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Posted Date: 4 August 2024

doi: 10.20944/preprints202408.0216.v1

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Article

Generalized Core-EP Inverse for Triangular Operator Matrices

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Abstract: We investigate the existence and representation of the generalized core-EP inverse of some triangular matrices over a Banach algebra. Further, the general representations of the generalized core-EP inverse of a triangular matrix over a C^*-algebra are presented. As applications, the generalized core-EP inverses of some block operator matrices over Hilbert spaces are given.

Keywords: core inverse; core-EP inverse; generalized core-EP inverse; triangular matrix; Block operator matrix; Banach algebra

MSC: 15A09; 16U90; 16W10

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1. Introduction

Let $\mathcal{A}$ be a Banach *-algebra. An element $a \in \mathcal{A}$ has Drazin inverse provided that there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, ax = xa, xa^{k+1} = a^k,$$

where k is the index of a (denoted by $ind(a)$), i.e., the smallest k such that the previous equations are satisfied. Such x is unique if exists, denoted by a^D, and called the Drazin inverse of a. We say that $a \in \mathcal{A}$ has group inverse x if $ind(a) = 1$, i.e., there exists a unique $x \in \mathcal{A}$ such that

$$ax^2 = x, ax = xa, xa^2 = a.$$

We denote the group inverse x by $a^\#$. Evidently, a square complex matrix A has group inverse if and only if $rank(A) = rank(A^2)$.

An element $a \in \mathcal{A}$ has core-EP inverse (i.e., pseudo core inverse) if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$ax^2 = x, (ax)^* = ax, xa^{k+1} = a^k,$$

where the smallest k is the index of a (denoted by $i(a)$). If such x exists, it is unique, and denote it by $a^{\textcircled{D}}$. We say that $a \in \mathcal{A}$ has core inverse x if $i(a) = 1$, i.e., there exists a unique $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, xa^2 = a.$$

We denote the core inverse x by $a^{\textcircled{\#}}$. As is well known, an element $a \in \mathcal{A}$ has core inverse x if and only if

$$a = axa, x\mathcal{A} = a\mathcal{A}, \mathcal{A}x = \mathcal{A}a^*.$$

As a natural generalization of core-EP invertibility, the authors introduced the generalized core-EP inverse in a Banach algebra $\mathcal{A}$ with an involution $*$. An element $a \in \mathcal{A}$ has generalized core-EP inverse if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{d}}$.

The generalized inverses mentioned above are powerful tools in linear algebra and operator algebra for dealing with matrices and operators that do not have a traditional inverse. They are used

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in various applications and provide a means to find solutions to linear systems and has applications across various scientific and engineering disciplines. Recently, many authors have studied them from many different views, e.g., [2,4,5,7–11,13,17,20,21,24,25].

Recall that $a \in \mathcal{A}$ has generalized Drazin inverse if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, ax = xa, a - a^2x \in \mathcal{A}^{qnil}.$$

Here, $\mathcal{A}^{qnil} = \{a \in \mathcal{A} \mid 1 + \lambda a \in \mathcal{A}^{-1}\}$. Such x is unique, if exists, and denote it by a^d. We use $\mathcal{A}^d, \mathcal{A}^{\circledast}$ and $\mathcal{A}^{\textcircled{d}}$ to denote the sets of all generalized Drazin inverse, core and generalized core-EP invertible elements in $\mathcal{A}$, respectively. If a and x satisfy the equations $a = axa$ and $(ax)^* = ax$, then x is called $(1,3)$-inverse of a and is denoted by $a^{(1,3)}$. We use $\mathcal{A}^{(1,3)}$ to stand for sets of all $(1,3)$-invertible elements in $\mathcal{A}$. We list several characterizations of generalized core-EP inverse.

Theorem 1.1. *(see [3,6])Let $\mathcal{A}$ be a Banach *-algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{d}}$.
(2) There exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = yx = 0, x \in \mathcal{A}^{\circledast}, y \in \mathcal{A}^{qnil}.$$

(3) There exists a projection $p \in \mathcal{A}$ such that

$$a + p \in \mathcal{A}^{-1}, pa = pap \in \mathcal{A}^{qnil}.$$

(4) $xax = x, im(x) = im(x^*) = im(a^d)$.
(5) $a \in \mathcal{A}^d$ and $a^d \in \mathcal{A}^{\circledast}$. In this case, $a^{\textcircled{d}} = (a^d)^2(a^d)^{\circledast}$.
(6) $a \in \mathcal{A}^d$ and $a^d \in \mathcal{A}^{(1,3)}$. In this case, $a^{\textcircled{d}} = (a^d)^2(a^d)^{(1,3)}$.
(7) $a \in \mathcal{A}^d$ and there exists a projection $q \in \mathcal{A}$ such that $a^d\mathcal{A} = q\mathcal{A}$. In this case, $a^{\textcircled{d}} = a^dq$.

The motivation of this paper is to investigate the generalized core-EP inverse for the triangular matrices over a Banach *-algebra.

In Section 2, we establish necessary and sufficient conditions under which the block operator triangular matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ over a Banach algebra has the generalized core-EP inverse with upper triangular form.

A C^*-algebra is a Banach algebra equipped with an involution operation $*$ that satisfies satisfies the C^*-identity: $||x^*x|| = ||x||^2$ for all $x \in \mathcal{A}$. In Section 3, we particularly investigate the generalized core-EP inverse of a triangular block operator matrices over a C^*-algebra. We prove that every triangular operator matrix $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ over a C^*-algebra with generalized core-EP invertible diagonal entries has the generalized core-EP inverse and its representation of generalized core-EP inverse is presented.

The set of all bounded linear operators on a Hilbert space H, denoted $\mathcal{B}(H)$, forms a C^*-algebra with the operator norm and the adjoint operation. Lex X and Y be Hilbert spaces. We use $\mathcal{B}(X,Y)$ to stand for the set of all bounded linear operators from X to Y. Finally, in Section 4, we apply our results and study the generalized core-EP inverse for the block operator matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathcal{B}(X)^{\textcircled{d}}, B \in \mathcal{B}(X,Y), C \in \mathcal{B}(Y,X), D \in \mathcal{B}(Y)^{\textcircled{d}}$. Here, M is a linear operator on Hilbert space $X \oplus Y$.

Throughout the paper, all Banach *-algebras are complex with an identity. An element $p \in \mathcal{A}$ is a projection if $p^2 = p = p^*$. $\mathcal{A}^D, \mathcal{A}^{\textcircled{d}}$ and $\mathcal{A}^{nil}$ denote the sets of all Drazin, generalized core-EP invertible and nilpotent elements in $\mathcal{A}$ respectively. Let $a \in \mathcal{A}^d$. We use a^π to stand for the spectral idempotent $a^\pi = 1 - aa^d$.

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2. Triangular Operator Matrices over Banach *-Algebras

Let $\mathcal{A}$ be a Banach *-algebra. Then $M_2(\mathcal{A})$ is a Banach *-algebra with *-transpose as the involution. We come now to generalized EP-inverse of a triangular matrix over $\mathcal{A}$. To prove the main results, some lemmas are needed. We begin with

Lemma 2.1. *Let $a \in \mathcal{A}^{\textcircled{d}}$ and $b \in \mathcal{A}$. Then the following are equivalent:*

(1) $(1 - a^{\textcircled{d}}a)b = 0$.
(2) $(1 - aa^{\textcircled{d}})b = 0$.
(3) $a^{\pi}b = 0$.

Proof. (1) $\Rightarrow$ (3) Since $(1 - a^{\textcircled{d}}a)b = 0$, we have $b = a^{\textcircled{d}}ab$. In view of ????, $a^{\textcircled{d}} = (a^d)^2(a^d)^{\textcircled{\#}}$. Thus, $(1 - aa^d)b = (1 - aa^d)(a^d)^2(a^d)^{\textcircled{\#}}ab = 0$.

(3) $\Rightarrow$ (2) Since $a^d = (a^d)^2a = a^d[a^d(a^d)^{\textcircled{\#}}a^d]a = [(a^d)^2(a^d)^{\textcircled{\#}}]aa^d = a^{\textcircled{d}}aa^d$. Then $b = aa^db = a^{\textcircled{d}}a^2a^db$; and so $(1 - aa^{\textcircled{d}})b = (1 - aa^{\textcircled{d}})a^{\textcircled{d}}a^2a^db = 0$, as required.

(2) $\Rightarrow$ (1) Since $(1 - aa^{\textcircled{d}})b = 0$, we get $b = aa^{\textcircled{d}}b$. Therefore $(1 - a^{\textcircled{d}}a)b = (1 - a^{\textcircled{d}})aa^{\textcircled{d}}b = (a - a^{\textcircled{d}}a^2)a^{\textcircled{d}}b = 0$, as asserted. □

Let $\mathcal{A}$ be a Banach *-algebra. Then $M_2(\mathcal{A})$ is a Banach *-algebra with *-transpose as the involution. We come now to generalized EP-inverse of a triangular matrix over $\mathcal{A}$.

Lemma 2.2. *Let* $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$.

(1) If $a, d \in \mathcal{A}^d$, then $x \in M_2(\mathcal{A})^d$ and $x^d = \begin{pmatrix} a^d & z \\ 0 & d^d \end{pmatrix}$, where

$$z = \sum_{i=0}^{\infty}(a^d)^{i+2}bd^id^{\pi} + \sum_{i=0}^{\infty}a^ia^{\pi}b(d^d)^{i+2} - a^dbd^d.$$

(2) If $a, d \in \mathcal{A}^{\textcircled{\#}}$ and $a^{\pi}b = 0$, then $x \in M_2(\mathcal{A})^{\textcircled{\#}}$ and

$$x^{\textcircled{\#}} = \begin{pmatrix} a^{\textcircled{\#}} & -a^{\textcircled{\#}}bd^{\textcircled{\#}} \\ 0 & d^{\textcircled{\#}} \end{pmatrix}.$$

Proof. See [26][Lemma 2.1] and [23][Theorem 2.5]. □

We are ready to prove:

Theorem 2.3. *Let* $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathcal{A})$ *with* $a, d \in \mathcal{A}^d$. *Then the following are equivalent:*

(1) $x \in M_2(\mathcal{A})$ has upper triangular generalized core-EP inverse.
(2) $a, d \in \mathcal{A}^{\textcircled{d}}$ and

$$\sum_{i=0}^{\infty}a^ia^{\pi}b(d^d)^{i+2} = 0.$$

In this case,

$$x^{\textcircled{d}} = \begin{pmatrix} a^{\textcircled{d}} & z \\ 0 & d^{\textcircled{d}} \end{pmatrix},$$

where $z = -a^dbd^{\textcircled{d}}$.

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Proof. By virtue of Lemma 2.2, we have

$$x^d = \begin{pmatrix} a^d & s \\ 0 & d^d \end{pmatrix},$$

where

$$s = \sum_{i=0}^{\infty}(a^d)^{i+2}bd^id^\pi + \sum_{i=0}^{\infty} a^ia^\pi b(d^d)^{i+2} - a^dbd^d.$$

(1) $\Rightarrow$ (2) By virtue of Theorem 1.1, x^d has core inverse and that $x^{\textcircled{d}} = (x^d)^2(x^d)^{\textcircled{\#}}$. Hence,

$$(x^d)^{\textcircled{\#}} = x^2[(x^d)^2(x^d)^{\textcircled{\#}}],$$

and so $(x^d)^{\textcircled{\#}}$ is a upper triangular matrix. Write

$$(x^d)^{\textcircled{\#}} = \begin{pmatrix} \alpha & \delta \\ 0 & \beta \end{pmatrix}.$$

Then

$$x^d((x^d)^{\textcircled{\#}})^2 = (x^d)^{\textcircled{\#}}, (x^d)^{\textcircled{\#}}(x^d)^2 = x^d, (x^d(x^d)^{\textcircled{\#}})^* = x^d(x^d)^{\textcircled{\#}}.$$

This implies that

$$a^d\alpha^2 = \alpha, \alpha(a^d)^2 = a^d, (a^d\alpha)^* = a^d\alpha.$$

Hence, $a^d \in \mathcal{A}^{\textcircled{\#}}$. By using Theorem 1.1 again, $a \in \mathcal{A}^{\textcircled{d}}$. Likewise, $d \in \mathcal{A}^{\textcircled{d}}$. In view of Lemma 2.2, $(a^d)^\pi s = 0$. This implies that

$$\begin{aligned} (a^d)^\pi s &= a^\pi s \\ &= \sum_{i=0}^{\infty} a^ia^\pi b(d^d)^{i+2} \\ &= 0. \end{aligned}$$

Therefore

$$\sum_{i=0}^{\infty} a^ia^\pi b(d^d)^{i+2} = 0,$$

as asserted.

(2) $\Rightarrow$ (1) By hypothesis, we get

$$(a^d)^\pi s = (1 - a^da^2a^d)s = a^\pi s = a^\pi[\sum_{i=0}^{\infty}(a^d)^{i+2}bd^id^\pi - a^dbd^d] = 0.$$

Then it follows by Lemma 2.2 that

$$(x^d)^{\textcircled{\#}} = \begin{pmatrix} (a^d)^{\textcircled{\#}} & t \\ 0 & (d^d)^{\textcircled{\#}} \end{pmatrix},$$

where $t = -(a^d)^{\textcircled{\#}}s(d^d)^{\textcircled{\#}}$. Hence, $t = -(a^d)^{\textcircled{\#}}[\sum_{i=0}^{\infty}(a^d)^{i+2}bd^id^\pi - a^dbd^d](d^d)^{\textcircled{\#}} = (a^d)^{\textcircled{\#}}a^dbd^d(d^d)^{\textcircled{\#}}$.
Then we have

$$(x^d)^2 = \begin{pmatrix} (a^d)^2 & w \\ 0 & (d^d)^2 \end{pmatrix},$$

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where $w = \sum_{i=0}^{\infty}(a^d)^{i+3}bd^id^\pi - (a^d)^2bd^d - a^db(d^d)^2$. Therefore

$$\begin{aligned} x^{\textcircled{d}} &= (x^d)^2(x^d)^{\#} \\ &= \begin{pmatrix} (a^d)^2 & w \\ 0 & (d^d)^2 \end{pmatrix}\begin{pmatrix} (a^d)^{\#} & t \\ 0 & (d^d)^{\#} \end{pmatrix} \\ &= \begin{pmatrix} a^{\textcircled{d}} & z \\ 0 & d^{\textcircled{d}} \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} z &= (a^d)^2t + w(d^d)^{\#} \\ &= (a^d)^2[(a^d)^{\#}a^dbd^d(d^d)^{\#}] - [(a^d)^2bd^d + a^db(d^d)^2](d^d)^{\#} \\ &= (a^d)^2bd^d(d^d)^{\#} - a^d(a^db + bd^d)d^d(d^d)^{\#} \\ &= (a^d)^2bd^d(d^d)^{\#} - (a^d)^2bd^d(d^d)^{\#} - a^d[b(d^d)^2(d^d)^{\#}] \\ &= -a^dbd^{\textcircled{d}} \end{aligned}$$

This completes the proof. □

Corollary 2.4. *Let $\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathcal{A})$ with $a, d \in \mathcal{A}^{\textcircled{d}}$. If $a^\pi bd^{\textcircled{d}} = 0$, then $\alpha \in M_2(\mathcal{A})^{\textcircled{d}}$ and*

$$\alpha^{\textcircled{d}} = \begin{pmatrix} a^{\textcircled{d}} & -a^{\textcircled{d}}bd^{\textcircled{d}} \\ 0 & d^{\textcircled{d}} \end{pmatrix}.$$

Proof. Since $a^\pi bd^{\textcircled{d}} = 0$, it follows by Theorem 1.1 that $a^\pi b(d^d)^2(d^d)^{\#} = 0$; hence,

$$a^\pi bd^d = [a^\pi b(d^d)^2(d^d)^{\#}]b^db = 0.$$

By using Lemma 2.1, we have $(1 - aa^{\textcircled{d}})bd^{\textcircled{d}} = 0$, and so $bd^{\textcircled{d}} = aa^{\textcircled{d}}bd^{\textcircled{d}}$. Then

$$a^dbd^{\textcircled{d}} = a^d(aa^{\textcircled{d}})bd^{\textcircled{d}} = a^{\textcircled{d}}bd^{\textcircled{d}}.$$

In light of Theorem 2.3,

$$\alpha^{\textcircled{d}} = \begin{pmatrix} a^{\textcircled{d}} & -a^{\textcircled{d}}bd^{\textcircled{d}} \\ 0 & d^{\textcircled{d}} \end{pmatrix},$$

as asserted. □

It is very hard to determine the core-EP inverse of a triangular complex matrix (see [10]). As a consequence of Theorem 2.3, we now derive the following.

Corollary 2.5. *Let $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, $A, B, D \in \mathbb{C}^{n\times n}$. If*

$$\sum_{i=0}^{i(A)} A^iA^\pi B(D^D)^{i+2} = 0,$$

then

$$M^{\textcircled{D}} = \begin{pmatrix} A^{\textcircled{D}} & Z \\ 0 & D^{\textcircled{D}} \end{pmatrix},$$

where $Z = -A^DBD^{\textcircled{D}}$.

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Proof. Since the generalized core-EP inverse and generalized core-EP inverse coincide with each other for a complex matrix, we obtain the result by Theorem 2.3. □

Corollary 2.6. *Let* $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, $A, B, D \in \mathbb{C}^{n\times n}$. *If* A *is invertible, then*

$$M^{\textcircled{d}} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{\textcircled{d}} \\ 0 & D^{\textcircled{d}} \end{pmatrix}.$$

Proof. Straightforward. □

The condition "$x^{\textcircled{d}} \in M_2(\mathcal{A})$ is upper triangular" in Theorem 2.3 is necessary as the following shows.

Example 2.7. *Let* σ *and* τ *be linear operators, acting on separable Hilbert space* $l_2(\mathbb{N})$ *with the conjugate adjoint as an involution, defined as follows respectively:*

$$\begin{array}{rcl} \sigma(x_1, x_2, x_3, x_4, \cdots) & = & (0, x_1, x_2, x_3, \cdots), \\ \tau(x_1, x_2, x_3, x_4, \cdots) & = & (x_2, x_3, x_4, x_5, \cdots). \end{array}$$

Then $\tau\sigma = 1$. *Take* $M = \begin{pmatrix} \sigma & 1-\sigma\tau \\ 0 & \tau \end{pmatrix}$. *Then*

$$M^{\textcircled{d}} = M^{-1} = \begin{pmatrix} \tau & 0 \\ 1-\sigma\tau & \sigma \end{pmatrix}.$$

In this case, M is upper triangular matrix, but its generalized core-EP inverse is lower triangular.

3. Triangular Matrices with C^*-Algebra Entries

The aim of this section is to investigate the generalized core-EP inverse of triangular matrices over a C^*-algebra. Throughout this section, $\mathcal{A}$ is always a C^*-algebra. We start by

Lemma 3.1. *Let* $\mathcal{A}$ *be a* C^**-algebra and let* $a \in \mathcal{A}^{\textcircled{\#}} \bigcap \mathcal{A}^{\dagger}$. *Then* $a^{\dagger}aa^{\textcircled{\#}} = a^{\dagger}$.

Proof. Since $(1-aa^d)a^d = 0$, by virtue of [23][Lemma 2.4], we have $(1-aa^{\textcircled{\#}})a^d = 0$. This implies that $a^{\dagger}[(1-aa^{\textcircled{\#}})a^d(a^2a^{\dagger})]^* = 0$, and so $a^{\dagger}[(1-aa^{\textcircled{\#}})(aa^{\dagger})]^* = 0$. Hence, $a^{\dagger}(aa^{\dagger})^*(1-aa^{\textcircled{\#}})^* = 0$. Therefore $a^{\dagger}(1-aa^{\textcircled{\#}}) = 0$, as required. □

Set $e_a = 1 - aa^{\dagger}$ and $f_d = 1 - d^{\dagger}d$. Then we derive

Lemma 3.2. *Let* $\mathcal{A}$ *be a* C^**-algebra and let* $a, d \in \mathcal{A}^{\textcircled{\#}} \bigcap \mathcal{A}^{\dagger}$. *Then*

$$dd^{\textcircled{\#}}[1 + (e_abd^{\dagger})^*(e_abd^{\dagger})] = [1 + (e_abd^{\dagger})^*(e_abd^{\dagger})]dd^{\textcircled{\#}}.$$

Proof. It is easy to check that

$$(e_abd^{\dagger}dd^{\textcircled{\#}})^*e_abd^{\dagger} = (e_abd^{\dagger})^*(e_abd^{\dagger}) = (e_abd^{\dagger})^*(e_abd^{\dagger}dd^{\textcircled{\#}}).$$

Then

$$dd^{\textcircled{\#}}(e_abd^{\dagger})^*e_abd^{\dagger} = (e_abd^{\dagger})^*e_abd^{\dagger}dd^{\textcircled{\#}}.$$

Therefore

$$dd^{\textcircled{\#}}[1 + (e_abd^{\dagger})^*(e_abd^{\dagger})] = [1 + (e_abd^{\dagger})^*(e_abd^{\dagger})]dd^{\textcircled{\#}},$$

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as asserted. □

In [15], Li and Du investigate the core inverse of a triangular block complex matrix. We now extend Li and Du's result to block operator matrices over a C^*-algebra by a new route.

Lemma 3.3. *Let $\mathcal{A}$ be a C^*-algebra and let $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. If $a, d \in \mathcal{A}^{\text{\#}}, a^{\pi}bd^{\pi} = 0$ and $e_abf_d = 0$, then $x \in \mathcal{A}^{\text{\#}}$. In this case, $x^{\text{\#}} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where*

$$\begin{array}{rcl}\alpha & = & a^{\text{\#}} + [a^{\pi}bd^{\dagger} - a^{\text{\#}}b]d^{\text{\#}}[1 + (e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*, \\ \beta & = & (a^{\pi}bd^{\dagger} - a^{\text{\#}}b)d^{\text{\#}}[1 + (e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}, \\ \gamma & = & d^{\text{\#}}[1 + (e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*, \\ \delta & = & d^{\text{\#}}[1 + (e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}.\end{array}$$

Proof. Since $a \in \mathcal{A}^{\text{\#}}$, by virtue of [23][Lemma 2.1], it has group inverse, and so a is regular. As $\mathcal{A}$ is a C^*-algebra, it follows by [14][Theorem 2.8] that $a \in \mathcal{A}^{\dagger}$. Likewise, $d \in \mathcal{A}^{\dagger}$. Since every C^*-algebra has the symmetry property, we have $1 + (e_abd^{\dagger})^*(e_abd^{\dagger}) \in \mathcal{A}^{-1}$.

Let $z = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, where α, β, γ and δ as defined above.

Claim 1. $(xz)^* = xz$.

Since $e_abf_d = 0$, by virtue of Lemma 3.1, we have

$$\begin{array}{rcl}(1 - aa^{\text{\#}})bd^{\text{\#}} & = & (1 - aa^{\dagger}aa^{\text{\#}})bd^{\text{\#}} \\ & = & (1 - aa^{\dagger})bd^{\text{\#}} \\ & = & (1 - aa^{\dagger})bd^{\dagger}dd^{\text{\#}} \\ & = & e_abd^{\dagger}.\end{array}$$

Hence,

$$\begin{array}{rl} & [(1 - aa^{\text{\#}})bd^{\text{\#}}(1 + (e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}(e_abd^{\dagger})^*]^* \\ = & [e_abd^{\dagger}](1 + (e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}[(1 - aa^{\text{\#}})bd^{\text{\#}}]^* \\ = & (1 - aa^{\text{\#}})bd^{\text{\#}}(1 + (e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}(e_abd^{\dagger})^*.\end{array}$$

Therefore

$$\begin{array}{rl} & a\alpha + b\gamma \\ = & aa^{\text{\#}} + (1 - aa^{\text{\#}})bd^{\text{\#}}(1 + (e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}(e_abd^{\dagger})^* \\ = & aa^{\text{\#}} + e_abd^{\dagger}(1 + (e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}(e_abd^{\dagger})^*.\end{array}$$

Hence,

$$(a\alpha + b\gamma)^* = a\alpha + b\gamma.$$

By virtue of Lemma 3.1, we have

$$\begin{array}{rcl}dd^{\text{\#}}(e_abd^{\dagger})^*(e_abd^{\dagger}) & = & (e_abd^{\dagger}dd^{\text{\#}})^*(e_abd^{\dagger}) \\ & = & (e_abd^{\dagger})^*(e_abd^{\dagger}) \\ & = & (e_abd^{\dagger})^*(e_abd^{\dagger})dd^{\text{\#}}.\end{array}$$

Hence,

$$\begin{array}{rcl}dd^{\text{\#}}[1 + (e_abd^{\dagger})^*(e_abd^{\dagger})] & = & dd^{\text{\#}} + dd^{\text{\#}}(e_abd^{\dagger})^*(e_abd^{\dagger}) \\ & = & dd^{\text{\#}} + (e_abd^{\dagger})^*(e_abd^{\dagger})dd^{\text{\#}} \\ & = & [1 + (e_abd^{\dagger})^*(e_abd^{\dagger})]dd^{\text{\#}}.\end{array}$$

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Thus, we derive that

$$dd^{\#}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1} = [1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}dd^{\#}.$$

Since $d\delta = dd^{\#}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}$, we have $(d\delta)^* = d\delta$.

In view of Lemma 3.2, we verify that

$$\begin{array}{rl} & a\beta + b\delta \\ = & -aa^{\#}bd^{\#}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1} + bd^{\#}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1} \\ = & [1-aa^{\#}]bd^{\#}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1} \\ = & [1-aa^{\#}]b(d^{\#}dd^{\#})[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1} \\ = & [1-aa^{\#}]bd^{\#}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}dd^{\#} \\ = & e_abd^{\dagger}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}dd^{\#} \\ = & e_abd^{\dagger}(dd^{\#})[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1} \\ = & e_abd^{\dagger}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1} \end{array}$$

Therefore

$$\begin{array}{rl} & (a\beta + b\delta)^* \\ = & [(1-aa^{\#})bd^{\#}(1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}dd^{\#}]^* \\ = & dd^{\#}(1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}[(1-aa^{\#})bd^{\#}]^* \\ = & dd^{\#}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^* \\ = & [1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^* \\ = & d\gamma. \end{array}$$

We compute that

$$\begin{array}{rcl} xz & = & \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ & = & \begin{pmatrix} a\alpha + b\gamma & a\beta + b\delta \\ d\gamma & d\delta \end{pmatrix} \end{array}$$

Therefore $(xz)^* = xz$.

Claim 2. $xz^2 = z$.

We compute that

$$\begin{array}{rl} & (1-xz)z \\ = & \begin{pmatrix} 1-(a\alpha+b\gamma) & -(a\beta+b\delta) \\ -d\gamma & 1-d\delta \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ = & \begin{pmatrix} [1-(a\alpha+b\gamma)]\alpha-(a\beta+b\delta)\gamma & [1-(a\alpha+b\gamma)]\beta-(a\beta+b\delta)\delta \\ -d\gamma\alpha+(1-d\delta)\gamma & -d\gamma\beta+(1-d\delta)\delta \end{pmatrix}. \end{array}$$

Obviously, we have

$$a\alpha + b\gamma = aa^{\#} + (a\beta + b\delta)(e_ab\delta^{\dagger})^*.$$

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Then we compute that

$$\begin{aligned}
& [1-(a\alpha+b\gamma)]\alpha \\
= & [1-aa^{\circledast}-(a\beta+b\delta)(e_abd^{\dagger})^*][a^{\circledast}+(a^{\pi}bd^{\dagger}-a^{\circledast}b)\gamma] \\
= & [1-aa^{\circledast}-(a\beta+b\delta)(e_ab\delta^{\dagger})^*][(a^{\pi}bd^{\dagger}-a^{\circledast}b)\gamma] \\
= & [1-aa^{\circledast}-(a\beta+b\delta)(e_ab\delta^{\dagger})^*][(a^{\pi}bd^{\dagger}]\gamma \\
= & [1-aa^{\circledast}-(a\beta+b\delta)(e_ab\delta^{\dagger})^*](e_abd^{\dagger})\gamma \\
= & e_abd^{\dagger}-(a\beta+b\delta)(e_ab\delta^{\dagger})^*(e_abd^{\dagger})\gamma \\
= & e_abd^{\dagger}-e_abd^{\dagger}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_ab\delta^{\dagger})^*(e_abd^{\dagger})\gamma \\
= & e_abd^{\dagger}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}[1+(e_abd^{\dagger})^*e_abd^{\dagger}-(e_ab\delta^{\dagger})^*e_abd^{\dagger}]\gamma \\
= & e_abd^{\dagger}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}\gamma \\
= & (a\beta+b\delta)\gamma;
\end{aligned}$$

hence,

$$[1-(a\alpha+b\gamma)]\alpha-(a\beta+b\delta)\gamma=0.$$

Analogously, we derive that

$$\begin{aligned}
& [1-(a\alpha+b\gamma)]\beta \\
= & [1-aa^{\circledast}-(a\beta+b\delta)(e_ab\delta^{\dagger})^*][(a^{\pi}bd^{\dagger}-a^{\circledast}b)d^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1} \\
= & [1-aa^{\circledast}-(a\beta+b\delta)(e_ab\delta^{\dagger})^*][(a^{\pi}bd^{\dagger}-a^{\circledast}b)\delta \\
= & (a\beta+b\delta)\delta;
\end{aligned}$$

hence,

$$[1-(a\alpha+b\gamma)]\beta-(a\beta+b\delta)\delta=0.$$

Also we check that

$$\begin{aligned}
& -d\gamma\alpha+(1-d\delta)\gamma \\
= & -dd^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*[a^{\circledast}+[a^{\pi}bd^{\dagger}-a^{\circledast}b] \\
& d^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*] \\
+ & [1-dd^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}]d^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^* \\
= & -[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*[a^{\pi}bd^{\dagger}-a^{\circledast}b]d^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*] \\
+ & [1-dd^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}]d^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^* \\
= & -[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*a^{\pi}bd^{\dagger}d^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*] \\
+ & [1-[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}]d^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^* \\
= & [1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}[-(e_abd^{\dagger})^*a^{\pi}bd^{\dagger}+(e_abd^{\dagger})^*e_abd^{\dagger}] \\
& d^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*] \\
= & [1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}[(e_abd^{\dagger})^*e_a(1-a^{\pi})bd^{\dagger}]d^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*] \\
= & 0.
\end{aligned}$$

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Furthermore, we verify that

$$\begin{aligned}
& -d\gamma\beta + (1-d\delta)\delta \\
= & -dd^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*[(a^{\pi}bd^{\dagger} - a^{\circledast}b)d^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}] \\
+ & [1-dd^{\circledast}\left(1+(e_abd^{\dagger})^*(e_abd^{\dagger})\right)^{-1}]d^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1} \\
= & -[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*(a^{\pi}bd^{\dagger} - a^{\circledast}b)d^{\circledast}(1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1} \\
+ & [1-\left(1+(e_abd^{\dagger})^*(e_abd^{\dagger})\right)^{-1}]d^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1} \\
= & -[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*a^{\pi}bd^{\dagger}d^{\circledast}(1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1} \\
+ & [1-\left(1+(e_abd^{\dagger})^*(e_abd^{\dagger})\right)^{-1}]d^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1} \\
= & [1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}[-(e_abd^{\dagger})^*a^{\pi}bd^{\dagger} + (e_abd^{\dagger})^*(e_abd^{\dagger})]d^{\circledast} \\
& (1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1} \\
= & [1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}[-(e_abd^{\dagger})^*a^{\pi}bd^{\dagger} + (e_abd^{\dagger})^*(e_abd^{\dagger})]d^{\circledast} \\
& (1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1} \\
= & [1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*e_a(1-a^{\pi})bd^{\dagger} + (e_abd^{\dagger})^*(e_abd^{\dagger})]d^{\circledast} \\
& (1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1} \\
= & 0.
\end{aligned}$$

Therefore $xz^2 = z$.

Claim 3. $xzx = x$.

$$\begin{aligned}
& (1-xz)x \\
= & \begin{pmatrix} 1-(a\alpha+b\gamma) & -(a\beta+b\delta) \\ -d\gamma & 1-d\delta \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \\
= & \begin{pmatrix} [1-(a\alpha+b\gamma)]a & [1-(a\alpha+b\gamma)]b-(a\beta+b\delta)d \\ -d\gamma a\gamma & -d\gamma b+(1-d\delta)d \end{pmatrix}.
\end{aligned}$$

Obviously, we have $(e_a)^*a = (1-aa^{\dagger})a = 0$. Then

$$\begin{aligned}
& [1-(a\alpha+b\gamma)]a \\
= & [1-aa^{\circledast} - e_abd^{\dagger}(1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}(e_abd^{\dagger})^*]a \\
= & [(1-aa^{\circledast})a - e_abd^{\dagger}(1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}(e_abd^{\dagger})^*(e_a)^*a)] \\
= & 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
-d\gamma a &= -dd^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*a \\
&= -dd^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*(e_a)^*a \\
&= 0.
\end{aligned}$$

Clearly, $(1-aa^{\circledast})b = (1-aa^{\#}aa^{\dagger})b = e_ab = e_ab(f_d + d^{\dagger}d) = e_abd^{dag}d$. Then we have

$$\begin{aligned}
& [1-(a\alpha+b\gamma)]b \\
= & [1-aa^{\circledast} - e_abd^{\dagger}(1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}(e_abd^{\dagger})^*]b \\
= & e_ab - e_abd^{\dagger}(1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}(e_abd^{\dagger})^*(e_ab) \\
= & e_abd^{\dagger}d - e_abd^{\dagger}(1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}(e_abd^{\dagger})^*(e_abd^{dag})d \\
= & e_abd^{\dagger}[1-(1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}(e_abd^{\dagger})^*(e_abd^{dag})]d \\
= & e_abd^{\dagger}(1+(e_abd^{\dagger})^*(e_abd^{\dagger}))^{-1}[1+(e_abd^{\dagger})^*(e_abd^{\dagger}) - (e_abd^{\dagger})^*(e_abd^{dag})]d \\
= & e_abd^{\dagger}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}d \\
= & (a\beta+b\delta)d.
\end{aligned}$$

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Finally, we verify that

$$\begin{aligned}
d\gamma b &= dd^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*b \\
&= dd^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*(e_a)^*b \\
&= dd^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}(e_abd^{\dagger})^*e_ab(f_d+d^{\dagger}d) \\
&= dd^{\circledast}[1-\big(1+(e_abd^{\dagger})^*(e_abd^{\dagger})\big)^{-1}][(e_abd^{\dagger})^*(e_abd^{\dagger})]d \\
&= dd^{\circledast}[1-\big(1+(e_abd^{\dagger})^*(e_abd^{\dagger})\big)^{-1}]d \\
&= dd^{\circledast}d-dd^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}d \\
&= d-dd^{\circledast}[1+(e_abd^{\dagger})^*(e_abd^{\dagger})]^{-1}d \\
&= d(1-\delta d) \\
&= (1-d\delta)d.
\end{aligned}$$

Therefore $xzx = x$. In light of [24][Theorem 3.3], $x \in \mathcal{A}^{\circledast}$ and $x^{\circledast} = z$, as asserted. □

We come now to the demonstration for which this section has been developed.

Theorem 3.4. *Let $\mathcal{A}$ be a C^*-algebra and $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. If $a, d \in \mathcal{A}^{\textcircled{d}}$, then $x \in \mathcal{A}^{\textcircled{d}}$ and*

$$x^{\textcircled{d}} = \begin{pmatrix} (a^d)^2\alpha + w\gamma & (a^d)^2\beta + w\delta \\ (d^d)^2\gamma & (d^d)^2\delta \end{pmatrix},$$

where

$$\begin{aligned}
w &= \sum_{i=0}^{\infty}(a^d)^{i+3}bd^id^{\pi} + \sum_{i=0}^{\infty}a^ia^{\pi}b(d^d)^{i+3} - (a^d)^2bd^d - a^db(d^d)^2, \\
s &= \sum_{i=0}^{\infty}(a^d)^{i+2}bd^id^{\pi} + \sum_{i=0}^{\infty}a^ia^{\pi}b(d^d)^{i+2} - a^dbd^d, \\
\alpha &= a^2a^{\textcircled{d}} + [a^{\pi}s[d^d]^{\dagger} - a^2a^{\textcircled{d}}s]d^2d^{\textcircled{d}}\big(1+(e_{a^d}s[d^d]^{\dagger})^*(e_{a^d}s[d^d]^{\dagger})\big)^{-1}(e_{a^d}s[d^d]^{\dagger})^*, \\
\beta &= (a^{\pi}s[d^d]^{\dagger} - a^2a^{\textcircled{d}}s)d^2d^{\textcircled{d}}[1+(e_{a^d}s[d^d]^{\dagger})^*(e_{a^d}s[d^d]^{\dagger})]^{-1}, \\
\gamma &= d^2d^{\textcircled{d}}[1+(e_{a^d}s[d^d]^{\dagger})^*(e_{a^d}s[d^d]^{\dagger})]^{-1}(e_{a^d}s[d^d]^{\dagger})^*, \\
\delta &= d^2d^{\textcircled{d}}[1+(e_{a^d}s[d^d]^{\dagger})^*(e_{a^d}s[d^d]^{\dagger})]^{-1}.
\end{aligned}$$

Proof. In view of Theorem 1.1, $a, d \in \mathcal{A}^d$ and $a^d, d^d \in \mathcal{A}^{\circledast}$. By virtue of Lemma 2.2, we have

$$x^d = \begin{pmatrix} a^d & s \\ 0 & d^d \end{pmatrix},$$

where $s = \sum_{i=0}^{\infty}(a^d)^{i+2}bd^id^{\pi} + \sum_{i=0}^{\infty}a^ia^{\pi}b(d^d)^{i+2} - a^dbd^d$.

Since $a^d = a^daa^d$, i.e., a^d is regular. As $\mathcal{A}$ is a C^*-algebra, it follows by [14][Theorem 2.8], $a^d \in \mathcal{A}^{\dagger}$. Likewise, $d^d \in \mathcal{A}^{\dagger}$. Let $e_{a^d} = 1 - a^d(a^d)^{\dagger}$ and $f_{d^d} = 1 - [d^d]^{\dagger}d^d$. Then we check that

$$\begin{aligned}
(a^d)^{\pi}s(d^d)^{\pi} &= a^{\pi}[\sum_{i=0}^{\infty}(a^d)^{i+2}bd^id^{\pi} + \sum_{i=0}^{\infty}a^ia^{\pi}b(d^d)^{i+2} - a^dbd^d]d^{\pi} \\
&= a^{\pi}[\sum_{i=0}^{\infty}a^ia^{\pi}b(d^d)^{i+2}]d^{\pi} \\
&= 0
\end{aligned}$$

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and

$$\begin{aligned} e_{a^d} s f_{d^d} &= [1 - a^d(a^d)^\dagger][\sum_{i=0}^{\infty} (a^d)^{i+2} b d^i d^\pi + \sum_{i=0}^{\infty} a^i a^\pi b (d^d)^{i+2} \\ &- a^d b d^d][1 - [d^d]^\dagger d^d] \\ &= [1 - a^d(a^d)^\dagger][\sum_{i=0}^{\infty} a^i a^\pi b (d^d)^{i+2}][1 - [d^d]^\dagger d^d] \\ &= 0. \end{aligned}$$

It follows by Lemma 3.3 that

$$(x^d)^{\#} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{aligned} \alpha &= [a^d]^{\#} + [a^\pi s[d^d]^\dagger - [a^d]^{\#} s][d^d]^{\#}[1 + (e_{a^d} s[d^d]^\dagger)^*(e_{a^d} s[d^d]^\dagger)]^{-1}(e_{a^d} s[d^d]^\dagger)^*, \\ \beta &= (a^\pi s[d^d]^\dagger - [a^d]^{\#} s)[d^d]^{\#}[1 + (e_{a^d} s[d^d]^\dagger)^*(e_{a^d} s[d^d]^\dagger)]^{-1}, \\ \gamma &= [d^d]^{\#}[1 + (e_{a^d} s[d^d]^\dagger)^*(e_{a^d} s[d^d]^\dagger)]^{-1}(e_{a^d} s[d^d]^\dagger)^*, \\ \delta &= [d^d]^{\#}[1 + (e_{a^d} s[d^d]^\dagger)^*(e_{a^d} s[d^d]^\dagger)]^{-1}. \end{aligned}$$

Set

$$\begin{aligned} w &:= a^d s + s d^d \\ &= \sum_{i=0}^{\infty} (a^d)^{i+3} b d^i d^\pi + \sum_{i=0}^{\infty} a^i a^\pi b [d^d]^{i+3} - (a^d)^2 b d^d - a^d b (d^d)^2. \end{aligned}$$

Accordingly,

$$\begin{aligned} x^{Ⓓ} &= (x^d)^2 (x^d)^{\#} \\ &= \begin{pmatrix} (a^d)^2 & w \\ 0 & (d^d)^2 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \\ &= \begin{pmatrix} (a^d)^2 \alpha + w\gamma & (a^d)^2 \beta + w\delta \\ (d^d)^2 \gamma & (d^d)^2 \delta \end{pmatrix}. \end{aligned}$$

In view of Theorem 1.1, $a^{Ⓓ} = (a^d)^2 [a^d]^{\#}$. Hence, $(a^d)^{\#} = a^2[(a^d)^2[a^d]^{\#}] = a^2 a^{Ⓓ}$. Similarly, we have $(d^d)^{\#} = d^2 d^{Ⓓ}$. Therefore we verify the formulas of α, β, γ and δ mentioned before. □

Corollary 3.5. *Let $\mathcal{A}$ be a C^*-algebra and $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. If $a \in \mathcal{A}^{qnil}, d \in \mathcal{A}^{Ⓓ}$, then $x \in \mathcal{A}^{Ⓓ}$ and*

$$x^{Ⓓ} = \begin{pmatrix} s d^d \gamma & s d^d \delta \\ (d^d)^2 \gamma & (d^d)^2 \delta \end{pmatrix},$$

where

$$\begin{aligned} s &= \sum_{i=0}^{\infty} a^i b (d^d)^{i+2}, \\ \alpha &= s[d^d]^\dagger d^2 d^{Ⓓ} \left(1 + (s[d^d]^\dagger)^*(s[d^d]^\dagger)\right)^{-1} (s[d^d]^\dagger)^*, \\ \beta &= (s[d^d]^\dagger) d^2 d^{Ⓓ} [1 + (s[d^d]^\dagger)^*(s[d^d]^\dagger)]^{-1}, \\ \gamma &= d^2 d^{Ⓓ} [1 + (s[d^d]^\dagger)^*(s[d^d]^\dagger)]^{-1}(s[d^d]^\dagger)^*, \\ \delta &= d^2 d^{Ⓓ} [1 + (s[d^d]^\dagger)^*(s[d^d]^\dagger)]^{-1}. \end{aligned}$$

Proof. Since $a \in \mathcal{A}^{qnil}$, we see that $a^d = 0$, and therefore we obtain the result by Theorem 3.4. □

 doi:10.20944/preprints202408.0216.v1

Corollary 3.6. *Let $\mathcal{A}$ be a C^*-algebra and $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. If $a \in \mathcal{A}^{\textcircled{d}}, d \in \mathcal{A}^{qnil}$, then $x \in \mathcal{A}^{\textcircled{d}}$ and*

$$x^{\textcircled{d}} = \begin{pmatrix} a^{\textcircled{d}} & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. Since $(a^d)^2 a^2 a^{\textcircled{d}} = aa^d a^{\textcircled{d}} = a^2 a^{\textcircled{d}}$ nd $d^d = 0$, we complete the proof by Theorem 3.4. □

Since the algebra $\mathbb{C}^{n\times n}$ of all $n \times n$ complex matrices is a C^*-algebra, the following result gives a simpler formula to compute the core-EP inverse of a block triangular complex matrices (see [10]).

Corollary 3.7. *Let $\mathcal{A}$ be a C^*-algebra and $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$. If $a, d \in \mathcal{A}^{\textcircled{D}}$, then $x \in \mathcal{A}^{\textcircled{D}}$ and*

$$x^{\textcircled{D}} = \begin{pmatrix} (a^D)^2\alpha + w\gamma & (a^D)^2\beta + w\delta \\ (d^D)^2\gamma & (d^D)^2\delta \end{pmatrix},$$

where

$$\begin{aligned}
w &= \sum_{i=0}^{m} (a^D)^{i+3} b d^i d^\pi + \sum_{i=0}^{m} a^i a^\pi b (d^D)^{i+3} - (a^D)^2 b d^D - a^D b (d^D)^2, \\
s &= \sum_{i=0}^{m} (a^D)^{i+2} b d^i d^\pi + \sum_{i=0}^{m} a^i a^\pi b (d^D)^{i+2} - a^d b d^D, \\
\alpha &= a^2 a^{\textcircled{D}} + [a^\pi s[d^D]^\dagger - a^2 a^{\textcircled{D}} s] d^2 d^{\textcircled{D}} \left(1 + (e_{a^D} s[d^D]^\dagger)^* (e_{a^D} s[d^D]^\dagger)\right)^{-1} \\
&\quad (e_{a^D} s[d^D]^\dagger)^*, \\
\beta &= (a^\pi s[d^D]^\dagger - a^2 a^{\textcircled{D}} s) d^2 d^{\textcircled{D}} [1 + (e_{a^D} s[d^D]^\dagger)^* (e_{a^D} s[d^D]^\dagger)]^{-1}, \\
\gamma &= d^2 d^{\textcircled{D}} [1 + (e_{a^D} s[d^D]^\dagger)^* (e_{a^D} s[d^D]^\dagger)]^{-1} (e_{a^D} s[d^D]^\dagger)^*, \\
\delta &= d^2 d^{\textcircled{D}} [1 + (e_{a^D} s[d^D]^\dagger)^* (e_{a^D} s[d^D]^\dagger)]^{-1}, \\
m &= \max\{ind(a), ind(d)\}.
\end{aligned}$$

Proof. Since $a, d \in \mathcal{A}^{\textcircled{D}}$, it follows by [3][Corollary 3.4] that $a, d \in \mathcal{A}^{\textcircled{d}} \bigcap \mathcal{A}^D$. By virtue of Theorem 3.4, $x \in \mathcal{A}^{\textcircled{d}}$. Clearly, $x \in \mathcal{A}^D$. By using [3][Corollary 3.4] again, $x \in \mathcal{A}^{\textcircled{D}}$ and $x^{\textcircled{D}} = x^{\textcircled{d}}$, as required. □

4. Applications

Lex X and Y be Hilbert spaces, and let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, where $A \in \mathcal{B}(X)^{\textcircled{d}}, B \in \mathcal{B}(X, Y), C \in \mathcal{B}(Y, X), D \in \mathcal{B}(Y)^{\textcircled{d}}$. Choose $p = \begin{pmatrix} I_X & 0 \\ 0 & I_Y \end{pmatrix}$. Then M can be regarded as the Pierce matrix $\begin{pmatrix} pMp & pMp^\pi \\ p^\pi Mp & p^\pi Mp^\pi \end{pmatrix}_p$. Here, every subblock matrices can be seen as the bounded linear operators on Hilbert space $X \oplus Y$. Throughout this section, without loss the generality, we consider M as the block operator matrix in a specifical case $X = Y$. In this case, $\mathcal{B}(X \oplus X)$ is indeed a C^*-algebra. The following lemma is crucial.

Lemma 4.1. *Let $a \in \mathcal{A}^{\textcircled{d}}$ and $b \in \mathcal{A}^{qnil}$. If $a^*b = 0$ and $ba = 0$, then $a + b \in \mathcal{A}^{\textcircled{d}}$. In this case,*

$$(a + b)^{\textcircled{d}} = a^{\textcircled{d}}.$$

Proof. Since $a \in \mathcal{A}^{\textcircled{d}}$, by virtue of Theorem 1.1, there exist $x \in \mathcal{A}^{\textcircled{\#}}$ and $y \in \mathcal{A}^{qnil}$ such that $a = x + y, x^*y = 0, yx = 0$. As in the proof of [3, Theorem 2.1], $x = aa^{\textcircled{d}}a$ and $y = a - aa^{\textcircled{d}}a$. Then

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$a = x + (y + b)$. Since $by = b(a - aa^{\textcircled{d}}a) = 0$, it follows by [26][Lemma 2.10] that $y + b \in \mathcal{A}^{qnil}$. We directly verify that

$$\begin{array}{rcl} x^*(y+b) & = & x^*y + x^*b = (a^{\textcircled{d}}a)^*(a^*b) = 0, \\ (y+b)x & = & yx + (ba)a^{\textcircled{d}}a = 0. \end{array}$$

In light of Theorem 1.1, $a + b \in \mathcal{A}^{\textcircled{d}}$. In this case,

$$(a+b)^{\textcircled{d}} = x^{\#} = a^{\textcircled{d}},$$

as asserted. □

Theorem 4.2. *If $CA = 0, CB = 0$ and $D^*C = 0$, then M has generalized core-EP inverse. In this case,*

$$M^{\textcircled{d}} = \begin{pmatrix} (A^d)^2\Lambda + W\Gamma & (A^d)^2\Sigma + W\Delta \\ (D^d)^2\Gamma & (D^d)^2\Delta \end{pmatrix},$$

where

$$\begin{array}{rcl} W & = & \sum\limits_{i=0}^{\infty}(A^d)^{i+3}BD^iD^{\pi} + \sum\limits_{i=0}^{\infty}A^iA^{\pi}B(D^d)^{i+3} - (A^d)^2BD^d - A^dB(D^d)^2, \\ S & = & \sum\limits_{i=0}^{\infty}(A^d)^{i+2}BD^iD^{\pi} + \sum\limits_{i=0}^{\infty}A^iA^{\pi}B(D^d)^{i+2} - A^dBD^d, \\ \Lambda & = & A^2A^{\textcircled{d}} + [A^{\pi}S[D^d]^{\dagger} - A^2A^{\textcircled{d}}S]D^2D^{\textcircled{d}}\big(I + (e_{A^d}S[D^d]^{\dagger})^*(e_{A^d}S[D^d]^{\dagger})\big)^{-1} \\ & & (e_{A^d}S[D^d]^{\dagger})^*, \\ \Sigma & = & (A^{\pi}S[D^d]^{\dagger} - A^2A^{\textcircled{d}}S)D^2D^{\textcircled{d}}[I + (e_{A^d}S[D^d]^{\dagger})^*(e_{A^d}S[D^d]^{\dagger})]^{-1}, \\ \Gamma & = & D^2D^{\textcircled{d}}[I + (e_{A^d}S[D^d]^{\dagger})^*(e_{A^d}S[D^d]^{\dagger})]^{-1}(e_{A^d}S[D^d]^{\dagger})^*, \\ \Delta & = & D^2D^{\textcircled{d}}[I + (e_{A^d}S[D^d]^{\dagger})^*(e_{A^d}S[D^d]^{\dagger})]^{-1}. \end{array}$$

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

We easily check that

$$\begin{array}{rcl} P^*Q & = & \begin{pmatrix} A^* & 0 \\ B^* & D^* \end{pmatrix}\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ D^*C & 0 \end{pmatrix} = 0, \\ QP & = & \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ CA & CB \end{pmatrix} = 0 \end{array}$$

Since A and D have generalized core-EP inverses, it follows by Theorem 3.4 that P has generalized core-EP inverse, and that

$$P^{\textcircled{d}} = \begin{pmatrix} (A^d)^2\Lambda + W\Gamma & (A^d)^2\Sigma + W\Delta \\ (D^d)^2\Gamma & (D^d)^2\Delta \end{pmatrix},$$

where $W, S, \Lambda, \Sigma, \Gamma, \Delta$ as defined before. Obviously, Q is nilpotent, and so it is quasinilpotent. According to Lemma 4.1, $M^{\textcircled{d}} = P^{\textcircled{d}}$, required. □

Corollary 4.3. *If $BD = 0, BC = 0$ and $A^*B = 0$, then M has generalized core-EP inverse. In this case,*

$$M^{\textcircled{d}} = \begin{pmatrix} (A^d)^2\Delta & (A^d)^2\Gamma \\ (D^d)^2\Sigma + W\Delta & (D^d)^2\Lambda + W\Gamma \end{pmatrix},$$

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where

$$\begin{aligned}
W &= \sum_{i=0}^{\infty}(D^d)^{i+3}CA^iA^{\pi} + \sum_{i=0}^{\infty} D^iD^{\pi}C(A^d)^{i+3} - (D^d)^2CA^d - D^dC(A^d)^2, \\
S &= \sum_{i=0}^{\infty}(D^d)^{i+2}CA^iA^{\pi} + \sum_{i=0}^{\infty} D^iD^{\pi}C(A^d)^{i+2} - D^dCA^d, \\
\Lambda &= D^2D^{\circledS} + [D^{\pi}S[A^d]^{\dagger} - D^2D^{\circledS}S]A^2A^{\circledS}\left(I + (e_{D^d}S[A^d]^{\dagger})^*(e_{D^d}S[A^d]^{\dagger})\right)^{-1} \\
&\quad (e_{D^d}S[A^d]^{\dagger})^*, \\
\Sigma &= (D^{\pi}S[A^d]^{\dagger} - D^2D^{\circledS}S)A^2A^{\circledS}[I + (e_{D^d}S[A^d]^{\dagger})^*(e_{D^d}S[A^d]^{\dagger})]^{-1}, \\
\Gamma &= A^2A^{\circledS}[I + (e_{D^d}S[A^d]^{\dagger})^*(e_{D^d}S[A^d]^{\dagger})]^{-1}(e_{D^d}S[A^d]^{\dagger})^*, \\
\Delta &= A^2A^{\circledS}[I + (e_{D^d}S[A^d]^{\dagger})^*(e_{D^d}S[A^d]^{\dagger})]^{-1}.
\end{aligned}$$

Proof. Obviously, $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} M \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} D & C \\ B & A \end{pmatrix}$. Applying Theorem 4.2 to the matrix $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$, we see that $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$ has generalized core-EP inverse. This implies that M has generalized core-EP inverse. Additionally,

$$M^{\circledS} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} D & C \\ B & A \end{pmatrix}^{\circledS}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Therefore we complete the proof by Theorem ???. □

Lemma 4.4. *Let $a \in \mathcal{A}^{qnil}, b \in \mathcal{A}^{\circledS}$. If $ab = ba, a^*b = ba^*$ and $b^*(ab) = (ba)b^*$, then $a + b \in \mathcal{A}^{\circledS}$. In this case,*

$$(a+b)^{\circledS} = (1 + ab^d)^{-1}b^{\circledS}.$$

Proof. Since $ab = ba$, it follows by [26][Theorem 2.3] that $ab^d = b^da$. Likewise, we have $a(b^d)^* = a(b^*)^d = (b^*)^da = (b^d)^*a$. Since $(ab^*)b = (b^*a)b = b^*(ab) = (ba)b^* = b(ab^*)$, by using [26][Theorem 2.3] again, $(ab^*)b^d = b^d(ab^*)$. Then $b^*(ab^d) = (b^*a)b^d = (ab^*)b^d = b^d(ab^*) = (ab^d)b^*$. Hence, $(b^d)^*(ab^d) = (ab^d)(b^d)^*$. In view of Theorem 1.1 and [7, Corollary 3.4], $b^{\circledS}(ab^d) = (ab^d)b^{\circledS}$. We directly verify that

$$\begin{aligned}
(a+b)b^d(1+ab^d)^{-1} &= (ab^d + bb^d)(1+ab^d)^{-1} \\
&= bb^d(1+ab^d)(1+ab^d)^{-1} = bb^d \\
&= (1+ab^d)^{-1}(1+ab^d)bb^d \\
&= (1+ab^d)^{-1}b^d(a+b) \\
&= b^d(1+ab^d)^{-1}(a+b), \\
(a+b)[b^d(1+ab^d)^{-1}]^2 &= (bb^d)b^d(1+ab^d)^{-1} = b^d(1+ab^d)^{-1}, \\
(a+b) - [b^d(1+ab^d)^{-1}](a+b)^2 &= (a+b) - bb^d(a+b) \\
&= (b - b^db^2) + (1 - bb^d)a \\
&\in \mathcal{A}^{qnil}.
\end{aligned}$$

This implies that $(a+b)^d = b^d(1+ab^d)^{-1}$. We easily verify that

$$b^d(1+ab^d) = (1+ab^d)b^d, (b^d)^*(1+ab^d) = (1+ab^d)(b^d)^*.$$

Then we derive that

$$b^d(1+ab^d)^{-1} = (1+ab^d)^{-1}b^d, (b^d)^*(1+ab^d)^{-1} = (1+ab^d)^{-1}(b^d)^*.$$

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According to [7][Theorem 3.5], $(a+b)^d \in \mathcal{A}^{\#}$ and

$$[(a+b)^d]^{\#} = (b^d)^{\#}[(1+ab^d)^{-1}]^{\#}.$$

$$\begin{aligned}(a+b)^{\circledD} &= [(a+b)^d]^2[(a+b)^d]^{\#}\\ &= [(a+b)^d]^2(b^d)^{\#}(1+ab^d)\\ &= [(a+b)^d]^2b^2(b^d)^2(b^d)^{\#}(1+ab^d)\\ &= [b^d(1+ab^d)^{-1}b^d(1+ab^d)^{-1}]b^2b^{\circledD}(1+ab^d)\\ &= (1+ab^d)^{-2}b^{\circledD}(1+ab^d)\\ &= (1+ab^d)^{-1}b^{\circledD}.\end{aligned}$$

□

Theorem 4.5. *If $BC=0, CB=0, CA=DC, CA^*=D^*C$ and $D^*CA=DCA^*$, then M has generalized core-EP inverse. In this case,*

$$M^{\circledD} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{aligned}\alpha &= (A^d)^2\Lambda + W\Gamma,\\ \beta &= (A^d)^2\Sigma + W\Delta,\\ \gamma &= -CA^d[(A^d)^2\Lambda + W\Gamma] + (I-CS)(D^d)^2\Gamma,\\ \delta &= -CA^d[(A^d)^2\Sigma + W\Delta] + (I-CS)(D^d)^2\Delta\end{aligned}$$

and $W, S, \Lambda, \Sigma, \Gamma$ and Δ constructed as in Theorem 4.2.

Proof. Write $M = P + Q$, where

$$P = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}.$$

Then

$$P^d = \begin{pmatrix} A^d & S \\ 0 & D^d \end{pmatrix},$$

where $S = \sum\limits_{i=0}^{\infty}(A^d)^{i+2}BD^iD^{\pi} + \sum\limits_{i=0}^{\infty}A^iA^{\pi}B(D^d)^{i+2} - A^dBD^d$. Hence, we have

$$\begin{aligned}I - QP^d &= I - \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} A^d & S \\ 0 & D^d \end{pmatrix}\\ &= \begin{pmatrix} I & 0 \\ -CA^d & I-CS \end{pmatrix}.\end{aligned}$$

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We easily check that

$$\begin{aligned} PQ &= \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} = \begin{pmatrix} BC & 0 \\ DC & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ CA & CB \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = QP, \\ P^*Q &= \begin{pmatrix} A^* & 0 \\ B^* & D^* \end{pmatrix}\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ D^*C & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ CA^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}\begin{pmatrix} A^* & 0 \\ B^* & D^* \end{pmatrix} = QP, \\ P^*(QP) &= \begin{pmatrix} A^* & 0 \\ B^* & D^* \end{pmatrix}\begin{pmatrix} 0 & 0 \\ CA & CB \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ D^*CA & D^*CB \end{pmatrix} \\ &= \begin{pmatrix} BCA^* & 0 \\ DCA^* & 0 \end{pmatrix} = \begin{pmatrix} BC & 0 \\ DC & 0 \end{pmatrix}\begin{pmatrix} A^* & 0 \\ B^* & D^* \end{pmatrix} = (PQ)P^*. \end{aligned}$$

Since A and D have generalized core-EP inverses, it follows by Theorem 3.4 that P has generalized core-EP inverse, and that

$$P^{\circledD} = \begin{pmatrix} (A^d)^2\Lambda + W\Gamma & (A^d)^2\Sigma + W\Delta \\ (D^d)^2\Gamma & (D^d)^2\Delta \end{pmatrix},$$

where $W, S, \Lambda, \Sigma, \Gamma, \Delta$ as defined before. Obviously, $Q^2 = 0$, and so it is quasinilpotent. According to Lemma 4.4,

$$\begin{aligned} M^{\circledD} &= (I + QP^d)^{-1}P^{\circledD} \\ &= (I - QP^d)P^{\circledD} \\ &= \begin{pmatrix} I & 0 \\ -CA^d & I - CS \end{pmatrix}\begin{pmatrix} (A^d)^2\Lambda + W\Gamma & (A^d)^2\Sigma + W\Delta \\ (D^d)^2\Gamma & (D^d)^2\Delta \end{pmatrix} \\ &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \end{aligned}$$

where

$$\begin{aligned} \alpha &= (A^d)^2\Lambda + W\Gamma, \\ \beta &= (A^d)^2\Sigma + W\Delta, \\ \gamma &= -CA^d[(A^d)^2\Lambda + W\Gamma] + (I - CS)(D^d)^2\Gamma, \\ \delta &= -CA^d[(A^d)^2\Sigma + W\Delta] + (I - CS)(D^d)^2\Delta. \end{aligned}$$

as required. □

Corollary 4.6. *If $BC = 0, CB = 0, AB = BD, A^*B = BD^*$ and $A^*BD = ABD^*$, then M has generalized core-EP inverse. In this case,*

$$M^{\circledD} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

where

$$\begin{aligned} \alpha &= -BD^d[(D^d)^2\Sigma + W\Delta] + (I - BS)(A^d)^2\Delta, \\ \beta &= -BD^d[(D^d)^2\Lambda + W\Gamma] + (I - BS)(A^d)^2\Gamma, \\ \gamma &= (D^d)^2\Sigma + W\Delta, \\ \delta &= (D^d)^2\Lambda + W\Gamma \end{aligned}$$

and $W, S, \Lambda, \Sigma, \Gamma$ and Δ constructed as in Corollary 4.3.

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Proof. Applying Theorem 4.5 to the matrix $\begin{pmatrix} D & C \\ B & A \end{pmatrix}$, we see that it has generalized core-EP inverse. Analogously to Corollary 4.3, we have

$$M^{\textcircled{d}} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} D & C \\ B & A \end{pmatrix}^{\textcircled{d}} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Therefore we obtain the result by Theorem 4.5. □

Conflicts of Interest: The authors declare there is no conflicts of interest.

Data Availability Statement: The data used to support the findings of this study are included within the article.

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