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Posted Date: 25 July 2024

doi: 10.20944/preprints202407.2043.v1

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On The Incompleteness Of Einstein-De Sitter Spacetime

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Abstract: his paper presents a mathematically rigorous investigation into the incompleteness of Einstein-de Sitter spacetime. The incompleteness of the spacetime is proved using results regarding isometries between connected Semi-Riemannian manifolds following the computation of the Einstein tensor of the Einstein-de Sitter spacetime. Various properties of the Einstein-de Sitter spacetime apart from its incompleteness is also discussed based on its incompleteness with the help of various results of Semi-Riemannian manifolds. Also, few properties of spacetime which cannot be concluded from the incompleteness of the spacetime is also discussed

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1. Introduction

The Einstein-de Sitter universe or spacetime was jointly proposed by Albert Einstein and Willem De Sitter as a relativistic model for expanding universe by setting cosmological constant and curvature of space to zero. A brief note throwing light on the historical and philosophical aspects of the Einstein-de Sitter model can be studied in [12]. From a topological point of view, the Einstein-de Sitter spacetime universe has the topology $R^1 \times S^3$ representing time and 3-dimensional space respectively. It is the maximally symmetric Lorentzian manifold and is also a simply connected topological space. A detailed analysis of Einstein-de Sitter spacetime from a topological and geometrical point of view has been done in [8,2]. In the recent past, various studies has been conducted on the black hole in the Einstein-de Sitter universe [15,9]. Despite the ongoing attempts to integrate and incorporate quantum theories with the classical theories of gravity, various attempts have been made to explain the instabilities and impacts of quantum theory of gravity in association with the Einstein-de Sitter universe [6,18]. Moreover, the Einstein–de Sitter universe has showed a reasonable fit to recent observations [13,17].
Completeness is an important topological aspect of spacetime. In fact, in physics, one prefers to work with complete and maximal spacetime. Completeness accounts for various geometrical properties of the spacetime, including the ones such as spacetime singularities which have gained attention among the researchers in the antecedent [7,16]. At our current level of understanding in the pertinent physics literature, it appears widely acknowledged that a comprehensive definition of a spacetime singularity remains elusive. Despite this, recent studies have shown the rigid involvement of completeness of spacetime to act as a key feature to describe the notion of singularity and its relevance in general relativity. Apart from mathematical aspects like singularity, research works on completeness of spacetime have also been successful in formulation of various theorems and results providing insights into the perpetual research on big bang, theoretical astrophysics, cosmology etc. Theorems like the Collapse theorems [14] which provide relationships completeness of spacetime, associated Cauchy surface etc. have been worked out in the recent past which has provided more deeper theoretical conclusion that there must have been a big bang if the galaxies are now diverging. Intriguingly, the

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allure of incompleteness of spacetime has captivated the attention of numerous scholars, leading to a plethora of investigations and scholarly contributions in this domain [1].

2. Preliminaries

We give some basic concepts that we will use in this paper. Most of the mathematical notations used are similar to in [14].
Let (M, g) denote a connected Lorentzian manifold and TM be its tangent bundle with the projection $\Pi : TM \to M$. $\mathcal{T} \subset TM$ be the set of timelike points. We denote the tangent space of M at $x \in M$ as M_x. It is clear that TM is the set $\{(x, X)|x \in M \text{ and } X \in M_x\}$ with its standard C^∞ manifold structure. The causal character of $(x, X) \in TM$ is the causal character of $X \in M_x$.
Theorem 1: $\mathcal{T}$ is an open sub manifold and has either one or two connected components.
Proof: We first prove that $\mathcal{T}$ is open and then prove that it has either one connected components or two. To prove $\mathcal{T}$ is open, we construct $B : TM \to R$ by defining $B(x, X) = g(X, X)$.
$\Rightarrow B$ is C^∞
$\mathcal{T}$ is the complete inverse image of $(-\infty, 0)$ under the C^∞ map B.
$\Rightarrow \mathcal{T}$ is open.
Let $\mathcal{W}$ be a component of $\mathcal{T}$. Then $\psi\mathcal{W}$ is also a component of $\mathcal{T}$ if $\psi : \mathcal{T} \to \mathcal{T}$ is an homeomorphism given by $\psi(x, X) = (x, -X)$.
Let $\mathcal{N} = \mathcal{W} \cup \psi\mathcal{W}$, $\mathcal{C} = \mathcal{T} - \mathcal{N}$. $\mathcal{N}$ is both open and closed in $\mathcal{T}$.
$\Rightarrow \mathcal{C}$ is also both open and closed in $\mathcal{T}$. Also both $\mathcal{C}$ and $\mathcal{N}$ are open in TM.
Π being the projection of the tangent bundle TM as mentioned above, we claim $\Pi\mathcal{N} \cap \Pi\mathcal{C} = \phi$. If our claim is false, then there should exist $(x, Z) \in \mathcal{N}$ and $(x, Y) \in \mathcal{C}$ for some $x \in M$. Out of the two components of $M_x \cap \mathcal{T}$, let us consider $\mathcal{P} \subset M_x$ be that component in which (x, Y) lies.
$\Rightarrow \mathcal{C} \cap \mathcal{P} \neq \phi$.
$\mathcal{C}$ is a union of components of $\mathcal{T} \Rightarrow \mathcal{P} \subset \mathcal{C}$.
Now, either $(x, Z) \in \mathcal{P}$ or $(x, -Z) \in \mathcal{P}$ while both are in $\mathcal{N}$ by the definition of $\mathcal{N}$. Thus, $\mathcal{N} \cap \mathcal{P} \neq \phi$ and hence also $\mathcal{P} \subset \mathcal{N}$ since $\mathcal{N}$ is a union of components.
$\Rightarrow \mathcal{N} \cap \mathcal{C} \neq \phi$. This is a contradiction. We therefore have $\Pi\mathcal{N} \cap \Pi\mathcal{C} = \phi$, $\Pi\mathcal{N} \cup \Pi\mathcal{C} = M$.
We know that M is connected and $\Pi\mathcal{C} = \phi \Rightarrow \mathcal{C} = \prec \Rightarrow \mathcal{T} = \mathcal{W} \cup \psi\mathcal{W}$. Hence we can conclude that, $\mathcal{W} \cap \psi\mathcal{W} = \phi \Rightarrow \mathcal{T}$ has two components and $\mathcal{W} \cap \psi\mathcal{W} \neq \phi \Rightarrow \mathcal{W} = \psi\mathcal{W} \Rightarrow \mathcal{T}$ has only one component.

Definition: The connected Lorentzian manifold (M, g) is called time orientable iff $\mathcal{T}$ has two components.
Suppose (M, g) is time orientable. One component of $\mathcal{T}$ is labelled $\mathcal{T}^*$ and called the future. The complement of $\mathcal{T}^*$ in $\mathcal{T}$ is denoted by $\mathcal{T}^-$ and called the past. Suppose there is a causal vector field X on (M, g). Then $(x, W) \to g(W, X)$ defines a C^∞ onto function $\psi : \mathcal{T} \to (-\infty, 0) \cup (0, \infty)$. Hence (M, g) is time orientable since $\mathcal{T}$ is not connected. If we nominate $\psi^{-1}(-\infty, 0)$ as $\mathcal{T}^+$, we say (M, g) is time oriented by X.

There exist myriad definitions of spacetime, and a multitude of researchers have harnessed these diverse conceptual frameworks and definitions to demonstrate a wide spectrum of results. For instance, in [5], spacetime is considered as an affine space endowed with invariant notions to explain recent developments in the study of non-Lorentzian spacetimes and geometries. In [10], a vector space structure along with the inner product notion of the spacetime is harnessed in order to study the geometry and mathematics of Minkowski spacetime. The comprehensive exploration of alternative mathematical and intuitive definitions of spacetime lies beyond the scope of this paper. In this paper, mathematically, spacetime is viewed as a Lorentzian manifold admitting certain other conditions and structures as given below.
Definition: A spacetime (M, g, D) is a connected 4-dimensional, oriented and time oriented Lorentzian

manifold (M, g) together with the Levi-Civita connection D of g on M.
Minkowski spacetime, schwarzschild spacetime, Vaidya spacetime [4] etc. A general relativistic gravitational field $[(M, g)]$ is defined as an equivalence class of spacetimes where the equivalence is defined by orientation and time orientation preserving isometries. A wide range of proposals for the modification of the above definition of spacetime have appeared in research papers. For example, claims have been made to replace the Levi-Civita connection with metric connection along with torsion, include stable causality in the definition of spacetime etc. But we will stick on to our basic definition for proving the rest of the results.

Some results:
The following computational results will aid in proving the results and assertions made in this paper. Here (M, g) is a 4-dimensional Lorentzian manifold, D is the Levi-Civita connection, $\{X_i$ is the dual basis and $\{\omega^i$ be a local basis of 1-forms on M. The connection forms for $\{\omega\}^i$ denoted by $\{\omega\}^i_j$ are given by

$$D_{x_i} X_j = \sum_{k=1}^{4} \omega^k_j(X_i) X_k, \quad \forall i, j = 1, ..., 4. \tag{2.1}$$

1. If X and Y are vector fields on M, $D_X Y$ can be computed as follows:

$$D_X Y = \sum_{i=1}^{4} \{X(\omega^i Y) + \sum_{j=1}^{4} \omega^i_j(X)\omega^j(Y)\} X_i. \tag{2.2}$$

2. The curvature forms Ω^i_j for (M, g) are defined by

$$\Omega^i_j = 2(d\omega^i_j + \sum_{k=1}^{4} \omega^i_k \wedge \omega^k_j) \quad \forall i, j = 1, ..., 4. \tag{2.3}$$

3. The curvature tensor R can be computed as follows:

$$R = \sum_{i,j=1}^{4} X_i \otimes \omega^j \otimes \Omega^i_j. \tag{2.4}$$

4. If $\omega\{i\}$ is an orthonormal basis, the connection forms are uniquely determined by the following two conditions:

$$d\omega^i = -\sum_{j=1}^{4} \omega^i_j \wedge \omega^j \quad i = 1, ..., 4 \tag{2.5}$$

$$\omega^4_4 = 0, \omega^4_\mu = \omega^\mu_4, \omega^\mu_\nu = -\omega^\nu_\mu \quad \nu, \mu = 1, 2, 3. \tag{2.6}$$

The last two equations are very useful for computations.

Theorem 2: Let (A, g) and $(B, \tilde{g})$ be two spacetimes and $(B, \tilde{g})$ contains (A, g), but for all lightlike geodesic $\theta : \epsilon \to B$ such that $(\theta\epsilon) \cap A \neq \phi$, $\theta\epsilon \subset A$. Then $A = B$.
The detailed proof of this theorem is not incorporated in this paper due to the requirement of certain advanced mathematical tools. The mathematical techniques and ideas for the detailed proof can be referred from [14].

Theorem 3: Let (M, g) denote a Lorentzian manifold, X denote a timelike vector in the tangent space M_x, and $\omega \in M^*_x$ be a timelike one-form where M^*_x denote the dual space. Then $|\omega X| \geq |\omega||X|$ and equality holds if and only if ω and kX are equivalent for some $k \in \mathbb{R}$.
The norm $|\omega|$ is taken with respect to the innerproduct $\hat{g}$ on the tangent space M^*_x defined as

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$\hat{g} : M_x^* \times M_x^* \to R$ by $\hat{g}(\omega, \omega') = g(\phi^{-1}\omega, \phi^{-1}\omega')$, $\forall \omega, \omega' \in M_x^*$, where ϕ is an isomorphism defined as $\phi : M_x \to M_x^*$ by $\phi v(w) = g(v, w)$ $\forall v, w \in M_x$.

3. Einstein-De Sitter spacetime

Notations:

We give certain notations which we will be using in the coming sections.

1. $\mathbb{R}^n = \mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R}$ n times.
2. $u^P : \mathbb{R}^n \to \mathbb{R}$ is the projection onto Pth factor.
3. $\{du^P\}$ denotes a basis of one-forms on some open submanifold of $\mathbb{R}^n$ and $\{\partial_P\}$ denotes the dual basis.

Let M be the manifold defined as $M = \mathbb{R}^3 \times F$, where F is an open interval in $\mathbb{R}$. Let $R : F \to (0, \infty)$ be a function. Considering M as a subset of $\mathbb{R}^4$, define g on M by

$$g = \{(R \circ u^4)^2 \sum_{\mu=1}^{3} du^\mu \otimes du^\mu\} - du^4 \otimes du^4.$$

Then g is a Lorentzian metric on M and ∂_4 is a timelike vector field on (M, g). (M, g, D) when time oriented by ∂_4 and oriented by $du^1 \wedge \cdots \wedge du^4$ is a spacetime where D is the Levi-Civita connection of g on M. This spacetime is called a simple cosmological spacetime. If $F = (0, \infty)$ and $R(u) = u^{2/3}$, such a spacetime is Einstein-de Sitter spacetime. The gravitational field which contains the Einstein-de Sitter spacetime is called the Einstein-de Sitter gravitational field.

The Einstein tensor of Einstein-de Sitter spacetime

Let (M, g) be Einstein-de Sitter spacetime. The forthcoming proclamations are mandated $\forall i, j = 1, ..., 4$ and $\forall \mu, \nu = 1, 2, 3$. Define $\omega^4 = du^4$, $\omega^\mu = (u^4)^{2/3} du^\mu$. Then $\{\omega^i\}$ will be an orthonormal basis of 1-forms om M.

$d\omega^4 = 0$ and $d\omega^\mu = (2/3)(u^4)^{-1/3} du^4 \wedge du^\mu = (2/3)(u^4)^{-1}\omega^4 \wedge \omega^\mu$.

Now let us define $\{\omega^i_j\}$ values by $\omega^4_4 = \omega^\mu_\nu = 0$, $\omega^4_\mu = \omega^\mu_4 = (2/3)(u^4)^{-1}\omega^\mu$. Then $\{\omega^i_j\}$ is the set of connection forms for $\{\omega^i\}$ since $\{\omega^i_j\}$ obey equations (2.5) and (2.6). We have $d\omega^4_4 = 0 = d\omega^\mu_\nu$, and

$$d\omega^4_\mu = -(2/3)(u^4)^{-2} du^4 \wedge \omega^\mu + (2/3)(u^4)^{-1} d\omega^\mu$$

$$= -(2/9)(u^4)^{-2}\omega^4 \wedge \omega^\mu = d\omega^\mu_4.$$

Now, for computing curvature forms, according to equation (2.3) $\Omega^4_4 = 0$, $\Omega^4_\mu = -(4/9)(u^4)^{-2}\omega^4 \wedge \omega^\mu = \omega^\mu_4$, and $\Omega^\mu_\nu = (8/9)\omega^\mu \wedge \omega^\nu = -\Omega^\nu_\mu$. Consequently, the curvature tensor can be calculated according to equation 1.0.3e as

$$R = (4/9)(u^4)^{-2}\left\{\sum_{\rho.\sigma=1}^{3} 2X_\rho \otimes \omega^\sigma \otimes (\omega^\rho \wedge \omega^\sigma) - \sum_{\rho=1}^{3}(X_\rho \otimes \omega^4 + X_4 \otimes \omega^\rho) \otimes (\omega^4 \wedge \omega^\rho)\right\},$$

where $\{X_i\}$ is the vector field basis dual to $\{\omega^i\}$.

Ric can be calculated by contraction. For example, the relevant contraction of $X_4 \otimes \omega^\mu \otimes (\omega^4 \wedge \omega^\mu) = (1/2)X_4 \otimes \omega^\mu \otimes (\omega^4 \otimes \omega^\mu - \omega^\mu \otimes \omega^4)$ is $(1/2)\{\omega^4(X_4)\omega^\mu \otimes \omega^\mu - \omega^\mu(X_4)\omega^\mu \otimes \omega^4\} = (1/2)\omega^\mu \otimes \omega^\mu$.

Algebra computes $Ric = (2/3)(u^4)^{-2}\sum_{k=1}^{4}\omega^k \otimes \omega^k$. Hence the physically equivalent (1,1) tensor field to Ric is

$$(2/3)(u^4)^{-2}\left\{-X_4 \otimes \omega^4 + \sum_{\rho=1}^{3} X_\rho \otimes \omega^\rho\right\}.$$

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Thus, the scalar curvature by contraction is $S = (4/3)(u^4)^{-2}$. Henceforth, algebra yields $G = (4/3)(u^4)^{-2} du^4 \otimes du^4$.
Next, we will examine a pivotal corollary and finally prove the incompleteness of Enistein-de Sitter spacetime using the above computed results and few results of Riemannian manifolds.

Corollary 4:

Let Z be a future-pointing unit timelike vector field on Einstein-de Sitter spacetime which is an eigenvector of G in the following sense: $G(Z,.) = fg(Z,.)$ for some function f. Then $Z = \partial_4$.
Proof: Suppose $G(Z,\cdot) = fg(Z,\cdot)$. At every point, $du^4(Z)du^4 = ag(Z,\cdot)$ for some $a \in \mathbb{R}$. According to theorem 3, $du^4(Z)$ is not equal to zero because both Z and du^4 are timelike. Thus $du^4 = bg(Z,\cdot)$ for some $b \neq 0$ and $b \in \mathbb{R}$.
$\Rightarrow\ Z = e\partial_4$ for some $e \in \mathbb{R}$.
Since Z is future pointing and unit, $Z = \partial_4$.
Thus, $Z = \partial_4$ is primordially delineated in the sense that it can be determined exclusively based on the time orientation and g without alluding to frameworks that $\mathbb{R}^4$ has but M does not. The ideas and results regarding comoving reference frames can be formulated, studied and analysed with the help of this concept.

4. Proving the incompleteness of Einstein-de Sitter Spacetime

With the acquaintance of all the lemmas, results and theorems we discussed above, we are now prepared to solve and discuss the incompleteness of Einstein-de Sitter spacetime using two important theorems of semi-Riemannian manifolds and spacetimes. In the following section, the notations used are same as in corollary 4 and previous sections of this paper.

Theorem 5:

Let (A,g) and $(B,\tilde{g})$ be two semi-Riemannian or manifolds. Then $\phi : A \to B$ is a local isometry if and only if each point $x \in A$ has an open neighborhood $U \subseteq A$ such that $(\phi(U), \tilde{g}|_{\phi(U)})$ is isometric to $(U, g|_U)$ under $\phi|_U$. Moreover, if $\gamma : \lambda \to A$ is a geodesic of (A,g), then $\tilde{\gamma} = \phi \circ \gamma$ will be a geodesic of $(B,\tilde{g})$.
Proof: Considering the complexity and the requirement of additional advanced mathematical tools, we are excluding the detailed proof of the theorem in this paper. Details regarding proof can be referenced from [14] and [11].

Theorem 6:

For semi-Riemannian or Riemannian manifolds, the relation "is isometric to" is an equivalence relation.
Proof:
To show that "is isomorphic to" is an equivalence relation, we have to show that the relation is reflexive, symmetric and transitive. Throughout this proof, we will be using the following definition and notation: An isometry between two semi-Riemannian manifolds (M,g) and (N,h) is a diffeomorphism $f : M \to N$ such that $f^*h = g$. Here, f^*h is the pullback of h by f, defined by $f^*h(X,Y) = h(f_*X, f_*Y)$ for all tangent vectors X, Y on M, where f_* is the differential of f.
To show reflexivity, we need to prove that every semi-Riemannian manifold (M,g) is isometric to itself. Consider the identity map $id : M \to M$. This map is obviously a diffeomorphism. We need to show that it preserves the metric, i.e., $id^*g = g$.
For any tangent vectors X, Y at a point $p \in M$, $id^*g(X,Y) = g(id_*X, id_*Y)$. Since $id_*X = X$ and $id_*Y = Y$, we have $g(id_*X, id_*Y) = g(X,Y)$. Thus $id^*g = g$ which shows that the identity map is an isometry. Therefore (M,g) is isometric to itself, proving the relation is reflexive.
To prove symmetry, we need to prove that if (M,g) is isometric to (N,h) then (N,h) is isometric to (M,g). Suppose there exists an isometry $f : M \to N$. This means f is a diffeomorphism and $f^*h = g$.

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We need to show that the inverse map $f^{-1} : N \to M$ is also an isometry.
First, f^{-1} is a diffeomorphism because f is a diffeomorphism. Now, consider the pullback of g by f^{-1}. For any tangent vectors X, Y at a point $q \in N : (f^{-1})^* g(X,Y) = g((f^{-1})_* X, (f^{-1})_* Y)$.
Let $p = f^{-1}(q)$. Since f is an isometry, for tangent vectors $\tilde{X} = (f^{-1})_* X$ and $\tilde{Y} = (f^{-1})_* Y$ at p, we have $g(\tilde{X}, \tilde{Y}) = h(f_* \tilde{X}, f_* \tilde{Y})$. But $f_* \tilde{X} = X$ and $f_* \tilde{Y} = Y$, so $g((f^{-1})_* X, (f^{-1})_* Y) = h(X,Y)$. Thus $f_*^{-1} g = h$, which shows that f^{-1} is an isometry. Therefore, (N,h) is isometric to (M,g), proving symmetry.
To show transitivity, we need to prove that if (M,g) is isometric to (N,h) and (N,h) is isometric to (P,k), then (M,g) is isometric to (P,k).
Suppose there exist isometries $f : M \to N$ and $g : N \to P$. This means f and g are diffeomorphisms, and $f^* h = g$ and $g^* k = h$. We need to show that the composition $g \circ f : M \to P$ is an isometry.
First, $g \circ f$ is a composition of diffeomorphisms, so it is a diffeomorphism. Now, consider the pullback of k by $g \circ f$, $(g \circ f)^* k = f^*(g^* k)$. Since $g^* k = h$, we have $f^*(g^* k) = f^* h$. And since $f^* h = g$, we have $(g \circ f)^* k = g$. Thus, $g \circ f$ is an isometry. Therefore, (M,g) is isometric to (P,k), proving transitivity.
Since the relation is reflexive, symmetric and transitive, it is an equivalence relation on the set of semi-Riemannian and Riemannian manifolds.

Theorem 7:

Let (M,g) be the Einstein-de Sitter spacetime. Let $(\tilde{M}, \tilde{g})$ be a spacetime and $\phi : M \to \tilde{M}$ an isometry of (M,g) onto $(\phi M, \tilde{g}|_{\phi M})$. Then $(\tilde{M}, \tilde{g})$ is incomplete.
Proof:
By the equations from eq (2.1) to eq (2.2) in the section 2 and derivation of Einstein tensor of the Einstein-de Sitter spacetime,

$$D_z Z = \sum_{i=1}^{4} Z(\omega^i Z) X_i + \sum_{i,j=1}^{4} \omega^i_j(Z) \omega^j(Z) X_i = (Z1) X_4 + 0 = 0.$$

Hence each integral curve of Z is a geodesic. Now, consider an integral curve of Z, $\gamma : (0, \infty) \to M$ defined by $\gamma u = (0,0,0,u) \in M$. The scalar curvature obeys $lim_{u \to 0} S(\gamma u) = \infty$.
If $(\tilde{M}, \tilde{g})$ were complete, then by theorem 5, $\phi \circ \gamma : (0, \infty) \to \tilde{M}$ could be extended to a geodesic $\psi : (-\infty, \infty) \to \tilde{M}$. Then if we analyse the scalar curvature of $(\tilde{M}, \tilde{g})$ denoted by $\tilde{S}$,

$$lim_{u \to 0^+} \tilde{S}(\psi u) = lim_{u \to 0^+} (S \circ \phi^{-1}) \psi u = lim_{u \to 0} S(\gamma u) = \infty.$$

This is a clear contradiction. Hence, it proves the incompleteness of $(\tilde{M}, \tilde{g})$ and establishes the theorem.

Now, by theorem 6, we have proved that the isometry relation is an equivalence relation for semi-Riemannian manifolds or spacetimes and hence is a reflexive relation. Hence, if (M,g) is the Einstein-de Sitter spacetime, then (M,g) is isometric to (M,g) itself and there exists an isometry $\phi : M \to M$ of (M,g) onto $(\phi M, g|_{\phi M})$ as defined in theorem 7. In other words, we are replacing $(\tilde{M}, \tilde{g})$ with the Einstein-de Sitter spacetime (M,g) itself in theorem 7. By theorem 7, (M,g) is incomplete and hence we can come to the conclusion that Einstein-de Sitter spacetime is incomplete.

5. Discussions:

In this section, we discuss few properties of Einstein-de Sitter spacetime based on its incompleteness. These properties can be derived or proved using alternative ways, but knowing the completeness or incompleteness of a spacetime gives a direct, easy and straight forward conclusion regarding these properties. Some theorems discussed in this section will be accompanied by proof while some will be stated without detailed proofs since many proofs regarding semi-Riemannian manifolds and spacetimes are complicated and lengthy and incorporating each and every proofs and

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minute details are beyond the scope of this paper. For detailed definitions and alternative proofs of the properties being discussed in the section, [14], [11] can be used as reference. It is worth noting certain important definitions and terms before discussig the theorems and related results for the Einstein-de Sitter spacetime.

Definition: A semi-Riemannian symmetric space is a connected semi-Riemannian manifold M such that for each $x \in M$, there is a unique isometry $\zeta_x : M \to M$ with identity differential map $-id$ on the tangent space of M at $x \in M$, denoted M_x as usual. The isometry ζ_p is termed the golbal symmetry of M at x.
Definition: Let (A, p) and (B, q) be two spacetimes. (A, p) is contained in (B, q) if and only if A is an open submanifold of B, $q|_A = p$, and (A, p) has the time orientation and induced orientation. (A, p) is defined as inextendible if and only if each spacetime that contains (A, p) is (A, p). The term maximal is also used as an alternative for inextendible. From an intuitive view point or a physical perspective, the theorem 2 (of preliminaries section) says that a spacetime is inextendible if and only if one cannot see out of it or into it.
Definition: A semi-Riemannian manifold M is called Misner complete if there does not exist any geodesic that races to infinity, that is, given every geodesic $\lambda : [0, a) \to M$, $a < \infty$, lies in a compact set.

Theorem 8: A semi-Riemannian symmetric space is complete.
Proof: Let $x \in M$ be an arbitrary point and let $v \in M_x$ be a tangent vector at x. Consider the geodesic γ with initial conditions $\gamma(0) = x$ and $\gamma'(0) = v$. Due to the symmetric property of M, there exists an isometry ζ_x such that $\zeta_x(x) = x$ and $d\zeta_x|_x = -id$.
The isometry ζ_x implies that if $\gamma(t)$ is a geodesic starting at x, then $\zeta_x(\gamma(t))$ is also a geodesic. Specifically, the geodesic symmetry ensures that for any point q on the geodesic γ, the segment can be extended by reflecting it about q. This reflection effectively extends the geodesic beyond any finite segment.
To show geodesic completeness, we need to establish that any geodesic γ can be extended to arbitrary values of its affine parameter t. Given any finite segment of the geodesic, the local geodesic symmetry around any point on this segment allows us to extend the geodesic beyond this segment. By repeatedly applying this extension process, we can extend the geodesic to cover any desired range of the affine parameter.
A semi-Riemannian manifold is geodesically complete if every geodesic can be extended to arbitrary values of its affine parameter. The symmetric properties of (M, g) guarantee that every geodesic can be indefinitely extended using the isometries ζ_x. Hence (M, g) is geodesically complete. Hence we have proved the theorem.
Corollary 9: An incomplete spacetime or a semi-Riemannian manifold can never be a semi-Riemannian symmetric space.
Result 1: In the context of Einstein-de Sitter spacetime, it is topologically incomplete. Hence, by corollary 1, we can easily establish the fact that it is not a semi-Riemannian symmetric space.
Proving this result using the fundamental definition of semi-Riemannian symmetric space is very complicated and tedious. But with the help of corollary 1 and the fact that Einstein-de Sitter spacetime is incomplete, we can easily establish the result that the spacetime is not a semi-Riemannian symmetric space.

Theorem 10: A spacetime (M, g, D) is in the trivial gravitational field if and only if (M, g) is complete.
Result 2: By theorem 2, it can be easily concluded that the Einstein-de Sitter gravitational field is not the trivial one.
In fact, a more strong theorem which provides a relation between flatness and triviality states that a spacetime (M, g, D) is in the trivial gravitational field if and only if (M, g) is flat. Also, we know that

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the Einstein-de Sitter spacetime is not ricci flat and hence not flat. Hence by this theorem also, we can deduce the fact that the Einstein-de Sitter gravitational field is not the trivial one.

Theorem 11: A spacetime is complete if and only if it is its universal covering [3].
Result 3: Einstein-de Sitter spacetime is incomplete and hence it is not its universal covering.

Theorem 12: On an incomplete spacetime, Killing vector fields may or may not be incomplete [11].
Result 4: An incomplete spacetime admits both complete as well as incomplete Killing vector fields. For example, if we consider the unit disk in 2-dimensional Euclidean space $\mathbb{R}^2$, the infinitesimal rotation $-v\partial_u + u\partial_v$ is complete where as infinitesimal translations are incomplete. Similarly, on Einstein-de Sitter spacetime, Killing vector fields may or may not be incomplete.

Now we discuss certain results and properties of the Einstein-de Sitter spacetime that cannot be directly deduced or concluded from the incompleteness of the spacetime, but certain intimations and allusions can be obtained from the incompleteness.
Theorem 13: A complete spacetime is inextendible.
Discussion 1: If we had a complete spacetime, we could have immediately deduced the result that the spacetime is maximal or inextendible. For example, since the Minkowski spacetime is complete [14], we can easily come to the conclusion that the spacetime is also inextendible. The inexistence of such a constraint for incomplete spacetimes acts as an hindrance for commenting whether the Einstein-de Sitter spacetime is inextendible or not. But, once we compute the lightlike geodesics, it can be established using Theorem 2 mentioned in the preliminaries section that Einstein-de Sitter spacetime is in fact maximal.

Theorem 14: A complete spacetime is Misner complete.
Discussion 2: Just like the case of inextendiblity, a complete spacetime would have guaranteed the existence of Misner completeness, but the incomplete Einstein-de Sitter spacetime is not capable of providing a solid conclusion on its Misner completeness. Commenting on the Misner completeness of Einstein-de Sitter spacetime or in general any incomplete spacetime is very challenging due the mathematical complexity involved and currently no results exist which can give a direct answer regarding whether the incomplete spacetime is Misner complete or not, other than investigating the spacetime for Misner completeness from its fundamental definition.
In fact, theorems 4 and 5 can be contracted into a single relationship, complete spacetime $\Rightarrow$ Misner complete spacetime $\Rightarrow$ inextendible spacetime.

6. conclusion

Einstein-de Sitter spacetime is mathematically a special case of a simple cosmological spacetime. We have computed the Einstein tensor G of Einstein-de Sitter spacetime as $G = (4/3)(u^4)^{-2}du^4 \otimes du^4$. The incompleteness of Einstein de-Sitter spacetime is proved using the property of isometry to be an equivalence relation for semi-Riemannian manifolds along with few other theoremes. Using the result that the Eintein-de Sitter spacetime is incomplete, we prove certain other properties of the spacetime which include, Einstein de-Sitter spacetime is not a semi-Riemannian symmetric space, it is not its own universal covering and Killing vector fields may or may not be incomplete on Einstein-de Sitter spacetime. We also deduced the fact that Einstein-de Sitter spacetime is not the trivial one just like Schwarzschild spacetimes. Apart from this, we have also discussed certain geometric properties of spacetime which can't be concluded from its incompleteness. The maximality or inextendiblity of a spacetime is not identifiable from its incompleteness, hence the maximality of the Einstein-de Sitter spacetime can't be concluded from just the fact that it is incomplete. But Einstein-de Sitter spacetime is maximal, which can be proved following the computation of lightlike geodesics just like Minkowski spacetime. Just like the case of maximality, Misner completeness is also a property which is not identifiable from its incompleteness. But unlike the case of maximality, the unavailability of explicit results, proofs and theorems which can prove whether the Einstein-de Sitter spacetime is Misner complete or not makes computations difficult and no proper conclusion can be made on this property. Hence it remains as a scope for further research works in determining whether topologically incomplete spacetimes which hold importance in physics including Einstein de-Sitter spacetime are Misner complete or not, which might unveil lot more interesting properties of Einstein-de Sitter spacetime. Also, exact details regarding compactness of the spacetime is not discussed in the paper due to absence of an alternative in the context of semi-Riemannian manifolds for stronger theorems like Hopf-Rinow theorem, corollary of whose states that compact Riemannian manifolds are complete, or indirectly, incomplete Riemannian manifolds are non compact. Once a detailed investigation has been made on the compactness of Einstein-de Sitter spacetime, it might be possible to comment on whether the spacetime is simply connected or not as compact spacetimes cannot be simply connected which can be proved using Poincare duality, which can be key in deducing many more interesting properties of spacetime. According to comments, theorems and results we have made and proved in this paper, it can be understood that undisturbed particles can enter Einstein-de Sitter spacetime.

References

1. Beem, J. K. (1976). Some examples of incomplete space-times. General Relativity and Gravitation, 7:501–509.
2. Bengtsson, I. and Holst, S. (1999). de sitter space and spatial topology. Classical and Quantum Gravity, 16(11):3735.
3. Candela, A. M., Sanchez, M., et al. (2008). Geodesics in semi-riemannian manifolds: geometric ´ properties and variational tools. Recent developments in pseudo-Riemannian geometry, 4:359.
4. Coudray, A. and Nicolas, J.-P. (2021). Geometry of vaidya spacetimes. General Relativity and Gravitation, 53:1–23
5. Figueroa-O'Farrill, J. (2022). Non-lorentzian spacetimes. Differential Geometry and its Applications, 82:101894.
6. Ford, L. (1985). Quantum instability of de sitter spacetime. Physical Review D, 31(4):710.
7. Joshi, P. S. and Malafarina, D. (2011). Recent developments in gravitational collapse and spacetime singularities. International Journal of Modern Physics D, 20(14):2641–2729.
8. Kim, Y., Oh, C. Y., and Park, N. (2002). Classical geometry of de sitter spacetime: An introductory review. arXiv preprint hep-th/0212326.
9. McClure, M. and Dyer, C. (2006). Asymptotically einstein–de sitter cosmological black holes and the problem of energy conditions. Classical and Quantum Gravity, 23(6):1971.
10. Naber, G. L. (2012). The geometry of Minkowski spacetime. Springer.
11. O'neill, B. (1983). Semi-Riemannian geometry with applications to relativity. Academic press.
12. O'Raifeartaigh, C., O'Keeffe, M., and Mitton, S. (2021). Historical and philosophical reflections on the einstein-de sitter model. The European Physical Journal H, 46:1–10.
13. Rubio, R. M. (2014). Complete constant mean curvature spacelike hypersurfaces in the einstein-de sitter spacetime. Reports on Mathematical Physics, 74(1):127–133.
14. Sachs, R. K. and Wu, H.-H. (2012). General relativity for mathematicians, volume 48. Springer Science & Business Media.
15. Sultana, J. and Dyer, C. C. (2005). Cosmological black holes: A black hole in the einstein-de sitter universe. General Relativity and Gravitation, 37:1347–1370.
16. Uggla, C. (2013). Spacetime singularities: recent developments. International Journal of Modern Physics D, 22(03):1330002.
17. Vishwakarma, R. (2005). Einstein-de sitter model re-examined for newly discovered type ia supernovae. Monthly Notices of the Royal Astronomical Society, 361(4):1382–1386.
18. Witten, E. (2001). Quantum gravity in de sitter space. arXiv preprint hep-th/0106109.

doi:10.20944/preprints202407.2043.v1