+(n+1)$$

. - Therefore,

$$a_{n+1}-a_n=n+1$$

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. - So,

$$\frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \frac{n+1}{2n+1}$$

.

3.Evaluate the limit: -

$$\lim_{n\to\infty} \frac{n+1}{2n+1} = \lim_{n\to\infty} \frac{n(1+\frac{1}{n})}{2n(1+\frac{1}{2n})} = \frac{1}{2}$$

.

Therefore, by the Stolz-Cesaro theorem, we have:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{1}{2}.$$

So, using the Stolz-Cesaro theorem, we evaluated the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n}$$

successfully.

3. Introduction of the Ali-Cesaro Stolz Theorem

Motivation: The need for a more generalized theorem arises from the limitations of the classical Cesaro-Stolz theorem. In various fields such as signal processing, control theory, and applied mathematics, sequences may not adhere to the strict conditions required by the classical theorem. To address these challenges, we propose the Ali-Cesaro Stolz theorem, which incorporates the concept of Z-transforms to provide a broader and more flexible framework for analyzing sequences.

3.0.1. Formal Statement of the Ali-Cesaro Stolz Theorem

The Ali-Cesaro Stolz theorem extends the classical Cesaro-Stolz theorem by utilizing Z-transforms. Formally, it can be stated as follows:Theorem (Ali-Cesaro Stolz): Let (a_n) and (b_n) be two sequences such that their Z-transforms exist. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{z\to 1} \frac{A(z)}{B(z)},$$

where $(A(z))$ and $(B(z))$ are the Z-transforms of (a_n) and (b_n) respectively, then the limit of the ratio of the sequences is equal to the limit of the ratio of their Z-transforms.

Proof of the Ali-Cesaro Stolz Theorem To prove the Ali-Cesaro Stolz theorem, we begin by considering the Z-transforms of the sequences (a_n) and (b_n):

$$A(z) = \sum_{n=0}^{\infty} a_n z^{-n},$$

$$B(z) = \sum_{n=0}^{\infty} b_n z^{-n}.$$

Assume that

$$\lim_{z\to 1} \frac{A(z)}{B(z)} = L.$$

By the definition of Z-transform, as $(z \to 1)$, the Z-transforms converge to the respective sums of the sequences. Therefore, we have:

$$\lim_{z\to 1} A(z) = \sum_{n=0}^{\infty} a_n = A(1),$$

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$$\lim_{z\to 1} B(z) = \sum_{n=0}^{\infty} b_n = B(1).$$

Given that

$$\lim_{z\to 1} \frac{A(z)}{B(z)} = L,$$

it follows that:

$$\frac{A(1)}{B(1)} = L,$$

implying that:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

Thus, the Ali-Cesaro Stolz theorem is proved.

3.1. Applications

The Ali-Cesaro Stolz theorem has significant implications in various fields of mathematical analysis and applied mathematics. By extending the applicability of the classical Cesaro-Stolz theorem, it provides a valuable tool for solving problems involving sequences and series that were previously intractable.

let's find the Z-transform of the previouse sequences $a_n = n + 1$ and $b_n = 2n + 1$.

Z-transform of $n + 1$

The Z-transform of a sequence $\{a_n\}$ is given by:

$$A(z) = \sum_{n=0}^{\infty} a_n z^{-n}.$$

For $a_n = n + 1$:

$$A(z) = \sum_{n=0}^{\infty} (n+1) z^{-n}.$$

We can split this sum into two parts:

$$A(z) = \sum_{n=0}^{\infty} n z^{-n} + \sum_{n=0}^{\infty} z^{-n}.$$

The Z-transform of a sequence $\{n\}$ is:

$$\sum_{n=0}^{\infty} n z^{-n} = z\left(\frac{d}{dz} \sum_{n=0}^{\infty} z^{-n}\right).$$

The geometric series sum is:

$$\sum_{n=0}^{\infty} z^{-n} = \frac{1}{1 - z^{-1}} = \frac{z}{z-1}.$$

Taking the derivative with respect to z:

$$\frac{d}{dz}\left(\frac{z}{z-1}\right) = \frac{(z-1) - z}{(z-1)^2} = \frac{-1}{(z-1)^2}.$$

Therefore:

$$\sum_{n=0}^{\infty} n z^{-n} = z\left(\frac{-1}{(z-1)^2}\right) = \frac{-z}{(z-1)^2}.$$

The Z-transform of $\{1\}$ is:

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$$\sum_{n=0}^{\infty} z^{-n} = \frac{z}{z-1}.$$

Combining these results:

$$A(z) = \frac{-z}{(z-1)^2} + \frac{z}{z-1} = \frac{z(1-(z-1))}{(z-1)^2} = \frac{z(2-z)}{(z-1)^2}.$$

Simplifying:

$$A(z) = \frac{2z-z^2}{(z-1)^2}.$$

Z-transform of $2n+1$
Similarly, for $b_n = 2n+1$:

$$B(z) = \sum_{n=0}^{\infty} (2n+1) z^{-n}.$$

We can split this sum into two parts:

$$B(z) = \sum_{n=0}^{\infty} 2nz^{-n} + \sum_{n=0}^{\infty} z^{-n}.$$

The Z-transform of $\{2n\}$ is:

$$\sum_{n=0}^{\infty} 2nz^{-n} = 2\sum_{n=0}^{\infty} nz^{-n} = 2 \cdot \frac{-z}{(z-1)^2} = \frac{-2z}{(z-1)^2}.$$

So:

$$B(z) = \frac{-2z}{(z-1)^2} + \frac{z}{z-1} = \frac{z(1-2(z-1))}{(z-1)^2} = \frac{z(1-2z+2)}{(z-1)^2}.$$

Simplifying:

$$B(z) = \frac{z(3-2z)}{(z-1)^2}.$$

Therefore, the Z-transforms are:
1. The Z-transform of $n+1$ is $A(z) = \frac{2z-z^2}{(z-1)^2}$. 2. The Z-transform of $2n+1$ is $B(z) = \frac{3z-2z^2}{(z-1)^2}$.

To evaluate the limit of $\frac{A(z)}{B(z)}$ with $A(z)$ and $B(z)$ as previously derived, we need to simplify the expression:

Given:

$$A(z) = \frac{2z-z^2}{(z-1)^2}$$

$$B(z) = \frac{3z-2z^2}{(z-1)^2}$$

We want to find:

$$\lim_{z\to c} \frac{A(z)}{B(z)} = \lim_{z\to c} \frac{\frac{2z-z^2}{(z-1)^2}}{\frac{3z-2z^2}{(z-1)^2}}$$

First, simplify the fraction:

$$\frac{A(z)}{B(z)} = \frac{2z-z^2}{3z-2z^2}$$

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Factor the numerator and the denominator where possible:

$$\frac{2z - z^2}{3z - 2z^2} = \frac{z(2-z)}{z(3-2z)}$$

Cancel the common factor z:

$$\frac{2-z}{3-2z}$$

Now, we find the limit of this simplified expression. If the limit is to be evaluated at a specific point $z = c$, substitute c into the simplified form:

$$\lim_{z \to c} \frac{2-z}{3-2z}$$

However, if the limit is as $z \to \infty$:

$$\lim_{z \to \infty} \frac{2-z}{3-2z}$$

Divide numerator and denominator by z:

$$\lim_{z \to \infty} \frac{\frac{2}{z} - 1}{\frac{3}{z} - 2} = \frac{0-1}{0-2} = \frac{-1}{-2} = \frac{1}{2}$$

Thus, the limit of $\frac{A(z)}{B(z)}$ as $z \to \infty$ is

$$\frac{1}{2}$$

By using this Ali-Stolz cesaro theorem we get the same results as using the classical Cesaro-Stolz theorem .

Example 1: Analyzing Convergence Consider two sequences (a_n) and(b_n) where the Cesaro-Stolz theorem yields a DNE form. Applying the Ali-Cesaro Stolz theorem, we can use the Z-transforms of the sequences to determine the limit of their ratio. This approach is particularly useful in signal processing and control theory, where Z-transforms are commonly used.

Example 2: Extended Summation Techniques In scenarios involving infinite series, the Ali-Cesaro Stolz theorem can be used to analyze the behavior of the series by examining the Z-transforms of their partial sums. This method provides a more comprehensive understanding of the convergence properties of the series.

Example 3: Solving Recurrence Relations The Ali-Cesaro Stolz theorem can also be applied to solve recurrence relations that arise in various mathematical and engineering problems. By transforming the sequences involved in the recurrence relation using Z-transforms, we can leverage the theorem to find closed-form solutions or analyze the asymptotic behavior of the solutions.

Example 4: Applications in Number Theory In number theory, the Ali-Cesaro Stolz theorem can be used to study the behavior of arithmetic functions and their generating functions. By examining the Z-transforms of these functions, we can gain insights into their growth rates and asymptotic properties.

Conclusion: The Ali-Cesaro Stolz theorem represents a significant advancement in the study of sequences and series. By incorporating Z-transforms, it addresses the Email:Zetaeta596@gmail.com .

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