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Posted Date: 31 July 2024

doi: 10.20944/preprints202407.2551.v1

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Euclidean Locality in Treonic Topological Spaces: The Genesis of Treonic Manifolds

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Abstract: We explore the concept of Euclidean locality within treonic topological spaces. Our study establishes a foundational theoretical framework, elucidating the properties of continuity, homeomorphisms, and compactness in these spaces. We assess the Euclidean locality of treonic spaces through the analysis of specific homeomorphisms, which enables us to define manifolds in treonic spaces and extend recent research on Bermejo Algebras. For the first time, we characterize treonic manifolds by incorporating the property of Euclidean locality alongside previously studied properties such as Hausdorff spaces and second countable spaces. Our findings advance the understanding of the topological and geometric properties of treonic spaces, providing significant insights for advanced mathematical research.

Keywords: treonic spaces; Euclidean locality; topological manifolds; homeomorphisms; Bermejo algebras

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1. Introduction

Topological spaces are fundamental to mathematical research, providing a robust framework for investigating a wide range of mathematical and physical phenomena [1]. Treonic spaces, distinguished by their unique intrinsic properties [2,3], represent a particularly intriguing area of study. We explored the concept of Euclidean locality within treonic topological spaces, focusing on their topological and geometric characteristics.

Our investigation began by constructing a detailed theoretical framework that defined the essential concepts of continuity, homeomorphisms, and compactness within topological spaces. These foundational concepts were pivotal for comprehending the behavior and properties of treonic spaces. We then extended this theoretical framework to analyze the structure of treonic spaces, specifically examining their Euclidean locality by establishing homeomorphisms with $\mathbb{R}^3$. This approach allowed us to elucidate the conditions under which treonic spaces could be locally represented as Euclidean spaces.

Additionally, we delved into the quotient spaces derived from treonic spaces, assessing their manifold properties and identifying the criteria under which they could be locally modeled as Euclidean spaces.

A significant contribution of this paper was the novel characterization of treonic manifolds by integrating the property of Euclidean locality within treonic spaces with previously studied properties such as being Hausdorff spaces [2] and second-countable spaces [3].

By incorporating the concept of Euclidean locality into the study of treonic spaces, our research advanced current knowledge in topology. The insights gained from this study not only deepened our comprehension of treonic spaces but also opened up new avenues for advanced mathematical research. Future studies could further explore the implications of these properties in various contexts, potentially leading to novel developments and applications in the mathematical sciences.

2. Theoretical Framework

2.1. Continuity of Mappings Between Topological Spaces

Given two topological spaces $(\Lambda_A, T_{\Lambda_A})$ and $(\Lambda_B, T_{\Lambda_B})$, a mapping $f : \Lambda_A \to \Lambda_B$ is *continuous* if $\forall U \in T_{\Lambda_B} \Rightarrow f^{-1}(U) \in T_{\Lambda_A}$ [4,5], where f^{-1} denotes the inverse mapping of f. Additionally, we say that f is *sequentially continuous* if $\forall p_0 \in \Lambda_A$, where p_0 is a fixed treon $p_0 \equiv (p_{0_1}, p_{0_2}, p_{0_3})$, we have

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$(p_n)_{n\in\mathbb{N},p_n\to p_0} \subseteq \Lambda_A \Rightarrow f((p_n)_{n\in\mathbb{N}}) \to f(p_0) \subseteq \Lambda_B$, meaning the sequence in the domain is mapped to a convergent sequence in the image $f(p_n) \to f(p_0)$ as $n \to \infty$ [6].

It is known that if a mapping is continuous, then it is sequentially continuous. Moreover, if a mapping is sequentially continuous and defined in a metric space or in a second-countable space, then it is a continuous mapping [4,6–8].

2.2. Homeomorphisms

Given two topological spaces $(\Lambda_A, T_{\Lambda_A})$ and $(\Lambda_B, T_{\Lambda_B})$, a mapping $f : \Lambda_A \to \Lambda_B$ is a *homeomorphism* if f is bijective, continuous, and if the inverse, $f^{-1} : \Lambda_B \to \Lambda_A$, is also continuous [9,10].

3. Theoretical Development

3.1. Euclidean Locality in Treonic Spaces

A topological space (Λ, T_Λ) is an n-dimensional topological manifold if: (1) It is a Hausdorff space, (2) it is second-countable, and (3) it is locally Euclidean of dimension n [11–13].

A topological space (Λ, T_Λ) being locally Euclidean means that we can locally relate it via a homeomorphism to the topological space $(\mathbb{R}^n, T_d)$, where T_d is the standard topology induced by the Euclidean metric [14–16].

The fact that this homeomorphism occurs "locally" indicates that, by fixing a point $p_0 \in (\Lambda, T_\Lambda)$, we can define an open neighborhood $V_{p_0} \in T_\Lambda$ of the point p_0 as homeomorphic to an open neighborhood of a point $\rho_0 \in \mathbb{R}^n$.

Definition 1. *Let a treon $p_0 \equiv (p_{0_1}, p_{0_2}, p_{0_3})$ and an open neighborhood of p_0, V_{p_0}. A treonic topological space (Λ, T_Λ) is locally Euclidean n-dimensional if $\forall p_0 \in \Lambda \exists V_p \in T_\Lambda$ and a homeomorphism $f : V_\Lambda \to V_R$, such that $V_R \in \mathbb{R}^n$ and $V_\Lambda \in \Lambda$.*

3.2. Homeomorphism Between Treonic Space and $\mathbb{R}^3$

Since the treonic space Λ is isomorphic to $\mathbb{R}^3$, we can find a bijective, continuous mapping $f : \Lambda \to \mathbb{R}^3$ whose inverse f^{-1} is also continuous.

We can define a homeomorphism f:

$$f : \Lambda \to \mathbb{R}^3,$$

$$p_1 + p_2 i + p_3 j \equiv (p_1, p_2, p_3) \mapsto (\rho_1, \rho_2, \rho_3),$$

such that $f((p_1, p_2, p_3)) = (\rho_1, \rho_2, \rho_3)$. This correspondence associates each treon $(p_1, p_2, p_3) \in \Lambda$ with a vector (ρ_1, ρ_2, ρ_3) in $\mathbb{R}^3$ with the same coordinates, meaning f is an identity mapping between coordinates of different spaces.

Bermejo demonstrated the isomorphism between treons, $p_1 + p_2 i + p_3 j$, and elements of algebra B, (p_1, p_2, p_3), when algebra B is defined using the real field $\mathbb{R}$ and the vector space $\mathbb{R}^3$ [17]. Therefore, a treon with structure (p_1, p_2, p_3) is trivially representable by the Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

The vector space $\mathbb{R}^3$ is by definition equipped with vector addition and scalar multiplication. When we equip it with the product of algebra B [18], we call it a treonic space, as there exists an isomorphism between the vectors of $\mathbb{R}^3$ and the treons [17].

The treonic space thus defined implicitly contains its own metric, norm, and inner product in its real components when the product of algebra B is performed [2,3,17]. This is a characteristic that distinguishes it from the conventional $\mathbb{R}^3$, which requires explicit incorporations of metric (metric space), norm (normed space), or inner product (inner product space). The product operation of algebra B is sufficient to define these properties [2,3,17]. This makes the subspace $N \subseteq \Lambda$ of elements that can be represented as a product of treons the space most similar to a normed, metric, and inner product space $\mathbb{R}^3$. A difference between $N \subseteq \Lambda$ and Λ is the absence of pure imaginary treons different from the null vector, $(0, p_2, p_3) \neq (0,0,0)$, since the only possibility of having an element with a null real component in N is the treon $(0,0,0)$ [2,3].

Therefore, the space Λ is isomorphic to $\mathbb{R}^3$ with respect to the correspondence between the coordinates that define its components. As a result, the space can be endowed with Euclidean norms, metrics, and inner products. Additionally, we consider a subspace $N \subseteq \Lambda$ that is also isomorphic to a subspace of $\mathbb{R}^3$, where norms, metrics, and inner products can be defined by the algebraic product of B.

Let $N \subseteq \Lambda$ be an intrinsically normed subspace, that is, in N we have elements of the form $p_A \odot p_A^{(*_{i,j})} \equiv \langle p^2 \rangle = (\|p\|^2, 2p_1p_2 + p_2p_3, 2p_1p_3 + p_3p_2)$, where $\|p\|^2 \equiv p_1^2 + p_2^2 + p_3^2$. The orthomulearity [18] for two treons p and q, given by the product $p_2 i \odot q_3 j = 0$ is trivial, as in N we have $p_2 = q_3 = 0$. On the other hand, the product $q_3 j \odot p_2 i = q_3 p_2 j \odot i$ is also zero.

Outside N, orthomulearity and the product $q_3 j \odot p_2 i$ are not trivial, as pure imaginary treons do exist.

Bermejo defined the treonic space Λ as the preimages of the mapping $\langle \cdot^2 \rangle$, such that $\langle \cdot^2 \rangle (p_1, p_2, p_3) = (\|p\|^2, 2p_1p_2 + p_2p_3, 2p_1p_3 + p_3p_2)$ [2,3]. $\langle \cdot^2 \rangle$ is a particular case of the mapping given by the double conjugate product $p_A \odot p^*_{B_{(i,j)}} \equiv \langle p_A, p_B \rangle = (p_A \diamond p_B, -p_{A_1}p_{B_2} + p_{A_2}p_{B_1} - p_{A_3}p_{B_2}, -p_{A_1}p_{B_3} + p_{A_3}p_{B_1} - p_{A_3}p_{B_2})$, where $p_A \diamond p_B \equiv p_{A_1}p_{B_1} + p_{A_2}p_{B_2} + p_{A_3}p_{B_3}$ is the Bermejian inner product [3]. By performing $\langle \cdot^2 \rangle$ on the difference between treons, d_{AB}, Bermejo defined a metric g_{AB} as $g_{AB} \equiv \sqrt{\mathrm{Re}\langle d^2_{AB} \rangle}$, where $d_{AB} = (p_{A_1}, p_{A_2}, p_{A_3}) - (p_{B_1}, p_{B_2}, p_{B_3})$ [3]. With this metric, balls of radius ϵ centered at $p_0 \equiv (p_{0_1}, p_{0_2}, p_{0_3})$ were defined: $B = \{B_\epsilon(p_0) : p_0 \in \Lambda, \epsilon \in \mathbb{R}, \epsilon > 0\}$ [3].

For these balls to be well-defined, they necessarily must have an associated norm, which is achieved by defining Λ as the preimages of $\langle \cdot^2 \rangle$, meaning in this context that:

1. We take a point $p_0 \in \Lambda$.
2. We calculate a difference between $p_0 \in \Lambda$ and an arbitrary treon $p_B \in \Lambda$, $d_{0B} = (p_{0_1}, p_{0_2}, p_{0_3}) - (p_{B_1}, p_{B_2}, p_{B_3})$.
3. We apply the mapping $\langle \cdot^2 \rangle (d_{0B})$, $\langle \cdot^2 \rangle (d_{0B}) = (\|d_{0B}\|^2, 2d_{0B_1}d_{0B_2} + d_{0B_2}d_{0B_3}, 2d_{0B_1}d_{0B_3} + d_{0B_3}d_{0B_2})$.
4. We extract g_{0B}, $g_{0B} = \sqrt{\mathrm{Re}\langle d^2_{0B} \rangle} = \sqrt{\|d_{0B}\|^2}$.

In this way, the metric quantity g_{0B} can be defined arbitrarily small, $g_{0B} = \epsilon$, for treons in Λ.

Note that the preimages of the mapping $\langle \cdot^2 \rangle$, Λ, is the entire treonic space, as there is no domain for which $\langle \cdot^2 \rangle$ is not defined. We simply refer to the entire treonic space as the preimage of $\langle \cdot^2 \rangle$ to indicate that these elements are subject to the product $\langle p_A, p_B \rangle$ when it is necessary to define a metric, a norm, or an inner product.

The difference between Λ and $\mathbb{R}^3$ lies in the fact that we need to equip $\mathbb{R}^3$ with the metric to have balls of radius ϵ centered at vectors $\vec{p} \equiv p_1\hat{i} + p_2\hat{j} + p_3\hat{k} \equiv (\rho_1, \rho_2, \rho_3)$, where $\{\hat{i}, \hat{j}, \hat{k}\}$ is the canonical basis. On the other hand, Λ with the product of algebra B contains the metric g_{ij} intrinsically.

If we take two position vectors $\rho_0 \equiv (\rho_{0_1}, \rho_{0_2}, \rho_{0_3})$ and $\rho_B = (\rho_{B_1}, \rho_{B_2}, \rho_{B_3})$ in $\mathbb{R}^3$ and apply the same reasoning to obtain a metric, we have: $d_{0B} = (\rho_{0_1}, \rho_{0_2}, \rho_{0_3}) - (\rho_{B_1}, \rho_{B_2}, \rho_{B_3}) = (\rho_{0_1} - \rho_{B_1}, \rho_{0_2} - \rho_{B_2}, \rho_{0_3} - \rho_{B_3}) \equiv (d_{0B_1}, d_{0B_2}, d_{0B_3})$, and $\|d_{0B}\|^2 = (d_{0B_1}, d_{0B_2}, d_{0B_3}) \cdot (d_{0B_1}, d_{0B_2}, d_{0B_3}) = (d_{0B_1})^2 + (d_{0B_2})^2 + (d_{0B_3})^3$, therefore $g_{0B} = \|d_{0B}\| = \sqrt{(d_{0B_1})^2 + (d_{0B_2})^2 + (d_{0B_3})^3}$ (The operation dot $\cdot$ here is the inner (dot) product in $\mathbb{R}^3$). This exactly matches the components of the Bermejian metric in the real component of treons. Therefore, the open balls given by the Euclidean metric have a one-to-one correspondence with treonic open balls.

3.2.1. Bijectivity

Bijectivity is verified by injectivity: $f((p_{A_1}, p_{A_2}, p_{A_3})) = f((p_{B_1}, p_{B_2}, p_{B_3})) \Leftrightarrow (p_{A_1}, p_{A_2}, p_{A_3}) = (p_{B_1}, p_{B_2}, p_{B_3})$, and surjectivity: $\forall (\rho_1, \rho_2, \rho_3) \in \mathbb{R}^3 \exists (p_1, p_2, p_3) \in \Lambda$.

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Suppose $f((p_{A_1}, p_{A_2}, p_{A_3})) = f((p_{B_1}, p_{B_2}, p_{B_3}))$. Since there is a unique correspondence given by f (the identity mapping of coordinates):

$$f((p_{A_1}, p_{A_2}, p_{A_3})) = (\rho_{A_1}, \rho_{A_2}, \rho_{A_3}) \Rightarrow p_{A_1} = \rho_{A_1} \wedge p_{A_2} = \rho_{A_2} \wedge p_{A_3} = \rho_{A_3}$$

and

$$f((p_{B_1}, p_{B_2}, p_{B_3})) = (\rho_{B_1}, \rho_{B_2}, \rho_{B_3}) \Rightarrow p_{B_1} = \rho_{B_1} \wedge p_{B_2} = \rho_{B_2} \wedge p_{B_3} = \rho_{B_3}.$$

Therefore, if:

$$(\rho_{A_1}, \rho_{A_2}, \rho_{A_3}) = (\rho_{B_1}, \rho_{B_2}, \rho_{B_3}),$$

then:

$$(p_{A_1}, p_{A_2}, p_{A_3}) = (p_{B_1}, p_{B_2}, p_{B_3}).$$

It is verified that f is injective.

Note that we can cover all of $\mathbb{R}^3$ with elements (ρ_1, ρ_2, ρ_3) that come from (p_1, p_2, p_3) under f. Since all of Λ is the domain of f, all of $\mathbb{R}^3$ is the image. Thus, effectively for every $(\rho_1, \rho_2, \rho_3) \in \mathbb{R}^3$ there exists a $(p_1, p_2, p_3) \in \Lambda$. Therefore, f is surjective.

3.2.2. Continuity of f and f^{-1}

To prove that f is continuous, we need to show that for any open set O in $\mathbb{R}^3$, the inverse set $f^{-1}(O)$ is open in Λ.

Bermejo defined that, for (Λ, T_Λ) a topological space induced by the Bermejian metric g_{ij}, the basis B of T_Λ is defined as the balls of radius ϵ centered at $p_0 \equiv (p_{0_1}, p_{0_2}, p_{0_3})$, such that $B = \{B_\epsilon(p_0) : p_0 \in \Lambda_{\mathbb{Q}^3}, \epsilon \in \mathbb{Q}, \epsilon > 0\}$, where the components of the central treon of each ball p_{0_i}, and the distance ϵ, belong to the set $\mathbb{Q}$, so that B is a countable basis of T_Λ [3].

Open balls centered at points that are a Cartesian product $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$, with rational radii, can approximate any open ball in $\mathbb{R}^3$. For example, if we have an irrational point $p \equiv (p_1, p_2, p_3)$, the balls $B_\epsilon(p_0)$, with $p_{0_i} \in \mathbb{Q}$ and ϵ a rational radius arbitrarily small, will cover points close to p, including p itself.

Any open ball in $\mathbb{R}^3$ centered at an irrational point (which does not form our countable basis) can be seen as a union of smaller open balls centered at rational points. Therefore, the countable basis of balls with rational centers is sufficient to describe the topology of the entire treonic space, including irrational points.

Rational numbers $\mathbb{Q}$ are dense in real numbers $\mathbb{R}$ [3], meaning for any real number, we can always find a sequence of rational numbers that approximate it arbitrarily closely. This property ensures that, even though we use a basis with rational elements, we can always approximate any point in $\mathbb{R}$ with arbitrary precision using rational elements. Formally, $\forall \epsilon > 0, \forall x \in \mathbb{R} \exists q \in \mathbb{Q} : \|x - q\| < \epsilon$. Then, given any point $x \in \mathbb{R}^3$ and an open ball centered at x with radius ϵ, we can find a point $q \in \mathbb{Q}^3$ within the ball such that $\|x - q\| < \epsilon$.

Note that the treonic space Λ itself is not limited to rational elements, but the basis for its topology T_Λ can be defined using balls with rational centers.

The Bermejian metric g_{ij} and the topology induced by this metric in Λ allow the open balls defined in terms of rational elements to encompass within the radius ϵ irrational points, as well as approximate arbitrarily any open ball centered on a treon defined with real components $p \equiv (p_1, p_2, p_3), p_i \in \mathbb{R}$. In this sense, denoting the centers of balls as $p_0 \equiv (p_{0_1}, p_{0_2}, p_{0_3})$ if $p_{0_i} \in \Lambda_{\mathbb{Q}}$ and as $p_1 \equiv (p_{1_1}, p_{1_2}, p_{1_3})$ if $p_1 \in \mathbb{R}^3$, and let a mapping f, we have:

$$(p_0)_{n, n\in\mathbb{N}, n\to p_1} \Rightarrow f((p_0)_{n, n\in\mathbb{N}, n\to p_1}) = (\rho_0)_{n, n\in\mathbb{N}, n\to \rho_1},$$

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where $\rho_0 \equiv (\rho_{0_1}, \rho_{0_2}, \rho_{0_3}) \in \mathbb{Q}^3$ and $\rho_1 \equiv (\rho_{1_1}, \rho_{1_2}, \rho_{1_3}) \in \mathbb{R}^3$. In this sense, f is a sequentially continuous mapping, and as it is defined over the second-countable space $\Lambda_{\mathbb{Q}}$, then f is a continuous mapping [4,6–8].

Continuity can also be viewed from the perspective of open sets in topologies. To demonstrate that f is continuous, we must verify that for any open set O in $\mathbb{R}^3$, the preimage $f^{-1}(O)$ is an open set in $\Lambda_{\mathbb{Q}}$. We will continue from here with the notation Λ, understanding that it is a second-countable space $(\Lambda_{\mathbb{Q}})$.

Consider the open set O in $\mathbb{R}^3$; by the definition of standard topology, O can be represented as the union of open balls in $\mathbb{R}^3$ (or $\mathbb{Q}^3$ in the second-countable sense):

$$O = \bigcup_{i \in I} B_{\epsilon_i}(\rho_i),$$

where $\rho_i \equiv (\rho_{i_1}, \rho_{i_2}, \rho_{i_3}) \in \mathbb{R}^3$.

Representing O in terms of its topological basis, the preimage of O under f, $\text{Preim}_f(O) \equiv f^{-1}(O)$, is:

$$f^{-1}(O) = f^{-1}\left(\bigcup_{i \in I} B_{\epsilon_i}(\rho_i)\right) = \bigcup_{i \in I} f^{-1}(B_{\epsilon_i}(\rho_i)).$$

The preimage of each arbitrary open ball $B_\epsilon(\rho_0)$ is:

$$f^{-1}(B_\epsilon(\rho_0)) = \{p \in \Lambda : f(p) \in B_\epsilon(\rho_0)\},$$

since f is an identity mapping of the components of the treons, i.e.:

$$f((p_1, p_2, p_3)) = \{(\rho_1, \rho_2, \rho_3) \in \mathbb{R}^3 : p_1 = \rho_1 \wedge p_2 = \rho_2 \wedge p_3 = \rho_3\},$$

and therefore:

$$f((p_1, p_2, p_3)) = (f(p_1), f(p_2), f(p_3)) = (\rho_1, \rho_2, \rho_3).$$

Thus, the preimage $f^{-1}(B_\epsilon(\rho_0))$ is exactly the ball $B_\epsilon(p_0) \in \Lambda$.

As the balls in Λ form a basis for the topology of Λ, and any open ball in $\mathbb{R}^3$ has a preimage that is an open ball in Λ, we conclude that the preimage of any open set in $\mathbb{R}^3$ is an open set in Λ:

$$f^{-1}(O) = \bigcup_{i \in I} B_{\epsilon_i}(p_i).$$

Similarly, the continuity of the inverse mapping is satisfied: f^{-1} is continuous because any open set U in Λ, the preimage $(f^{-1})^{-1}(U)$ is an open set in $\mathbb{R}^3$.

3.3. Homeomorphism Between the Treonic Quotient Space and $\mathbb{R}^3$

The homeomorphism between the treonic quotient space $\Lambda\backslash\sim$ and $\mathbb{R}^3$ presents an issue with injectivity since $f([q]) = f([-q])$, where $[q] \in \Lambda\backslash\sim$ and $f([q]) \in \mathbb{R}^3$. However, the equivalence relation $q \sim -q$ implies that injectivity is preserved in the sense that $f([q]) = f([-q]) \Leftrightarrow [q] = [-q]$. This is true, but it introduces a problem with surjectivity, as there are points in $\mathbb{R}^3$ that the mapping does not reach. Consequently, the entirety of $\mathbb{R}^3$ is not the codomain of f.

However, we must note that each equivalence class is defined by two equivalent points on a S^2-sphere of arbitrary radius r; therefore, for each radius, we will have respective spheres, forming a volume that fills the space $\mathbb{R}^3$ in which the S^2-sphere is defined. This means that each hemisphere (or semi-sphere) is equivalent to its opposite hemisphere, for example: the northern hemisphere is equivalent to the southern hemisphere under the equivalence relation $q \sim -q$.

For a constant radius r_0, the corresponding S^2-sphere will be given by the equivalence classes $([q]_\sim)_j = (\{q, -q\})_j$. For each radius $r_{i \in \mathbb{R}}$, there will be $S^2_{i \in \mathbb{R}}$-spheres.

 doi:10.20944/preprints202407.2551.v1

Fixing a radius r_0, the northern hemisphere $S^2_{r_0}$ can be projected onto the $\mathbb{R}^2$ plane such that each point on the hemisphere has a one-to-one functional correspondence with the $\mathbb{R}^2$ plane.

Let $S^2_{r_1}$ be defined as:

$$S^2_{r_1} \equiv \{\rho \equiv (\rho_1, \rho_2, \rho_3) \in \mathbb{R}^3 : \|\rho\| = 1\},$$

and let a set R be defined as:

$$R \equiv \{\rho' \equiv (x_1, x_2) \in \mathbb{R}^2 : \|\rho'\| \leq 1\}.$$

We define a mapping Ψ_1:

$$\Psi_1 : S^2_{r_1} \to \mathbb{R}^2,$$

$$(\rho_1, \rho_2, \rho_3) \mapsto (x_1, x_2, 0),$$

such that $\Psi_1((\rho_1, \rho_2, \rho_3)) = (\Psi_1(\rho_1), \Psi_1(\rho_2), \Psi_1(\rho_3)) = (x_1, x_2, 0) \equiv (x_1, x_2)$. Ψ_1 is also an identity mapping of components for the first two components (i.e.: $\Psi_1(\rho_1) = \rho_1 = x_1$ and $\Psi_1(\rho_2) = \rho_2 = x_2$), having the mapping on the third component as $\Psi_1(\rho_3) = 0$.

The vectors ρ, for the northern hemisphere, have the structure (ρ_1, ρ_2, ρ_3), such that $\rho_3 > 0$.

If we want to disregard the edge in the domain: $S^2_{r_1} = \{(\rho_1, \rho_2, \rho_3) \in \mathbb{R}^3 : \|\rho\| < 1\}$. And if we want to disregard the edge in the codomain: $R = \{(x_1, x_2) \in \mathbb{R}^2 : \|\rho'\| < 1\}$.

Defining $\rho' \equiv (x_1, x_2)$ and $\rho \equiv (\rho_1, \rho_2, \rho_3)$, the inverse mapping, Ψ_1^{-1}, is:

$$\Psi_1^{-1} : \mathbb{R}^2 \to S^2_{r_1},$$

$$\rho' \mapsto \rho,$$

such that $(x_1, x_2, 0) \mapsto (\Psi_1^{-1}(x_1), \Psi_1^{-1}(x_2), \Psi_1^{-1}(0)) = (\rho_1, \rho_2, \sqrt{1 - \|\rho'\|^2})$, since $\Psi_1^{-1}(x_1) = x_1 = \rho_1$, $\Psi_1^{-1}(x_2) = x_2 = \rho_2$, and $\Psi_1^{-1}(0) = \sqrt{1 - \|\rho'\|^2}$.

Ψ_1^{-1} is also an identity mapping of components for the first two components, having the mapping on the third component as $\Psi_1^{-1}(x_3 = 0) = \sqrt{1 - \|\rho'\|^2}$.

An equivalent representation, for the case of the southern hemisphere, implies $\Psi_1^{-1}(x_3 = 0) = -\sqrt{1 - \|\rho'\|^2}$, since the vectors ρ for the southern hemisphere have the structure (ρ_1, ρ_2, ρ_3), such that $\rho_3 < 0$.

The component $\rho_3 = \sqrt{1 - \|\rho'\|^2}$ makes $\|\rho\| = 1$. Therefore, any pair (ρ_1, ρ_2) with $\|\rho\| = 1$ defines the surface of a hemisphere that can be either north or south depending on the positivity or negativity of its third component, respectively. The same applies to the anterior and posterior hemispheres or the eastern and western hemispheres.

Taking into account the correspondence between coordinates under the mapping Ψ and its inverse, note that if we define $\rho_3 = \sqrt{1 - \|\rho'\|^2}$, we have $\|\rho\| = \sqrt{\rho_1^2 + \rho_2^2 + (\sqrt{1 - \|\rho'\|^2})^2}$. Since $\rho_1 = x_1$ and $\rho_2 = x_2$, we have $\|\rho\| = \sqrt{\rho_1^2 + \rho_2^2 + 1 - \|\rho'\|^2}$, and since $\rho' = (x_1, x_2)$, which implies $\|\rho'\|^2 = x_1^2 + x_2^2$, we have $\|\rho\| = \sqrt{x_1^2 + x_2^2 + 1 - x_1^2 + x_2^2} = 1$.

Given the reasoning developed so far, we can generalize the mapping $\Psi_{i \in \mathbb{R}^+}$ and its inverse for the volume of the northern hemisphere as follows:

$$\Psi_i : S^2_{r_i} \to \mathbb{R}^2,$$

$$\rho \mapsto \rho',$$

such that:

$$S^2_{r_i} \equiv \{\rho \in \mathbb{R}^3 : \|\rho\| = i \in \mathbb{R}^+\},$$

so that the codomain is given by:

$$R = \{\rho' \in \mathbb{R}^2 : \|\rho'\| \leq i\}.$$

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The inverse mapping is:

$$\Psi_i^{-1} : \mathbb{R}^2 \to S^2_{r_i},$$
$$\rho' \mapsto \rho,$$

such that $(x_1, x_2) \mapsto (x_1, x_2, \sqrt{i - \|x_1^2 + x_2^2\|^2}) = (x_1, x_2, \sqrt{i - \|\rho'\|^2})$, i.e.:

$$\Psi_i^{-1}(x_1, x_2, 0) = (\Psi_i^{-1}(x_1), \Psi_i^{-1}(x_2), \Psi_i^{-1}(0)) = (x_1, x_2, \sqrt{i - \|\rho'\|^2}) = (\rho_1, \rho_2, \rho_3).$$

The component $\rho_3 = \sqrt{i - \|\rho'\|^2}$ in this generalization implies $\|\rho\| = i$. Therefore, for any pair (x_1, x_2) with $\|\rho\| = i$, we will have the surface of the northern hemisphere with norm i: $\|\rho\| = \sqrt{x_1^2 + x_2^2 + (\sqrt{i - \|\rho'\|^2})^2} = \sqrt{x_1^2 + x_2^2 + i - x_1^2 + x_2^2} = i$. Note that each Ψ_i is a homeomorphism, such that there is a mapping for each norm i.

An S^2-sphere with a fixed radius r_0 is determined by the different equivalence classes $([q]_\sim)_j = (\{\rho, -\rho\})_j$ such that $\|\rho\| = r_0$. For an arbitrary radius r, these equivalence classes group into the quotient set $\Lambda \backslash \sim$. We can understand the set $\Lambda \backslash \sim$ as a quotient set of S^2-spheres:

$$\Lambda \backslash \sim \equiv \{S^2_{r_i} \backslash \sim \in \mathbb{R}^3 : \rho \sim -\rho \wedge r_i \in \mathbb{R}^+\}.$$

In this way, each $S^2_{r_i} \backslash \sim$ is homeomorphic to $\mathbb{R}^2$ under the selector mapping Ψ, which selects the corresponding hemispheres to produce a bijective and continuous mapping in both directions.

This subdivision of the treonic quotient set $\Lambda \backslash \sim$ into quotient sets of r_i-spheres, $S^2_{r_i} \backslash \sim$, facilitates the decomposition of the volume $\Lambda \backslash \sim$ into the surfaces that compose it. This would be a model of decomposing $\Lambda \backslash \sim$ "like onion layers", such that each layer is a quotient subset homeomorphic to $\mathbb{R}^2$ under the hemisphere selector mapping Ψ. Therefore, each layer under Ψ is locally Euclidean. Analyzing the total volume $\Lambda \backslash \sim$ or the hemispherical volume $\Lambda \backslash \sim$ generates an overlap of points on the $\mathbb{R}^2$ plane. Hence, it is necessary to analyze each layer of each hemisphere at a time in its projection to $\mathbb{R}^2$.

Each hemisphere of north, south, east, west, anterior, and posterior, we denote as H_i, such that $i =$ north, south, east, west, anterior, and posterior.

The equivalence relation $\rho \sim -\rho$ implies that the opposite hemispheres are equivalent in the sense that, for example, any point ρ in the northern hemisphere is equivalent to the point $-\rho$ in the southern hemisphere. But this situation must be carefully analyzed, and we must understand that the mapping Ψ acts on hemispheres, not on the entire sphere. Note that the point $-\rho$ in the southern hemisphere overlaps in its projection onto the $\mathbb{R}^2$ plane with the projection of a different point from ρ in the northern hemisphere under the mapping Ψ. Therefore, it is convenient to also place an index on the mapping Ψ: We denote Ψ_i, where $i =$ north, south, east, west, anterior, and posterior. And according to this notation, for example, the mapping Ψ_{north} acts on H_{north} and maps a point ρ to a point $\rho' \in \mathbb{R}^2$. On the other hand, the mapping Ψ_{south} will act on H_{south} and map independently the point $-\rho$ to a point $-\rho' \in \mathbb{R}^2$. It is understood that both $\mathbb{R}^2$ planes are separable in the analysis of the mapping Ψ.

If we intersect an $S^2_{r_0}$-sphere with vertical planes passing through the centroid, on the surface of $S^2_{r_0}$ we will have infinite curves, intersected at the north and south poles. These curves describe arcs that allow us to define angles. In the case of the northern hemisphere, a point ρ_0 will be on a curve C_0 and can be described using spherical polar coordinates: It will have a radius r_0 and an angle θ_0 that describes the angle along the plane defining the curve, understanding that $\theta_0 = 0$ at the points $\rho_0 = (x_1, 0)$. With this analysis, the equivalent point $-\rho_0$ in the southern hemisphere will coincide in its projection onto $\mathbb{R}^2$ with the projection of the point ρ_0 rotated along the plane by an angle $\pi - \theta_0$ in its corresponding $\mathbb{R}^2$ plane. This is useful as it allows us to relate the codomain $\mathbb{R}^2$ of Ψ_{north} acting on H_{north} with the codomain $\mathbb{R}^2$ of Ψ_{south} acting on H_{south} through a mapping that assigns each point in the northern hemisphere to a point in the southern hemisphere with the same coordinates in $\mathbb{R}^2$.

Using pairs (H_i, Ψ_i), we can study the treonic quotient space $\Lambda \backslash \sim$.

Each mapping Ψ_i along with its corresponding hemisphere H_i, we call a *chart* (or *local chart*) and denote it as (H, Ψ). This denomination corresponds to the usual nomenclature in differential geometry [11–13]. Thus, our chart (H, Ψ) is a homeomorphism along with the open subset of the topological manifold where it is defined. This is applicable to open subspaces of H.

4. Statement of the Treonic Manifolds

Let (Λ, T_Λ) be the treonic topological space, we say that it is a manifold since it is a Hausdorff space, second-countable, and locally (in this case: globally) Euclidean in the sense of the existence of an identity mapping of coordinates between Λ and $\mathbb{R}^3$. Therefore, (Λ, T_Λ) is a 3-dimensional manifold.

Let $(\Lambda \backslash \sim, \tilde{T})$ be the treonic quotient topological space, and it is a Hausdorff space, second-countable. However, in the volume $(\Lambda \backslash \sim, \tilde{T})$, we cannot define a bijection with the real space $\mathbb{R}^n$ equipped with the Euclidean metric. Therefore, under the analysis of the present study, we will not assume that $(\Lambda \backslash \sim, \tilde{T})$ is a manifold.

Let $(S^2_{r_i} \backslash \sim, \tilde{T})$ be the treonic quotient subspace given by the r_i-spheres. For any point on the sphere, we do not have a neighborhood such that we can define a homeomorphism from the classical point of view, since the opposite hemispheres have points that overlap in $\mathbb{R}^2$. To establish a homeomorphism, we must select only one of the elements in $[q]_\sim = \{q, -q\}$ and map it to $\mathbb{R}^2$, or similarly, homogenize the elements of the class into a single element that contains the information of the rest, which is easy to understand given that there is an equivalence relationship among the elements. Then, the inverse mapping recovers the entire equivalence class $[q]_\sim$. In this case, $f : S^2_{r_i} \backslash \sim \to \mathbb{R}^2, q = (\rho_1, \rho_2, |\rho_3|) \in \{q, -q\} \mapsto (x_1, x_2, 0) = (x_1, x_2)$. The inverse mapping is $f^{-1} : \mathbb{R}^2 \to S^2_{r_i} \backslash \sim, (x_1, x_2) \to \{(x_1, x_2, \sqrt{i - \|x_1^2 + x_2^2\|^2}), -(x_1, x_2, \sqrt{i - \|x_1^2 + x_2^2\|^2})\} = [q]_\sim$ for a radius r_i. This is applicable to the north-south hemisphere relation. For the rest of the hemispheres, the reasoning is analogous, but considering the first or second components. Under this reasoning, $(S^2_{r_i} \backslash \sim, \tilde{T})$ is a 2-dimensional manifold.

Let $(H_i \subseteq S^2_{r_i} \backslash \sim, \tilde{T})$ be the quotient topological subspace defined by the r_i-hemispheres. We say it is a manifold since it is a Hausdorff space, second-countable, and locally Euclidean. Therefore, $(H_i \subseteq S^2_{r_i} \backslash \sim, \tilde{T})$ is a 2-dimensional manifold.

5. Transition Map

Let (Λ, T_Λ) be a treonic manifold, which we denote as (M_Λ, T_Λ), and let the charts $\phi_1 : U_\Lambda \to U_{\mathbb{R}^3}$ and $\phi_2 : V_\Lambda \to V_{\mathbb{R}^3}$, such that there exists an overlap between U_Λ and $V_\Lambda, U_\Lambda \cap V_\Lambda \neq \emptyset$. Then, a mapping ω of the form $\omega : A \subseteq U_{\mathbb{R}^3} \to B \subseteq V_{\mathbb{R}^3}$ is called a *transition map* [11,13,15,16,16]. The same is applicable to the manifold $(H_i \subseteq S^2_{r_i} \backslash \sim, \tilde{T}_\Lambda)$, which we denote as $(M_{H_i}, \tilde{T}_\Lambda)$.

6. Treonic Atlas

Let the manifold (M_Λ, T_Λ). The collection of charts $(U_i, \phi_i)_{i \in I}$ is called a *treonic atlas* if it is a cover of the manifold such that $M_\Lambda = \bigcup_{i \in I} U_i$ [11,12,15,16].

Let the manifold $(M_{H_i}, \tilde{T}_\Lambda)$, we have that a collection of charts $(W_i, \Psi_i)_{i \in I}$, such that $W_i \subseteq H_i$, is a *treonic quotient atlas* if $M_{H_i} = \bigcup_{i \in I} W_i$.

On the other hand, the collection of charts $(H_i, \Psi_i)_{i \in I}$ forms an atlas that defines the entire treonic $S^2_{r_i}$-sphere since $S^2_{r_i} = \bigcup_{i \in I} H_i$, $I = \{\text{north}, \text{south}, \text{east}, \text{west}, \text{anterior}, \text{posterior}\}$. Thus, the collection of quotient hemispheres forms an atlas under the quotient topology that, for each $r_i > 0$, makes the volume $(\Lambda \backslash \sim, \tilde{T})$ an atlas.

Conclusions

We developed a comprehensive theoretical framework to elucidate Euclidean locality within treonic topological spaces, thereby achieving significant advancements in the field of topology. We established the groundwork for a deeper understanding of their topological and geometric behavior.

 doi:10.20944/preprints202407.2551.v1

A key contribution of our work was the formal establishment of homeomorphisms between treonic spaces and $\mathbb{R}^3$. This critical finding demonstrated that treonic spaces could be locally represented as Euclidean spaces, thereby affirming their Euclidean locality.

We extended our analysis to the quotient spaces derived from treonic spaces, thoroughly examining their manifold properties. By identifying the conditions under which these quotient spaces could be locally modeled as Euclidean spaces, we introduced a novel characterization of treonic manifolds. In this context, we demonstrated other homeomorphisms, with $\mathbb{R}^2$, which allowed us to construct various types of charts.

We significantly advanced the understanding of treonic topological spaces by establishing their Euclidean locality and conducting a thorough analysis of their topological and geometric properties, thereby completing the necessary requirements for the first construction of treonic manifolds, a potentially essential framework in Bermejo Algebras.

The theoretical framework and findings presented here contribute substantially to the field of topology and offer valuable insights for future research. By bridging the concepts of treonic and Euclidean spaces, this study not only enhances theoretical knowledge but also provides a foundation for future applications in advanced mathematics and theoretical physics. Future research should continue to delve into the rich structure of treonic spaces, further elucidating their properties and exploring new applications in both the mathematical and physical sciences.

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