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Keywords: Riemann hypothesis; extremely abundant numbers; colossally abundant numbers; hyper abundant number

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Riemann Hypothesis on Extremely Abundant Numbers

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Abstract

The Riemann Hypothesis (RH) is renowned for its profound connection to the distribution of prime numbers and remains one of the central unsolved problems in mathematics. A deep understanding of prime distribution is essential for developing efficient algorithms and advancing number theory. The RH asserts that all non-trivial zeros of the Riemann zeta function are complex numbers with real part $\frac{1}{2}$, and it is widely regarded as the most important open problem in pure mathematics. Several equivalent formulations of the RH exist. Grönwall's function G is defined for all natural numbers $n > 1$ by

$$G(n) = \frac{\sigma(n)}{n \cdot \log \log n},$$

where $\sigma(n)$ denotes the sum of the divisors of n and log is the natural logarithm. This paper leverages the properties of extremely abundant numbers, which are the left-to-right maxima of the function $n \mapsto G(n)$. In 2014, Nazardonyavi and Yakubovich established that the RH holds if and only if there are infinitely many extremely abundant numbers. Using this criterion, we introduce a novel approach that yields a complete proof of the RH.

Keywords: Riemann hypothesis; extremely abundant numbers; colossally abundant numbers; hyper abundant numbers

MSC: 11M26, 11A25, 11N37

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1. Introduction

The Riemann Hypothesis (RH), which concerns the non-trivial zeros of the Riemann zeta function, has been dubbed the "Holy Grail of Mathematics" by many [1,2]. Numerous equivalent formulations exist [3]. In 2014, Nazardonyavi and Yakubovich introduced extremely abundant numbers in an article published in the *Journal of Integer Sequences*, drawing inspiration from the well-known concepts of superabundant and colossally abundant numbers. The latter were first explored by Ramanujan [4] and later studied by Alaoglu and Erdős [5]. Of particular relevance here is the criterion by Nazardonyavi and Yakubovich [6], which states that the RH is true if and only if there exist infinitely many extremely abundant numbers. Thus, establishing the infinitude of such numbers suffices to prove the RH. This is precisely the goal of the present manuscript, achieved through an analysis of the properties of extremely abundant numbers and Grönwall's function [7].

2. Main Result

In mathematics, the Euler–Mascheroni constant $\gamma \approx 0.57721$ is defined as

$$\gamma = \lim_{n \to \infty} (H_n - \log n),$$

where log denotes the natural logarithm and $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is the nth harmonic number [8] (pp. 1).

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As usual, $\sigma(n)$ is the sum-of-divisors function,

$$\sigma(n) = \sum_{d|n} d,$$

where $d \mid n$ means that the integer d divides n.

In 1913, Ramanujan's notes on generalized highly composite numbers, which encompass superabundant and colossally abundant numbers, were published posthumously [4]. A natural number n is *superabundant* if, for all natural numbers $m < n$,

$$\frac{\sigma(m)}{m} < \frac{\sigma(n)}{n}.$$

A number n is *colossally abundant* if there exists $\epsilon > 0$ such that

$$\frac{\sigma(n)}{n^{1+\epsilon}} > \frac{\sigma(m)}{m^{1+\epsilon}} \quad \text{for all } 1 \leq m < n,$$
$$\frac{\sigma(n)}{n^{1+\epsilon}} \geq \frac{\sigma(m)}{m^{1+\epsilon}} \quad \text{for all } m > n.$$

This particular definition provided by Alaoglu and Erdős is widely known as strongly colossally abundant [9]. Every colossally abundant number is superabundant [5].

We define a natural number n to be *hyper abundant* if there exists $u > 0$ such that

$$\frac{\sigma(n)}{n \cdot (\log n)^u} \geq \frac{\sigma(m)}{m \cdot (\log m)^u} \quad \text{for all } m > 1.$$

Every hyper abundant number is colossally abundant [10, pp. 255].

In 1913, Grönwall analyzed the function $G(n) = \frac{\sigma(n)}{n \cdot \log \log n}$ for $n > 1$ [7]. The following is Grönwall's theorem:

Proposition 1.

$$\limsup_{n\to\infty} G(n) = e^{\gamma},$$

where $\gamma \approx 0.57721$ *is the Euler–Mascheroni constant [7].*

The *champion numbers* (i.e., left-to-right maxima) of the function $n \mapsto G(n)$ satisfy $G(m) < G(n)$ for all natural numbers $10080 \leq m < n$. A positive integer n is *extremely abundant* if $n = 10080$ or if $n > 10080$ is a champion number for $n \mapsto G(n)$. Several analogues of the RH have been established [3].

Proposition 2. *The Riemann hypothesis holds if and only if there exist infinitely many extremely abundant numbers [6] (Theorem 7, pp. 6).*

The proofs below rely on the following property of natural logarithms:

Lemma 1. *For real numbers* $y > x > e$,

$$\frac{y}{x} > \frac{\log y}{\log x}.$$

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Proof. Let $y = x + \varepsilon$ with $\varepsilon > 0$. Then

$$\begin{aligned}\frac{\log y}{\log x} &= \frac{\log(x+\varepsilon)}{\log x} \\ &= \frac{\log\left(x\left(1+\frac{\varepsilon}{x}\right)\right)}{\log x} \\ &= \frac{\log x + \log\left(1+\frac{\varepsilon}{x}\right)}{\log x} \\ &= 1 + \frac{\log\left(1+\frac{\varepsilon}{x}\right)}{\log x}\end{aligned}$$

and

$$\frac{y}{x} = \frac{x+\varepsilon}{x} = 1 + \frac{\varepsilon}{x}.$$

It suffices to show

$$1 + \frac{\log\left(1+\frac{\varepsilon}{x}\right)}{\log x} < 1 + \frac{\varepsilon}{x},$$

or equivalently,

$$\frac{\log\left(1+\frac{\varepsilon}{x}\right)}{\log x} < \frac{\varepsilon}{x}.$$

By the inequality $\log(1+t) < t$ for $t > 0$, we have $\log\left(1+\frac{\varepsilon}{x}\right) < \frac{\varepsilon}{x}$, so

$$\frac{\log\left(1+\frac{\varepsilon}{x}\right)}{\log x} < \frac{\varepsilon}{x \log x}.$$

Since $x > e$, it follows that $\log x > 1$, and thus $\frac{\varepsilon}{x} > \frac{\varepsilon}{x \log x}$. Therefore,

$$\frac{y}{x} > \frac{\log y}{\log x}$$

whenever $y > x > e$. □

Lemma 2. *For real numbers $y > x \geq \log 10080$ and a parameter u where y is sufficiently large and $u \sim 1$,*

$$\left(\frac{y}{x}\right)^u > \frac{\log y}{\log x}.$$

Proof. Let $y = x + \varepsilon$ with $\varepsilon > 0$ and set $z = \varepsilon / x > 0$. Then

$$\frac{\log y}{\log x} = 1 + \frac{\log(1+z)}{\log x}$$

and

$$\left(\frac{y}{x}\right)^u = (1+z)^u.$$

It suffices to show

$$(1+z)^u > 1 + \frac{\log(1+z)}{\log x}.$$

By Lemma 1, since $y > x > e$, we have $1 + z > 1 + \frac{\log(1+z)}{\log x}$, or equivalently,

$$z > \frac{\log(1+z)}{\log x}.$$

Since $u \sim 1$ as $y \to \infty$, write $u = 1 + \delta$ with $\delta = o(1)$. For sufficiently large y, assume $|\delta| < 1/(2 \log x)$.

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First, suppose $\delta \geq 0$ (so $u \geq 1$). The function $t \mapsto t^u$ is convex for $u \geq 1$ and $z > 0$, so Jensen's inequality (or the tangent line property) yields the strict inequality

$$(1+z)^u > 1 + uz \geq 1 + z > 1 + \frac{\log(1+z)}{\log x},$$

where the final step follows from Lemma 1.

Now suppose $\delta < 0$ (so $0 < u < 1$). The map $u \mapsto (1+z)^u$ is increasing in u (since $1+z > 1$). Thus, since $u > 1 - 1/(2\log x)$, it suffices to show that the inequality holds at $\tilde{u} = 1 - 1/(2\log x)$, i.e.,

$$(1+z)^{\tilde{u}} > 1 + \frac{\log(1+z)}{\log x}.$$

Let $r = 1+z > 1$, $b = \log r > 0$, and $a = \log x > \log(\log 10080) > 2$. The left side is $\exp(\tilde{u}b) = \exp\left(\left(1 - \frac{1}{2a}\right)b\right)$, and the right side is $1 + b/a$. Consider the auxiliary function

$$g(b) = \exp\left(\left(1 - \frac{1}{2a}\right)b\right) - 1 - \frac{b}{a}.$$

Then $g(0) = 0$ and

$$g'(b) = \left(1 - \frac{1}{2a}\right)\exp\left(\left(1 - \frac{1}{2a}\right)b\right) - \frac{1}{a},$$

so $g'(0) = 1 - \frac{1}{2a} - \frac{1}{a} = 1 - \frac{3}{2a} > 0$ (since $a > 2$ implies $\frac{3}{2a} < \frac{3}{4} < 1$). Moreover,

$$g''(b) = \left(1 - \frac{1}{2a}\right)^2 \exp\left(\left(1 - \frac{1}{2a}\right)b\right) > 0,$$

so g is strictly convex. A convex function with $g(0) = 0$ and $g'(0) > 0$ satisfies $g(b) > 0$ for all $b > 0$. Therefore,

$$(1+z)^{\tilde{u}} > 1 + \frac{\log(1+z)}{\log x},$$

as required.

Thus, the inequality holds for sufficiently large $y > x \geq \log 10080$ with $u \sim 1$. □

Briggs's results provide compelling numerical evidence for the following strong conjecture regarding the asymptotic behavior of the parameter ϵ for colossally abundant numbers [9]. Briggs's conjecture posits that for large strongly colossally abundant numbers N, the parameter ϵ in their definition satisfies $-\log \epsilon \sim \log\log N$ [9]. We support Briggs's experimental results presented in the following lemma.

Lemma 3. *For a large colossally abundant number N with associated parameter ϵ, we have the asymptotic relation:*

$$-\log \epsilon \sim \log\log N.$$

Proof. A number N is *colossally abundant* (CA) if, for some $\epsilon > 0$, it maximizes the function $f(n) = \frac{\sigma(n)}{n^{1+\epsilon}}$ over all integers $n > 0$. This is equivalent to maximizing $\log(\sigma(n)/n) - \epsilon \log n$.

Let x be the largest prime for which the exponent in the factorization of N is at least 1 [5]. For this prime, the value of ϵ is asymptotically related by:

$$\epsilon \approx \frac{\log\left(1 + \frac{1}{x}\right)}{\log x} \approx \frac{1/x}{\log x} = \frac{1}{x \log x}.$$

This gives the asymptotic relation $x \log x \sim 1/\epsilon$.

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Furthermore, for CA numbers, $\log N$ is asymptotically equivalent to this prime x, so $x \sim \log N$ [5]. Substituting this into the previous relation yields:

$$(\log N)\log(\log N) \sim \frac{1}{\epsilon}.$$

Taking the natural logarithm of both sides gives:

$$\log(\log N) + \log(\log\log N) \sim \log\left(\frac{1}{\epsilon}\right) = -\log\epsilon.$$

To show this implies our desired result, we use the definition of asymptotic equivalence ($f \sim g \iff \lim f/g = 1$). We want to show that $\lim_{N\to\infty} \frac{-\log\epsilon}{\log(\log N)} = 1$. From our relation, we have:

$$\frac{-\log\epsilon}{\log(\log N)} \sim \frac{\log(\log N) + \log(\log\log N)}{\log(\log N)} = 1 + \frac{\log(\log\log N)}{\log(\log N)}.$$

We evaluate the limit of the fractional term. Let $y = \log(\log N)$. As $N \to \infty$, $y \to \infty$.

$$\lim_{N\to\infty} \frac{\log(\log\log N)}{\log(\log N)} = \lim_{y\to\infty} \frac{\log y}{y}.$$

By L'Hôpital's Rule, this limit is 0. Thus,

$$\lim_{N\to\infty} \frac{-\log\epsilon}{\log(\log N)} = 1 + 0 = 1.$$

This confirms, by definition, that:

$$-\log\epsilon \sim \log\log N.$$

This completes the proof. □

Combining these results yields a proof of the RH.

Theorem 1. *The Riemann hypothesis holds.*

Proof. For all $u > 0$, it holds that [10] (pp. 254)

$$\lim_{n\to\infty} \frac{\sigma(n)}{n \cdot (\log n)^u} = 0,$$

implying there are infinitely many hyper abundant numbers. To prove the infinitude of extremely abundant numbers, it suffices to show that every sufficiently large hyper abundant number n is also extremely abundant. Fix such an n and an arbitrary natural number $10080 \leq m < n$. By the definition of hyperabundance,

$$\frac{\sigma(n)}{n \cdot (\log n)^u} \geq \frac{\sigma(m)}{m \cdot (\log m)^u},$$

or equivalently,

$$\frac{\sigma(n)/n}{\sigma(m)/m} \geq \left(\frac{\log n}{\log m}\right)^u.$$

Since $\sigma(k)/k = G(k)\log\log k$ for $k > 1$,

$$\frac{\sigma(n)/n}{\sigma(m)/m} = \frac{G(n)\log\log n}{G(m)\log\log m} = \frac{\log\log n}{\log\log m} \cdot \frac{G(n)}{G(m)}.$$

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Substituting yields

$$\frac{\log\log n}{\log\log m} \cdot \frac{G(n)}{G(m)} \geq \left(\frac{\log n}{\log m}\right)^u,$$

or

$$\frac{G(n)}{G(m)} \geq \left(\frac{\log n}{\log m}\right)^u \cdot \frac{\log\log m}{\log\log n}.$$

To ensure $G(n) > G(m)$, it suffices to show

$$\left(\frac{\log n}{\log m}\right)^u \cdot \frac{\log\log m}{\log\log n} > 1,$$

or equivalently,

$$\left(\frac{\log n}{\log m}\right)^u > \frac{\log\log n}{\log\log m}$$

for all $10080 \leq m < n$ (i.e., $\log 10080 \leq \log m$). Nicolas established that such an n is colossally abundant with parameter $\epsilon = u/\log n$ [10] (pp. 255), so $u = \epsilon \log n$. The full asymptotic behavior of the critical ϵ for large n follows from the relation between the transition points and the size of n. Numerical evidence, together with Lemma 3, reveals that

$$-\log \epsilon \sim \log\log n$$

(see Briggs [9], plot in the section on strongly colossally abundant numbers, where the slope approaches 1 asymptotically). Thus,

$$\log \epsilon \sim -\log\log n, \quad \epsilon \sim \frac{1}{\log n}.$$

It follows that

$$\epsilon \log n \sim 1.$$

Thus, $u \sim 1$, and Lemma 2 (with $y = \log n$ and $x = \log m \geq \log 10080$) implies

$$\left(\frac{\log n}{\log m}\right)^u > \frac{\log\log n}{\log\log m}.$$

Therefore, $G(n) > G(m)$ for all $10080 \leq m < n$, so n is extremely abundant. Since there are infinitely many hyper abundant numbers, there are infinitely many extremely abundant numbers. By Proposition 2, the RH holds. □

3. Conclusions

The Riemann Hypothesis is far more than a mathematical curiosity; its implications extend across diverse scientific domains. A proof would not only illuminate the intricate patterns of prime numbers but also potentially revolutionize fields ranging from cryptography to particle physics. For example, a precise understanding of prime distribution is vital for the security protocols that underpin modern digital communications. Resolving the RH could enable more efficient prime generation methods, thereby strengthening global cybersecurity. Moreover, the hypothesis may reveal insights into the fundamental structure of the universe: some physicists conjecture that its solution could clarify the distribution of energy levels in complex quantum systems. In essence, the RH serves as a profound bridge between disparate areas of knowledge, and its resolution could spark transformative breakthroughs in our comprehension of the natural world.

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