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Singular Value Decomposition of Elliptic Quaternion Matrices: Theory and Algorithms

Hidayet Hüda Kösal *, Emre Kişi , Mahmut Akyiğit and Beyza Çelik

Faculty of Sciences, Department of Mathematics, Sakarya University, Sakarya, TR-54050, TÜRKİYE; ekisi@sakarya.edu.tr (E.K); makyigit@sakarya.edu.tr (M.A); beyza.celik6@ogr.sakarya.edu.tr (B.Ç)

* Correspondence: hhkosal@sakarya.edu.tr

Abstract: In this study, we obtained results for the computation of eigen-pairs, singular value decomposition, pseudo-inverse, and the least square problem for elliptic quaternion matrices. Moreover, we established algorithms based on these results and provided illustrative numerical experiments to substantiate the accuracy of our conclusions. In the experiments, it was observed that the p-value in the algebra of elliptic quaternions and elliptic numbers directly affects the performance of the problem under consideration. Selecting the optimal p-value for problem-solving and the elliptic behavior of many physical systems make this number system advantageous in applied sciences.

Keywords: elliptic quaternion matrix; optimal p-value; Eigen-pairs; singular value decomposition; pseudo-inverse; least square solution

MSC: 11R52; 15A60; 15A18

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1. Introduction

The singular value decomposition is a foundational aspect of matrix theory. In contrast to eigenvalue decomposition, which is limited to certain square matrices, singular value decomposition is applied to matrices of any size. Numerous results related to matrices, e.g., pseudo-inverses of matrices, solutions for least squares problems, and invariant norms under unitary transformations, can be derived from the principles of singular value decomposition. As a result, singular value decomposition is pivotal in the computation and analysis of matrices. In addition to its importance in the theoretical field, it appears in the solution of many problems in the applied fields, e.g., image processing, signal processing, principal component analysis, data compression, machine learning, deep learning, and computational mathematics, etc. Many researchers studied in these applied fields examine singular value decomposition methods. For example, Dian et al. presented a novel hyperspectral image and multispectral image fusion method based on the subspace representation and convolutional neural network denoiser. They obtained the subspaces via singular value decomposition of a high-resolution hyperspectral image [1]. Hashemipour et al. proposed a new lossy data compression framework centred on optimal singular value decomposition for big data compression [2]. Wang and Zhu focused on the implementation of data reduction algorithms such as SVD and principal component analysis in machine learning [3].

There is a generalization that includes three classes of 2-dimensional hypercomplex numbers [4]. The following is the definition of these numbers, known as generalized complex numbers:

$$q_{(g)} = q_{(g),r} + q_{(g),i}i,$$

where $q_{(g),r}, q_{(g),i} \in \mathbb{R}$, $i \notin \mathbb{R}$ and $i^2 = p$ $(p \in \mathbb{R})$. Generalized Segre quaternions are generalized complex numbers extended to 4 dimensions. Generalized Segre quaternions are defined as follows:

$$q_{(GS)} = q_{(GS),r} + q_{(GS),i}i + q_{(GS),j}j + q_{(GS),k}k,$$

where $q_{(GS),r}, q_{(GS),i}, q_{(GS),j}, q_{(GS),k} \in \mathbb{R}$, $i, j, k \notin \mathbb{R}$. The multiplication rules for i, j and k units are given in the table below:

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Table 1. Multiplication rules of i, j and k units.

	i	j	k
i	p	k	pj
j	k	1	i
k	pj	i	p

Based on the value of p, generalized complex numbers and Segre quaternions are classified as follows [5]:

Every number system in Table 2 has various scientific and technological applications. Problems in non-Euclidean geometries are solved by hyperbolic complex numbers and hyperbolic quaternions [6]. In domains like robotic control and spatial mechanics, parabolic complex (dual) numbers and parabolic quaternions are employed [7]. On the other hand, as numerous physical systems demonstrate elliptical behavior, the practical applications of elliptic complex numbers and elliptic quaternions in applied science are noteworthy. For example, Ozdemir defined elliptic quaternions (non-commutative) and generated an elliptical rotation matrix for the motion of a point on an ellipse through some angle about a vector using those quaternions [8]. Dundar et al. studied elliptical harmonic motion, which is the superposition of two simple harmonic motions in perpendicular directions with the same angular frequency and phase difference of $\frac{\pi}{2}$ by using elliptic complex numbers [9]. Derin and Gungor proposed the generalization of gravity, including the Proca-type and gravitomagnetic monopole by means of elliptic biquaternions [10]. Catoni et al. introduced algebraic properties and the differential conditions of elliptic quaternionic systems [5]. Additionally, Catoni et al. studied the constant curvature spaces associated with the geometry generated by elliptic quaternions. They formulated geodesic equations within the context of Riemann geometry [11]. Gua et al. defined the elliptic quaternionic canonical transform and investigated Parseval's theorem with the help of this transform [12]. Yuan et al. obtained the Hermitian solutions of the elliptic quaternion matrix equation $(AXB, CXD) = (E, G)$ [13]. Tosun and Kosal characterized the existence of the solution to Sylvester s-conjugate elliptic quaternion matrix equations. They obtained the solution explicitly using a real representation of an elliptic quaternion matrix [14]. Gai and Huang developed a new convolutional neural network with elliptic quaternion values. They conducted extensive experiments on colour image classification and colour image denoising to evaluate the performance of the proposed convolutional neural network [15]. Guo et al. studied the problem of solutions to Maxwell's equations of elliptic quaternions using a real representation of elliptic quaternion matrices [16]. Atali et al. obtained the elliptic quaternionic least-squares solution with the minimum norm of the elliptic quaternion matrix equation $AX = B$. Furthermore, leveraging the insights derived from their theories, they developed a novel color image restoration model known as the elliptical quaternionic least squares restoration filter [17].

Table 2. Classification of numbers.

	$p > 0$	$p = 0$	$p < 0$
Generalized complex numbers	Hyperbolic complex numbers	Parabolic complex numbers	Elliptic complex numbers
Generalized segre quaternions	Hyperbolic quaternions	Parabolic quaternions	Elliptic quaternions

As observed, the elliptic quaternions and their matrices find numerous practical applications in various branches of applied sciences. Thus, further study of the theoretical properties and numerical computations of elliptic quaternions and their matrices is becoming increasingly necessary. In this regard, we have derived outcomes concerning the computation of eigen-pairs, singular value decomposition, pseudo-inverse, and least square solutions with the minimum norm for elliptic quaternion matrices. Additionally, algorithms have been formulated based on these results, accompanied by illustrative numerical experiments to validate our findings' precision empirically.

Within the context of this paper, the following notations will be employed. Let $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}_p$, and $\mathbb{H}_p$ denote the sets of real numbers, complex numbers, elliptic complex numbers, and elliptic quaternions, respectively. $\mathbb{R}^{m\times n}$, $\mathbb{C}^{m\times n}$, $\mathbb{C}_p^{m\times n}$, and $\mathbb{H}_p^{m\times n}$ denote the set of all $m \times n$ matrices on $\mathbb{R}$, $\mathbb{C}$, $\mathbb{C}_p$ and $\mathbb{H}_p$,

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respectively. We will also denote elliptic complex numbers as *EC* numbers and elliptic quaternions as *EQs* for short. The notation to be used to represent numbers throughout the study is given in the table below.

Table 3. Notations.

	$\square = c$	$\square = e$	$\square = E$
$q_{(\square)}$	Complex number	Elliptic complex number	Elliptic quaternion
$Q_{(\square)}$	Complex matrix	Elliptic complex matrix	Elliptic quaternion matrix

Moreover, all computations in this study are performed using the MATLAB 2024a (64bit) on an Intel(R) Xeon(R) CPU E5-1650 v4 @3.60GHz (12 CPUs)/16GB (DDR3) RAM computer.

2. Preliminaries

An *EC* number $q_{(e)}$ is denoted by $q_{(e)} = q_{(e),r} + q_{(e),i}i$ where $i^2 = p < 0$ and $q_{(e),r}, q_{(e),i}, p \in \mathbb{R}$. The conjugate and norm of $q_{(e)} \in \mathbb{C}_p$ are defined as $\overline{q_{(e)}} = q_{(e),r} - q_{(e),i}i$ and $\left\|q_{(e)}\right\|_p = \sqrt{q_{(e),r}^2 - pq_{(e),i}^2}$, respectively [18]. The multiplication of *EC* numbers $q_{1,(e)} = q_{1,(e),r} + q_{1,(e),i}i$ and $q_{2,(e)} = q_{2,(e),r} + q_{2,(e),i}i$ is defined as

$$q_{1,(e)}q_{2,(e)} = \left(q_{1,(e),r}q_{2,(e),r} + pq_{1,(e),i}q_{2,(e),i}\right) + i\left(q_{1,(e),r}q_{2,(e),i} + q_{2,(e),r}q_{1,(e),i}\right).$$

An *EC* matrix $Q_{(e)}$ is denoted as $Q_{(e)} = Q_{(e),r} + Q_{(e),i}i$ where $i^2 = p < 0$, $p \in \mathbb{R}$, and $Q_{(e),r}$, $Q_{(e),i} \in \mathbb{R}^{m\times n}$. The conjugate, transpose, conjugate transpose and Frobenius norm of $Q_{(e)}$ in $\mathbb{C}_p^{m\times n}$ are defined by $\overline{Q_{(e)}} = Q_{(e),r} - Q_{(e),i}i$, $Q_{(e)}^T = Q_{(e),r}^T + Q_{(e),i}^T i$, $Q_{(e)}^* = Q_{(e),r}^T - Q_{(e),i}^T i$, and $\|Q_{(e)}\|_p = \sqrt{\|Q_{(e),r}\|^2 - p\|Q_{(e),i}\|^2}$, respectively. The multiplication of the *EC* matrices $Q_{1,(e)} = Q_{1,(e),r} + Q_{1,(e),i}i$ and $Q_{2,(e)} = Q_{2,(e),r} + Q_{2,(e),i}i$ is defined as

$$Q_{1,(e)}Q_{2,(e)} = \left(Q_{1,(e),r}Q_{2,(e),r} + pQ_{1,(e),i}Q_{2,(e),i}\right) + i\left(Q_{1,(e),r}Q_{2,(e),i} + Q_{2,(e),r}Q_{1,(e),i}\right).$$

An *EQ* $q_{(E)}$ is denoted as $q_{(E)} = q_{(E),r} + q_{(E),i}i + q_{(E),j}j + q_{(E),k}k$ where $i^2 = k^2 = p < 0$, $j^2 = 1$, $ij = ji = k$, $jk = kj = i$, $ki = ik = pj$, and $q_{(E),r}, q_{(E),i}, q_{(E),j}, q_{(E),k}, p \in \mathbb{R}$ [5,18]. An *EQ* $q_{(E)}$ is denoted in the forms:

$$q_{(E)} = \left(q_{(E),r} + iq_{(E),i}\right) + \left(q_{(E),j} + iq_{(E),k}\right)j = q_{(e),1}e_1 + q_{(e),2}e_2$$

where

$$q_{(e),1} = \left(q_{(E),r} + q_{(E),j}\right) + \left(q_{(E),i} + q_{(E),k}\right)i$$

and

$$q_{(e),2} = \left(q_{(E),r} - q_{(E),j}\right) + \left(q_{(E),i} - q_{(E),k}\right)i,$$

are an *EC* numbers and $e_1 = \frac{1+j}{2}$, $e_2 = \frac{1-j}{2}$. Clearly, $e_1e_2 = 0$, $e_1 + e_2 = 1$, $e_1^2 = e_1$ and $e_2^2 = e_2$. As a result, e_1 and e_2 are disjoint idempotent units.

The multiplication of the two *EQs* $q_{1,(E)} = q_{1,(e),1}e_1 + q_{1,(e),2}e_2$ and $q_{2,(E)} = q_{2,(e),1}e_1 + q_{2,(e),2}e_2$ is defined by

$$q_{1,(E)}q_{2,(E)} = \left(q_{1,(e),1}q_{2,(e),1}\right)e_1 + \left(q_{1,(e),2}q_{2,(e),2}\right)e_2.$$

The conjugate and norm of the *EQ* $q_{(E)} = q_{(e),1}e_1 + q_{(e),2}e_2$ are defined by $\overline{q_{(E)}} = \overline{q_{(e),1}}e_1 + \overline{q_{(e),2}}e_2$ and $\left\|q_{(E)}\right\|_p = \frac{1}{\sqrt{2}}\sqrt{\left(\left\|q_{(e),1}\right\|_p^2 + \left\|q_{(e),2}\right\|_p^2\right)}$, respectively.

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An EQ matrix $Q_{(E)}$ is represented

$$Q_{(E)} = Q_{(E),r} + Q_{(E),i}i + Q_{(E),j}j + Q_{(E),k}k = \left(Q_{(E),r} + Q_{(E),i}i\right) + j\left(Q_{(E),j} + Q_{(E),k}i\right) \\ = Q_{(e),1}e_1 + Q_{(e),2}e_2 \tag{1}$$

where

$$Q_{(e),1} = \left(Q_{(E),r} + Q_{(E),j}\right) + \left(Q_{(E),i} + Q_{(E),k}\right)i$$

and

$$Q_{(e),2} = \left(Q_{(E),r} - Q_{(E),j}\right) + \left(Q_{(E),i} - Q_{(E),k}\right)i$$

are EC matrices and $Q_{(E),r}, Q_{(E),i}, Q_{(E),j}, Q_{(E),k}k \in \mathbb{R}^{m\times n}$ [14,17]. The multiplication of the two EQ matrices $Q_{1,(E)} = Q_{1,(e),1}e_1 + Q_{1,(e),2}e_2$ and $Q_{2,(E)} = Q_{2,(e),1}e_1 + Q_{2,(e),2}e_2$ is defined by

$$Q_{1,(E)}Q_{2,(E)} = \left(Q_{1,(e),1}Q_{2,(e),1}\right)e_1 + \left(Q_{1,(e),2}Q_{2,(e),2}\right)e_2.$$

The conjugate, transpose, conjugate transpose, and Frobenius norm of EQ matrix $Q_{(E)} = Q_{(e),1}e_1 + Q_{(e),2}e_2 \in \mathbb{H}_p^{m\times n}$ are defined by $\overline{Q_{(E)}} = \overline{Q_{(e),1}}e_1 + \overline{Q_{(e),2}}e_2$, $Q_{(E)}^T = Q_{(e),1}^T e_1 + Q_{(e),2}^T e_2$, $Q_{(E)}^* = \overline{Q_{(e),1}^T}e_1 + \overline{Q_{(e),2}^T}e_2$ and

$$\left\|Q_{(E)}\right\|_p = \frac{1}{\sqrt{2}}\sqrt{\left(\left\|Q_{(e),1}\right\|_p^2 + \left\|Q_{(e),2}\right\|_p^2\right)}.$$

3. Eigenvalues and Eigenvectors, Singular Value Decomposition, Pseudo-Inverse, and Least Squares Problem for EQ Matrices

3.1. EC Matrices

The lemmas that will be given in this subsection and will be used in the proofs of theorems regarding EQ matrices have been obtained through the following isomorphisms and their inverses:

$$\begin{gathered} \eta_p : \mathbb{C}_p \to \mathbb{C} \\ q_{(e)} = q_{(e),r} + iq_{(e),i} \to \eta_p\left(q_{(e)}\right) = q_{(c)} = q_{(e),r} + I\sqrt{-p}q_{(e),i} \end{gathered}$$

and

$$\begin{gathered} \mathrm{H}_p : \mathbb{H}_p^{m\times n} \to \mathbb{H}^{m\times n} \\ Q_{(e)} = Q_{(e),r} + Q_{(e),i} \to \mathrm{H}_p\left(Q_{(e)}\right) = Q_{(c)} = Q_{(e),r} + I\sqrt{-p}Q_{(e),i} \end{gathered}$$

where I represents the complex unit ($I^2 = -1$) [4].

Lemma 1. *A polynomial function of degree N with EC number coefficients presented by*

$$f_p(x_{(e)}) = x_{(e)}^N + q_{(e),N-1}x_{(e)}^{N-1} + \ldots + q_{(e),1}x_{(e)} + q_{(e),0}$$

has exactly N zeros in the set of EC numbers.

Lemma 2. *Let $Q_{(e)} \in \mathbb{C}_p^{n\times n}$. An EC matrix $Q_{(e)}$ is nonsingular if and only if the complex matrix $\mathrm{H}_p(Q_{(e)})$ are nonsingular. If $\mathrm{H}_p(Q_{(e)})$ are nonsingular, then*

$$Q_{(e)}^{-1} = \mathrm{Re}\left(\left(\mathrm{H}_p\left(Q_{(e)}\right)\right)^{-1}\right) + \frac{i}{\sqrt{-p}}\mathrm{Im}\left(\left(\mathrm{H}_p\left(Q_{(e)}\right)\right)^{-1}\right).$$

Lemma 3. *Let $Q_{(e)} \in \mathbb{C}_p^{m\times n}$. Then $rank\left(Q_{(e)}\right) = rank\left(\mathrm{H}_p\left(Q_{(e)}\right)\right)$.*

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Lemma 4. *Let the eigenvalues of an complex matrix $\mathrm{H}_p\left(Q_{(e)}\right)$ be denoted by $\lambda_{\mathrm{H}_p(Q_{(e)})}$, and let the corresponding eigenvectors be represented by $x_{\mathrm{H}_p(Q_{(e)})}$. Then, the eigenvalues of the EC matrix $Q_{(e)} \in \mathbb{C}_p^{n\times n}$ are given by*

$$\lambda_{(e)} = \mathrm{Re}(\lambda_{\mathrm{H}_p(Q_{(e)})}) + \frac{i}{\sqrt{-p}}\mathrm{Im}(\lambda_{\mathrm{H}_p(Q_{(e)})})$$

and the corresponding eigenvectors are given by

$$x_{(e)} = \mathrm{Re}(x_{\mathrm{H}_p(Q_{(e)})}) + \frac{i}{\sqrt{-p}}\mathrm{Im}(x_{\mathrm{H}_p(Q_{(e)})}).$$

The converse of this lemma is also true.

Lemma 5. *Let $Q_{(e)} \in \mathbb{C}_p^{m\times n}$ be an EC matrix. The pseudo-inverse of $Q_{(e)}$, denoted by $\left(Q_{(e)}\right)^{\dagger}$, is given by*

$$\left(Q_{(e)}\right)^{\dagger} = \mathrm{Re}\left(\left(\mathrm{H_p}\left(Q_{(e)}\right)\right)^{\dagger}\right) + \frac{i}{\sqrt{-p}}\mathrm{Im}\left(\left(\mathrm{H_p}\left(Q_{(e)}\right)\right)^{\dagger}\right)$$

where $\left(\mathrm{H_p}\left(Q_{(e)}\right)\right)^{\dagger}$ is the pseudo-inverse of the complex matrix $\mathrm{H_p}\left(Q_{(e)}\right)$.

Lemma 6. *Let $Q_{(e)} \in \mathbb{C}_p^{m\times n}$. Suppose that the singular value decomposition of the complex matrix $\mathrm{H}_p\left(Q_{(e)}\right)$ is given by $\mathrm{H_p}\left(Q_{(e)}\right) = U_{(c)}\Sigma V_{(c)}^*$. Then, the singular value decomposition of the EC matrix $Q_{(e)}$ is*

$$Q_{(e)} = \left(\mathrm{Re}\left(U_{(c)}\right) + \frac{i}{\sqrt{-p}}\mathrm{Im}\left(U_{(c)}\right)\right)\Sigma\left(\mathrm{Re}\left(V_{(c)}\right) + \frac{i}{\sqrt{-p}}\mathrm{Im}\left(V_{(c)}\right)\right)^*$$

where Σ is a real matrix. The converse of this statement is also true.

Lemma 7. *Let $Q_{(e)} \in \mathbb{C}_p^{m\times n}$. Suppose that $\mathrm{H_p}\left(Q_{(e)}\right) = U_{(c)}\Sigma V_{(c)}^*$. In this case, the pseudo-inverse of EC matrix $Q_{(e)}$ is given by*

$$\left(Q_{(e)}\right)^{\dagger} = \left(\mathrm{Re}\left(V_{(c)}\right) + \frac{i}{\sqrt{-p}}\mathrm{Im}\left(V_{(c)}\right)\right)\Sigma^{\dagger}\left(\mathrm{Re}\left(U_{(c)}\right) + \frac{i}{\sqrt{-p}}\mathrm{Im}\left(U_{(c)}\right)\right)^*.$$

Lemma 8. *Let $Q_{1,(e)} \in \mathbb{C}_p^{m\times n}$ and $Q_{2,(e)} \in \mathbb{C}_p^{m\times q}$. Suppose that $\mathrm{H_p}\left(Q_{1,(e)}\right) = U_{(c)}\Sigma V_{(c)}^*$. In this case, the least squares solution with the minimum norm $X_{(e)}$ of the EC matrix equation $Q_{1,(e)}X_{(e)} = Q_{2,(e)}$ is given by*

$$X_{(e)} = \left(\mathrm{Re}\left(V_{(e)}\right) + \frac{i}{\sqrt{-p}}\mathrm{Im}\left(V_{(e)}\right)\right)\Sigma^{\dagger}\left(\mathrm{Re}\left(U_{(e)}\right) + \frac{i}{\sqrt{-p}}\mathrm{Im}\left(U_{(e)}\right)\right)^* Q_{2,(e)}.$$

3.2. EQ Matrices

Let $Q_{(E)} = Q_{(e),1}e_1 + Q_{(e),2}e_2 \in \mathbb{H}_p^{m\times n}$. Since e_1 and e_2 are disjoint idempotent units here, the mathematical properties associated with *EQ* matrices are closely related to *EC* matrices $Q_{(e),1}$, $Q_{(e),2}$. In this subsection, results related to eigen-pairs, singular value decomposition, pseudo-inverse, and least squares solution with the minimum norm for *EQ* matrices have been derived from this fact.

Definition 1. *A monic polynomial of degree N with EQ variables $x_{(E)}$ is expressed as*

$$f_p\left(x_{(E)}\right) = x_{(E)}^N + q_{(N-1),(E)}x_{(E)}^{N-1} + \ldots + q_{1,(E)}x_{(E)} + q_{0,(E)},$$

where $q_{(N-1),(E)}, q_{(N-2),(E)}, \ldots, q_{1,(E)}, q_{0,(E)}$ are EQ coefficients.

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Theorem 1. *A polynomial function $f_p\left(x_{(E)}\right)$ with degree N has exactly N^2 zeros in the set of EQs.*

Proof. *The polynomial $f_p\left(x_{(E)}\right)$ can be written in the form*

$$\begin{aligned}
f_p\left(x_{(E)}\right) &= \left(x_{(e),1}e_1 + x_{(e),2}e_2\right)^N + \left(q_{(N-1),(e),1}e_1 + q_{(N-1),(e),2}e_2\right)\left(x_{(e),1}e_1 + x_{(e),2}e_2\right)^{N-1} \\
&\quad + \cdots + \left(q_{1,(e),1}e_1 + q_{1,(e),2}e_2\right)\left(x_{(e),1}e_1 + x_{(e),2}e_2\right) + \left(q_{0,(e),1}e_1 + q_{0,(e),2}e_2\right) \\
&= \left(\left(x_{(e),1}\right)^N + \left(q_{(N-1),(e),1}\right)\left(x_{(e),1}\right)^{N-1} + \ldots + \left(q_{1,(e),1}\right)\left(x_{(e),1}\right) + q_{0,(e),1}\right)e_1 + \\
&\quad \left(\left(x_{(e),2}\right)^N + \left(q_{(N-1),(e),2}\right)\left(x_{(e),2}\right)^{N-1} + \ldots + \left(q_{1,(e),2}\right)\left(x_{(e),2}\right) + q_{0,(e),2}\right)e_2 \\
&= f_p\left(x_{(e),1}\right)e_1 + f_p\left(x_{(e),2}\right)e_2,
\end{aligned}$$

where $f_p\left(x_{(e),1}\right)$ and $f_p\left(x_{(e),2}\right)$ are polynomials of degree N with EC number coefficients and values. Since these polynomials have at most N zeros each, the EQ-valued polynomial $f_p\left(x_{(E)}\right)$ has at most N^2 roots.

Theorem 2. *An EQ matrix $Q_{(E)} = Q_{(e),1}e_1 + Q_{(e),2}e_2 \in \mathbb{H}_p^{n\times n}$ is nonsingular if and only if the EC matrices $Q_{(e),1}, Q_{(e),2} \in \mathbb{C}_p^{n\times n}$ are nonsingular. If $Q_{(e),1}, Q_{(e),2} \in \mathbb{C}_p^{n\times n}$ are nonsingular, then*

$$Q_{(E)}^{-1} = Q_{(e),1}^{-1}e_1 + Q_{(e),2}^{-1}e_2.$$

Proof. *Let's assume that $Q_{(E)} = Q_{(e),1}e_1 + Q_{(e),2}e_2 \in \mathbb{H}_p^{n\times n}$ is nonsingular and $Q_{(E)}^{-1} = P_{(e),1}e_1 + P_{(e),2}e_2$ is inverse of $Q_{(E)}$. In this case,*

$$Q_{(E)}Q_{(E)}^{-1} = \left(Q_{(e),1}P_{(e),1}e_1 + Q_{(e),2}P_{(e),2}e_2\right) = I_n e_1 + I_n e_2$$

holds. By this fact,

$$Q_{(e),1}P_{(e),1} = I_n \text{ and } Q_{(e),2}P_{(e),2} = I_n$$

are obtained. Then, we get

$$Q_{(e),1}^{-1} = P_{(e),1} \text{ and } Q_{(e),2}^{-1} = P_{(e),2}.$$

Conversely, let's assume that $Q_{(e),1}, Q_{(e),2} \in \mathbb{C}_p^{n\times n}$ are nonsingular EC matrices. In this case,

$$\left(Q_{(e),1}e_1 + Q_{(e),2}e_2\right)\left(Q_{(e),1}^{-1}e_1 + Q_{(e),2}^{-1}e_2\right) = I_n e_1 + I_n e_2$$

holds. Consequently, $Q_{(E)} = Q_{(e),1}e_1 + A_{(e),2}e_2 \in \mathbb{H}_p^{n\times n}$ is nonsingular and

$$Q_{(E)}^{-1} = Q_{(e),1}^{-1}e_1 + Q_{(e),2}^{-1}e_2$$

is valid.

Theorem 3. *Let $Q_{(E)} = Q_{(e),1}e_1 + Q_{(e),2}e_2 \in \mathbb{H}_p^{n\times n}$. Suppose that $\lambda_{(e),1}$ and $\lambda_{(e),2}$ are eigenvalues of EC matrices $Q_{(e),1}$ and $Q_{(e),2}$ corresponding to the eigenvectors $x_{(e),1}$ and $x_{(e),2}$, respectively. Then $\lambda_{(E)} = \lambda_{(e),1}e_1 + \lambda_{(e),2}e_2$ is an eigenvalue of $Q_{(E)}$ corresponding to the eigenvector $x_{(E)} = x_{(e),1}e_1 + x_{(e),2}e_2$ and its converse is also true.*

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Proof. *Suppose* $\left(\lambda_{(e),1}, x_{(e),1}\right)$ *and* $\left(\lambda_{(e),2}, x_{(e),2}\right)$ *are eigen-pairs of EC matrices* $Q_{(e),1}$ *and* $Q_{(e),2}$*, respectively. Then,*

$$\begin{aligned} Q_{(E)}x_{(E)} = Q_{(E)}\left(x_{(e),1}e_1 + x_{(2),2}e_2\right) &= \left(Q_{(e),1}e_1 + Q_{(e),2}e_2\right)\left(x_{(e),1}e_1 + x_{(e),2}e_2\right) \\ &= Q_{(e),1}x_{(e),1}e_1 + Q_{(e),2}x_{(e),2}e_2 \\ &= \lambda_{(e),1}x_{(e),1}e_1 + \lambda_{(e),2}x_{(e),2}e_2 \\ &= \left(\lambda_{(e),1}e_1 + \lambda_{(e),2}e_2\right)\left(x_{(e),1}e_1 + x_{(e),2}e_2\right) = \lambda_{(E)}x_{(E)}. \end{aligned}$$

Thus, $\left(\lambda_{(E)}, x_{(E)}\right)$ *is an eigen-pairs of* $Q_{(E)}$*. Conversely, assume that* $\left(\lambda_{(E)}, x_{(E)}\right)$ *is an eigen-pairs of* $Q_{(E)}$*. Then, we get* $Q_{(E)}x_{(E)} = \lambda_{(E)}x_{(E)}$ *and*

$$Q_{(e),1}x_{(e),1}e_1 + Q_{(e),2}x_{(e),2}e_2 = \lambda_{(e),1}x_{(e),1}e_1 + \lambda_{(e),2}x_{(e),2}e_2$$

which implies

$$Q_{(e),1}x_{(e),1} = \lambda_{(e),1}x_{(e),1} \text{ and } Q_{(e),2}x_{(e),2} = \lambda_{(e),2}x_{(e),2}.$$

Corollary 1. *Let* $Q_{(E)} = Q_{(e),1}e_1 + Q_{(e),2}e_2 \in \mathbb{H}_p^{n\times n}$*. Since EC matrices* $Q_{(e),1}$ *and* $Q_{(e),2}$ *have at most* n *eigenvalues, the EQ matrix* $Q_{(E)}$ *has at most* n^2 *eigenvalues.*

Theorem 4. *Let* $Q_{(E)} = Q_{(e),1}e_1 + Q_{(e),2}e_2 \in \mathbb{H}_p^{m\times n}$*. Suppose that singular value decompositions of* $Q_{(e),1}$ *and* $Q_{(e),2}$ *are* $Q_{(e),1} = U_{(e),1}\Sigma_1 V^*_{(e),1}$ *and* $Q_{(e),2} = U_{(e),2}\Sigma_2 V^*_{(e),2}$*, respectively. Then, the singular value decomposition of EQ matrix* $Q_{(E)}$ *is given by*

$$Q_{(E)} = U_{(E)}\Sigma_{(E)}V^*_{(E)}$$

where $\Sigma_{(E)} = \Sigma_1 e_1 + \Sigma_2 e_2$*,* $U_{(E)} = U_{(e),1}e_1 + U_{(e),2}e_2$ *and* $V_{(E)} = V_{(e),1}e_1 + V_{(e),2}e_2$ *such that* $U_{(E)}$ *and* $V_{(E)}$ *are unitary matrices.*

Proof. Let the singular value decompositions of $Q_{(e),1}$ and $Q_{(e),2}$ be $Q_{(e),1} = U_{(e),1}\Sigma_1 V^*_{(e),1}$ and $Q_{(e),2} = U_{(e),2}\Sigma_2 V^*_{(e),2}$, respectively. Then, the singular value decomposition of $Q_{(E)}$ is as follows:

$$\begin{aligned} Q_{(E)} = Q_{(e),1}e_1 + Q_{(e),2}e_2 &= \left(U_{(e),1}\Sigma_1 V^*_{(e),1}\right)e_1 + \left(U_{(e),2}\Sigma_2 V^*_{(e),2}\right)e_2 \\ &= \left(U_{(e),1}e_1 + U_{(e),2}e_2\right)(\Sigma_1 e_1 + \Sigma_2 e_2)\left(V_{(e),1}e_1 + V_{(e),2}e_2\right)^* \\ &= U_{(E)}\Sigma_{(E)}V^*_{(E)} \end{aligned}$$

where

$$\begin{aligned} U_{(E)}U^*_{(E)} &= \left(U_{(e),1}e_1 + U_{(e),2}e_2\right)\left(U_{(e),1}e_1 + U_{(e),2}e_2\right)^* \\ &= U_{(e),1}U^*_{(e),1}e_1 + U_{(e),2}U^*_{(e),2}e_2 \\ &= I_n e_1 + I_n e_2 = I_n, \end{aligned}$$

$$\begin{aligned} V_{(E)}V^*_{(E)} &= \left(V_{(e),1}e_1 + V_{(e),2}e_2\right)\left(V_{(e),1}e_1 + V_{(e),2}e_2\right)^* \\ &= V_{(e),1}V^*_{(e),1}e_1 + V_{(e),2}V^*_{(e),2}e_2 \\ &= I_n e_1 + I_n e_2 = I_n, \end{aligned}$$

and $\Sigma_{(E)}$ is hyperbolic matrix. ($\Sigma_{(E)}$ is real matrix if and only if $\Sigma_1 = \Sigma_2$.) □

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Corollary 2. *Let* $Q_{(E)} = Q_{(e),1}e_1 + Q_{(e),2}e_2 \in \mathbb{H}_p^{m\times n}$. *Then*

$$rank\left(Q_{(E)}\right) = \max\left(rank\left(Q_{(e),1}\right), rank\left(Q_{(e),1}\right)\right) = \max(rank(\Sigma_1), rank(\Sigma_2)).$$

Corollary 3. *Let* $Q_{(E)} = Q_{(e),1}e_1 + Q_{(e),2}e_2 \in \mathbb{H}_p^{m\times n}$ *be the singular value decomposition of an EQ matrix in the form* $Q_{(E)} = U_{(E)}\Sigma_{(E)}V^*_{(E)}$. *Then, the pseudo-inverse of* $Q_{(E)}$ *is* $Q^\dagger_{(E)} = V_{(E)}\Sigma^\dagger_{(E)}U^*_{(E)}$ *where* $\Sigma^\dagger = \Sigma_1^\dagger e_1 + \Sigma_2^\dagger e_2$ *and* $\Sigma_1, \Sigma_2 \in \mathbb{R}^{m\times n}$.

Theorem 5. *The least squares solution with the minimum norm of the EQ matrix equation* $Q_{1,(E)}X_{(E)} = Q_{2,(E)}$ *is*

$$X_{(E)} = Q^\dagger_{1,(E)}Q_{2,(E)} = V_{(E)}\Sigma^\dagger_{(E)}U^*_{(E)}Q_{2,(E)}.$$

where $Q_{1,(E)} \in \mathbb{H}_p^{m\times n}$ *and* $Q_{2,(E)} \in \mathbb{H}_p^{m\times q}$.

Proof. *The least squares solution with the minimum norm of the EQ matrix equation* $Q_{1,(E)}X_{(E)} = Q_{2,(E)}$ *is* $X_{(E)}$. *Then we get* $\left\|Q_{1,(E)}X_{(E)} - Q_{2,(E)}\right\|_p = \min$ *and*

$$\begin{aligned}
\left\|Q_{1,(E)}X_{(E)} - Q_{2,(E)}\right\|_p^2 &= \left\|\left(Q_{1,(e),1}e_1 + Q_{1,(e),2}e_2\right)\left(X_{(e),1}e_1 + X_{(e),2}e_2\right) - \left(Q_{2,(e),1}e_1 + Q_{2,(e),2}e_2\right)\right\|_p^2 \\
&= \left\|\left(Q_{1,(e),1}X_{(e),1}e_1 + Q_{1,(e),2}X_{(e),2}e_2\right) - \left(Q_{2,(e),1}e_1 + Q_{2,(e),2}e_2\right)\right\|_p^2 \\
&= \left\|\left(Q_{1,(e),1}X_{(e),1} - Q_{2,(e),1}\right)e_1 + \left(Q_{1,(e),2}X_{(e),2} - Q_{2,(e),2}\right)e_2\right\|_p^2.
\end{aligned}$$

From Eq. (2), we have

$$\left\|Q_{1,(E)}X_{(E)} - Q_{2,(E)}\right\|_p^2 = \frac{1}{2}\left(\left\|Q_{(e),1}X_{(e),1} - Q_{2,(e),1}\right\|_p^2 + \left\|Q_{1,(e),2}X_{(e),2} - Q_{2,(e),2}\right\|_p^2\right).$$

Hence, $\left\|Q_{1,(E)}X_{(E)} - Q_{2,(E)}\right\|_p = \min.$ *if and only if*

$$\left\|Q_{1,(e),1}X_{(e),1} - Q_{2,(e),1}\right\|_p = \min., \quad and \quad \left\|Q_{1,(e),2}X_{(e),2} - Q_{2,(e),2}\right\|_p = \min.$$

If $\left\|Q_{1,(e),1}X_{(e),1} - Q_{2,(e),1}\right\|_p = \min$, *then* $X_{(e),1} = Q^\dagger_{1,(e),1}Q_{2,(e),1}$ *and similarly,* $X_{(e),2} = Q^\dagger_{1,(e),2}Q_{2,(e),2}$ *where* $Q^\dagger_{1,(e),1} = V_{(e),1}\Sigma_1U^*_{(e),1}$ and $Q^\dagger_{1,(e),2} = V_{(e),2}\Sigma_2U^*_{(e),2}$. *Therefore, the least squares solution of the equation* $Q_{1,(E)}X_{(E)} = Q_{2,(E)}$ *is*

$$\begin{aligned}
X_{(E)} &= \left(V_{(e),1}\Sigma_1U^*_{(e),1}Q_{2,(e),1}\right)e_1 + \left(V_{(e),2}\Sigma_2U^*_{(e),2}Q_{2,(e),2}\right)e_2 \\
&= \left(\left(V_{(e),1}\Sigma_1U^*_{(e),1}\right)e_1 + \left(V_{(e),2}\Sigma_2U^*_{(e),2}\right)e_2\right)\left(Q_{2,(e),1}e_1 + Q_{2,(e),2}e_2\right) \\
&= \left(V_{(e),1}e_1 + V_{(e),2}e_2\right)(\Sigma_1e_1 + \Sigma_2e_2)\left(U^*_{(e),1}e_1 + U^*_{(e),2}e_2\right)\left(Q_{2,(e),1}e_1 + Q_{2,(e),2}e_2\right) \\
&= V_{(E)}\Sigma^\dagger_{(E)}U^*_{(E)}Q_{2,(E)} = Q^\dagger_{1,(E)}Q_{2,(E)}
\end{aligned}$$

which completes the proof.

3.2.1. Algorithms

The following algorithms delineate the computational procedures for determining eigen-pairs, singular value decomposition, pseudo-inverse computation, and the derivation of least squares solution with the minimum norm for *EQ* matrices.

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Algorithm 1 Eigen-pairs of $Q_{(E)} \in \mathbb{H}_p^{n\times n}$

Start
Input $Q_{(E),r}, Q_{(E),i}, Q_{(E),j}, Q_{(E),k}$ and p
Form $Q_{(e),1}$ and $Q_{(e),2}$ according to Eq. (1)
Compute $\lambda_{(e),1}$ and $\lambda_{(e),2}$ according to Lemma 4 and Theorem 3
Compute $x_{(e),1}$ and $x_{(e),2}$ according to Lemma 4 and Theorem 3
Form $\lambda_{(E)} = \lambda_{(e),1}e_1 + \lambda_{(e),2}e_2$
Form $x_{(E)} = x_{(e),1}e_1 + x_{(e),2}e_2$
Output $\lambda_{(E)}$ and $x_{(E)}$
End

Algorithm 2 Singular Value Decomposition of $Q_{(E)} \in \mathbb{H}_p^{m\times n}$

Start
Input $Q_{(E),r}, Q_{(E),i}, Q_{(E),j}, Q_{(E),k}$ and p
Form $Q_{(e),1}$ and $Q_{(e),2}$ according to Eq. (1)
Compute $Q_{(e),1} = U_{(e),1}\Sigma_1 V^*_{(e),1}$ and $Q_{(e),2} = U_{(e),2}\Sigma_2 V^*_{(e),2}$ according to Lemma 2
Form $U_{(E)} = \left(U_{(e),1}e_1 + U_{(e),2}e_2\right)$, $\Sigma_{(E)} = (\sum_1 e_1 + \sum_2 e_2)$, and $V_{(E)} = \left(V_{(e),1}e_1 + V_{(e),2}e_2\right)$
Output $U_{(E)}$, $\Sigma_{(E)}$, $V_{(E)}$
End

Algorithm 3 Pseudo-inverse of $Q_{(E)} \in \mathbb{H}_p^{m\times n}$

Start
Run the Algorithm 2 for *EQ* matrix $Q_{(E)}$
Form $Q_{(E)} = U_{(E)}\Sigma_{(E)}V^*_{(E)}$
Compute $Q^\dagger_{(E)} = V_{(E)}\Sigma^\dagger_{(E)}U^*_{(E)}$
Output $Q^\dagger_{(E)}$
End

Algorithm 4 Least Squares Solution with the Minimum Norm of $Q_{1,(E)}X_{(E)} = Q_{2,(E)}$

Start
Input $Q_{1,(E),r}, Q_{1,(E),i}, Q_{1,(E),j}, Q_{1,(E),k}, Q_{2,(E),r}, Q_{2,(E),i}, Q_{2,(E),j}, Q_{2,(E),k}$ and p
Run the Algorithm 3 for *EQ* matrix $Q_{1,(E)}$
Compute $X_{(E)} = Q^\dagger_{1,(E)}Q_{2,(E)} = V_{(E)}\Sigma^\dagger_{(E)}U^*_{(E)}Q_{2,(E)}$
Output $X_{(E)}$
End

3.2.2. Numerical Examples

Example 1. *Given the EQ matrices $Q_{1,(E)}$ and $Q_{2,(E)}$ as follows:*

$$Q_{1,(E)} = \begin{pmatrix} 9+i-7j+2k & 2-7i+j+5k & 8-4i+7j-4k \\ 5+8i-j+4k & 4+7i+j-k & 9-6i-3j+8k \\ 8+9i+3j-4k & 9-2i+8j+2k & 6-5i+9j-6k \end{pmatrix},$$

and

$$Q_{2,(E)} = \begin{pmatrix} 7+2i+3j+k \\ 7-6i+j+4k \\ 3+3i-5j-8k \end{pmatrix}.$$

Let's find the least squares solution with the minimum norm by using Algorithms 2, 3, and 4 for $p = -0.5$. By the Algorithm 2, we get the singular value decomposition of EQ matrix $Q_{1,(E)}$ as follows:

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$$\Sigma = \begin{pmatrix} 27.1507 + 5.1850j & 0 & 0 \\ 0 & 13.3956 - 3.6604j & 0 \\ 0 & 0 & 3.5511 + 0.0727j \end{pmatrix},$$

$$U = \begin{pmatrix} -0.5887 - 0.0895i + 0.1905j - 0.1797k & 0.5737 + 0.2198i + 0.2880j - 0.0213k & -0.1504 - 0.0811i + 0.3531j + 0.1408k \\ -0.2573 - 0.4628i + 0.0895j - 0.0090k & -0.2014 - 0.8270i + 0.0050j + 0.3648k & 0.1490 + 0.6752i + 0.1013j + 0.4671k \\ -0.4969 - 0.4443i - 0.2639j + 0.0273k & -0.1356 + 0.2676i - 0.1680j - 0.2529k & -0.1172 + 0.4220i - -0.326j - 0.7217k \end{pmatrix},$$

and

$$V = \begin{pmatrix} -0.6674 + 0.2309j & -0.1204 - 0.3768j & 0.5534 + 0.1965j \\ -0.3280 - 0.3659i - 0.1703j + 0.0553k & -0.4857 + 0.1440i - 0.0426j + 0.0036k & -0.3596 - 0.6220i - 0.2807j + 0.5391k \\ -0.2692 - 0.5713i - 0.2100j - 0.1815k & 0.4859 - 0.3190i + 0.1157j + 0.7682k & -0.1476 - 0.1702i + 0.2676j + 0.0299k \end{pmatrix}.$$

By Algorithms 2 and 3, the least squares solution and minimum norm are found as

$$\begin{aligned} X_{(E)} &= Q_{1,(E)}^{\dagger} Q_{2,(E)} = V_{(E)} \Sigma_{(E)}^{\dagger} U_{(E)}^{*} Q_{2,(E)} \\ &= \begin{pmatrix} 0.7612 - 1.5582i - 0.1979j - 0.5332k \\ -1.4757 - 0.2366i + 0.1496j + 1.2282k \\ 0.0461 + 0.2919i + 0.4652j - 0.0282k \end{pmatrix} \end{aligned}$$

and

$$\left\| Q_{1,(E)} X_{(E)} - Q_{2,(E)} \right\| = 1.1801 \times 10^{-14},$$

respectively.

Example 2. *Let's define the dimensions of the EQ matrices $Q_{1,(E)}$ and $Q_{2,(E)}$ given by:*

$$\begin{aligned} & g = 1:1:20, \;\; m = 50g, \\ & Q_{1,(E)} = rand(m,m) + rand(m,m)i + rand(m,m)j + rand(m,m)k, \\ & Q_{2,(E)} = rand(m,1) + rand(m,1)i + rand(m,1)j + rand(m,1)k. \end{aligned}$$

Then, the errors (or minimum norms) corresponding p and m-values are shown in Figure 1. Here, Algorithm 4 was executed for each m-value, iterating over each p-value in the range $-1 \leq p \leq -0.1$ with a step size of 0.1. The minimal errors were identified and highlighted on the surface plot with red dots.

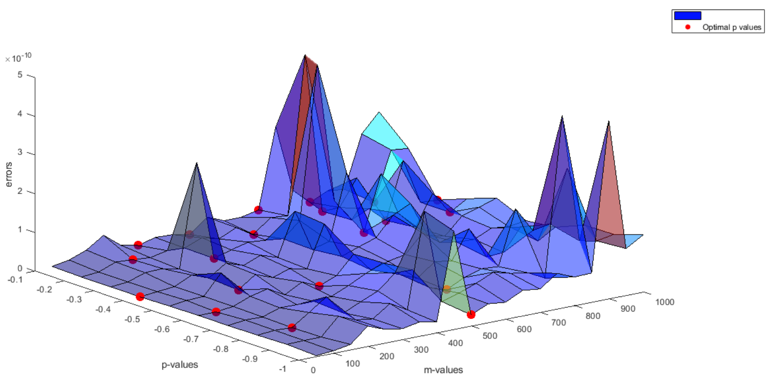

Figure 1. Errors corresponding p and m-values.

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Also, we compare our new proposed Algorithm 4 and the Algorithm documented by Atali et al. in [17], focusing on CPU time and error metrics. The experimental results of this comparison are depicted in Figure 2 (CPU times) and Figure 3 (Errors).

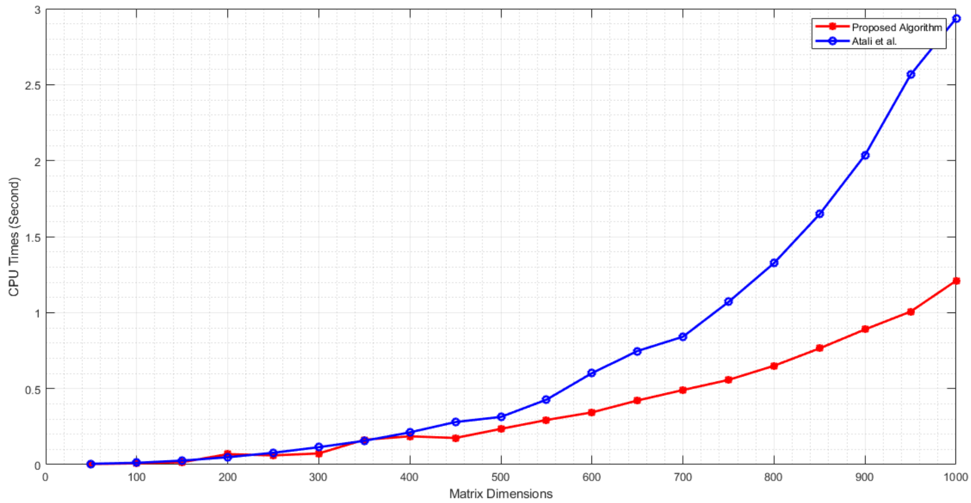

Figure 2. CPU times according to m and optimal $p-$ values.

and

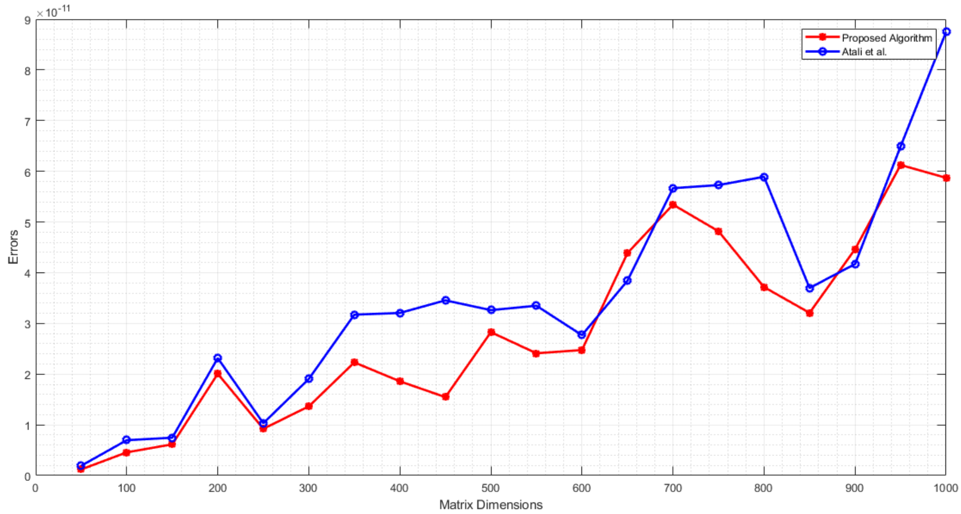

Figure 3. Errors according to m and optimal $p-$ values.

Figures 2 and 3 show that our proposed algorithm outperforms the algorithm presented by Atali et al. in [17] regarding computational efficiency and accuracy.

4. Conclusion

In this research, we derived outcomes for determining eigen-pairs, performing singular value decomposition, obtaining pseudo-inverse, and finding the least squares solution with the minimum norm for EQ matrices. Additionally, we developed algorithms grounded on these outcomes and presented illustrative numerical instances to validate our results.

This number system is more useful in the applied sciences since it allows one to select the ideal p-value suited for the type of problem in EQs, considering the elliptical behaviour of many physical

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systems. As a result, the use of *EQ*s in today's critical technology fields - information security, data analytics, simulation technologies, robotics, signal processing, image processing, artificial intelligence, and machine learning - will effectively solve many problems related to time, memory, and performance.

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