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A Fibration of Fermat Surface in Characteristic

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A Fibration of Fermat Surface in Characteristic

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Abstract: Let S be a Fermat surface of degree 5 and let k be an algebraically closed field of characteristic $p = 2$. We give a fibration $f\colon S \to \mathbb{P}^1_k$ and investigate the geometric properties of fiber of f. Furthermore, we investigate the fibration-preserving automorphisms group $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$ of f. We obtain that $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$ is a p-group of order 256.

Keywords: positive characteristic; fibrations; automorphisms of surfaces

MSC: 14G17; 14D06; 14J50

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1. Introduction

Let k be an algebraically closed field and let S be the hypersurface in $\mathbb{P}^3_k$ defined by the equation

$$z_0^5 - z_1^5 = z_2^5 - z_3^5, \tag{1}$$

where z_0, z_1, z_2, z_3 are the homogeneous coordinates.

The morphism $f\colon S \to \mathbb{P}^1_k$ is defined as follows:

$$f\colon (z_0 : z_1 : z_2 : z_3) \mapsto \begin{cases} z_2^4/z_0^4, & \text{if } z_0 = z_1 \text{ and } z_2 = z_3, \\ (z_0 - z_1)/(z_2 - z_3), & \text{otherwise.} \end{cases} \tag{2}$$

When k is the complex numbers $\mathbb{C}$, f is a fibration. It is well-known that the general fiber of f is a smooth curve. (see [1], Chapter III, Corollary 10.7). In [2–4], they investigate the fibration $f\colon S \to \mathbb{P}^1_{\mathbb{C}}$. For example, the general fiber of f is a smooth curve of genus 3. On the other hand, let S_λ denote the fiber over a point $\lambda \in \mathbb{P}^1_{\mathbb{C}}$, they obtain that S_λ is a singular fiber if and only if λ belongs to the following set

$$E := \{\lambda \,|\, \lambda^5 = -1/4, 1, or\ -4\} \cup \{0, \infty\}. \tag{3}$$

However, when k is an algebraically closed field of positive characteristic, the situation will be different. The general fiber of a fibration may be singular (see [5]).

In this paper, k will be an algebraically closed field of characteristic 2. Although f remains a fibration, the general fiber of f is no longer smooth. In Section 2, we recall some necessary preliminaries. In Section 3, we investigate the geometric properties of fiber of f. We obtain that the following theorem.

Theorem 1. (see Proposition 1, Proposition 2). *Let k be an algebraically closed field of characteristic 2 and let S be a Fermat surface of degree 5 defined by the equation (1). Let $f\colon S \to \mathbb{P}^1_k$ be a fibration defined by (2). We obtain that*
(i) for any $\lambda \in k$, the fiber S_λ has arithmetic genus 3.
(ii) the fiber S_λ is reducible if and only if

$$\lambda \in \{c \,|\, c^{15} = 1\} \cup \{0, \infty\}. \tag{4}$$

(iii) if $\lambda \notin \{c \,|\, c^{15} = 1\} \cup \{0, \infty\}$, then the fiber S_λ is a rational curve with a unique singular point $(1 : 1 : \lambda^{\frac{1}{4}} : \lambda^{\frac{1}{4}})$.

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In Section 4, we investigate the fibration-preserving automorphisms group of f. More specifically, let

$$G := \{\sigma \in \mathrm{Aut}_k(S) \,|\, \exists\, \varphi \in \mathrm{Aut}_k(\mathbb{P}^1_k) \text{ such that } f \circ \sigma = \varphi \circ f\} \tag{5}$$

and $\mathrm{Aut}_{\mathbb{P}^1_k}(S) := \{\sigma \in G \,|\, f \cdot \sigma = f\}$. We have the following theorem.

Theorem 2. (see Theorem 5, Theorem 6). *Let H be the Sylow p-group of $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$. We have*
(i) for any $\sigma \in H$, there are some elements a, b, c and d in k such that

$$\sigma = \begin{pmatrix} a & 1+a & b & b \\ 1+a & a & b & b \\ c & c & d & 1+d \\ c & c & 1+d & d \end{pmatrix}, \tag{6}$$

$a^4 = a$, $c = b^4 = c^{16}$ and $d^4 = d$. Conversely, if σ is of the form as in (6), then $\sigma \in H$.
(ii) the order of H is 256.
(iii) $\mathrm{Aut}_{\mathbb{P}^1_k}(S) = H$.

2. Preliminaries

In this section, let k be an algebraically closed field. Let S be a smooth projective surface, and C a smooth projective curve over k.

2.1. Fibration

We say $f : S \to C$ is a fibration if f is a surjective morphism and $f_*\mathcal{O}_S = \mathcal{O}_C$.

Definition 1. *Let $f : S \to C$ be a fibration and let $c \in C$ be a point. Let $k(c)$ be the residue field of c. Then the fiber of f over the point c is defined to be the fiber product $S_c := S \times_C \operatorname{Spec} k(c)$.*

It is a fact that the topological space of S_c is homeomorphic to $f^{-1}(c)$ with the induced topology. (see [1], Chapter II, exercise 3.10)

2.2. Automorphism group of $\mathbb{P}^1_k$

Let $\mathrm{GL}_2(k)$ be the group of invertible 2×2 matrices with entries in k, and $\mathrm{PGL}_2(k) := \mathrm{GL}_2(k)/k^*$. It is known that $\mathrm{PGL}_2(k) \cong \mathrm{Aut}_k(\mathbb{P}^1_k)$.

Theorem 3. ([6], Lemma 3.2). Let $s \in \mathrm{PGL}_2(k)$ be nontrivial and of finite order. Then $s^p = \mathrm{id}$ if and only if s has a unique fixed point in $\mathbb{P}^1_k$.

Corollary 1. *Let H be a finite subgroup of $\mathrm{PGL}_2(k)$. If H fixes at least two points of $\mathbb{P}^1_k$, then H is a cyclic group with order prime to p.*

Proof. For any $s \in H$, if H fixes at least two points of $\mathbb{P}^1_k$, then s fixes at least two points of $\mathbb{P}^1_k$. By Theorem 3, the order of s is prime to p. This implies that the order of H is prime to p. Furthermore, H is a cyclic group (refer to [6], Proposition 4.5). □

Theorem 4. ([6], Proposition 4.7). If D is a finite subgroup of $\mathrm{PGL}_2(k)$, then D fixes a unique point of $\mathbb{P}^1_k$ if and only if $D \cong H \rtimes M$, where $H \subseteq D$ is a nontrivial Sylow p-group and $M \subseteq D$ is a cyclic group with order prime to p.

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2.3. Automorphism group of Fermat surface of degree 5

Let k be an algebraically closed field of characteristic $p = 2$ and $q := p^2$. Let S be the Fermat surface defined by the equation (1). Then

$$\mathrm{Aut}_k(S) = PU(4,q) := \{(a_{ij}) \in GL(4,\mathbb{F}_{q^2}) | (a_{ij})^t(a_{ij}^4) = I\}/\{scalars\} \tag{7}$$

(see [7], page 100, Theorem). Furthermore, one obtains that the order of $PU(4,q)$ is $4^6 \cdot (4^2-1)(4^3+1)(4^4-1)$ (see [8], page 8, Table 1).

3. The fiber of f

In this section, let k be an algebraically closed field of characteristic $p = 2$ and let S be the Fermat surface defined by the equation (1). Note that the characteristic of k is 2, so $a + b = a - b$ for any $a, b \in k$.

3.1. Reducible fiber

Proposition 1. *For any $\lambda \in \mathbb{P}^1_k$, the fiber S_λ is reducible if and only if*

$$\lambda \in \{c \,|\, c^{15} = 1\} \cup \{0, \infty\}. \tag{8}$$

Proof. When $\lambda \in \{0, \infty\}$, we obtain that S_λ is a union of four projective lines meeting in a point (see 3.1.1).

Now, let us suppose $\lambda \notin \{0, \infty\}$. Let

$$F(z_0, z_1, z_2) := z_2^4 + \frac{1}{\lambda^3}(z_0+z_1)^3 \cdot z_2 + (\lambda + \frac{1}{\lambda^4})(z_0+z_1)^4 + \lambda(z_0^3 z_1 + z_0^2 z_1^2 + z_0 z_1^3). \tag{9}$$

By definition, $(z_0 : z_1 : z_2 : z_3) \in f^{-1}(\lambda)$ if and only if

$$\begin{cases} F(z_0, z_1, z_2) = 0, \\ z_0 + z_1 = \lambda(z_2 + z_3) \end{cases} \tag{10}$$

Therefore, if $(z_0 : z_1 : z_2 : z_3) \in f^{-1}(\lambda)$, then $F(z_0, z_1, z_2) = 0$.

If $F(z_0, z_1, z_2)$ is reducible, then either

$$F(z_0, z_1, z_2) = (a_0 + a_1 z_2 + a_2 z_2^2 + z_2^3)(b_0 + z_2) \tag{11}$$

for some $a_0, a_1, a_2, b_0 \in k[z_0, z_1]$, or

$$F(z_0, z_1, z_2) = (\alpha_0 + \alpha_1 z_2 + z_2^2)(\beta_0 + \beta_1 z_2 + z_2^2) \tag{12}$$

for some $\alpha_0, \alpha_1, \beta_0, \beta_1 \in k[z_0, z_1]$.

In 3.1.2 and 3.1.3, we obtain that the necessary condition for S_λ to be reducible is that $\lambda^{15} = 1$. In 3.1.4, we obtain that S_λ is still a union of four projective lines meeting in a point if $\lambda^{15} = 1$.

In summary, the proof is complete. □

3.1.1. Case $\lambda \in \{0, \infty\}$

Let $E := \{(z_0 : z_1 : z_2 : z_3) \in S \,|\, z_0 = z_1, z_2 \neq z_3\}$.

By definition, $f^{-1}(0) = E \cup \{(1:1:0:0)\}$. More specifically, if $(z_0 : z_1 : z_2 : z_3) \in E$, then $z_2^5 + z_3^5 = z_0^5 + z_1^5 = 0$. Hence $0 = z_2^5 + z_3^5 = (z_2+z_3)(z_2^4 + z_2^3 z_3 + z_2^2 z_3^2 + z_2 z_3^3 + z_3^4)$. Combining this with the inequation that $z_2 \neq z_3$, we obtain $z_2^4 + z_2^3 z_3 + z_2^2 z_3^2 + z_2 z_3^3 + z_3^4 = 0$. Hence

$$E = \{(z_0 : z_0 : z_2 : z_3) \in \mathbb{P}^3_k \,|\, z_2^4 + z_2^3 z_3 + z_2^2 z_3^2 + z_2 z_3^3 + z_3^4 = 0 \text{ and } z_2 \neq z_3\}. \tag{13}$$

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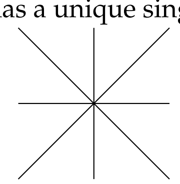
Let $E' := \{(z_0 : z_0 : z_2 : z_3) \in \mathbb{P}^3_k \,|\, z_2^4 + z_2^3 z_3 + z_2^2 z_3^2 + z_2 z_3^3 + z_3^4 = 0\}$. Note that $E' = E \cup \{(1 : 1 : 0 : 0)\}$, so $f^{-1}(0) = E'$. Let $\{\theta_1, \theta_2, \theta_3, \theta_4\}$ be the solution set of the equation $z^4 + z^3 + z^2 + z = 1$. We have $z_2^4 + z_2^3 z_3 + z_2^2 z_3^2 + z_2 z_3^3 + z_3^4 = (z_2 + \theta_1 z_3)(z_2 + \theta_2 z_3)(z_2 + \theta_3 z_3)(z_2 + \theta_4 z_3)$. Hence

$$f^{-1}(0) = \bigcup_{i=1}^{4} \{(z_0 : z_0 : z_2 : z_3) \in \mathbb{P}^3_k \,|\, z_2 + \theta_i z_3 = 0\} \tag{14}$$

Therefore, $f^{-1}(0)$ is reducible and it has a unique singular point $(1 : 1 : 0 : 0)$. See the figure below.



Similarly,

$$f^{-1}(\infty) = \bigcup_{i=1}^{4} \{(z_0 : z_1 : z_2 : z_2) \in \mathbb{P}^3_k \,|\, z_0 + \theta_i z_1 = 0\}. \tag{15}$$

It is reducible and it has a unique singular point $(0 : 0 : 1 : 1)$.

3.1.2. Case $F(z_0, z_1, z_2) = (a_0 + a_1 z_2 + a_2 z_2^2 + z_2^3)(b_0 + z_2)$

Suppose $\lambda \notin \{0, \infty\}$. Note that

$$\begin{aligned} F(z_0, z_1, z_2) &= (a_0 + a_1 z_2 + a_2 z_2^2 + z_2^3)(b_0 + z_2) \\ &= z_2^4 + (b_0 + a_2) z_2^3 + (b_0 a_2 + a_1) z_2^2 + (b_0 a_1 + a_0) z_2 + b_0 a_0. \end{aligned} \tag{16}$$

Combining this with (9), we obtain

$$\begin{cases} a_2 = b_0, \\ a_1 = b_0 a_2 = b_0^2, \\ a_0 = b_0 a_1 + \dfrac{1}{\lambda^3}(z_0 + z_1)^3 = b_0^3 + \dfrac{1}{\lambda^3}(z_0 + z_1)^3, \\ a_0 b_0 = (\lambda + \dfrac{1}{\lambda^4})(z_0^4 + z_1^4) + \lambda(z_0^3 z_1 + z_0^2 z_1^2 + z_0 z_1^3). \end{cases} \tag{17}$$

Hence $a_0 b_0 = b_0 \cdot [b_0^3 + \frac{1}{\lambda^3}(z_0 + z_1)^3] = b_0^4 + \frac{1}{\lambda^3} b_0 (z_0 + z_1)^3$. We may assume $b_0 = c_0 z_0 + c_1 z_1$, where $c_0, c_1 \in k$. Thus

$$\begin{aligned} a_0 b_0 &= b_0^4 + \frac{b_0}{\lambda^3}(z_0 + z_1)^3 \\ &= (c_0^4 + \frac{c_0}{\lambda^3}) z_0^4 + (c_1^4 + \frac{c_1}{\lambda^3}) z_1^4 + \frac{c_0 + c_1}{\lambda^3}(z_0^3 z_1 + z_0^2 z_1^2 + z_0 z_1^3). \end{aligned} \tag{18}$$

Combining this with (17), we obtain

$$\begin{aligned} &(\lambda + \frac{1}{\lambda^4})(z_0^4 + z_1^4) + \lambda(z_0^3 z_1 + z_0^2 z_1^2 + z_0 z_1^3) \\ =\ &(c_0^4 + \frac{c_0}{\lambda^3}) z_0^4 + (c_1^4 + \frac{c_1}{\lambda^3}) z_1^4 + \frac{c_0 + c_1}{\lambda^3}(z_0^3 z_1 + z_0^2 z_1^2 + z_0 z_1^3). \end{aligned} \tag{19}$$

Therefore,

$$\begin{cases} c_0^4 + \dfrac{c_0}{\lambda^3} = c_1^4 + \dfrac{c_1}{\lambda^3} = \lambda + \dfrac{1}{\lambda^4}, \\ \dfrac{1}{\lambda^3}(c_0 + c_1) = \lambda. \end{cases} \tag{20}$$

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This implies that $c_0 + c_1 = \lambda^4$, hence $0 = c_0^4 + \frac{c_0}{\lambda^3} + c_1^4 + \frac{c_1}{\lambda^3} = \lambda^{16} + \lambda$. Therefore, $\lambda^{15} = 1$.

3.1.3. Case $F(z_0, z_1, z_2) = (\alpha_0 + \alpha_1 z_2 + z_2^2)(\beta_0 + \beta_1 z_2 + z_2^2)$

Suppose $\lambda \notin \{0, \infty\}$. Note that

$$\begin{aligned} F(z_0, z_1, z_2) &= (\alpha_0 + \alpha_1 z_2 + z_2^2)(\beta_0 + \beta_1 z_2 + z_2^2) \\ &= z_2^4 + (\alpha_1 + \beta_1) z_2^3 + (\alpha_0 + \beta_0 + \alpha_1\beta_1) z_2^2 + (\alpha_0\beta_1 + \alpha_1\beta_0) z_2 + \alpha_0\beta_0. \end{aligned} \tag{21}$$

Combining this with (9), we obtain

$$\begin{cases} \alpha_1 = \beta_1, \\ \alpha_0 = \alpha_1\beta_1 + \beta_0 = \beta_1^2 + \beta_0, \\ \alpha_0\beta_1 + \alpha_1\beta_0 = (\beta_1^2 + \beta_0)\beta_1 + \beta_0\beta_1 = \beta_1^3 = \dfrac{1}{\lambda^3}(z_0 + z_1)^3, \\ \alpha_0\beta_0 = \beta_0\beta_1^2 + \beta_0^2 = (\lambda + \dfrac{1}{\lambda^4})(z_0^4 + z_1^4) + \lambda(z_0^3 z_1 + z_0^2 z_1^2 + z_0 z_1^3). \end{cases} \tag{22}$$

By (22), $\beta_1^3 = \frac{1}{\lambda^3}(z_0 + z_1)^3$. Therefore, we may assume $\beta_1 = \frac{\theta}{\lambda}(z_0 + z_1)$, where $\theta \in \{z \in k \,|\, z^3 = 1\}$. We may assume $\beta_0 = c_0 z_0^2 + c_1 z_0 z_1 + c_2 z_1^2$, where $c_0, c_1, c_2 \in k$. Thus

$$\begin{aligned} \alpha_0\beta_0 &= \beta_0\beta_1^2 + \beta_0^2 \\ &= (c_0 z_0^2 + c_1 z_0 z_1 + c_2 z_1^2) \cdot \frac{\theta^2}{\lambda^2}(z_0 + z_1)^2 + (c_0 z_0^2 + c_1 z_0 z_1 + c_2 z_1^2)^2 \\ &= (c_0^2 + \frac{c_0\theta^2}{\lambda^2}) z_0^4 + (c_2^2 + \frac{c_2\theta^2}{\lambda^2}) z_1^4 + \frac{c_1\theta^2}{\lambda^2}(z_0^3 z_1 + z_0 z_1^3) + [\frac{(c_0 + c_2)\theta^2}{\lambda^2} + c_1^2] z_0^2 z_1^2. \end{aligned} \tag{23}$$

Combining this with (22), we obtain

$$\begin{cases} c_0^2 + \dfrac{c_0\theta^2}{\lambda^2} = c_2^2 + \dfrac{c_2\theta^2}{\lambda^2} = \lambda + \dfrac{1}{\lambda^4}, \\ \dfrac{c_1\theta^2}{\lambda^2} = \dfrac{(c_0 + c_2)\theta^2}{\lambda^2} + c_1^2 = \lambda. \end{cases} \tag{24}$$

Hence $0 = c_0^2 + \frac{c_0\theta^2}{\lambda^2} + c_2^2 + \frac{c_2\theta^2}{\lambda^2} = (c_0 + c_2)(c_0 + c_2 + \frac{\theta^2}{\lambda^2})$. This implies that either $c_0 + c_2 = 0$ or $c_0 + c_2 + \frac{\theta^2}{\lambda^2} = 0$.

If $c_0 + c_2 = 0$, then $\lambda = \frac{c_1\theta^2}{\lambda^2} = \frac{(c_0+c_2)\theta^2}{\lambda^2} + c_1^2 = c_1^2$ by (24). It follows that $\lambda^3 = c_1\theta^2$ and $\lambda = c_1^2$. Note that $\theta^3 = 1$, so $\lambda^6 = c_1^2\theta^4 = \lambda\theta$. Hence $\lambda^{15} = \theta^3 = 1$.

If $c_0 + c_2 + \frac{\theta^2}{\lambda^2} = 0$, then $\lambda = \frac{c_1\theta^2}{\lambda^2} = \frac{(c_0+c_2)\theta^2}{\lambda^2} + c_1^2 = \frac{\theta^4}{\lambda^2} + c_1^2 = \frac{\theta}{\lambda^2} + c_1^2$ by (24). In other words, we have $\lambda = \frac{c_1\theta^2}{\lambda^2} = \frac{\theta}{\lambda^2} + c_1^2$. This implies that $c_1 = \frac{\lambda^3}{\theta^2}$, hence $c_1^2 = \frac{\lambda^6}{\theta^4} = \frac{\lambda^6}{\theta}$. Substituting this into $\lambda = \frac{\theta}{\lambda^2} + c_1^2$, we obtain $\lambda^{10} + \lambda^5\theta + \theta^2 = 0$. Note that $0 = \theta^3 + 1 = (\theta + 1)(\theta^2 + \theta + 1)$. If $\theta = 1$, then $\lambda^{10} + \lambda^5 + 1 = 0$, and hence $\lambda^{15} + 1 = (\lambda^5 + 1)(\lambda^{10} + \lambda^5 + 1) = 0$. Otherwise $\theta^2 + \theta + 1 = 0$, as $\lambda^{10} + \lambda^5\theta + \theta^2 = 0$, we obtain that either $\lambda^5 = 1$ or $\lambda^5 = 1 + \theta$. Therefore, $\lambda^{15} = 1$.

3.1.4. Case $\lambda^{15} = 1$

Note that $\lambda^{15} + 1 = (\lambda^5 + 1)(\lambda^{10} + \lambda^5 + 1)$. If $\lambda^{15} + 1 = 0$, then we obtain that either $\lambda^5 + 1 = 0$ or $\lambda^{10} + \lambda^5 + 1 = 0$.

(i) Suppose $\lambda^5 + 1 = 0$. Let $\{\theta_0, \theta_1\}$ be the solution set of the equation $z^2 + z + 1 = 0$. Then we have

$$F(z_0, z_1, z_2) = (z_2 + \lambda^4 z_0)(z_2 + \lambda^4 z_1)(z_2 + \lambda^4\theta_0 z_0 + \lambda^4\theta_1 z_1)(z_2 + \lambda^4\theta_1 z_0 + \lambda^4\theta_0 z_1). \tag{25}$$

Hence

$$f^{-1}(\lambda) = \bigcup_{i=1}^{4} \{(z_0 : z_1 : z_2 : z_3) \in \mathbb{P}_k^3 \,|\, F_i = 0 \text{ and } z_0 + z_1 = \lambda(z_2 + z_3)\}, \tag{26}$$

where $F_1 := z_2 + \lambda^4 z_0, F_2 := z_2 + \lambda^4 z_1, F_3 := z_2 + \lambda^4\theta_0 z_0 + \lambda^4\theta_1 z_1$, and $F_4 := z_2 + \lambda^4\theta_1 z_0 + \lambda^4\theta_0 z_1$. Therefore, $f^{-1}(\lambda)$ is reducible and it has a unique singular point $(1 : 1 : \lambda^4 : \lambda^4)$.

(ii) Suppose $\lambda^{10} + \lambda^5 + 1 = 0$. Let $\{\delta_0, \delta_1\}$ be the solution set of the equation $z^2 + \lambda^9 z + \lambda^{13} = 0$. Let $F_1 := z_2 + \delta_0(z_0 + z_1) + \lambda^4 z_1, F_2 := z_2 + \delta_1(z_0 + z_1) + \lambda^4 z_1, F_3 := z_2 + \delta_0(z_0 + z_1) + \lambda^4 z_0$, and $F_4 := z_2 + \delta_1(z_0 + z_1) + \lambda^4 z_0$. Then we have $F(z_0, z_1, z_2) = \prod_{i=1}^{i=4} F_i$. Hence

$$f^{-1}(\lambda) = \bigcup_{i=1}^{4} \{(z_0 : z_1 : z_2 : z_3) \in \mathbb{P}_k^3 \,|\, F_i = 0 \text{ and } z_0 + z_1 = \lambda(z_2 + z_3)\}. \tag{27}$$

Therefore, $f^{-1}(\lambda)$ is reducible and it has a unique singular point $(1 : 1 : \lambda^4 : \lambda^4)$.

3.2. *Irreducible fiber*

By Proposition 1, if $\lambda \notin \{c \,|\, c^{15} = 1\} \cup \{0, \infty\}$, then the fiber S_λ is irreducible. S_λ is defined by

$$\begin{cases} z_0 + z_1 = \lambda(z_2 + z_3) \\ z_2^4 + \dfrac{1}{\lambda^3} z_2(z_0 + z_1)^3 + (\lambda + \dfrac{1}{\lambda^4})(z_0^4 + z_1^4) + \lambda(z_0^3 z_1 + z_0^2 z_1^2 + z_0 z_1^3) = 0 \end{cases} \tag{28}$$

Hence S_λ is isomorphic to the curve C_λ in $\mathbb{P}_k^2$, where C_λ is defined by the equation

$$h_\lambda := z_2^4 + \frac{1}{\lambda^3} z_2(z_0 + z_1)^3 + (\lambda + \frac{1}{\lambda^4})(z_0^4 + z_1^4) + \lambda(z_0^3 z_1 + z_0^2 z_1^2 + z_0 z_1^3) = 0. \tag{29}$$

Proposition 2. *(i) If $\lambda \notin \{c \,|\, c^{15} = 1\} \cup \{0, \infty\}$, then the fiber S_λ is a rational curve with a unique singular point $(1 : 1 : \lambda^{\frac{1}{4}} : \lambda^{\frac{1}{4}})$.*
(ii) For any λ, the arithmetic genus $p_a(S_\lambda)$ is equal to 3.

Proof. (i) Let us consider the curve C_λ in $\mathbb{P}_k^2$. We have

$$\begin{cases} \dfrac{\partial h_\lambda}{\partial z_2} = \dfrac{1}{\lambda^3}(z_0 + z_1)^3 \\ \dfrac{\partial h_\lambda}{\partial z_1} = (\dfrac{1}{\lambda^3} z_2 + \lambda z_0)(z_0 + z_1)^2 \\ \dfrac{\partial h_\lambda}{\partial z_0} = (\dfrac{1}{\lambda^3} z_2 + \lambda z_1)(z_0 + z_1)^2 \end{cases} \tag{30}$$

By Jacobian criterion, $(1 : 1 : \lambda^{\frac{1}{4}})$ is the unique singular point of C_λ. In other words, $(1 : 1 : \lambda^{\frac{1}{4}} : \lambda^{\frac{1}{4}})$ is the unique singular point of S_λ.

Note that the arithmetic genus $p_a(C_\lambda) = \frac{(4-1)(4-2)}{2} = 3$, and $(1 : 1 : \lambda^{\frac{1}{4}})$ is a triple singular point of C_λ. Hence the normalization of C_λ is a rational curve. Since S_λ is isomorphic to C_λ, we obtain that S_λ is a rational curve with a unique singular point $(1 : 1 : \lambda^{\frac{1}{4}} : \lambda^{\frac{1}{4}})$.

(ii) According to chapter III, corollary 9.10 in [1], the arithmetic genus of S_λ are all independent of λ. □

4. Subgroups of $\mathrm{Aut}_k(S)$

In this section, let k be an algebraically closed field of characteristic $p = 2$ and $q := p^2$. Let S be the Fermat surface defined by the equation (1). Note that $\mathrm{Aut}_k(S) = PU(4, q^2)$. Denoting $G := \{\sigma \in \mathrm{Aut}_k(S) \,|\, \exists\, \varphi \in \mathrm{Aut}_k(\mathbb{P}_k^1) \text{ such that } f \circ \sigma = \varphi \circ f\}$ and $\mathrm{Aut}_{\mathbb{P}_k^1}(S) := \{\sigma \in G \,|\, f \circ \sigma = f\}$.

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For any $\sigma = (a_{ij}) \in \mathrm{Aut}_{\mathbb{P}^1_k}(S)$, as σ preserves the fiber of f, we have $\sigma(f^{-1}(0)) = f^{-1}(0)$. In particular, σ fixes the unique singular point $(1:1:0:0)$ of $f^{-1}(0)$. Hence

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{31}$$

We obtain that

$$\begin{cases} a_{11} + a_{12} = 1 \\ a_{21} + a_{22} = 1 \\ a_{31} + a_{32} = 0 \\ a_{41} + a_{42} = 0 \end{cases} \tag{32}$$

Similarly, σ fixes the unique singular point $(0:0:1:1)$ of $f^{-1}(\infty)$. Hence

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \tag{33}$$

We obtain that

$$\begin{cases} a_{13} + a_{14} = 0 \\ a_{23} + a_{24} = 0 \\ a_{33} + a_{34} = 1 \\ a_{43} + a_{44} = 1 \end{cases} \tag{34}$$

Combining (32) and (34), we have

$$\sigma = \begin{pmatrix} a_{11} & 1 + a_{11} & a_{13} & a_{13} \\ 1 + a_{22} & a_{22} & a_{23} & a_{23} \\ a_{31} & a_{31} & a_{33} & 1 + a_{33} \\ a_{41} & a_{41} & 1 + a_{44} & a_{44} \end{pmatrix} \tag{35}$$

4.1. Sylow p-group of $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$

Let H be the Sylow p-group of $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$. Let F be a general fiber of f. We have the following exact sequences:

$$1 \to \mathrm{Aut}_{\mathbb{P}^1_k}(S) \to \mathrm{Aut}_k(F). \tag{36}$$

By Proposition 2, F is a rational curve with a unique singular point. Hence $\mathrm{Aut}_k(F) \subset \mathrm{Aut}_k(\mathbb{P}^1_k) \cong \mathrm{PGL}(2,k)$. By (36), we may regard $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$ as a subgroup of $\mathrm{Aut}_k(F)$ (hence of $\mathrm{Aut}_k(\mathbb{P}^1_k)$). By Theorem 3 and Corollary 1, we obtain that $\sigma^2 = \mathrm{id}$ for any $\sigma \in H$.

Let $\sigma = (a_{ij}) \in H$. As $\sigma^2 = \mathrm{id}$, we obtain that $(a_{ij})(a_{ij}) = I$, and hence $(a_{ij})^t(a_{ij})^t = I$. On the other hand, $\sigma \in H$, hence $\sigma \in \mathrm{Aut}_k(S) = PU(4,q^2)$. This implies that $(a_{ij})^t(a_{ij}^4) = I$. Therefore, we have $(a_{ij}^4) = (a_{ij})^t$. In other words, we obtain that

$$\begin{cases} a_{ij} = a_{ji}^4 = a_{ij}^{16}, & \text{if } i \neq j, \\ a_{ii} = a_{ii}^4, & \text{otherwise.} \end{cases} \tag{37}$$

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Note that σ is also an element of $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$, so σ has the form of (35). We obtain that $(1+a_{11})^4 = 1+a_{22}$ and $a_{11}^4 = a_{11}$. This implies that $a_{11} = a_{11}^4 = a_{22}$. Similarly, we have $a_{33} = a_{44}$, $a_{13} = a_{23}$ and $a_{31} = a_{41}$. Therefore, we may assume

$$\sigma = \begin{pmatrix} a & 1+a & b & b \\ 1+a & a & b & b \\ c & c & d & 1+d \\ c & c & 1+d & d \end{pmatrix} \tag{38}$$

where $a^4 = a$, $c = b^4 = c^{16}$ and $d^4 = d$.

Conversely, as

$$\sigma \cdot \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} a(z_0+z_1)+b(z_2+z_3)+z_1 \\ a(z_0+z_1)+b(z_2+z_3)+z_0 \\ c(z_0+z_1)+d(z_2+z_3)+z_3 \\ c(z_0+z_1)+d(z_2+z_3)+z_2 \end{pmatrix}, \tag{39}$$

we obtain that σ preserves the fiber of f. Hence any element of the p-group of $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$ is of the form as in (38). We have proved the following theorem.

Theorem 5. *Let H be the Sylow p-group of $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$. We have*
(i) for any $\sigma \in H$, there are some elements a, b, c and d in k such that

$$\sigma = \begin{pmatrix} a & 1+a & b & b \\ 1+a & a & b & b \\ c & c & d & 1+d \\ c & c & 1+d & d \end{pmatrix}, \tag{40}$$

$a^4 = a$, $c = b^4 = c^{16}$ and $d^4 = d$. Conversely, if σ be of the form as in (40), then $\sigma \in H$.
(ii) the order of H is 256.

Remark 1. *In particular,*

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{41}$$

is a nontrivial element of H.

Remark 2. *Let F be a general fiber of f and let P be the unique singular point of F. By (36), we may regard $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$ as a subgroup of $\mathrm{Aut}_k(F)$. By Corollary 1 and the fact that H is nontrivial, we obtain that $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$ fixes a unique point P of F. By Theorem 4, we have $\mathrm{Aut}_{\mathbb{P}^1_k}(S) \cong H \rtimes M$, where M is a cyclic group with order prime to p.*

4.2. Subgroups of $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$ with order prime to p

By Remark 4, we have $\mathrm{Aut}_{\mathbb{P}^1_k}(S) \cong H \rtimes M$, where M is a cyclic group with order prime to p.

Theorem 6. *Let M as above. We obtain that*
(i) M is trivial.
(ii) $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$ is a p-group of order 256.

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Proof. (i) We may assume the order of M is r. Suppose $r \neq 1$. Let $\psi := (a_{ij})$ be a generator of M. Note that $\psi^r = (a_{ij})^r = I$ and $(a_{ij})^t(a_{ij}^4) = I$, so $(a_{ij})^{r-1} = (a_{ij}^4)^t$. As ψ is an element of $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$, we may assume ψ has the form of (35). Therefore,

$$\psi^{r-1} = \begin{pmatrix} a_{11}^4 & 1+a_{11}^4 & a_{13}^4 & a_{13}^4 \\ 1+a_{22}^4 & a_{22}^4 & a_{23}^4 & a_{23}^4 \\ a_{31}^4 & a_{31}^4 & a_{33}^4 & 1+a_{33}^4 \\ a_{41}^4 & a_{41}^4 & 1+a_{44}^4 & a_{44}^4 \end{pmatrix}^t \tag{42}$$

On the other hand, by Lemma 5, we may assume

$$\psi^{r-1} = \begin{pmatrix} c_{11} & 1+c_{11} & c_{13} & c_{13} \\ 1+c_{22} & c_{22} & c_{23} & c_{23} \\ c_{31} & c_{31} & c_{33} & 1+c_{33} \\ c_{41} & c_{41} & 1+c_{44} & c_{44} \end{pmatrix} \tag{43}$$

Combining (42) and (43), we obtain that $a_{11}^4 = c_{11}$ and $1 + a_{22}^4 = 1 + c_{11}$. This implies that $a_{11}^4 = c_{11} = a_{22}^4$, hence $a_{11} = a_{22}$. Similarly, we have $a_{33} = a_{44}$, $a_{13} = a_{23}$, and $a_{31} = a_{41}$. Therefore, we may assume

$$\psi = \begin{pmatrix} a & 1+a & b & b \\ 1+a & a & b & b \\ c & c & d & 1+d \\ c & c & 1+d & d \end{pmatrix} \tag{44}$$

where $a, b, c, d \in k$, so $\psi^2 = I$. Hence the order of M is 2, which is a contradiction.

Therefore, $r = 1$ and M is trivial.

(ii) As M is trivial, we obtain that $\mathrm{Aut}_{\mathbb{P}^1_k}(S) = H$ is a p-group. By Theorem 5, the order of $\mathrm{Aut}_{\mathbb{P}^1_k}(S)$ is 256. □

Lemma 1. *Let*

$$A = \begin{pmatrix} a_{11} & 1+a_{11} & a_{13} & a_{13} \\ 1+a_{22} & a_{22} & a_{23} & a_{23} \\ a_{31} & a_{31} & a_{33} & 1+a_{33} \\ a_{41} & a_{41} & 1+a_{44} & a_{44} \end{pmatrix} \tag{45}$$

and

$$B = \begin{pmatrix} b_{11} & 1+b_{11} & b_{13} & b_{13} \\ 1+b_{22} & b_{22} & b_{23} & b_{23} \\ b_{31} & b_{31} & b_{33} & 1+b_{33} \\ b_{41} & b_{41} & 1+b_{44} & b_{44} \end{pmatrix}. \tag{46}$$

Then

$$AB = \begin{pmatrix} c_{11} & 1+c_{11} & c_{13} & c_{13} \\ 1+c_{22} & c_{22} & c_{23} & c_{23} \\ c_{31} & c_{31} & c_{33} & 1+c_{33} \\ c_{41} & c_{41} & 1+c_{44} & c_{44} \end{pmatrix} \tag{47}$$

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where

$$\begin{cases} c_{11} = a_{11}b_{11} + (1+a_{11})(1+b_{22}) + a_{13}(b_{31}+b_{41}), \\ c_{13} = a_{11}b_{13} + b_{23}(1+a_{11}) + a_{13}(1+b_{33}+b_{44}), \\ c_{22} = a_{22}b_{22} + (1+a_{22})(1+b_{11}) + a_{23}(b_{31}+b_{41}), \\ c_{23} = a_{22}b_{23} + b_{13}(1+a_{22}) + a_{23}(1+b_{33}+b_{44}), \\ c_{33} = a_{33}b_{33} + (1+a_{33})(1+b_{44}) + a_{31}(b_{13}+b_{23}), \\ c_{31} = a_{33}b_{31} + b_{41}(1+a_{33}) + a_{31}(1+b_{11}+b_{22}), \\ c_{44} = a_{44}b_{44} + (1+a_{44})(1+b_{33}) + a_{41}(b_{13}+b_{23}), \\ c_{41} = a_{44}b_{41} + b_{31}(1+a_{44}) + a_{41}(1+b_{11}+b_{22}). \end{cases} \tag{48}$$

Remark 3. *By Theorem 6, for any nontrivial element $\sigma \in \mathrm{Aut}_k(S)$ with order prime to p, we obtain that σ does preserve the fiber of f.*

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