Short Note

Not peer-reviewed version

Functional van Lint-Seidel Relative and Gerzon Universal Bounds

K. Mahesh Krishna *

Posted Date: 12 August 2024

doi: 10.20944/preprints202408.0799.v1

Keywords: Equiangular lines.

doi:10.20944/preprints202408.0799.v1

Short Note

Functional van Lint-Seidel Relative and Gerzon Universal Bounds

K. Mahesh Krishna

School of Mathematics and Natural Sciences, Chanakya University Global Campus, NH-648, Haraluru Village, Devanahalli Taluk, Bengaluru Rural District, Karnataka State 562 110 India, kmaheshak@gmail.com

Abstract: We introduce the notion of functional equiangular lines in finite rank modules over subrings of $\mathbb{R}$. We show that van Lint-Seidel relative and Gerzon universal bounds hold for this most general lines.

Keywords: equiangular lines

MSC: 52A40; 05C50

---

1. Introduction

Let $d \in \mathbb{N}$ and $\gamma \in [0,1]$. Recall that a collection $\{\tau_j\}_{j=1}^n$ of unit vectors in $\mathbb{R}^d$ is said to be γ**-equiangular lines** [1,2] if

$$|\langle \tau_j, \tau_k \rangle| = \gamma, \quad \forall 1 \leq j,k \leq n, j \neq k.$$

A fundamental problem associated with equiangular lines is the following.

Problem 1.1. *Given $d \in \mathbb{N}$ and $\gamma \in [0,1]$, what is the upper bound on n such that there exists a collection $\{\tau_j\}_{j=1}^n$ of γ-equiangular lines in $\mathbb{R}^d$?*

Two answers to Problem (1.1) which are driving forces in the study of equiangular lines is the following relative bound of van Lint and Seidel [2,3] and universal bound of Gerzon [4].

Theorem 1.2. *[2,3] (**van Lint-Seidel Relative Bound**) Let $\{\tau_j\}_{j=1}^n$ be γ-equiangular lines in $\mathbb{R}^d$. Then*

$$n(1-d\gamma^2) \leq d(1-\gamma^2).$$

In particular, if

$$\gamma < \frac{1}{\sqrt{d}},$$

then

$$n \leq \frac{d(1-\gamma^2)}{1-d\gamma^2}.$$

Theorem 1.3. *[4] (**Gerzon Universal Bound**) Let $\{\tau_j\}_{j=1}^n$ be γ-equiangular lines in $\mathbb{R}^d$. Then*

$$n \leq \frac{d(d+1)}{2}.$$

The notion of functional equiangular lines is hinted in [5]. In this paper, we define it in most general form and derive functional forms of Theorems 1.2 and 1.3.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 August 2024 doi:10.20944/preprints202408.0799.v1

2. Functional Equiangular Lines

In the paper, $\mathcal{R}$ denotes a subring of $\mathbb{R}$ and $\mathcal{M}$ is a rank d module over $\mathcal{R}$. By $\mathcal{M}^*$ we mean the module of all homomorphisms from $\mathcal{M}$ to $\mathcal{R}$. Module of all homomorphisms from $\mathcal{M}$ to $\mathcal{M}$ is denoted by $\operatorname{Mor}(\mathcal{M})$.

Definition 2.1. *Let $\{\tau_j\}_{j=1}^n$ be a collection in a module $\mathcal{M}$ of rank d over $\mathcal{R}$ and $\{f_j\}_{j=1}^n$ be a collection in $\mathcal{M}^*$. Let $\gamma \geq 0$. The pair $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ is said to be* ***functional γ-equiangular*** *if following conditions hold.*

(i) $f_j(\tau_j) = 1$ *for all* $1 \leq j \leq n$.
(ii) $|f_j(\tau_k) f_k(\tau_j)| = \gamma^2$ *for all* $1 \leq j,k \leq n, j \neq k$.
(iii) The operator

$$S_{f,\tau} : \mathcal{M} \ni x \mapsto \sum_{j=1}^{n} f_j(x)\tau_j \in \mathcal{M}$$

is similar (through invertible operator) to a diagonal operator.

Theorem 2.2. *(**Functional van Lint-Seidel Relative Bound**) If $(\{f_j\}_{j=1}^n, \{\tau_j\}_{j=1}^n)$ is functional γ-equiangular for rank d module $\mathcal{M}$, then*

$$n(1 - d\gamma^2) \leq d(1 - \gamma^2).$$

In particular, if

$$\gamma < \frac{1}{\sqrt{d}},$$

then

$$n \leq \frac{d(1-\gamma^2)}{1 - d\gamma^2}.$$

Proof. Define

$$S_{f,\tau} : \mathcal{M} \ni x \mapsto S_{f,\tau}x := \sum_{j=1}^{n} f_j(x)\tau_j \in \mathcal{M}.$$

Since $S_{f,\tau}$ is diagonalizable,

$$\begin{aligned} n^2 &= \left(\sum_{j=1}^{n} f_j(\tau_j)\right)^2 = (\operatorname{Tra}(S_{f,\tau}))^2 = \left(\sum_{k=1}^{d} \lambda_k\right)^2 \\ &\leq d\sum_{k=1}^{d} \lambda_k^2 = d\operatorname{Tra}(S_{f,\tau}^2) = d\sum_{j=1}^{n}\sum_{k=1}^{n} f_j(\tau_k)f_k(\tau_j) \\ &= d\sum_{j=1}^{n} f_j(\tau_j)^2 + d\sum_{j,k=1,j\neq k}^{n} f_j(\tau_k)f_k(\tau_j) = dn^2 + d\sum_{j,k=1,j\neq k}^{n} f_j(\tau_k)f_k(\tau_j) \\ &\leq dn^2 + d\sum_{j,k=1,j\neq k}^{n} |f_j(\tau_k)f_k(\tau_j)| = dn^2 + d(n^2-n)\gamma. \end{aligned}$$

Therefore

$$n \leq dn + d(n-1)\gamma.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 12 August 2024 doi:10.20944/preprints202408.0799.v1

□

We are unable to derive Gerzon bound for functional equiangular lines. However, we derive following Gerzon bound for functionals.

Theorem 2.3. (***Functional Gerzon Universal Bound***) *Let $\{\tau_j\}_{j=1}^n$ be a collection in a module $\mathcal{M}$ of rank d over $\mathcal{R}$ and $\{f_j\}_{j=1}^n$ be a collection in $\mathcal{M}^*$. Assume the following.*

(i) $f_j(\tau_j) = 1$ for all $1 \leq j \leq n$.
(ii) There is a $\gamma \neq 1$ such that $f_j(\tau_k)f_k(\tau_j) = \gamma^2$ for all $1 \leq j, k \leq n, j \neq k$.

Then

$$n \leq d^2.$$

Proof. For $1 \leq j \leq n$, define

$$\tau_j \otimes f_j : \mathcal{M} \ni x \mapsto (\tau_j \otimes f_j) := f_j(x)\tau_j \in \mathcal{M}$$

We show that the collection $\{\tau_j \otimes f_j\}_{j=1}^n$ in $\text{Mor}(\mathcal{M})$ is linearly independent over $\mathcal{R}$. Let $c_1, \ldots, c_n \in \mathcal{R}$ be such that

$$\sum_{j=1}^{n} c_j(\tau_j \otimes f_j) = 0.$$

Let $1 \leq k \leq n$ be fixed. Then previous equation gives

$$\sum_{j=1}^{n} c_j(\tau_j \otimes f_j)(tau_k \otimes f_k) = 0.$$

By taking trace we get

$$\begin{aligned}
0 &= \sum_{j=1}^{n} c_j \operatorname{Tra}((\tau_j \otimes f_j)(\tau_k \otimes f_k)) = \sum_{j=1}^{n} c_j f_j(\tau_k) f_k(\tau_j) \\
&= \sum_{j=1, j\neq k}^{n} c_j f_j(\tau_k) f_k(\tau_j) + c_k f_k(\tau_k)^2 = \sum_{j=1, j\neq k}^{n} c_j \gamma^2 + c_k \\
&= \gamma^2 \left(\sum_{j=1}^{n} c_j - c_k \right) + c_k = \gamma^2 \left(\sum_{j=1}^{n} c_j \right) + (1 - \gamma^2) c_k.
\end{aligned}$$

Therefore

$$c_k = \frac{\gamma^2}{\gamma^2 - 1} \sum_{j=1}^{n} c_j =: c, \quad \forall 1 \leq k \leq n.$$

Now

$$\begin{aligned}
0 &= \operatorname{Tra}\left(\sum_{j=1}^{n} c_j(\tau_j \otimes f_j) \right) = \operatorname{Tra}\left(\sum_{j=1}^{n} c(\tau_j \otimes f_j) \right) \\
&= \sum_{j=1}^{n} c \operatorname{Tra}(\tau_j \otimes f_j) = \sum_{j=1}^{n} c f_j(\tau_j) = cn.
\end{aligned}$$

Hence $c = 0$. Therefore $\{\tau_j \otimes f_j\}_{j=1}^n$ is linearly independent. Since the rank of $\text{Mor}(\mathcal{M})$ is d^2 we must have $n \leq d^2$. □

 doi:10.20944/preprints202408.0799.v1

References

1. Haantjes, J. Equilateral point-sets in elliptic two- and three-dimensional spaces. *Nieuw Arch. Wiskunde (2)* **1948**, *22*, 355–362.
2. Lemmens, P.W.H.; Seidel, J.J. Equiangular lines. *J. Algebra* **1973**, *24*, 494–512. doi:10.1016/0021-8693(73)90123-3.
3. van Lint, J.H.; Seidel, J.J. Equilateral point sets in elliptic geometry. *Indag. Math.* **1966**, *28*, 335–348.
4. Waldron, S.F.D. *An introduction to finite tight frames*; Applied and Numerical Harmonic Analysis, Birkhäuser/Springer, New York, 2018; pp. xx+587. doi:10.1007/978-0-8176-4815-2.
5. Krishna, K.M. Discrete and continuous Welch bounds for Banach spaces with applications. *J. Class. Anal.* **2023**, *22*, 81–111. doi:10.7153/jca-2023-22-07.