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*Article*

# Robin's Criterion on Superabundant Numbers

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**Abstract:** The Riemann hypothesis is the assertion that all non-trivial zeros are complex numbers with real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. There are several statements equivalent to the famous Riemann hypothesis. Robin's criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \cdot n \cdot \log \log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of $n$, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. We require the properties of superabundant numbers, that is to say left to right maxima of $n \mapsto \frac{\sigma(n)}{n}$. In this note, using Robin's criterion on superabundant numbers, we prove that the Riemann hypothesis is true.

**Keywords:** riemann hypothesis; Robin's criterion; superabundant numbers; prime numbers

**MSC:** 11M26; 11A41; 11A25

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## 1. Introduction

The Riemann hypothesis stands as one of the most formidable and celebrated unsolved problems in the realm of pure mathematics. Enunciated by Bernhard Riemann in 1859, it centers on the behavior of the Riemann zeta function $\zeta(s)$, a complex-valued function of profound significance in number theory. At its core, the hypothesis posits that all non-trivial zeros of the Riemann zeta function lie on a specific line in the complex plane. These non-trivial zeros, points where the function equals zero, are of paramount importance due to their intricate connection to the distribution of prime numbers, those fundamental building blocks of arithmetic.

Prime numbers, as the indivisible integers greater than one, exhibit an erratic and seemingly random pattern. The Riemann hypothesis offers a tantalizing prospect: a potential formula to predict the distribution of these primes with remarkable accuracy. Such a breakthrough would have far-reaching implications, not only for number theory but also for fields as diverse as cryptography, computer science, and physics. The hypothesis has captivated mathematicians for centuries, inspiring countless attempts at proof and generating a wealth of related research [1]. Its resolution would undoubtedly mark a monumental achievement in human understanding of the structure of numbers and the laws governing the universe.

Beyond its theoretical elegance, the Riemann hypothesis carries a substantial monetary reward. It is one of the Clay Mathematics Institute's Millennium Prize Problems, each offering a million-dollar prize for a correct solution. This financial incentive, combined with the problem's intrinsic allure, has fueled intense competition and innovation among mathematicians worldwide. While progress has been made in understanding the properties of the Riemann zeta function and its zeros, a definitive proof of the hypothesis remains elusive. Yet, the pursuit of this elusive goal continues to drive mathematical discovery and push the boundaries of human knowledge. We provide a solution to this problem based on the properties of the superabundant numbers over the Robin's criterion.

## 2. Background and Ancillary Results

In mathematics, the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant which is defined as

$$\gamma = \lim_{n \to \infty} (H_n - \log n)$$

where log is the natural logarithm and $H_n = \sum_{k=1}^{n} \frac{1}{k}$ is called the $n^{th}$ harmonic number [2, pp. 1]. The following property is based on this constant:

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**Proposition 1.** *By the Euler-Maclaurin formula,*

$$H_n = \log n + \gamma + \frac{1}{2 \cdot n} - \varepsilon_n$$

*where* $0 \leq \varepsilon_n \leq \frac{1}{8 \cdot n^2}$ *which approaches 0 as n goes to infinity [3].*

The following inequalities are based on natural logarithms:

**Proposition 2.** *For* $t > 0$ *[4]:*

$$\frac{1}{t + 0.5} < \log\left(1 + \frac{1}{t}\right).$$

**Proposition 3.** *For* $x > -1$ *[5, pp. 1]:*

$$\log(1 + x) \leq x.$$

As usual $\sigma(n)$ is the sum-of-divisors function of $n$

$$\sum_{d \mid n} d,$$

where $d \mid n$ means the integer $d$ divides $n$. Define $f(n)$ as $\frac{\sigma(n)}{n}$.

**Proposition 4.** *Let* $\prod_{i=1}^{r} q_i^{a_i}$ *be the representation of n as a product of prime numbers* $q_1 < \ldots < q_r$ *with natural numbers* $a_1, \ldots, a_r$ *as exponents. Then [6, Lemma 1 pp. 2],*

$$f(n) = \left(\prod_{i=1}^{r} \frac{q_i}{q_i - 1}\right) \cdot \prod_{i=1}^{r} \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

**Definition 1.** *We say that* $\mathsf{Robin}(n)$ *holds provided that*

$$f(n) < e^{\gamma} \cdot \log \log n.$$

The Ramanujan's Theorem stated that if the Riemann hypothesis is true, then $\mathsf{Robin}(n)$ holds for large enough $n$ [7]. Next, we have the Robin's Theorem:

**Proposition 5.** $\mathsf{Robin}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann hypothesis is true [8, Theorem 1 pp. 188].*

In 1997, Ramanujan's old notes were published where he defined the generalized highly composite numbers, which include the superabundant and colossally abundant numbers [7]. Superabundant numbers were also studied by Leonidas Alaoglu and Paul Erdős (1944) [9]. Let $q_1 = 2, q_2 = 3, \ldots, q_k$ denote the first $k$ consecutive primes, then an integer of the form $\prod_{i=1}^{k} q_i^{a_i}$ with $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ is called a Hardy-Ramanujan integer [10, pp. 367]. A natural number $n$ is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

We know the following property for the superabundant numbers:

**Proposition 6.** *If n is superabundant, then n is a Hardy-Ramanujan integer [9, Theorem 1 pp. 450].*

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In number theory, the *p-adic* order of an integer $n$ is the exponent of the highest power of the prime number $p$ that divides $n$. It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which $p$ appears in the prime factorization of $n$.

**Proposition 7.** *If $n$ is superabundant and $q$ is the largest prime factor of $n$, then $\nu_q(n) = 1$, except when $n \in \{4, 36\}$ [9, Theorem 3 pp. 450].*

Several analogues of the Riemann hypothesis have already been proved. Many authors expect (or at least hope) that it is true. However, there are some implications in case of the Riemann hypothesis could be false.

**Proposition 8.** *If $n > 5040$ is the smallest integer such that $\mathsf{Robin}(n)$ does not hold, then $n$ must be a superabundant number [11, Theorem 3 pp. 273].*

**Proposition 9.** *If $n > 5040$ is the smallest integer such that $\mathsf{Robin}(n)$ does not hold, then $q < \log n$ where $q$ is the largest prime factor of $n$ [10, Lemma 6.1 pp. 369].*

**Proposition 10.** *If $n > 5040$ is the smallest integer such that $\mathsf{Robin}(n)$ does not hold, then $q > e^{31.018189471}$ where $q$ is the largest prime factor of $n$ [12, Theorem 4.2 pp. 748].*

Putting all together yields the proof of the Riemann hypothesis.

## 3. Main Result

This is a trivial result.

**Lemma 1.** *For every prime $q$, the inequality*

$$\frac{1}{2 \cdot q} + \frac{1}{2 \cdot (q+1)} + \frac{1}{8 \cdot (q+1)^2} < \frac{1}{q} - \frac{3}{8 \cdot (q+1)^2}$$

*holds.*

**Proof.** The inequality

$$\frac{1}{2 \cdot q} + \frac{1}{2 \cdot (q+1)} + \frac{1}{8 \cdot (q+1)^2} < \frac{1}{q} - \frac{3}{8 \cdot (q+1)^2}$$

is the same as

$$\frac{1}{2 \cdot (q+1)^2} < \frac{1}{2 \cdot q \cdot (q+1)}$$

since

$$\frac{1}{q} - \frac{1}{2 \cdot q} - \frac{1}{2 \cdot (q+1)} = \frac{1}{2 \cdot q} - \frac{1}{2 \cdot (q+1)} = \frac{1}{2 \cdot q \cdot (q+1)}$$

and

$$\frac{1}{8 \cdot (q+1)^2} + \frac{3}{8 \cdot (q+1)^2} = \frac{4}{8 \cdot (q+1)^2} = \frac{1}{2 \cdot (q+1)^2}.$$

In this way, we obtain that

$$\frac{1}{(q+1)} < \frac{1}{q}$$

which is trivially true for every prime $q$. □

The following is a key Lemma.

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**Lemma 2.** *For every prime $q$, the inequalities*

$$\exp\left(-\frac{1}{2\cdot q}\right)\cdot\frac{\exp(H_q)}{q}\leq e^{\gamma}\leq\exp\left(\frac{1}{2\cdot(q+1)}+\frac{1}{8\cdot(q+1)^2}\right)\cdot\frac{\exp(H_q)}{q+1}$$

*hold.*

**Proof.** By Proposition 1, we have

$$H_q=\log q+\gamma+\frac{1}{2\cdot q}-\varepsilon_q$$

for every prime $q$ where $0\leq\varepsilon_q\leq\frac{1}{8\cdot q^2}$. Therefore, we have

$$H_q\leq\log q+\gamma+\frac{1}{2\cdot q}$$

and so,

$$H_q-\log q-\frac{1}{2\cdot q}\leq\gamma$$

which is

$$\exp\left(-\frac{1}{2\cdot q}\right)\cdot\frac{\exp(H_q)}{q}\leq e^{\gamma}$$

after of applying the exponentiation. By Proposition 1, we obtain

$$H_{q+1}=\log(q+1)+\gamma+\frac{1}{2\cdot(q+1)}-\varepsilon_{(q+1)}$$

for every prime $q$ where $0\leq\varepsilon_{(q+1)}\leq\frac{1}{8\cdot(q+1)^2}$. So, we could show

$$H_{q+1}\geq\log(q+1)+\gamma+\frac{1}{2\cdot(q+1)}-\frac{1}{8\cdot(q+1)^2}$$

and thus,

$$H_q+\frac{1}{q+1}-\log(q+1)-\frac{1}{2\cdot(q+1)}+\frac{1}{8\cdot(q+1)^2}\geq\gamma$$

which is

$$e^{\gamma}\leq\exp\left(\frac{1}{2\cdot(q+1)}+\frac{1}{8\cdot(q+1)^2}\right)\cdot\frac{\exp(H_q)}{q+1}$$

after of applying the exponentiation where $\frac{1}{q+1}-\frac{1}{2\cdot(q+1)}=\frac{1}{2\cdot(q+1)}$. □

This is the main insight.

**Lemma 3.** *Let $n>5040$ be the possible smallest integer such that $\mathsf{Robin}(n)$ does not hold. Let $q$ be the largest prime factor of $n$ such that $n=r\cdot q$. Then,*

$$\exp\left(\frac{1}{2\cdot q}+\frac{1}{2\cdot(q+1)}+\frac{1}{8\cdot(q+1)^2}\right)<\frac{\log\log n}{\log\log r}.$$

**Proof.** We know this number $n>5040$ must be a superabundant number by Proposition 8. By Proposition 6, let $\prod_{i=1}^{k}q_i^{a_i}$ be the representation of this superabundant number $n$ as the product of the

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first $k$ consecutive primes $q_1 < \ldots < q_k$ with the natural numbers $a_1 \geq a_2 \geq \ldots \geq a_k \geq 1$ as exponents. This follows as

$$\begin{aligned}\log\log r &= \log\log\left(\frac{n}{q_k}\right)\\ &= \log(\log(n) - \log q_k)\\ &= \log\left((\log(n)) \cdot \left(1 - \frac{\log q_k}{\log n}\right)\right)\\ &= \log\log(n) + \log\left(1 - \frac{\log q_k}{\log n}\right)\\ &\leq \log\log(n) - \frac{\log q_k}{\log n}\end{aligned}$$

since

$$\log\left(1 - \frac{\log q_k}{\log n}\right) \leq -\frac{\log q_k}{\log n}$$

by Proposition 3 because of

$$-\frac{\log q_k}{\log n} > -1$$

by Propositions 9 and 10. For that reason, we have

$$\frac{\log\log n}{\log\log r} \geq \frac{\log\log n}{\log\log(n) - \frac{\log q_k}{\log n}} = \frac{\frac{(\log n)\cdot\log\log(n)}{\log q_k}}{\frac{(\log n)\cdot\log\log(n)}{\log q_k} - 1}.$$

We know that

$$\begin{aligned}\log\left(\frac{\frac{(\log n)\cdot\log\log(n)}{\log q_k}}{\frac{(\log n)\cdot\log\log(n)}{\log q_k} - 1}\right) &= \log\left(1 + \frac{1}{\frac{(\log n)\cdot\log\log(n)}{\log q_k} - 1}\right)\\ &> \frac{1}{\frac{(\log n)\cdot\log\log(n)}{\log q_k} - 0.5}\\ &= \frac{\log q_k}{(\log n) \cdot \log\log(n) - 0.5 \cdot \log q_k}\end{aligned}$$

because of

$$\log\left(1 + \frac{1}{\frac{(\log n)\cdot\log\log(n)}{\log q_k} - 1}\right) > \frac{1}{\frac{(\log n)\cdot\log\log(n)}{\log q_k} - 0.5}$$

by Proposition 2 since

$$\frac{(\log n) \cdot \log\log(n)}{\log q_k} - 1 > \log(n) - 1 > 0$$

by Propositions 9 and 10. We arrive at:

$$\frac{\frac{(\log n)\cdot\log\log(n)}{\log q_k}}{\frac{(\log n)\cdot\log\log(n)}{\log q_k} - 1} > \exp\left(\frac{\log q_k}{(\log n) \cdot \log\log(n) - 0.5 \cdot \log q_k}\right).$$

Hence, it is enough to show that

$$\frac{\log q_k}{(\log n) \cdot \log\log(n) - 0.5 \cdot \log q_k} \geq \frac{1}{q_k} - \frac{3}{8 \cdot (q_k + 1)^2}$$

by Lemma 1. That is equivalent to

$$1 - \frac{(\log n) \cdot \log\log(n) - 1.5 \cdot \log q_k}{(\log n) \cdot \log\log(n) - 0.5 \cdot \log q_k} \geq \frac{1}{q_k} - \frac{3}{8 \cdot (q_k + 1)^2}$$

since

$$\begin{aligned}
&\frac{\log q_k}{(\log n) \cdot \log\log(n) - 0.5 \cdot \log q_k} \\
&= \frac{1.5 \cdot \log q_k - 0.5 \cdot \log q_k}{(\log n) \cdot \log\log(n) - 0.5 \cdot \log q_k} \\
&= \frac{-((\log n) \cdot \log\log(n) - 1.5 \cdot \log q_k) + (\log n) \cdot \log\log(n) - 0.5 \cdot \log q_k}{(\log n) \cdot \log\log(n) - 0.5 \cdot \log q_k} \\
&= 1 - \frac{(\log n) \cdot \log\log(n) - 1.5 \cdot \log q_k}{(\log n) \cdot \log\log(n) - 0.5 \cdot \log q_k}.
\end{aligned}$$

We only need to show that

$$1 - \frac{q_k \cdot \log(q_k) - 1.5 \cdot \log q_k}{q_k \cdot \log(q_k) - 0.5 \cdot \log q_k} \geq \frac{1}{q_k} - \frac{3}{8 \cdot (q_k + 1)^2}$$

on the basis that

$$1 - \frac{(\log n) \cdot \log\log(n) - 1.5 \cdot \log q_k}{(\log n) \cdot \log\log(n) - 0.5 \cdot \log q_k} \geq \frac{1}{q_k} - \frac{3}{8 \cdot (q_k + 1)^2}$$

and

$$\frac{(\log n) \cdot \log\log(n) - 1.5 \cdot \log q_k}{(\log n) \cdot \log\log(n) - 0.5 \cdot \log q_k} < \frac{q_k \cdot \log(q_k) - 1.5 \cdot \log q_k}{q_k \cdot \log(q_k) - 0.5 \cdot \log q_k}$$

by Proposition 9. This implies that

$$1 - \frac{2 \cdot q_k - 3}{2 \cdot q_k - 1} \geq \frac{1}{q_k} - \frac{3}{8 \cdot (q_k + 1)^2}$$

which is

$$(2 \cdot q_k - 1) - (2 \cdot q_k - 3) \geq \frac{2 \cdot q_k - 1}{q_k} - \frac{3 \cdot (2 \cdot q_k - 1)}{8 \cdot (q_k + 1)^2}$$

and

$$2 \geq 2 - \frac{1}{q_k} - \frac{3 \cdot (2 \cdot q_k - 1)}{8 \cdot (q_k + 1)^2}$$

due to

$$\frac{q_k \cdot \log(q_k) - 1.5 \cdot \log q_k}{q_k \cdot \log(q_k) - 0.5 \cdot \log q_k} = \frac{q_k - 1.5}{q_k - 0.5} = \frac{2 \cdot q_k - 3}{2 \cdot q_k - 1}$$

and

$$(2 \cdot q_k - 1) - (2 \cdot q_k - 3) = 2, \quad \frac{2 \cdot q_k - 1}{q_k} = 2 - \frac{1}{q_k}.$$

Since the inequality

$$2 \geq 2 - \frac{1}{q_k} - \frac{3 \cdot (2 \cdot q_k - 1)}{8 \cdot (q_k + 1)^2}$$

trivially holds, then the proof is done. □

This is the main theorem.

**Theorem 1.** *The Riemann hypothesis is true.*

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**Proof.** Let $n > 5040$ be the possible smallest integer such that $\mathsf{Robin}(n)$ does not hold. We know this number must be a superabundant number by Proposition 8. Let $q$ be the largest prime factor of $n$ such that $n = q \cdot r$. Under our assumption, we have

$$f(n) \geq e^{\gamma} \cdot \log \log n.$$

By Lemma 2, we have

$$\exp\left(-\frac{1}{2 \cdot q}\right) \cdot \frac{\exp(H_q)}{q} \leq e^{\gamma}$$

and so,

$$f(n) \cdot \exp\left(\frac{1}{2 \cdot q}\right) \geq \frac{\exp(H_q)}{q} \cdot \log \log n$$

after making a distribution. By Propositions 4 and 7, we deduce that

$$f(r) \cdot \frac{q+1}{q} \cdot \exp\left(\frac{1}{2 \cdot q}\right) \geq \frac{\exp(H_q)}{q} \cdot \log \log n.$$

where $f(q) = \frac{q+1}{q}$. That would be

$$f(r) \cdot \exp\left(\frac{1}{2 \cdot q}\right) \geq \frac{\exp(H_q)}{q+1} \cdot \log \log n.$$

By Lemma 2, we can see that

$$e^{\gamma} \cdot \exp\left(-\frac{1}{2 \cdot (q+1)} - \frac{1}{8 \cdot (q+1)^2}\right) \leq \frac{\exp(H_q)}{q+1}.$$

We arrive at:

$$f(r) \cdot \exp\left(\frac{1}{2 \cdot q} + \frac{1}{2 \cdot (q+1)} + \frac{1}{8 \cdot (q+1)^2}\right) \geq e^{\gamma} \cdot \log \log n$$

after making a distribution. We know that $\mathsf{Robin}(r)$ holds under the assumption that $n = q \cdot r$ and whenever $n > 5040$ would be the smallest integer such that $\mathsf{Robin}(n)$ does not hold. By our supposition, we get

$$(e^{\gamma} \cdot \log \log r) \cdot \exp\left(\frac{1}{2 \cdot q} + \frac{1}{2 \cdot (q+1)} + \frac{1}{8 \cdot (q+1)^2}\right) \geq e^{\gamma} \cdot \log \log n$$

which is

$$\exp\left(\frac{1}{2 \cdot q} + \frac{1}{2 \cdot (q+1)} + \frac{1}{8 \cdot (q+1)^2}\right) \geq \frac{\log \log n}{\log \log r}.$$

However, this contradicts the fact that

$$\exp\left(\frac{1}{2 \cdot q} + \frac{1}{2 \cdot (q+1)} + \frac{1}{8 \cdot (q+1)^2}\right) < \frac{\log \log n}{\log \log r}.$$

holds by Lemma 3. Consequently, we obtain a contradiction under the assumption of the existence of the smallest integer $n > 5040$ such that $\mathsf{Robin}(n)$ does not hold. By Reductio ad absurdum, we prove that the Riemann hypothesis is true according to Proposition 5. ☐

## 4. Conclusions

The Riemann hypothesis is far more than a mathematical curiosity. Its implications reverberate across diverse scientific domains. A proof would illuminate not only the mysterious patterns of prime numbers but could also revolutionize fields as disparate as cryptography and particle physics. For instance, understanding the precise distribution of primes is paramount for the security protocols

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underlying modern digital communication. A proven Riemann hypothesis could potentially unlock more efficient methods of prime number generation, bolstering the defenses of our digital world. Beyond this, the hypothesis may hold clues about the underlying structure of the universe. Some physicists believe that its resolution could shed light on the distribution of energy levels in complex systems, a fundamental question in their field. In essence, the Riemann hypothesis serves as a bridge connecting seemingly disparate areas of knowledge. Its solution could catalyze breakthroughs that reshape our understanding of the natural world.

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