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Keywords: uniform polytopes; 600-cell; grand antiprism

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The Symmetry Group of the Grand Antiprism

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Abstract: The grand antiprism **A** is an outlier among the uniform 4-polytopes, since it is not obtainable from Wythoff's construction. Its symmetry group $G(\mathbf{A})$ has been incorrectly described as $[[10,2^+,10]]$ or even as an 'ionic diminished Coxeter group'. In fact, $G(\mathbf{A})$ is another group of order 400, namely the group $\pm[D_{10}\times D_{10}]\cdot 2$, in the notation of Conway and Smith.

Keywords: uniform polytopes; 600-cell; grand antiprism

1. Introduction

A convex d-polytope **Q** in Euclidean space is *uniform* if its symmetry group $G(\mathbf{Q})$ is transitive on its vertices, and if, furthermore, each facet of **Q** is uniform. To initiate this recursive condition in a geometrically pleasing way, we agree that a uniform polygon should be *regular*.

It is easy to see that all edges of **Q** have the same length. However, for $d \geqslant 3$, **Q** may well have different kinds of facets. For example, the pentagonal antiprism $\mathbf{P}_5$ on the right in Figure 1 is bounded by two regular pentagons $\{5\}$ and ten equilateral triangles $\{3\}$. A *regular polytope* **Q**, which by definition has a symmetry group transitive on flags, is certainly uniform. Consider the regular tetrahedron $\{3,3\}$, also in Figure 1.

In ordinary space $\mathbb{E}^3$, the uniform convex polyhedra include the five Platonic solids, the thirteen Archimedean solids, as well as n-gonal prisms and antiprisms, for $n \geqslant 3$. There is a little redundancy here: the 3-gonal antiprism and 4-gonal prism have more symmetry than first expected, being the regular octahedron $\{3,4\}$ and cube $\{4,3\}$, respectively. For an excellent discussion of these polyhedra, their groups, as well as uniform tessellations of the plane, we refer to Coxeter's paper [7]. After a remarkable break starting with World War II, Coxeter explored uniform polytopes of higher dimension in two follow-up articles [9,10] appearing in the 1980s. An essential tool throughout is *Wythoff's construction* for uniform polytopes.

In [9, Section 2.8], we find a discussion of the *grand antiprism* **A**, discovered by J. H. Conway and M. Guy in 1965 [4]. This remarkable object is the only uniform 4-polytope which cannot be constructed by Wythoff's construction, even accepting Coxeter's extension of the method to rotation groups. Coxeter also described the symmetry group $G(\mathbf{A})$ as

$$[[10,2^+,10]] \simeq G^{4,4,10}.$$

In fact, this is the wrong group of order 400, an error which has percolated into the literature.

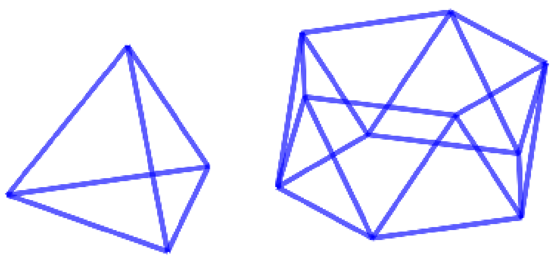

Figure 1. The tetrahedron and pentagonal antiprism.

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In Section 2 we use Wythoff's construction to construct the 600-cell **S**, then find **A** inscribed in it. A correct description of the symmetry group $G(\mathbf{A})$ (as a semidirect product $[5,2,5] \rtimes C_4$) appears in Proposition 1. Actually, this was already derived in a slightly different way in [17]. Nevertheless, that paper still seems to suggest $[[10,2^+,10]]$ as the group.

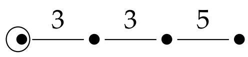
Now the finite subgroups of $GO_4(\mathbb{R})$ have been variously classified, but it seems that the catalogue recently appearing in [5, Chapter 4] is complete and corrects small errors or oversights in earlier attempts, such as that in [14]. In order to help the reader understand all this, we have reviewed in Section 4 how unit quaternions are used to describe isometries in $\mathbb{E}^3$ and $\mathbb{E}^4$. At the end of this long but necessary digression, we show in Example 3 that

$$G(\mathbf{A}) \simeq \pm[D_{10} \times D_{10}] \cdot 2,$$

here using the notation of [5, Table 4.3].

2. The 600-cell $\mathbf{S} = \{3,3,5\}$ and the Grand Antiprism A

A useful way to understand the grand antiprism **A** is to see it inscribed in the 600-cell $\mathbf{S} = \{3,3,5\}$, so we begin by describing the latter regular 4-polytope. The symmetry group $G(\mathbf{S})$ is the (linear) Coxeter group $H_4 = [3,3,5]$, with generating reflections r_0, r_1, r_2, r_3 corresponding to the nodes of the diagram



(1)

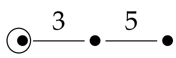
The ring decorating the first node is an instruction to perform *Wythoff's construction*. In this instance, we choose a non-zero *base vertex* **v** fixed by r_1, r_2, r_3. The regular polytope **S** is then the convex hull of the H_4-orbit of **v**.

If, as in [18, Section 5A], we identify an involutory isometry like r_j with its fixed space, or *mirror*, we see that **v** spans the *Wythoff space*

$$W = r_1 \cap r_2 \cap r_3 \tag{2}$$

corresponding to the unringed nodes in diagram (1).

A linear Coxeter group like H_4 has special properties which serve to make the construction recursive. In particular, the subgroup of H_4 which fixes W pointwise is generated by the reflections indicated in (2). Thus the number of vertices in **S** is the index of the subgroup $\langle r_1, r_2, r_3 \rangle$. Furthermore, this subgroup is itself the Coxeter group $[3,5]$ corresponding to the diagram obtained by deleting the first node:



(3)

We conclude that there are $14400/120 = 120$ vertices. The diagram in (3) arises by transferring the ring in (1) to the second node. This means that the vertex-figure at each vertex of $\{3,3,5\}$ is a regular icosahedron $\{3,5\}$. The orthogonal projection behind Figure 2 maps **v** to the centre of this isosahedron. The red edges **vu** and **vw** serve as a reminder that **v** lies outside the hyperplane supporting the vertex-figure. We shall soon see that ...**w v u**... is really part of a planar decagon.

One can read much more from the diagram (1). For instance, just by deleting the right-most node, we find that all facets of **S** are regular tetrahedra $\{3,3\}$, and that there are $600 = 14400/24$ of them.

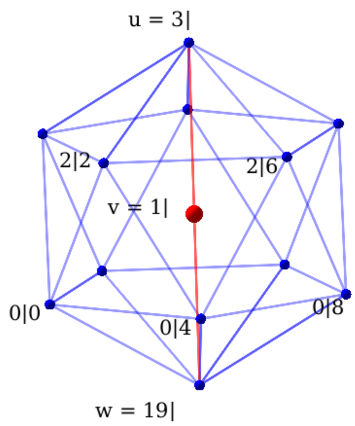


Figure 2. The vertex-figure for $\mathbf{v} = 1|\_\_$ in **S**.

We now draw on [11] and [8] to give a more explict description of both **S** and its group H_4 (as a subgroup of $GO_4(\mathbb{R})$). Depending on our algebraic needs, it will be useful at times to regard a point $\mathbf{x} = (x_0, x_1, x_2, x_3) \in \mathbb{E}^4$ as either a pair (u, v) of complex numbers (so $u = x_0 + x_1\imath, v = x_2 + x_3\imath$) or as a single quaternion $x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} = u + v\mathbf{j}$. In this spirit, we find in [11, Section 4.6] a description of the 120 vertices of **S** as pairs of complex numbers. We need $\epsilon = \exp(\pi/10)$ and the related angle $\lambda = \frac{1}{2}\arctan 2 \doteq 31.72^\circ$, so that $\cos\lambda = \tau^{\frac{1}{2}}5^{-\frac{1}{4}}$, $\sin\lambda = \tau^{-\frac{1}{2}}5^{-\frac{1}{4}}$, with $\tau = (1+\sqrt{5})/2$ the *Golden ratio*.

Here then are the 120 vertices of **S** in a slight modification of Coxeter's notation. The parameters μ, ν are residues modulo 10:

$$2\mu + 1|\_\_ = (\epsilon^{2\mu+1}, 0), \quad \_\_|2\nu + 1 = (0, \epsilon^{2\nu+1}) \tag{4a}$$

$$2\mu|2\nu = (\epsilon^{2\mu}\cos\lambda, \epsilon^{2\nu}\sin\lambda) \quad (\mu + \nu \text{ even}) \tag{4b}$$

$$2\mu|2\nu = (\epsilon^{2\mu}\sin\lambda, \epsilon^{2\nu}\cos\lambda) \quad (\mu + \nu \text{ odd}) \tag{4c}$$

Remark 1. *We have indeed* 120 *points of norm* 1 *in* $\mathbb{E}^4$*. Since* **S** *is centrally symmetric, the vertices occur in* 60 *antipodal pairs. A special property of* **S** *is that each pair is normal to a hyperplane of symmetry for the polytope. These* 60 *reflections comprise the single conjugacy class of reflections in* H_4*. Thus (in* 14400 *ways) we can extract from the vertices a simple system of roots for* H_4 *[15, Chapter 1.3]. That is, we can find four vertices to serve as 'outer' unit normals* $\mathbf{n}_j$ *for the mirrors of the generating reflections* r_j*,* $(j = 0, 1, 2, 3)$*. We choose*

$$\mathbf{n}_0 = 7|\_\_, \mathbf{n}_1 = 16|2, \mathbf{n}_2 = \_\_|9, \mathbf{n}_3 = \_\_|17\,. \tag{5}$$

Note, for instance, that

$$r_0 : (u, v) \mapsto (-\overline{u}\epsilon^{14}, v).$$

A suitable base vertex (fixed by r_1, r_2, r_3) is then $\mathbf{v} = 1|\_\_$. The base edge joins $\mathbf{v}$ to $\mathbf{u} = (\mathbf{v})r_0 = 3|\_\_$. Clearly, the angle between (vectors) $\mathbf{v}, \mathbf{u}$ is $2\pi/10$, and each edge of $\mathbf{S}$ has length

$$2\sin\frac{\pi}{10} = \tau^{-1}.$$

There is now enough algebraic detail in place for the reader to check, with effort, our subsequent calculations. (We often seek refuge in GAP [1].) □

First off, the central symmetry $\mathbf{x} \mapsto -\mathbf{x}$ factors as

$$z = (r_0 r_1 r_2 r_3)^{15}$$

in H_4 [8, p. 226]. The icosahedral vertex-figure at $\mathbf{v}$, say, has its own central symmetry $(r_1 r_2 r_3)^5$. Using Figure 2 and our earlier calculations, we see [11, Section 4.6] that

$$s = (r_1 r_2 r_3)^5 r_0 \tag{6}$$

cyclically moves ...**w v u**... one step along a planar convex decagon A (contained in the 1-skeleton of $\mathbf{S}$). We note that

$$s : (u, v) \mapsto (u\epsilon^2, -v).$$

Comparing (4a), we see that the vertices $2j+1|\_\_$ of A lie in the x_0x_1-plane, while the vertices $\_\_|2j+1$ of an orthogonal convex decagon B lie in the x_2x_3-plane.

Remark 2. *Since the icosahedron has 6 pairs of antipodal vertices, each vertex of $\mathbf{S}$ lies on 6 planar decagons, and altogether there are 72 such decagons. Furthermore, one can select 12 vertex-disjoint decagons to exhaust the vertices of $\mathbf{S}$. The 12 circumcircles belong to a Hopf fibration of $\mathbb{S}^3$ [11, Section 4.9].* □

Definition 1. *The grand antiprism $\mathbf{A}$ is the convex hull of the 100 vertices of $\mathbf{S}$ which remain after deleting two orthogonal decagons.*

Let us remove A and B, leaving the points $2\mu|2\nu$. Since $\mathbf{S}$ is inscribed in $\mathbb{S}^3$, these 100 points survive as the vertices of their convex hull $\mathbf{A}$. To survey the facets of $\mathbf{A}$, we consult [11, Section 4.6, Exercise 2].

Each edge of the decagon A is surrounded in $\mathbf{S}$ by 5 tetrahedral facets; and a vertex such as $\mathbf{v}$ is common to 10 further tetrahedra whose bases form a belt running in zig-zag fashion around the middle of the icosahedral vertex-figure, as in Figure 2. In this way A and B each meet 150 tetrahedra. These 300 facets of $\mathbf{S}$ are lost when we construct $\mathbf{A}$.

If we first remove $\mathbf{v}$ from $\mathbf{S}$, its icosahedral vertex-figure (Figure 2) becomes a facet of the new convex hull. If we next remove $\mathbf{u}, \mathbf{w}$ adjacent to $\mathbf{v}$, we further truncate this icosahedron back to the pentagonal antiprism $\mathbf{P}_5$ whose lateral triangles are those in the belt just described. In this way, the facets of $\mathbf{A}$ include a ring $\mathcal{R}_A$ of 10 copies of $\mathbf{P}_5$. One pentagonal face on an antiprism arising this way has vertices

$$2\mu|0, 2\mu|4, 2\mu|8, 2\mu|12, 2\mu|16 \ (\mu \text{ even}),$$

while the other pentagon is

$$2\mu + 2\alpha|2, 2\mu + 2\alpha|6, 2\mu + 2\alpha|10, 2\mu + 2\alpha|14, 2\mu + 2\alpha|18,$$

with α alternating ± 1 as we run round the ring. The 50 vertices of $\mathcal{R}_A$ are the points $2\mu|2\nu$ with $\mu + \nu$ even, from (4b). The symmetry s in (6) moves $\mathcal{R}_A$ one step along itself.

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The complementary ring $\mathcal{R}_B$ derived from B is disjoint from $\mathcal{R}_A$ and provides 10 more copies of $\mathbf{P}_5$. Its 50 vertices are the points $2\mu|2\nu$, with $\mu + \nu$ odd, found in (4c).

The 100 triangular faces in each ring form a non-regular toroidal map of Schläfli type $\{3, 6\}$ [11, Figures 4.6B, 4.6C]. Each triangle on $\mathcal{R}_A$ is the base of a tetrahedral facet of $\mathbf{S}$ whose apex is on $\mathcal{R}_B$. In this way, $\mathbf{A}$ inherits 100 tetrahedral facets, let us say of type A. In complementary fashion, $\mathbf{A}$ acquires from $\mathbf{S}$ the 100 tetrahedral facets of type B. The final 100 facets of $\mathbf{A}$ are tetrahedra of type AB. Each has one edge on $\mathcal{R}_A$ with the opposite edge on $\mathcal{R}_B$. Tetrahedra of type AB have vertices

$$2\mu|2\nu, 2\mu|2\nu + 2, 2\mu + 2|2\nu, 2\mu + 2|2\nu + 2 \text{ (any } \mu, \nu). \tag{7}$$

Altogether, $\mathbf{A}$ has 500 edges, 20 regular pentagons and 700 equilateral triangles as faces of lower rank. Each vertex-figure is non-uniform and arises as the convex hull of the 10 points which remain when an edge is deleted from an icosahedron $\{3, 5\}$.

It is still not quite clear that $\mathbf{A}$ is uniform, so we take a close look at its symmetry group $G = G(\mathbf{A})$. Notice that G is a subgroup of H_4. It coincides with the (set-wise) stabilizer of the decagons $\{A, B\}$.

Let $K \leqslant G$ be the subgroup that takes A into A (and thus B into B). First of all, K contains every reflection r in a hyperplane orthogonal to a pair of antipodal vertices of A. This r induces a reflection symmetry of A while fixing B pointwise; and the five reflections coming from A this way generate a dihedral group of order 10.

In addition, the central symmetry $z \in K$, so K contains rz, which also acts by reflection on A, though as a half-turn on B. (One can view rz as a half-turn about a vertex of A in the 3-space spanned by A and some vertex of B.)

Let us choose the new reflection r to have normal $\mathbf{n} = 15|\_$. Then $D(A) = \langle r_0, rz \rangle$ acts on A as the full dihedral symmetry group D_{20} of order 20, though half its elements act as half-turns on B. Similarly, we have $D(B) = \langle r_2 z, r_3 \rangle \simeq D_{20}$ acting on decagon B.

Note that $z = (r_0 rz)^5 = (r_2 z r_3)^5 \in D(A) \cap D(B)$. These two dihedral groups commute with one another and intersect in a centre $\langle z \rangle$ of order 2. Thus K has order 200.

In [9, p. 590], Coxeter observed that

$$K \simeq [10, 2^+, 10] = \langle a_0, a_1 a_2, a_3 \rangle, \tag{8}$$

the 'ionic' subgroup of the Coxeter group $[10, 2, 10] = \langle a_0, a_1, a_2, a_3 \rangle$ with diagram

(Compare [9, p. 569] and [16, p. 239]. The whimsical adjective 'ionic' comes from the fact that the reflections a_j have determinant -1, so that words of even length like $a_1 a_2$ give determinant $+1$, thereby reducing the 'negative charge'.)

To verify (8), first take $a_0 = r_0, a_3 = r_3$; but let a_1 be the reflection acting on A as rz but *fixing* B. Likewise let a_2 act as $r_2 z$ on B but fix A. (Note that a_1, a_2 do *not* belong to G.) We get (8) upon noting that $a_1 a_2 = r r_2 z$.

It is curious that with the involutory generators $a_0, a_1 a_2, a_3$, K is isomorphic to the full automorphism group of the regular map $\{10, 10 \,|\, 2\}$ [12, Section 8.5].

Any $g, h \in G$ which take A to B must also take B to A, so $gh \in K$. Thus G has order 400. The crucial question is how G extends K.

In [9, Section 2.8], Coxeter describes a half-turn t which is meant to do the job. Certainly various half-turns t swap A and B. However, no such t can lie in G (or in H_4)! To verify this, we note that the supposed half-turn would have to map (u, v) to either (vy^{-1}, uy) or $(\overline{v}y, \overline{u}y)$, for some complex number y of norm 1. But $0|0 = (\cos\lambda, \sin\lambda)$ must map to some $2\mu|2\nu = (\epsilon^{2\mu}\sin\lambda, \epsilon^{2\nu}\sin\lambda)$, with $\mu + \nu$ odd. We would need $\epsilon^{2\mu \pm 2\nu} = 1$, which is impossible for $\mu + \nu$ odd.

If we do move sideways and adopt the half-turn $t : (u, v) \mapsto (\bar{v}y, \bar{u}y)$, with $y = \epsilon^4$, then we have an involution which (by conjugation) swaps a_0, a_3 while fixing $a_1 a_2$. This is just what is needed to 'double' the group $K \simeq [10, 2^+, 10]$ and so arrive at

$$[[10, 2^+, 10]] \simeq G^{4,4,10}.$$

(See [16, pp. 255ff] and [9, p. 590].) The group on the left denotes the semidirect product $[10, 2^+, 10] \rtimes \langle t \rangle$, which indeed is isomorphic to $G^{4,4,10}$, one of a family of groups defined by a special sort of presentation [12, p. 96]. In this case, in terms of the generators $a = a_1 a_2 a_0, b = a_0 t, c = t a_0 a_1 a_2$, we have defining relations

$$a^{10} = b^4 = c^4 = (ab)^2 = (bc)^2 = (ca)^2 = (abc)^2 = 1$$

[9, Equation 2.39]. Note that $\langle t, a_2 \rangle \simeq D_8$. Since $G(\mathbf{A})$ has no such subgroup, we confirm once more that $G(\mathbf{A})$ cannot be $[[10, 2^+, 10]]$.

On the other hand, we can exhibit a symmetry $p \in H_4$ of period 4 which swaps A and B. Taking $\mu = 0, \nu = 1$ in (7), we see that

$$0|2, 0|4, 2|4, 2|2 \tag{9}$$

are vertices of a facet of type AB for $\mathbf{A}$. This regular tetrahedron is a facet of $\mathbf{S}$, so it admits the Petrie symmetry p which cyclically permutes the vertices as they appear in (9). Thus p has order 4, and in fact also permutes the roots $\mathbf{n}_0, \mathbf{n}_3, \mathbf{n}, \mathbf{n}_2$ in a 4-cycle. Moreover, p swaps A and B, and $p \in G(\mathbf{A})$.

It is now finally clear that $G(\mathbf{A})$ is vertex-transitive, so that $\mathbf{A}$ really is uniform!

Note that the subgroup $\langle r_0, r, r_2, r_3 \rangle$ of K is the linear Coxeter group $[5, 2, 5]$ of order 100. Conjugation by p in G will transform its generators in a 4-cycle (r_0, r_3, r, r_2). Furthermore, p^2 lies in K but not in its subgroup $[5, 2, 5]$. We have

Proposition 1. *The grand antiprism is uniform. Its symmetry group $G(\mathbf{A})$ is the semidirect product*

$$[5, 2, 5] \rtimes C_4.$$

Remark 3. *It is easy to check that $G(\mathbf{A})$ has defining relations*

$$r^2 = p^4 = (p^{-2} r p^2 r)^5 = (p^{-1} r p r)^2 = 1.$$

The group $G(\mathbf{A})$ was correctly described as such a semidirect product in [17, Section 2]. The authors there used quaternion methods, which we turn to in Section 4. However, they seem to continue the mislabelling of $G(\mathbf{A})$ as 'the ionic diminished Coxeter group $[10, 2^+, 10]$'.

Considering the toroidal maps on the surfaces of the rings $\mathcal{R}_A, \mathcal{R}_B$, it is quite natural that the symmetry p is induced by an affine function of the vertex symbols:

$$p : 2\mu|2\nu \mapsto (2\nu - 2)|(4 - 2\mu) \pmod{20}.$$

□

We conclude this section by describing the subgroups of $G(\mathbf{A})$ which preserve some substructures of $\mathbf{A}$.

The vertex $2|2$ is typical and is fixed in $G(\mathbf{A})$ by the subgroup $\langle r_0, r_3 \rangle \simeq C_2 \times C_2$.

The point $2|2$ belongs to 2 facets of type B. One of these has base triangle $0|2, 2|4, 4|2$ on $\mathcal{R}_B$ and is fixed in $G(\mathbf{A})$ by $\langle r_0 \rangle$. Each tetrahdron of type A or B in $\mathbf{A}$ has, in this way, a stabilizer generated by a single reflection.

However, a tetrahedron of type AB has a stabilizer of order 4 generated by a Petrie symmetry, just as p does for the tetrahedron with the vertices in (9).

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It is clear that $G(\mathbf{A})$ acts transitively and faithfully on the 20 pentagonal antiprisms. Thus each such facet must inherit its full symmetry group of order 20 from $G(\mathbf{A})$. For instance, the group of the pentagonal antiprism with vertices

$$\begin{array}{ccccccccccc} & 2|2 & & 2|6 & & 2|10 & & 2|14 & & 2|18 & \\ 0|0 & & 0|4 & & 0|8 & & 0|12 & & 0|16 & & 0|0 \end{array}$$

is generated by the reflection r_3 and the half-turn $h = p^2$ about the centre of the edge 2|2 0|4. (The reflection r_0 fixes the upper pentagon point-wise but maps the pentagonal antiprism itself to one of its neighbours in the ring $\mathcal{R}_A$.) We refer to Section 3 for more on the symmetry group $[5, 2^+]$ for a pentagonal antiprism.

From the action of $G(\mathbf{A})$ on the 20 antiprismatic facets we obtain this faithful permutation representation:

$$\begin{array}{rcl} r & \mapsto & (1,10)(2,9)(3,8)(4,7)(5,6) \\ p & \mapsto & (1,12)(2,11,10,13)(3,20,9,14)(4,19,8,15)(5,18,7,16)(6,17) \end{array}$$

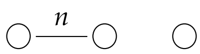
Note that

$$(rp)^2 \mapsto (1,2,3,4,5,6,7,8,9,10)(11,12,13,14,15,16,17,18,19,20)$$

simultaneously rotates each ring through a tenth of a turn.

3. More on Wythoff's Construction

In [6] Coxeter extended Wythoff's construction to general Coxeter groups G of finite or affine type, with nodes of the diagram ringed in any way. (See also [9, Section 2.4], [7, Section 1.5] and [11, Section 2.4].) All this is aimed at enumerating uniform polytopes and tessellations in Euclidean space. Actually, Coxeter also employed a variant of the construction based on just the rotation subgroup G^+, indicated by replacing all nodes in the diagram by empty rings. The choice of base vertex $\mathbf{x}$ required to guarantee uniformity is now trickier and sometimes impossible, although the construction always works in $\mathbb{E}^3$. In particular, the n-gonal antiprism $\mathbf{P}_n$ is produced by the diagram



(10)

The underlying Coxeter group $G = [n, 2]$ has order $4n$, so G^+ has order $2n$ and indeed is isomorphic to the dihedral group D_{2n} (of order $2n$). We have $G^+ = \langle q, h \rangle$, where q is a rotation through $2\pi/n$ about some axis l, while h is a half-turn about an axis m meeting l at right angles. A base vertex $\mathbf{x}$ can now be chosen to produce a pair of regular n-gons separated by a belt of $2n$ equilateral triangles running in zig-zag fashion. The pentagonal antiprism ($n = 5$) is illustrated in Figure 1. For $n > 3$ the full symmetry group of the antiprism does have order $4n$ but is not quite the group G. Instead, $G(\mathbf{P}_n)$ is the group $[2n, 2^+]$, once more following the notation of [16, Section 11.4] and [9]. Beginning with the larger Coxeter group $[2n, 2] = \langle u_0, u_1, u_2 \rangle$, we need the ionic subgroup

$$[2n, 2^+] = \langle u_0, u_1 u_0 u_1, u_1 u_2 \rangle,$$

of order $4n$. In fact, as we saw in Section 2, we find that this group is generated by the reflection u_0 and the half-turn $u_1 u_2$. When n is odd, this group actually is *isomorphic* to $[n, 2]$, though geometrically different.

Remark 4. *We noted earlier that Wythoff's construction, including the extension to rotation groups, gives all uniform polyhedra in $\mathbb{E}^3$. In [4], Conway and Guy apparently used a local approach to construct all uniform 4-polytopes [9, p. 588]: they computed the dihedral angles for all uniform polyhedra, then tried to assemble these facets so that around each edge the dihedral angles sum to less than 2π. They found that only one uniform*

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polytope in $\mathbb{E}^4$, the grand antiprism, eludes the more general Wythoff's construction. This is simply because $G(\mathbf{A})$ is neither a Coxeter group nor a rotation subgroup.

The grand antiprism is discussed in [3, pp. 402–403], with some useful figures. However, I could not find there a description of the group, although Conway surely knew all about it. Nor can I find that he wrote about the group elsewhere, so it was Coxeter who initiated the discussion in [9].

There are also some very fine illustrations in the Wikipedia article Grand Antiprism. However, at the time of writing, that article also mislabels the group $G(\mathbf{A})$. □

Remark 5. *One can actually use the method to manufacture a vertex-transitive polytope $\mathbf{Q}$ by applying the 'incorrect' group $G = [[10, 2^+, 10]]$ to the same base vertex $0|0$, as for $\mathbf{A}$. The orbit still has size 100, containing the vertices of ring $\mathcal{R}_A$ along with their images under the spurious half-turn t. We seem to obtain the points described in (4b) and (4c); however, we must now take $\mu + \nu$ even in both cases. The convex hull $\mathbf{Q}$ of this orbit still has the 100 vertices; but it cannot be uniform since it has edges of different lengths. For instance, there is an edge of length τ^{-1} in $\mathbf{Q}$ from $0|4$ to $0|0$, just as in $\mathbf{A}$. However, there is also a slightly longer edge of length $2^{\frac{1}{2}}\tau^{-\frac{1}{2}}5^{-\frac{1}{4}}$ from $0|4$ to the new vertex $(2|2)t$.*

The geometric effect of the spurious half-turn t is to map each pentagon from ring $\mathcal{R}_A$ to a pentagon coplanar and concentric with a pentagon in ring $\mathcal{R}_B$, but turned a half-turn with respect to the latter. We see again why t cannot lie in H_4.

Disappointed, we conclude that there is no undiscovered uniform 4-polytope missed by Conway and Guy!
□

4. Quaternions and Finite Isometry Groups in $\mathbb{E}^4$

In order to locate $G(\mathbf{A})$, or $[[10, 2^+, 10]]$ for that matter, in a catalogue of all finite isometry groups on $\mathbb{E}^4$, that is, within the finite subgroups of $GO_4(\mathbb{R})$, we need some tools from the algebra of quaternions. We follow [5, Chapter 4] and [11, Chapter 6].

Recall first that the conjugate of $\mathbf{x} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbb{E}^4$ is $\tilde{\mathbf{x}} = x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k}$, for which we have $\widetilde{\mathbf{xz}} = \tilde{\mathbf{z}}\tilde{\mathbf{x}}$. The *norm* or squared length of $\mathbf{x}$ is

$$N(\mathbf{x}) = \mathbf{x}\tilde{\mathbf{x}} = \tilde{\mathbf{x}}\mathbf{x},$$

which, crucially, is multiplicative. For a *unit quaternion* $\mathbf{x}$ we have $\mathbf{x}^{-1} = \tilde{\mathbf{x}}$.

The group $\mathbb{P}$ of *unit* quaternions (also known as $\mathbf{Spin}_3$) is a double cover of $SO_3(\mathbb{R})$ [11, 6.43]. To see this we first identify $\mathbb{E}^3$ with the space of *pure* quaternions $\mathbf{z}$ (for which $z_0 = 0$). Note that $\mathbf{z}^2 = -N(\mathbf{z})$.

For each $\mathbf{a} \in \mathbb{P}$, one can find a unit pure quaternion $\mathbf{u}$ and then a unique angle α $(0 \leqslant \alpha \leqslant \pi)$, so that

$$\mathbf{a} = \exp(\alpha\mathbf{u}) := \cos(\alpha) + \sin(\alpha)\mathbf{u}.$$

Next we observe that the mapping of pure quaternions given by

$$[\mathbf{a}] : \mathbf{z} \mapsto \tilde{\mathbf{a}}\mathbf{z}\mathbf{a}, \ \ (\mathbf{z} \in \mathbb{E}^3)$$

effects a rotation through angle 2α about the axis spanned by $\mathbf{u}$. Noting that we compose such mappings left to right, we have

Proposition 2. *There is a* $2:1$ *surjection*

$$\begin{array}{ccc} \mathbb{P} & \to & SO_3(\mathbb{R}) \\ \mathbf{a} & \mapsto & [\mathbf{a}] \end{array}$$

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The kernel of this epimorphism is ± 1.

A finite multiplicative group of quaternions must be a subgroup of $\mathbb{P}$. Using Proposition 2 and the known classification of finite rotation groups in $\mathbb{E}^3$, we easily verify that the finite groups of quaternions are those described in Table 1.

Table 1. The finite groups of quaternions

Name	Conway notation	Coxeter notation	Order	Convenient generators
cyclic (even order $2m$)	$2C_m$	$\langle m, m, 1\rangle = C_{2m}$	$2m$	$\exp(\pi\mathbf{i}/m)$
cyclic (odd order m)	$1C_m$	C_m	m	$\exp(2\pi\mathbf{i}/m)$
dicyclic	$2D_{2m}$	$\langle m, 2, 2\rangle$	$4m$	$\exp(\pi\mathbf{i}/m)$, $\mathbf{j}$
binary tetrahedral	$2T$	$\langle 3, 3, 2\rangle$	24	$\mathbf{a}$, $\mathbf{b}_T$
binary octahedral	$2O$	$\langle 4, 3, 2\rangle$	48	$\mathbf{a}$, $\mathbf{b}_O$
binary icosahedral	$2I$	$\langle 5, 3, 2\rangle$	120	$\mathbf{a}$, $\mathbf{b}_I$

From [5, Theorem 12] we have the generators $\mathbf{a} = (-1+\mathbf{i}+\mathbf{j}+\mathbf{k})/2$, $\mathbf{b}_T = \mathbf{i}$, $\mathbf{b}_O = (\mathbf{j}+\mathbf{k})/\sqrt{2}$ and $\mathbf{b}_I = (\mathbf{i}+\tau^{-1}\mathbf{j}+\tau\mathbf{k})/2$.

Example 1. *For the moment, let us view the vertices of* $\mathbf{S}$ *as unit quaternions. Since the identity quaternion is not one of these, we do not quite have a multiplicative group. However, if we premultiply vertices by* $\mathbf{v}^{-1}$ *(essentially* ϵ^{-1}*), then we do get the binary icosahedral group* $2I = \langle 5, 3, 2\rangle$*. The notation is a reminder that this group is a double cover of the icosahedral group* $(5, 3, 2) = [5, 3]^+$*, of order* 60*. Consider also quaternions* $\mathbf{u} = 3|\_\_$, $\mathbf{z} = 4|0$*. Then the two generators* $\mathbf{d} = \mathbf{v}^{-1}\mathbf{u}$, $\mathbf{e} = \mathbf{v}^{-1}\mathbf{z}$ *satisfy the defining relations*

$$\mathbf{d}^5 = \mathbf{e}^3 = (\mathbf{d}\,\mathbf{e})^2$$

for $\langle 5, 3, 2\rangle$ *[11, Chapter 6.5]. Derived as they are from (4), these alternate generators are a bit messier than* $\mathbf{a}$, $\mathbf{b}_I$ *from Table 1:*

$$\begin{aligned} \mathbf{d} &= \frac{1}{2}(\tau + 5^{\frac{1}{4}}\tau^{-\frac{1}{2}}\,\mathbf{i}), \\ \mathbf{e} &= \frac{1}{2}(1 + 5^{-\frac{1}{4}}\tau^{\frac{3}{2}}\mathbf{i} + \mathbf{j} - 5^{-\frac{1}{4}}\tau^{-\frac{3}{2}}\mathbf{k}). \end{aligned}$$

□

Let us move on to $\mathbb{E}^4$. The reflection in the hyperplane orthogonal to the unit quaternion $\mathbf{a}$ is described by the mapping

$$\mathbf{z} \mapsto -\mathbf{a}\tilde{\mathbf{z}}\mathbf{a},$$

which we denote by $*[-\tilde{\mathbf{a}}, \mathbf{a}]$. (Recall that $\tilde{\mathbf{a}} = \mathbf{a}^{-1}$.) It follows that any direct isometry on $\mathbb{E}^4$ can be described as

$$[\mathbf{l}, \mathbf{r}] : \mathbf{z} \mapsto \tilde{\mathbf{l}}\mathbf{z}\mathbf{r}, \quad \mathbf{z} \in \mathbb{E}^4.$$

The notation is meant to suggest a pair of left and right unit quaternions, and so we need the direct product

$$\Delta = \mathbb{P} \times \mathbb{P}.$$

Proposition 3. *There is a* $2:1$ *surjection*

$$\begin{aligned} \Delta &\rightarrow SO_4(\mathbb{R}) \\ (\mathbf{l}, \mathbf{r}) &\mapsto [\mathbf{l}, \mathbf{r}] \end{aligned}$$

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The kernel of this epimorphism is $\pm(1,1)$.

Any opposite symmetry is likewise described by

$$*[\mathbf{l},\mathbf{r}] : \mathbf{z} \mapsto \tilde{\mathbf{l}}\tilde{\mathbf{z}}\mathbf{r}, \quad \mathbf{z} \in \mathbb{E}^4.$$

For instance, ordinary conjugation is given by either $*[1,1]$ or $*[-1,-1]$. (This effects a central symmetry in the real subspace of pure quaternions.)

To put all this in one package, it is useful to extend Δ by an involution, which we label $*$ and which acts on Δ by swapping entries:

$$*(\mathbf{l},\mathbf{r})* = (\mathbf{l},\mathbf{r})^* = (\mathbf{r},\mathbf{l}).$$

Using the semidirect product

$$\Delta' = \Delta \rtimes C_2 = (\mathbb{P} \times \mathbb{P}) \rtimes \langle * \rangle\,,$$

we now have a $2:1$ epimorphism

$$\begin{array}{rcl} \Delta' & \to & GO_4(\mathbb{R}) \\ (\mathbf{l},\mathbf{r}) & \mapsto & [\mathbf{l},\mathbf{r}], \\ *(\mathbf{l},\mathbf{r}) & \mapsto & *[\mathbf{l},\mathbf{r}], \end{array}$$

still with kernel $\pm(1,1)$.

These results provide the first step to determining all geometrically distinct finite subgroups G of $GO_4(\mathbb{R})$. We must first find finite subgoups H of Δ if we seek subgroups G^+ of $SO_4(\mathbb{R})$. Clearly, H is a subdirect product of some $L_* \times R_*$, where the left and right groups L_*, R_* are, up to conjugacy, amongst the finite groups listed in Table 1. In fact, we could assume, if it helps, that L_*, R_* *are* just as given in the Table; and we can further assume $(-1,-1) \in L_* \times R_*$, though this need not be so for H. To organize the many possibilities, we can use Goursat's Theorems on subdirect products, as described for instance in [2]. The upshot is that the H's are parametrized by triples $(K_{L_*}, K_{R_*}, \theta)$ such that the normal subgroups $K_{L_*} \trianglelefteq L_*$ and $K_{R_*} \trianglelefteq R_*$ admit an isomorphism

$$\theta : L_*/K_{L_*} \to R_*/K_{R_*}.$$

Then

$$H = \{(\mathbf{l},\mathbf{r}) \in L_* \times R_* \ : \ (\mathbf{l}K_{L_*})\theta = \mathbf{r}K_{R_*}\}.$$

The group G^+ has order

$$\frac{|L_*|\cdot|R_*|}{e_* f_*},$$

where $f_* = [L_* : K_{L_*}] = [R_* : K_{R_*}]$ is the order of the common quotient, and $e_* \in \{1,2\}$ is the order of $H \cap \langle(-1,-1)\rangle$.

The actual cases are bewildering and are outlined in [5, Chapter 4, Tables 4.1 and 4.2]. In those Tables of groups G^+, typical entries look like

$$\pm\frac{1}{f}[A \times B] \text{ or } +\frac{1}{f}[A \times B],$$

for the so-called 'diploid' or 'haploid' cases, respectively, for which the central symmetry z does or does not lie in the group.

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The convention in the Tables is that A, B denote subgroups of $SO_3(\mathbb{R})$, *not* their quaternionic covers L_*, R_*. Likewise, we used f_* above rather than f, since in some cases (not of concern here), one has $f = f_*/2$.

If we seek a finite subgroup G of $GO_4(\mathbb{R})$ with opposite isometries, then we work in Δ' and adjoin to one of the subdirect products $H \leqslant L_* \times R_*$ some element $*(\mathbf{a}, \mathbf{b})$. Here there are simplications, mainly because $L_* \simeq R_*$ is forced. If desired, we can even take $L_* = R_*$. Up to conjugacy in Δ' there can be various choices for $*(\mathbf{a}, \mathbf{b})$, though often $*(1,1)$ is usable. The finite subgroups of this kind in $GO_4(\mathbb{R})$ are listed in Table 4.3 of [5].

We shall look more closely only at a few of the "diploid, achiral groups", which appear in [5, Table 4.3] as

$$G = \pm\frac{1}{f}[A \times A] \cdot 2,$$

though perhaps with some decorations to distinguish, for instance, choices for $*[a, b]$. In such cases, $L_* = 2A$, $e_* = 2$ and $f = f_*$, so the order is

$$\frac{4|A|^2}{f}. \tag{11}$$

We have reviewed all this machinery just so that the reader can make sense of the following brief results. It can take a great deal of work to fit a well-known linear group into the scheme underlying the Tables in [5].

Example 2. *From the very first entry in [5, Table 4.3] we have*

$$H_4 = [3,3,5] = \pm[I \times I] \cdot 2.$$

The "$\cdot 2$" indicates that we have doubled the order of the rotation group $[3,3,5]^+$ by adjoining an opposite symmetry, in fact, $*[1,1]$. We have $L_* = R_* = 2I$, so $A = B = I$, the icosahedral group of order 60. The parameter $f = f_* = 1$, and $K_{L_*} = K_{R_*} = 2I$, with θ trivial. The order of H_4 does indeed equal $4 \cdot 60^2 = 14400$ from (11).

Now compare this with our construction of H_4 in Section 2. There we chose the basic roots $\mathbf{n}_j$ in (5) for the generating reflections r_j. Thus $r_j = *[\mathbf{n}_j^{-1}, \mathbf{n}_j]$, so that the subgroup of direct isometries H_4^+ is generated by rotations

$$s_j := r_{j-1}r_j = [-\mathbf{n}_{j-1}\mathbf{n}_j^{-1}, -\mathbf{n}_{j-1}^{-1}\mathbf{n}_j], \ (j = 1,2,3).$$

We find that R_* is the binary icosahedral group generated by quaternions $\mathbf{d}, \mathbf{e}$ in Example 1. But now L_* is the conjugate subgroup $\mathbf{v}R_*\mathbf{v}^{-1}$ in $\mathbb{P}$. This has no effect on the conjugacy class of H_4 in $GO_4(\mathbb{R})$. □

Example 3. *The rotations $s_2 = r_1r_2, s_3 = r_2r_3$ in H_4 generate a copy of the icosahedral group $I = [3,5]^+$. From Figure 2, we see that I contains the dihedral group D_{10}, generated for instance by s_3, rotation through $2\pi/5$ about $\mathbf{u}$, and our half-turn $h = s_2s_3^{-2}s_2^{-1}s_3s_2^{-1}$ about the midpoint of edge $2|2\ 0|4$. (This edge belongs to the Petrie polygon preserved by s_3 in Figure 2.)*

Lift to the binary icosahedral group $R_* = \langle \mathbf{a}_2, \mathbf{a}_3 \rangle$ of Example 2, now generated by $\mathbf{a}_2 = -\mathbf{n}_1^{-1}\mathbf{n}_2, \mathbf{a}_3 = -\mathbf{n}_2^{-1}\mathbf{n}_3$. This group of order 120 contains the dicyclic group $\langle 5,2,2 \rangle = \langle \mathbf{a}_3, \mathbf{b} \rangle$ of order 20, where

$$\mathbf{b} = \mathbf{a}_2\mathbf{a}_3^{-2}\mathbf{a}_2^{-1}\mathbf{a}_3\mathbf{a}_2^{-1}.$$

But from the previous Example we now know that $L_* = \mathbf{v}R_*\mathbf{v}^{-1}$ contains its own copy $\langle \mathbf{v}\mathbf{a}_3\mathbf{v}^{-1}, \mathbf{v}\mathbf{b}\mathbf{v}^{-1} \rangle$ of the dicyclic group. Thus $L_* \times R_*$ contains the direct product of commuting dicyclic groups. This group of order 400 projects to the rotation group

$$\pm[D_{10} \times D_{10}]$$

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of order 200 *in* $SO_4(\mathbb{R})$ *[5, Table 4.2]. We can adjoin the opposite symmetry* $*[1,1]$ *to finally see that*

$$G(\mathbf{A}) \simeq \pm[D_{10} \times D_{10}] \cdot 2,$$

(so take $p = 5$ *in line 19 of [5, Table 4.3]).*

We see that $G(\mathbf{A})$ *appears as a subgroup of* H_4 *in a quite natural way. Indeed, this is essentially the approach taken in [17]. However, as mentioned earlier, the mislabelling of* $G(\mathbf{A})$ *is at least suggested there.* □

Example 4. *We will not include the details needed to correctly classify our unneeded 'ionic diminished Coxeter group':*

$$[[10, 2^+, 10]] \simeq \pm\frac{1}{4}[D_{20} \times \overline{D}_{20}] \cdot 2$$

(line 21 of [5, Table 4.3]). The adjoined opposite symmetry can again be taken to be $*[1,1]$*. The bar in* $\overline{D}_{20}$ *is merely a notational device signalling the fact that* $D_4 \simeq C_2 \times C_2$ *is special among dihedral groups in having automorphisms freely permuting the non-identity elements. See [5, p. 50 and footnote 3] for more.* □

Remark 6. *Conway and Smith describe in [5, Chapter 4.5] errors or omissions in previous catalogues of the isometry groups in* $\mathbb{E}^4$*. Perhaps the best known earlier enumeration of the groups is that of Du Val in [14, Sections 21–22]. Apparently, there are some redundancies to be found there.*

□

5. Some Final Comments and Thanks

The grand antiprism has been examined elsewhere, generally in wider discussions of uniform 4-polyopes. We mention, for instance, [19], [21] and [13]. This last paper employs subrootsystems for the group H_4, rather as in [17], but with a broader look that takes in other uniform polytopes, such as the snub 24-cell.

Our work on the grand antiprism is an offshoot of a more extensive investigation into abstract regular 4-polytopes whose automorphism groups are subgroups of low index in some orthogonal group $O(d, p, e)$ over a finite field $GF(p)$ [20]. (The parameter $e = \pm 1$ flags the Witt index for the corresponding d-dimensional orthogonal geometry.) For instance, $H_4 \simeq O_1(4, 5, +1)$, the subgroup generated by reflections whose roots have square spinor norm. From [20, Equation (14)], we find a similarly structured group and accompanying geometry whenever $p \equiv 1 \pmod 4$. Moreover, we get an abstract regular 4-polytope with tetrahedral facets and $p^3 - p$ vertices. Inscribed in it, we must find a relative of the grand antiprism. But, of course, this abstract 4-polytope will not have a familiar convex realization.

We still do not properly understand the presentations of such orthogonal groups, when ones hands are tied (as they will be!) by using just 4 generating reflections. This in turn is necessary for an understanding of the universal regular polytopes whose facets are tetrahedra and whose vertex-figures are certain naturally occuring maps of type $\{3, p\}$ [20, Conjectures 1,2,3].

Finally, let me here thank Peter McMullen for his input, in particular for suggesting the use of the subgroup K in Section 2 as a way to more easily understand the structure of $G(\mathbf{A})$.

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