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The Adomian Decomposition Method for a Class of First Order Fuzzy Dynamic Equations on Time Scales

Wafaa Salih Ramadan [1,*], Svetlin G. Georgiev [2] and Waleed Al-Hayani [3]

[1] Department of Mathematics, College of Education for Pure Sciences, University of Al-Hamdaniya, Iraq; E-mail address: wafaamath@uohamdaniya.edu.iq

[2] Department of Mathematics, Sorbonne University, Paris, France; e-mail: svetlingeorgiev1@gmail.com

[3] Department of Mathematics, College of Computer Science and Mathematics, University of Mosul, Iraq; E-mail address: waleedalhayani@uomosul.edu.iq or waleedalhayani@yahoo.es

[*] Correspondence: wafaamath@uohamdaniya.edu.iq

Abstract: In this paper we introduce the Adomian decomposition method for a class of first order fuzzy dynamic equations on arbitrary time scales for existence of solutions. The results are provided with suitable numerical examples.

Keywords: time scale; first order fuzzy dynamic equations; Adomian decomposition method

MSC: 35A01; 35F21; 47H10; 70H20

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1. Introduction

The theory of dynamic equations has many interesting applications in control theory, mathematical economics, mathematical biology, engineering and technology. In some cases, there exists uncertainty, ambiguity or vague factors in such problems, and fuzzy theory and interval analysis are powerful tools for modeling these equations on time scales.

In this paper, we introduce the Adomian decomposition method for the following class of first order fuzzy dynamic equations

$$\delta_H y = f(y), \quad t \in (t_0, T], \tag{1}$$

$$y(t_0) = y_0, \tag{2}$$

where

(A1) $f \in \mathcal{C}([t_0, T] \times F(\mathbb{R}))$, $f : F(\mathbb{R}) \to F(\mathbb{R})$, $t_0, T \in \mathbb{T}$, $\mathbb{T}$ is an arbitrary time scale with forward jump operator and delta differentiation operator σ and Δ, respectively.

Here $F(\mathbb{R})$ denotes the set of all real fuzzy numbers, $\widetilde{0}$ denotes the zero fuzzy number and δ_H denotes the first type fuzzy delta derivative on $\mathbb{T}$.

To the best of our knowledge, there is a gap in the references for investigations of the Adomian decomposition method for fuzzy dynamic equations on time scales. Here, in this paper we try to fill out this gap.

This paper is organized as follows. In the next section, we give some basic definitions and facts of fuzzy dynamic calculus on time scales. In Section 3, we give an exposition of the Adomian decomposition method on time scales. In Section 4 we introduce the Adomian decomposition method for the problem (1), (2). In Section 5, we give a numerical example.

2. Fuzzy Dynamic Calculus Essentials

In this section, we will give some basic definitions and fact of fuzzy dynamic calculus on time scales. For detailed study of fuzzy dynamic calculus on time scales we refer the reader to the book [2].

Suppose that $\mathbb{T}$ is a time scale with forward jump operator and delta differentiation operator σ and Δ, respectively. With $F(\mathbb{R})$ we will denote the space of the real fuzzy numbers and with $D(\cdot, \cdot)$ we will denote the Hausdorff distance between the real fuzzy numbers. For more details for fuzzy

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numbers and Hausdorff distance between the real fuzzy numbers we refer the reader to the appendix of the book [2].

Definition 1. *([2]) Assume that $f : \mathbb{T} \to F(\mathbb{R})$ is a fuzzy function and $t \in \mathbb{T}^\kappa$. Then f is said to be first type right fuzzy delta differentiable at t, shortly right δ_H-differentiable at t, if there exists an element $\delta_H^+ f(t) \in F(\mathbb{R})$ with the property that, for any given $\epsilon > 0$, there exists a neighbourhood $U_\mathbb{T}$ of t, i.e., $U_\mathbb{T} = (t-\delta, t+\delta) \bigcap \mathbb{T}$ for some $\delta > 0$, such that for all $t+h \in U_\mathbb{T}$ the H-difference $f(t+h) \ominus_H f(\sigma(t))$ exists and*

$$D\big(f(t+h) \ominus_H f(\sigma(t)), \delta_H^+ f(t)(h-\mu(t))\big) \le \epsilon |h-\mu(t)|$$

with $0 \le h < \delta$. In this case, $\delta_H^+ f(t)$ is said to be first type right fuzzy delta derivative of f at t, shortly right δ_H-derivative of f at t.

Definition 2. *([2]) Assume that $f : \mathbb{T} \to F(\mathbb{R})$ is a fuzzy function and $t \in \mathbb{T}^\kappa$. Then f is said to be first type left fuzzy delta differentiable at t, shortly left δ_H-differentiable at t, if there exists an element $\delta_H^- f(t) \in F(\mathbb{R})$ with the property that, for any given $\epsilon > 0$, there exists a neighbourhood $U_\mathbb{T}$ of t, i.e., $U_\mathbb{T} = (t-\delta, t+\delta) \bigcap \mathbb{T}$ for some $\delta > 0$, such that for all $t-h \in U_\mathbb{T}$ the H-difference $f(\sigma(t)) \ominus_H f(t-h)$ exists and*

$$D\big(f(\sigma(t)) \ominus_H f(t-h), \delta_H^- f(t)(h+\mu(t))\big) \le \epsilon (h+\mu(t))$$

with $0 \le h < \delta$.

Definition 3. *([2]) Let $f : \mathbb{T} \to F(\mathbb{R})$ be a fuzzy function and $t \in \mathbb{T}^\kappa$. Then f is said to be first type fuzzy delta differentiable at t, shortly δ_H-differentiable at t, if f is both first type left and right fuzzy delta differentiable at $t \in \mathbb{T}^\kappa$ and $\delta_H^- f(t) = \delta_H^+ f(t)$, and we will denote it by $\delta_H f(t)$. We call $\delta_H f(t)$ the first type fuzzy delta derivative of f at t, shortly δ_H-derivative of f at t. We say that f is first type fuzzy delta differentiable at t, shortly δ_H-differentiable at t, if its δ_H-derivative exists at t. We say that f is first type fuzzy delta differentiable on $\mathbb{T}^\kappa$, shortly δ_H-differentiable on $\mathbb{T}^\kappa$, if its δ_H-derivative exists at each $t \in \mathbb{T}^\kappa$. The fuzzy function $\delta_H f : \mathbb{T}^\kappa \to F(\mathbb{R})$ is then called first type fuzzy delta derivative, shortly δ_H-derivative of f on $\mathbb{T}^\kappa$.*

The defined δ_H-derivative has the following properties.

Theorem 1. *([2]) If the δ_H-derivative of f at $t \in \mathbb{T}^\kappa$ exists, then it is unique. Hence, δ_H-derivative is well-defined.*

Theorem 2. *([2]) Assume that $f : \mathbb{T} \to F(\mathbb{R})$ is a continuous function at $t_1 \in \mathbb{T}^\kappa$ and t_1 is right-scattered. Then f is δ_H-differentiable at t_1 and*

$$\delta_H f(t_1) = \frac{f(\sigma(t_1)) \ominus_H f(t_1)}{\mu(t_1)}.$$

Theorem 3. *([2]) Assume that $f : \mathbb{T} \to F(\mathbb{R})$ is δ_H-differentiable at $t \in \mathbb{T}^\kappa$. Then f is continuous at t.*

Theorem 4. *([2]) Let $f : \mathbb{T} \to F(\mathbb{R})$ be a fuzzy function and let $t \in \mathbb{T}^\kappa$ be right-dense. Then f is δ_H-differentiable at t if and only if the limits*

$$\lim_{h \to 0+} \frac{f(t+h) \ominus_H f(t)}{h} \quad \text{and} \quad \lim_{h \to 0+} \frac{f(t) \ominus_H f(t-h)}{h} \tag{3}$$

exist and satisfy the relations

$$\lim_{h \to 0+} \frac{f(t+h) \ominus_H f(t)}{h} = \lim_{h \to 0+} \frac{f(t) \ominus_H f(t-h)}{h} = \delta_H f(t).$$

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Theorem 5. *([2]) Let $f : \mathbb{T} \to F(\mathbb{R})$ is δ_H-differentiable at $t \in \mathbb{T}^\kappa$. Then*

$$f(\sigma(t)) = f(t) + \mu(t) \cdot \delta_H f(t)$$

or

$$f(t) = f(\sigma(t)) + (-1) \cdot (\mu(t) \cdot \delta_H f(t)).$$

Theorem 6. *([2]) Let $f, g : \mathbb{T} \to F(\mathbb{R})$ be δ_H-differentiable at $t \in \mathbb{T}^\kappa$. Then $f + g : \mathbb{T} \to F(\mathbb{R})$ is δ_H-differentiable at $t \in \mathbb{T}^\kappa$ and*

$$\delta_H(f+g)(t) = \delta_H f(t) + \delta_H g(t).$$

Theorem 7. *([2]) Let $f : \mathbb{T} \to F(\mathbb{R})$ be δ_H-differentiable at $t \in \mathbb{T}^\kappa$. Then for any $\lambda \in \mathbb{R}$ the function $\lambda \cdot f : \mathbb{T} \to F(\mathbb{R})$ is δ_H-differentiable at $t \in \mathbb{T}^\kappa$ and*

$$\delta_H(\lambda \cdot f)(t) = \lambda \cdot \delta_H f(t).$$

Theorem 8. *([2]) Let $t \in \mathbb{T}^\kappa$, $f : \mathbb{T} \to F(\mathbb{R})$ and $f_\alpha(t) = [f(t)]^\alpha$, $\alpha \in [0,1]$. If f is δ_H-differentiable at t, then f_α is δ_H-differentiable at t and*

$$\delta_H[f(t)]^\alpha = \delta_H f_\alpha(t), \quad \alpha \in [0,1].$$

Theorem 9. *([2]) Let $t \in \mathbb{T}^\kappa$, $f : \mathbb{T} \to F(\mathbb{R})$ is δ_H-differentiable at t. Let also,*

$$[f(t)]^\alpha = \left[\underline{f}^\alpha(t), \overline{f}^\alpha(t)\right], \quad \alpha \in [0,1].$$

Then $\underline{f}^\alpha$ and $\overline{f}^\alpha$ are Δ-differentiable at t and

$$[\delta_H f(t)]^\alpha = \left[\underline{f}^{\alpha\Delta}(t), \overline{f}^{\alpha\Delta}(t)\right], \quad \alpha \in [0,1].$$

Now, we introduce the conception for the first type fuzzy delta integration on time scales. Let $I \subset \mathbb{T}$.

Definition 4. *([2]) A function $f : \mathbb{T} \to \mathbb{R}$ is called a sector of the fuzzy function $F : I \to F(\mathbb{R})$ if $f(t) \in F(t)$ for all $t \in I$. The set of all rd-continuous sectors of F on I is denoted by $S_{HF}(I)$.*

Theorem 10. *([2]) Let $t_0, T \in \mathbb{T}$, $t_0 < T$, $F, G : [t_0, T] \to F(\mathbb{R})$ be δ_H-integrable. Then $F + G : [t_0, T] \to F(\mathbb{R})$ is δ_H-integrable and*

$$\int_{t_0}^{T} (F(s) + G(s))\delta_H s = \int_{t_0}^{T} F(s)\delta_H s + \int_{t_0}^{T} G(s)\delta_H s. \tag{4}$$

Theorem 11. *([2]) Let $t_0, T \in \mathbb{T}$, $t_0 < T$, $F : [t_0, T] \to F(\mathbb{R})$ be δ_H-integrable. Then $\lambda \cdot F : [t_0, T] \to F(\mathbb{R})$ is δ_H-integrable and*

$$\int_{t_0}^{T} \lambda \cdot F(s)\delta_H s = \lambda \cdot \int_{t_0}^{T} F(s)\delta_H s$$

for any $\lambda \in \mathbb{R}$.

Theorem 12. *([2]) Let* $t_0, T \in \mathbb{T}$, $t_0 < T$, *and* $F : [t_0, T] \to F(\mathbb{R})$ *be* δ_H*-integrable. Then*

$$\int_{t_0}^{T} F(s)\delta_H s = \int_{t_0}^{t} F(s)\delta_H s + \int_{t}^{T} F(s)\delta_H s$$

for any $t \in [t_0, T]$.

Theorem 13. *([2]) Let* $t_0, T \in \mathbb{T}$, $t_0 < T$, $F : [t_0, T] \to F(\mathbb{R})$ *is rd-continuous. If* $X_0 \in F(\mathbb{R})$ *and*

$$f(t) = X_0 + \int_{t_0}^{t} F(s)\delta_H s, \quad t \in [t_0, T],$$

then f *is* δ_H*-differentiable and*

$$\delta_H f(t) = F(t), \quad t \in [t_0, T].$$

Theorem 14. *([2]) If* $f : [a, b] \to F(\mathbb{R})$ *is* δ_H*-differentiable on* $[a, b]$, *then*

$$\int_{a}^{b} \delta_H f(t) = f(b) \ominus_H f(a).$$

Theorem 15. *([2]) Let* $f : [a, b] \to F(\mathbb{R})$ *be* δ_H*-integrable. Then*

$$\int_{a}^{b} f(t)\delta_H t = (-1) \cdot \int_{b}^{a} f(t)\delta_H t.$$

Theorem 16 (Integration by Parts)**.** *([2]) Let* $f, g : [a, b] \to F(\mathbb{R})$ *be* δ_H*-differentiable and* $f \circ g$ *is also* δ_H*-differentiable on* $[a, b]$. *If*

$$I^{\alpha,1}_{f,g}(t) \le 0, \quad I^{\alpha,1}_{f^\sigma,\delta_H g}(t) \le 0, \quad I^{\alpha,1}_{\delta_H f,g}(t) \le 0, \quad t \in [a, b]^\kappa, \tag{5}$$

then

$$\begin{aligned} \int_a^b f(\sigma(t)) \circ \delta_H g(t)\delta_H t \quad &= \quad (f(b) \circ g(b)) \ominus_H (f(a) \circ g(a)) \\ &\quad \ominus_H \int_a^b \delta_H f(t) \circ g(t)\delta_H t. \end{aligned} \tag{6}$$

3. The Adomian Decomposition Method on Time Scales

In this section, we will describe the Adomian decomposition method on arbitrary time scale $\mathbb{T}$ with forward jump operator and delta differentiation operator σ and Δ, respectively. The exposition in this section, follows the exposition in [3].

Denote the set consisting of all possible strings of length n, containing exactly k times σ and $n - k$ times Δ operators by $S_k^{(n)}$. Then we have

$$h_n(t, s)h_m(t, s) = \sum_{l=m}^{m+n} \left(\sum_{\Lambda_{l,m} \in S_m^{(l)}} h_n^{\Lambda_{l,m}}(s, s) \right) h_l(t, s)$$

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for every $t, s \in \mathbb{T}$. For $s \in \mathbb{T}$, $l, m, n \in \mathbb{N}_0$, set

$$A_{l,m,n,s} = \sum_{\Lambda_{l,m} \in S_m^{(l)}} h_n^{\Lambda_{l,m}}(s,s)$$

and for any $m, n \in \mathbb{N}_0$, we have

$$h_n(t,s)h_m(t,s) = \sum_{l=m}^{m+n} A_{l,m,n,s} h_l(t,s). \tag{7}$$

For $n \in \mathbb{N}_0$, $t, s \in \mathbb{T}$, define the polynomials

$$H_n^1(t,s) = (h_1(t,s))^n, \quad t,s \in \mathbb{T}.$$

Note that

$$H_n^1(t,s)H_m^1(t,s) = H_{n+m}^1(t,s), \quad t,s \in \mathbb{T}.$$

Note also that

$$H_1^1(t,s) = h_1(t,s), \tag{8}$$

and

$$H_2^1(t,s) \quad = \quad A_{1,1,1,s}H_1^1(t,s) + A_{2,1,1,s}h_2(t,s),$$

whereupon

$$h_2(t,s) = -\frac{A_{1,1,1,s}}{A_{2,1,1,s}}H_1^1(t,s) + \frac{1}{A_{2,1,1,s}}H_2^1(t,s),$$

and so on. Below we denote by B_i^j, $i, j \in \mathbb{N}$, the constants for which

$$H_n^1(t,s) = B_1^n h_1(t,s) + B_2^n h_2(t,s) + \cdots + B_n^n h_n(t,s), \quad t,s \in \mathbb{T}. \tag{9}$$

Suppose that $u : \mathbb{T} \to \mathbb{R}$ is a given function which has a convergent series expansion of the form

$$u = \sum_{j=0}^{\infty} u_j. \tag{10}$$

Suppose also that $f : \mathbb{R} \to \mathbb{R}$ is a given analytic function such that

$$f(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n), \tag{11}$$

where A_n, $n \in \mathbb{N}_0$, are given by

$$\begin{aligned} A_0 &= f(u_0) \\ A_n &= \sum_{\nu=1}^{n} c(\nu,n) f^{(\nu)}(u_0), \quad n \in \mathbb{N}. \end{aligned}$$

Here the functions $c(\nu, n)$ denote the sum of products of ν components u_j of u given in (10), whose subscripts sum up to n, divided by the factorial of the number of repeated subscripts, i.e.,

$$\begin{aligned}
A_0 &= f(u_0), \\
A_1 &= c(1,1)f'(u_0) \\
&= u_1 f'(u_0), \\
A_2 &= c(1,2)f'(u_0) + c(2,2)f''(u_0) \\
&= u_2 f'(u_0) + \frac{u_1^2}{2!} f''(u_0),
\end{aligned}$$

$$\begin{aligned}
A_3 &= c(1,3)f'(u_0) + c(2,3)f''(u_0) + c(3,3)f'''(u_0) \\
&= u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f'''(u_0), \\
A_4 &= c(1,4)f'(u_0) + c(2,4)f''(u_0) + c(3,4)f'''(u_0) \\
&\quad + c(4,4)f^{(4)}(u_0) \\
&= u_4 f'(u_0) + \left(u_1 u_3 + \frac{u_2^2}{2}\right) f''(u_0) + \frac{u_1^2 u_2}{2} f'''(u_0) \\
&\quad + \frac{u_1^4}{4!} f^{(4)}(u_0)
\end{aligned}$$

and so on. Suppose now that u is given by the convergent series

$$u = \sum_{n=0}^{\infty} c_n H_n^1(x, x_0). \tag{12}$$

We wish to find the respected transformed series for $f(u)$. From (10), we have

$$u = \sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} c_n H_n^1(x, x_0),$$

and hence,

$$u_n = c_n H_n^1(x, x_0), \quad n \in \mathbb{N}_0.$$

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Thus,

$$\begin{aligned} f(u) &= \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n) \\ &= f\left(\sum_{n=0}^{\infty} c_n H_n^1(x, x_0)\right) \\ &= \sum_{n=0}^{\infty} A^n(c_0, c_1, \ldots, c_n) H_n^1(x, x_0). \end{aligned}$$

Hence,

$$A_n(u_0, u_1, \ldots, u_n) = A^n(c_0, c_1, \ldots, c_n) H_n^1(x, x_0).$$

For $n = 0$, we have

$$\begin{aligned} u_0 &= c_0 H_0^1(x, x_0). \\ &= c_0. \end{aligned}$$

Thus,

$$\begin{aligned} A_0(u_0) &= A^0(c_0) H_0^1(x, x_0) \\ &= A^0(c_0). \end{aligned}$$

For $n = 1$, we find

$$\begin{aligned} A_1(u_0, u_1) &= u_1 f'(u_0) \\ &= A^1(c_0, c_1) H_1^1(x, x_0) \end{aligned}$$

or

$$c_1 H_1^1(x, x_0) f'(u_0) = A^1(c_0, c_1) H_1^1(x, x_0),$$

whereupon

$$\begin{aligned} A^1(c_0, c_1) &= c_1 f'(u_0) \\ &= c_1 f'(c_0) \\ &= A_1(c_0, c_1). \end{aligned}$$

For $n = 2$, we have

$$A_2(u_0, u_1, u_2) = A^2(c_0, c_1, c_2) H_2^1(x, x_0)$$

or

$$u_2 f'(u_0) + \frac{u_1^2}{2} f''(u_0) = A^2(c_0, c_1, c_2) H_2^1(x, x_0).$$

Then

$$c_2 H_2^1(x, x_0) f'(c_0) + \frac{c_1^2 (H_1^1(x, x_0))^2}{2} f''(c_0) = A^2(c_0, c_1, c_2) H_2^1(x, x_0),$$

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or

$$\left(c_2 f'(c_0) + \frac{c_1^2}{2} f''(c_0)\right) H_2^1(x, x_0) = A^2(c_0, c_1, c_2) H_2^1(x, x_0),$$

whereupon

$$\begin{aligned} A^2(c_0, c_1, c_2) &= c_2 f'(c_0) + \frac{c_1^2}{2} f''(c_0) \\ &= A_2(c_0, c_1, c_2). \end{aligned}$$

For $n = 3$, we find

$$\begin{aligned} u_3 f'(u_0) + u_1 u_2 f''(u_0) + \frac{u_1^3}{3!} f'''(u_0) &= A_3(u_0, u_1, u_2, u_3) \\ &= A^3(c_0, c_1, c_2, c_3) H_3^1(x, x_0) \end{aligned}$$

or

$$c_3 H_3^1(x, x_0) f'(c_0) + c_1 c_2 H_3^1(x, x_0) f''(x_0) + \frac{c_1^3}{3!} f'''(c_0) H_3^1(x, x_0) = A^3(c_0, c_1, c_2, c_3) H_3^1(x, x_0),$$

whereupon

$$\begin{aligned} c_3 f'(c_0) + c_1 c_2 f''(c_0) + \frac{c_1^3}{3!} f'''(c_0) &= A^3(c_0, c_1, c_2, c_3) \\ &= A_3(c_0, c_1, c_2, c_3), \end{aligned}$$

and so on. We get the following result.

Theorem 17 ([3])**.** *Let $u : \mathbb{T} \to \mathbb{R}$ be a function with a convergent expansion given in (12). Let $f : \mathbb{R} \to \mathbb{R}$ be an analytic function having the form (11). Then*

$$f(u) = f\left(\sum_{n=0}^{\infty} c_n H_n^1(x, x_0)\right) = \sum_{n=0}^{\infty} A_n(c_0, c_1, \dots, c_n) H_n^1(x, x_0).$$

4. The Adomian Decomposition Method for the Problem (1), (2)

In this section, we will introduce the Adomian decomposition method for the problem (1), (2). Firstly, note that the problem (1), (2) can be rewritten in the form

$$\begin{aligned} \left[\underline{y}^{\alpha\Delta}, \overline{y}^{\alpha\Delta}\right] &= \left[\underline{f}^{\alpha}(y), \overline{f}^{\alpha}(y)\right], \quad t \in (t_0, T], \\ \left[\underline{y}^{\alpha}(t_0), \overline{y}^{\alpha}(t_0)\right] &= \left[\underline{y}_0^{\alpha}, \overline{y}_0^{\alpha}\right], \quad \alpha \in [0, 1]. \end{aligned}$$

Consider the problem

$$\underline{y}^{\alpha\Delta} = \underline{f}^{\alpha}(y), \quad t > t_0, \quad y(t_0) = 0, \tag{13}$$

where $\underline{f}^{\alpha} : \mathbb{R} \to \mathbb{R}$ is an analytic function. We propose a solution of the IVP (13), in the form

$$\underline{y}^{\alpha}(t) = \sum_{j=0}^{\infty} \underline{c}_j H_j^1(t, t_0), \quad t \geq t_0.$$

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Like in the general case, we suppose that

$$\underline{f}^\alpha(y) = \sum_{j=0}^{\infty} \underline{A}_j(\underline{c}_0, \ldots, \underline{c}_j) H_j^1(t, t_0), \quad t \geq t_0.$$

We have

$$\underline{y}^\alpha(t) = \underline{c}_0 + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{c}_j \underline{B}_k^j h_k(t, t_0), \quad t \geq t_0. \tag{14}$$

and

$$\underline{f}^\alpha(y) = \underline{A}_0(\underline{c}_0) + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{A}_j(\underline{c}_0, \ldots, \underline{c}_j) \underline{B}_k^j h_k(t, t_0), \quad t \geq t_0. \tag{15}$$

Let

$$\mathcal{L}\left(\underline{y}^\alpha(t)\right)(z) = Y(z).$$

Then we have

$$\mathcal{L}\left(\underline{y}^{\alpha\Delta}(t)\right)(z) = zY(z) - \underline{y}^\alpha(t_0) = zY(z).$$

Taking the Laplace transform of both sides of the dynamic equation (13) we obtain

$$\begin{aligned} zY(z) &= \mathcal{L}\left(A_0(c_0) + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{A}_j(\underline{c}_0, \ldots, \underline{c}_j) \underline{B}_k^j h_k(t, t_0)\right)(z) \\ &= \underline{A}_0(\underline{c}_0)\frac{1}{z} + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{A}_j(\underline{c}_0, \ldots, \underline{c}_j) \underline{B}_k^j \frac{1}{z^{k+1}}. \end{aligned}$$

Then we obtain

$$Y(z) = \underline{A}_0(\underline{c}_0)\frac{1}{z^2} + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{A}_j(\underline{c}_0, \ldots, \underline{c}_j) \underline{B}_k^j \frac{1}{z^{k+2}}.$$

Now, by taking inverse Laplace transform of both sides, we get

$$\underline{y}^\alpha(t) = \underline{A}_0(\underline{c}_0) h_1(t, t_0) + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{A}_j(\underline{c}_0, \ldots, \underline{c}_j) B_k^j h_{k+1}(t, t_0).$$

Employing (18), we have

$$\underline{c}_0 + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{c}_j \underline{B}_k^j h_k(t, t_0) = \underline{A}_0(\underline{c}_0) h_1(t, t_0) + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \underline{A}_j(\underline{c}_0, \ldots, \underline{c}_j) \underline{B}_k^j h_{k+1}(t, t_0).$$

In order to equate the coefficients of the time scale monomials $h_k(t, t_0)$ on both sides, we reorder the sums as follows.

$$\begin{aligned} &\underline{c}_0 + \left(\sum_{j=1}^{\infty} \underline{c}_j \underline{B}_1^j\right) h_1(t, t_0) + \sum_{k=2}^{\infty} \left(\sum_{j=k}^{\infty} \underline{c}_j \underline{B}_k^j\right) h_k(t, t_0) \\ &= \underline{A}_0(\underline{c}_0) h_1(t, t_0) + \sum_{k=2}^{\infty} \sum_{j=k-1}^{\infty} \underline{A}_j(\underline{c}_0, \ldots, \underline{c}_j) \underline{B}_{k-1}^j h_k(t, t_0). \end{aligned}$$

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This results in the following nonlinear system for determining the constants $\underline{c}_j$, $j = 0, 1, \ldots$.

$$\begin{aligned} \underline{c}_0 &= 0, \\ \sum_{j=1}^{\infty} \underline{c}_j \underline{B}_1^j &= \underline{A}_0(\underline{c}_0) = \underline{f}^{\alpha}(0) \\ \sum_{j=k}^{\infty} \underline{c}_j \underline{B}_k^j &= \sum_{j=k-1}^{\infty} \underline{A}_j(\underline{c}_0, \ldots, \underline{c}_j) \underline{B}_{k-1}^j, \quad k \geq 2. \end{aligned} \tag{16}$$

Notice that the system is infinite and nonlinear in its unknowns. However, the nonlinearity is of polynomial type. This is a results of the nonlinear structure of the function $\underline{f}^{\alpha}$.

Now, consider the problem

$$\overline{y}^{\alpha\Delta} = \overline{f}^{\alpha}(y), \quad t > t_0, \quad y(t_0) = 0, \tag{17}$$

where $\overline{f}^{\alpha} : \mathbb{R} \to \mathbb{R}$ is an analytic function. We will search a solution of the IVP (17), in the form

$$\overline{y}^{\alpha}(t) = \sum_{j=0}^{\infty} \overline{c}_j H_j^1(t, t_0), \quad t \geq t_0.$$

Assume that

$$\overline{f}^{\alpha}(y) = \sum_{j=0}^{\infty} \overline{A}_j(\overline{c}_0, \ldots, \overline{c}_j) H_j^1(t, t_0), \quad t \geq t_0.$$

We have

$$\overline{y}^{\alpha}(t) = \overline{c}_0 + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \overline{c}_j \overline{B}_k^j h_k(t, t_0), \quad t \geq t_0. \tag{18}$$

and

$$\overline{f}^{\alpha}(y) = \overline{A}_0(\overline{c}_0) + \sum_{j=1}^{\infty} \sum_{k=1}^{j} \overline{A}_j(\overline{c}_0, \ldots, \overline{c}_j) \overline{B}_k^j h_k(t, t_0), \quad t \geq t_0. \tag{19}$$

As above, we get the following system for the constants $\overline{c}_j$, $j = 0, 1, \ldots$.

$$\begin{aligned} \overline{c}_0 &= 0, \\ \sum_{j=1}^{\infty} \overline{c}_j \overline{B}_1^j &= \overline{A}_0(\overline{c}_0) = \overline{f}^{\alpha}(0) \\ \sum_{j=k}^{\infty} \overline{c}_j \overline{B}_k^j &= \sum_{j=k-1}^{\infty} \overline{A}_j(\overline{c}_0, \ldots, \overline{c}_j) \overline{B}_{k-1}^j, \quad k \geq 2. \end{aligned} \tag{20}$$

5. A Numerical Example

Consider the initial value problem associated with the first order nonlinear fuzzy dynamic equation of the form

$$\left[\underline{y}^{\alpha\Delta}(t), \overline{y}^{\alpha\Delta}(t)\right] = \left[e^{\alpha y(t)}, e^{2\alpha y(t)}\right], \quad t \geq 0, \quad y(t_0) = [0, 0], \tag{21}$$

$\alpha \in [0, 1]$. Consider the IVP

$$\underline{y}^{\alpha\Delta}(t) = e^{\alpha y(t)}, \quad \overline{y}^{\alpha}(0) = 0.$$

Assume that the solution has the series representation

$$y(t) = \sum_{j=0}^{\infty} c_j H_j^1(t, 0), \quad t \geq 0,$$

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where c_j, $j \in \mathbb{N}_0$ are the coefficients to be determined.

$$f(y) = e^{\alpha y(t)} = \sum_{j=0}^{\infty} A_j(c_0, \ldots, c_j) H_j^1(t, 0), \quad t \geq 0,$$

where

$$\begin{aligned}
A_0 &= f(c_0) \\
&= e^{\alpha c_0} \\
A_1 &= c_1 f'(c_0) \\
&= \alpha c_1 e^{\alpha c_0} \\
A_2 &= c_2 f'(c_0) + \frac{c_1^2}{2!} f''(c_0) \\
&= \left(\alpha c_2 + \frac{(\alpha c_1)^2}{2!}\right) e^{\alpha c_0} \\
A_3 &= c_3 f'(c_0) + c_1 c_2 f''(c_0) + \frac{c_1^3}{3!} f'''(c_0) \\
&= \left(\alpha c_3 + \alpha^2 c_1 c_2 + \frac{(\alpha c_1)^3}{3!}\right) e^{\alpha c_0} \\
A_4 &= c_4 f'(c_0) + \left(c_1 c_3 + \frac{c_2^2}{2}\right) f''(c_0) + \frac{c_1^2 c_2}{2} f'''(c_0) + \frac{c_1^4}{4!} f^{(4)}(c_0) \\
&= \left(\alpha c_4 + \alpha^2 c_1 c_3 + \frac{(\alpha c_2)^2}{2} + \frac{\alpha^3 c_1^2 c_2}{2} + \frac{(\alpha c_1)^4}{4!}\right) e^{\alpha c_0} \\
&\ldots
\end{aligned} \tag{22}$$

The infinite nonlinear system for this example has the form

$$\begin{aligned}
c_0 &= 0, \\
c_1 B_1^1 + c_2 B_1^2 + c_3 B_1^3 + \cdots &= 1 \\
c_2 B_2^2 + c_3 B_2^3 + c_4 B_2^4 + \cdots &= \alpha c_1 B_1^1 + \left(\alpha c_2 + \frac{(\alpha c_1)^2}{2!}\right) B_1^2 + \cdots \\
c_3 B_3^3 + c_4 B_3^4 + c_5 B_3^5 + \cdots &= \left(\alpha c_2 + \frac{(\alpha c_1)^2}{2!}\right) B_2^2 + \cdots \\
&\ldots
\end{aligned} \tag{23}$$

Solving this nonlinear system one can approximately obtain c_i, $i \in \mathbb{N}$, and hence, the approximate solution of the initial value problem which is

$$\underline{y}^{\alpha}(t) = c_1 H_1^1(t, 0) + c_2 H_2^1(t, 0) + c_3 H_3^1(t, 0) + \cdots \tag{24}$$

As above,

$$\overline{y}^{\alpha}(t) = \overline{c}_1 H_1^1(t, 0) + \overline{c}_2 H_2^1(t, 0) + \overline{c}_3 H_3^1(t, 0) + \cdots \tag{25}$$

where

$$\begin{aligned}
\overline{c}_0 &= 0, \\
\overline{c}_1 B_1^1 + \overline{c}_2 B_1^2 + \overline{c}_3 B_1^3 + \cdots &= 1 \\
\overline{c}_2 B_2^2 + \overline{c}_3 B_2^3 + \overline{c}_4 B_2^4 + \cdots &= 2\alpha \overline{c}_1 B_1^1 + \left(2\alpha \overline{c}_2 + \frac{(2\alpha \overline{c}_1)^2}{2!}\right) B_1^2 + \cdots \\
\overline{c}_3 B_3^3 + \overline{c}_4 B_3^4 + \overline{c}_5 B_3^5 + \cdots &= \left(2\alpha \overline{c}_2 + \frac{(2\alpha \overline{c}_1)^2}{2!}\right) B_2^2 + \cdots \\
&\ldots
\end{aligned} \tag{26}$$

Let $\mathbb{T} = 2^{\mathbb{N}_0}$ and $t_0 = 1$. Then $\sigma(t) = 2t$, $t \in \mathbb{T}$, and

$$\begin{aligned}
h_1(t, t_0) &= h_1(t, 1) \\
&= t - 1, \quad t \in \mathbb{T}.
\end{aligned}$$

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Next,

$$h_2(t,t_0)=\frac{t^2}{3}-t+\frac{2}{3},\quad t\in\mathbb{T}.$$

Really,

$$\begin{aligned}h_2^{\Delta}(t,t_0) &= \frac{\sigma(t)+t}{3}-1\\ &= \frac{2t+t}{3}-1\\ &= t-1\\ &= h_1(t,t_0),\quad t\in\mathbb{T}.\end{aligned}$$

Moreover,

$$h_3(t,t_0)=\frac{t^3}{21}-\frac{t^2}{3}+\frac{2}{3}t-\frac{8}{21},\quad t\in\mathbb{T}.$$

Indeed,

$$\begin{aligned}h_3^{\Delta}(t,t_0) &= \frac{(\sigma(t))^2+t\sigma(t)+t^2}{21}-\frac{\sigma(t)+t}{3}+\frac{2}{3}\\ &= \frac{(2t)^2+t(2t)t^2}{21}-\frac{2t+t}{3}+\frac{2}{3}\\ &= \frac{4t^2+2t^2+t^2}{21}-\frac{3t}{3}+\frac{2}{3}\\ &= \frac{7t^2}{21}-t+\frac{2}{3}\\ &= \frac{1}{3}t^2-t+\frac{2}{3}\\ &= h_2(t,t_0),\quad t\in\mathbb{T}.\end{aligned}$$

Note that

$$\begin{aligned}H_n^1(t,t_0) &= (t-t_0)^n\\ &= (t-1)^n\quad t\in\mathbb{T}.\end{aligned}$$

Then

$$\begin{aligned}H_2^1(t,t_0) &= (t-1)^2\\ &= t^2-2t+1,\\ H_3^1(t,t_0) &= (t-1)^3\\ &= t^3-3t^2+3t-1,\quad t\in\mathbb{T}.\end{aligned}$$

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For $n = 1$, we get

$$H_1^1(t, t_0) = B_1^1 h_1(t, t_0), \quad t \in \mathbb{T},$$

whereupon

$$t - 1 = B_1^1(t - 1), \quad t \in \mathbb{T}.$$

Therefore $B_1^1 = 1$. For $n = 2$, we find

$$H_2^1(t, t_0) = B_1^2 h_1(t, t_0) + B_2^2 h_2(t, t_0), \quad t \in \mathbb{T},$$

or

$$(t - 1)^2 = B_1^2(t - 1) + B_2^2\left(\frac{t^2}{3} - t + \frac{2}{3}\right), \quad t \in \mathbb{T},$$

or

$$\begin{aligned} t^2 - 2t + 1 &= B_1^2 t - B_1^2 + \frac{B_2^2}{3}t^2 - B_2^2 t + \frac{2}{3}B_2^2 \\ &= \frac{B_2^2}{3}t^2 + (B_1^2 - B_2^2)t + \frac{2}{3}B_2^2 - B_1^2, \quad t \in \mathbb{T}, \end{aligned}$$

whereupon we get the system

$$\begin{aligned} \frac{B_2^2}{3} &= 1 \\ B_1^2 - B_2^2 &= -2 \\ \frac{2}{3}B_2^2 - B_1^2 &= 1, \end{aligned}$$

whose solutions are

$$\begin{aligned} B_1^2 &= 1 \\ B_2^2 &= 3. \end{aligned}$$

Next,

$$H_3^1(t, t_0) = B_1^3 h_1(t, t_0) + B_2^3 h_2(t, t_0) + B_3^3 h_3(t, t_0), \quad t \in \mathbb{T},$$

or

$$(t - 1)^3 = B_1^3(t - 1) + B_2^3\left(\frac{t^2}{3} - t + \frac{2}{3}\right) + B_3^3\left(\frac{t^3}{21} - \frac{t^2}{3} + \frac{2}{3}t - \frac{8}{21}\right),$$

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$t \in \mathbb{T}$, or

$$\begin{aligned}
t^3 - 3t^2 + 3t - 1 &= B_1^3 t - B_1^3 + \frac{B_2^3}{3} t^2 - B_2^3 t + \frac{2}{3} B_2^3 \\
&\quad + \frac{B_3^2}{21} t^3 - \frac{B_3^3}{3} t^2 + \frac{2}{3} B_3^3 t - \frac{8}{21} B_3^3 \\
&= \frac{B_3^3}{21} t^3 + \left(\frac{B_2^3}{3} - \frac{B_3^3}{3} \right) t^2 \\
&\quad + \left(B_1^3 - B_2^3 + \frac{2}{3} B_3^3 \right) t \\
&\quad + \left(-B_1^3 + \frac{2}{3} B_2^3 - \frac{8}{21} B_3^3 \right), \quad t \in \mathbb{T},
\end{aligned}$$

whereupon we get the system

$$\begin{aligned}
\frac{B_3^3}{21} &= 1 \\
\frac{B_2^3}{3} - \frac{B_3^3}{3} &= 1 \\
B_1^3 - B_2^3 + \frac{2}{3} B_3^3 &= 3 \\
-B_1^3 + \frac{2}{3} B_2^3 - \frac{8}{21} B_3^3 &= -1,
\end{aligned}$$

whose solutions are

$$\begin{aligned}
B_1^3 &= 1 \\
B_2^3 &= 12 \\
B_3^3 &= 21.
\end{aligned}$$

Now, we consider the first four equations of (23) with the following approximations

$$\begin{aligned}
c_0 &= 0 \\
c_1 B_1^1 + c_2 B_1^2 + c_3 B_1^3 &= 1 \\
c_2 B_2^2 + c_3 B_2^3 &= \alpha c_1 B_1^1 + \left(\alpha c_2 + \frac{(\alpha c_1)^2}{2} \right) B_1^2 \\
c_3 B_3^3 &= \left(\alpha c_2 + \frac{(\alpha c_1)^2}{2} \right) B_2^2
\end{aligned}$$

or

$$\begin{aligned} c_0 &= 0 \\ c_1+c_2+c_3 &= 1 \\ 3c_2+12c_3 &= \alpha c_1+\left(\alpha c_2+\frac{\alpha^2c_1^2}{2}\right) \\ 21c_3 &= 3\left(\alpha c_2+\frac{\alpha^2c_1^2}{2}\right), \end{aligned}$$

whereupon we get

$$\begin{aligned} (c_1)_{1,2} &= \frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}, \\ (c_2)_{1,2} &= \frac{1}{2(\alpha+7)}\Bigg(-\alpha^2\left(\frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\right)^2 \\ &\quad -14\left(\frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\right)+14\Bigg) \\ (c_3)_{1,2} &= \frac{1}{2(\alpha+7)}\Bigg(\alpha^2\left(\frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\right)^2 \\ &\quad -2\alpha\left(\frac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\right)+2\alpha\Bigg). \end{aligned}$$

Replacing α with 2α, we find

$$\begin{aligned} (\bar{c}_1)_{1,2} &= \frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}, \\ (\bar{c}_2)_{1,2} &= \frac{1}{2(2\alpha+7)}\Bigg(-4\alpha^2\left(\frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\right)^2 \\ &\quad -14\left(\frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\right)+14\Bigg) \\ (\bar{c}_3)_{1,2} &= \frac{1}{2(2\alpha+7)}\Bigg(4\alpha^2\left(\frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\right)^2 \\ &\quad -4\alpha\left(\frac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\right)+2\alpha\Bigg) \end{aligned}$$

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Therefore approximative solutions are

$$\begin{aligned}
\underline{y}^{\alpha}(t) &= c_0 + c_1 H_1^1(t,1) + c_2 H_2^1(t,1) + c_3 H_3^1(t,1) \\
&= \tfrac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}(t-1), \\
&+ \tfrac{1}{2(\alpha+7)}\Bigg(-\alpha^2\Bigg(\tfrac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\Bigg)^2 \\
&-14\Bigg(\tfrac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\Bigg)+14\Bigg)(t-1)^2 \\
&+ \tfrac{1}{2(\alpha+7)}\Bigg(\alpha^2\Bigg(\tfrac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\Bigg)^2 \\
&-2\alpha\Bigg(\tfrac{\alpha^2+12\alpha+21\pm\sqrt{(\alpha^2+12\alpha+21)^2-4\alpha^2(5\alpha+21)}}{2\alpha^2}\Bigg)+2\alpha\Bigg)(t-1)^3
\end{aligned}$$

and

$$\begin{aligned}
\overline{y}^{\alpha}(t) &= \overline{c}_0 + \overline{c}_1 H_1^1(t,1) + \overline{c}_2 H_2^1(t,1) + \overline{c}_3 H_3^1(t,1) \\
&= \tfrac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}(t-1) \\
&+ \tfrac{1}{2(2\alpha+7)}\Bigg(-4\alpha^2\Bigg(\tfrac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\Bigg)^2 \\
&-14\Bigg(\tfrac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\Bigg)+14\Bigg)(t-1)^2 \\
&+ \tfrac{1}{2(2\alpha+7)}\Bigg(4\alpha^2\Bigg(\tfrac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\Bigg)^2 \\
&-4\alpha\Bigg(\tfrac{4\alpha^2+24\alpha+21\pm\sqrt{(4\alpha^2+24\alpha+21)^2-16\alpha^2(10\alpha+21)}}{8\alpha^2}\Bigg)+2\alpha\Bigg)(t-1)^3.
\end{aligned}$$

In figures below are shown the solutions for $\alpha = \frac{1}{8}$, $\alpha = \frac{1}{4}$ and $\alpha = \frac{1}{2}$, respectively, at $t = 1, 2, 4, 8, 16, 32, 64$.

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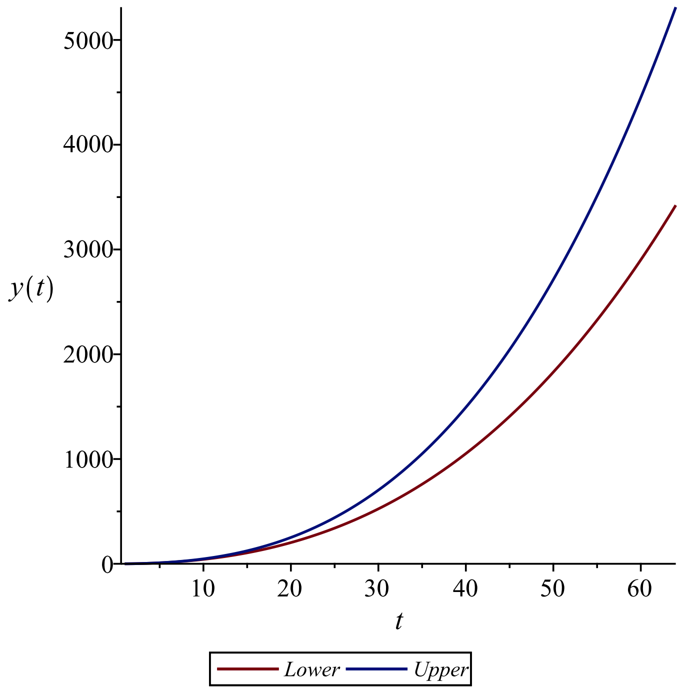

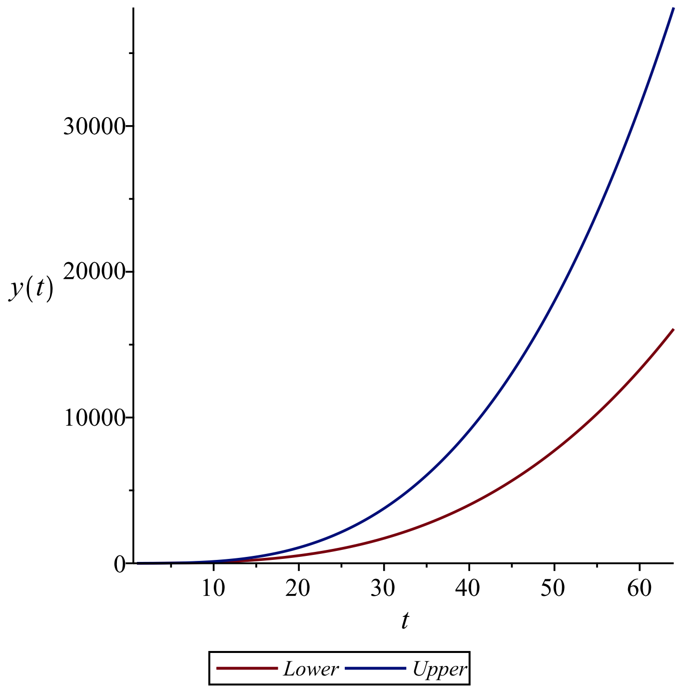

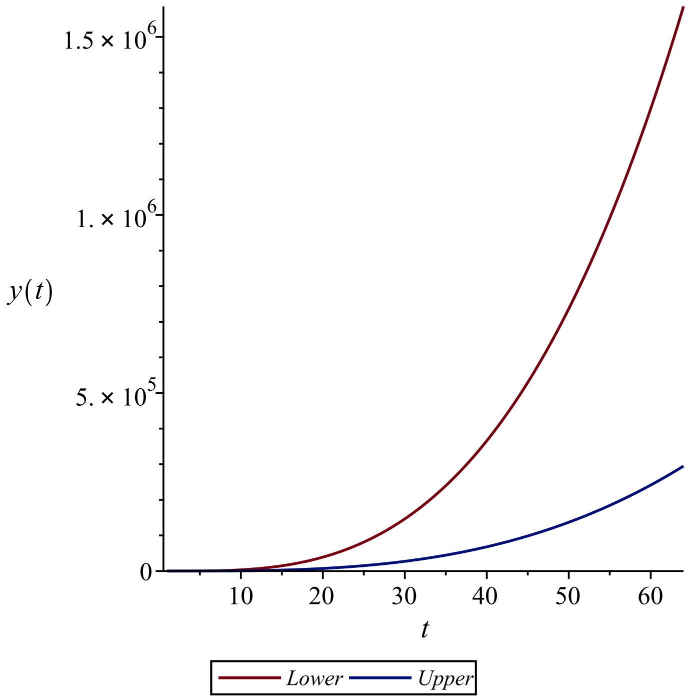

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