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On Čech-Completeness of the Space of τ-Smooth Idempotent Probability Measures

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Abstract: For the set of probability measures $I(X)$, where X is a compact Hausdorff space, we propose a new way to introduce the topology by using open subsets of the space X. Then, among other things, we give a new proof that for a compact Hausdorff space X the space $I(X)$ is also a compact Hausdorff space. For a Tychonoff space X, we consider the topological space $I_{\tau}(X)$ of τ-smooth idempotent probability measures on X,and show that the space $I_{\tau}(X)$ is Čech-complete if and only if the given space X is Čech-complete.

Keywords: Čech-complete space; compact space; probability measure; τ-smooth idempotent probability measure; neighbourhood system

MSC: 28C15; 28A33; 46E27; 60B05; 54D99

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1. Introduction

Idempotent mathematics is a branch of mathematical sciences, rapidly developing and gaining popularity over the last four decades. An important stage of development of the subject was presented in the book "Idempotency" [1] edited by J. Gunawardena (see also [2,3]). This book arose out of the well-known international workshop that was held in Bristol, England, in October 1994. Idempotent mathematics is based on replacing the usual arithmetic operations with a new set of basic operations, i. e., on replacing numerical fields by idempotent semirings and semifields. Typical example is the so-called max-plus algebra (in fact, an idempotent semifield) $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with operation $x \oplus y = \max\{x, y\}$ and $x \odot y = x + y$ ([1,4,14,18]).

M. Zarichnyi in 2010 in [7] investigated categorical properties of the space of idempotent probability measures. In [8] the theory was extended to the class of metric spaces. So, the space of idempotent probability measures is a new object. But it is already studied from different points of view in Measure Theory, Functional Analysis, Probability Theory, Topology and Category Theory. The study of spaces of idempotent probability measures leads to the problem its investigations on wider classes of topological classes than the class of compact Hausdorff spaces, in particular, the class of Tychonoff spaces.

T. Banakh [9], T. Banakh and T. Radul [10,11], carried out a systematic study on probability measures on Tychonoff spaces. In their studies, they fruitfully used the linearity of probability measures. Unlike probability measures, idempotent probability measures are not linear. In papers [12–14] the theory put forward, and in [15] some categorical properties of τ-smooth weakly additive (nonlinear) functionals were established.

The results obtained in [4,7,8,14,16,18] show that in order to establish "good" properties of the space of idempotent probability measures, methods are required that are very different from classical methods (i. e., from methods suitable for probability measures which have been productively used in [17,18] and others).

Unlike above mentioned papers [4,7,8,14,16,18], in this paper for a compact Hausdorff space X we introduce the notion of idempotent measures as a set-function on the family $\mathfrak{B}(X)$. Note that the

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works [19–21] and [22] also have this approach. Improving their theory in the current paper we note some types of open and closed subsets of the space of idempotent probability measures. In the set of idempotent probability measures, we introduce the base of the product topology and show that for a compact Hausdorff space X the topological space $\mathsf{I}(X)$ of idempotent probability measures is also a compact Hausdorff space.

Further, for a Tychonoff space X, we consider the space $I_\tau(X)$ of τ-smooth idempotent probability measures on X. Then we establish the Čech-completeness of the space of τ-smooth idempotent probability measures for the Čech-complete Tychonoff space. (Čech-complete spaces were introduced by Eduard Čech in 1937 to prove the Baire category theorem. Another important application of the Čech-completeness appears in the metrization of a topological space by a complete metric. Note that a topological space X is Čech-complete if X is a Tychonoff space and the remainder $\beta X \setminus X$ is an F_σ-set in the Stone-Čech compactification βX. Locally compact spaces are Čech-complete, but the inverse is not take place. The space of all irrational numbers with the topology of a subspace of the real line is an example of a Čech-complete space that is not locally compact [23].) From here, since Čech-completeness is hereditary with respect to G_δ-subsets and a Čech-complete Tychonoff space X is G_δ-subset in its Stone-Čech compactification βX, we conclude that a Tychonoff space X is Čech-complete if and only if $I_\tau(X)$ is Čech-complete. Note that a linear (in the classical sense) version of this result was established in [9]. One can see that the methods used in [9,10,12–14,17,18] are not suitable for the present case.

2. Preliminaries

Let X be a compact Hausdorff space and $\mathfrak{B}(X)$ the family of Borel subsets of X. We denote $\overline{\mathbb{R}}_+ = [0, +\infty) \cup \{+\infty\} = [0, +\infty]$. The symbol D denotes the directed set. Following [19], we enter the following notion.

Definition 1. A set function $\mu\colon \mathfrak{B}(X) \to \overline{\mathbb{R}}_+$ is said to be an *idempotent measure* on X if the following conditions hold:

1) $\mu(\varnothing) = 0$;
2) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$ for any $A, B \in \mathfrak{B}(X)$;
3) $\mu\left(\bigcup\limits_{\alpha\in D} A_\alpha\right) = \sup\limits_{\alpha\in D}\{\mu(A_\alpha)\}$ for every increasing net $\{A_\alpha, \alpha \in D\} \subset \mathfrak{B}(X)$ such that $\bigcup\limits_{\alpha\in D} A_\alpha \in \mathfrak{B}(X)$.

Remark 1. Every idempotent measure μ is increasing, i. e. for $A, B \in \mathfrak{B}(X)$ if $A \subset B$, then $\mu(A) \leq \mu(B)$.

The set of all idempotent measure on X we denote by $IM(X)$. If $\mu(X) = 1$, the idempotent measure μ is called an *idempotent probability measure* on X. We denote

$$\mathsf{I}(X) = \{\mu \in IM(X)\colon\ \mu(X) = 1\}.$$

A set

$$\text{supp } \mu = [X \setminus \cup\{A \in \mathfrak{B}(X) : \mu(A) = 0\}]_X \tag{1}$$

is said to be a *support* of the given idempotent measure μ.

The support of an idempotent measure μ can be defined by the following equality:

$$\text{supp } \mu = [\cap\{C\colon\ C \in \mathfrak{B}(X),\ \mu(X \setminus C) = 0\}]_X. \tag{2}$$

Lemma 1. *For every $\mu \in \mathsf{I}(X)$ we have $\mu \in I(\text{supp } \mu)$.*

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Proof. For a support we will apply (2). It is easy to see that

$$\begin{aligned}\mu(X \setminus \text{supp } \mu) &= \mu(X \setminus [\cap\{C\colon C \in \mathfrak{B}(X), \mu(X \setminus C) = 0\}]_X) \\ &= \mu(\text{int}(X \setminus \cap\{C\colon C \in \mathfrak{B}(X), \mu(X \setminus C) = 0\})) \\ &= \mu(\text{int}(\cup\{X \setminus C\colon C \in \mathfrak{B}(X), \mu(X \setminus C) = 0\})) \\ &\le \mu(\cup\{X \setminus C\colon C \in \mathfrak{B}(X), \mu(X \setminus C) = 0\}) \\ &= \sup\{\mu(X \setminus C) : C \in \mathfrak{B}(X), \mu(X \setminus C) = 0\} = 0.\end{aligned}$$

On the other side,

$$1 = \mu(X) = \mu((X \setminus \text{supp } \mu) \cup \text{supp } \mu) = \max\{\mu(X \setminus \text{supp } \mu), \mu(\text{supp } \mu)\},$$

i. e. $\mu(\text{supp } \mu) = 1$ and $\mu \in I(\text{supp } \mu)$. □

Let X be a compact Hausdorff space, $\mathcal{B}$ a base in X, $U_i \in \mathcal{B}$, $i = 1, \ldots, n$, and $\varepsilon > 0$. For an idempotent probability measure $\mu \in \mathsf{I}(X)$ we define a set

$$\langle \mu; U_1, \ldots, U_n; \varepsilon\rangle = \{\nu \in \mathsf{I}(X)\colon |\nu(U_i) - \mu(U_i)| < \varepsilon, i = 1, \ldots, n\}.$$

Gathering all of such sets construct a family

$$\mathcal{B}(\mu) = \{\langle \mu; U_1, \ldots, U_n; \varepsilon\rangle\colon U_i \in \mathcal{B}, i = 1, \ldots, n; \varepsilon > 0\}, \quad \mu \in I(X),$$

and put

$$\mathcal{B}_{\mathsf{I}(X)} = \bigcup_{\mu \in \mathsf{I}(X)} \mathcal{B}(\mu).$$

Proposition 1. *The built family $\mathcal{B}_{\mathsf{I}(X)}$ forms a base (or, a neighbourhoods system) for some topology in $\mathsf{I}(X)$.*

Proof. Since $\mathcal{B} \neq \emptyset$, there exists an open nonempty set $U \in \mathcal{B}$. On the other side, $U \in \mathfrak{B}(X)$, so $\mu(U)$ is well defined. That is why for every $\varepsilon > 0$ we have $\langle \mu; U; \varepsilon\rangle \in \mathcal{B}(\mu)$. Consequently, $\mathcal{B}(\mu) \neq \emptyset$. Clearly, $\mu \in \langle \mu; U_1, \ldots, U_n; \varepsilon\rangle$ for each $\langle \mu; U_1, \ldots, U_n; \varepsilon\rangle \in \mathcal{B}(\mu)$.

Suppose we get $\mu \in \langle \nu; V_1, \ldots, V_n; \delta\rangle \in \mathcal{B}(\nu)$. Note that $(\mu - \nu)(V) = \mu(V) - \nu(V)$, and designate

$$a = \min\{(\nu - \mu)(V_1) + \delta, (\mu - \nu)(V_1) + \delta, \ldots, (\nu - \mu)(V_n) + \delta, (\mu - \nu)(V_n) + \delta\}.$$

Obviously, $a > 0$. Then for every $\xi \in \langle \mu; V_1, \ldots, V_n; a\rangle$ one has

$$\begin{aligned}|\xi(V_i) - \nu(V_i)| &= |\xi(V_i) - \mu(V_i) + \mu(V_i) - \nu(V_i)| \le \\ &\le |\xi(V_i) - \mu(V_i)| + |\mu(V_i) - \nu(V_i)| < \\ &< a + |\mu(V_i) - \nu(V_i)|.\end{aligned}$$

Two cases are possible: $\mu(V_i) - \nu(V_i) \ge 0$ or $\mu(V_i) - \nu(V_i) \le 0$. In the front case we take into attention $a < \nu(V_i) - \mu(V_i) + \delta$, and the last case: $a < \mu(V_i) - \nu(V_i) + \delta$. Hence, $\xi \in \langle \nu; V_1, \ldots, V_n; \delta\rangle$, in the other words, $\langle \mu; V_1, \ldots, V_n; a\rangle \subset \langle \nu; V_1, \ldots, V_n; \delta\rangle$.

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Finally, consider an arbitrary couple of sets $\langle \mu;\, U_1,\, \ldots,\, U_n;\, \varepsilon\rangle$, $\langle \mu;\, V_1,\, \ldots,\, V_k;\, \delta\rangle$ belonging $\mathcal{B}(\mu)$. Denote $\theta = \min\{\varepsilon,\, \delta\}$, and we have

$$\begin{aligned} &\langle \mu;\, U_1,\, \ldots,\, U_n;\, \varepsilon\rangle \cap \langle \mu;\, V_1,\, \ldots,\, V_k;\, \delta\rangle \supset \\ \supset&\langle \mu;\, U_1,\, \ldots,\, U_n;\, \theta\rangle \cap \langle \mu;\, V_1,\, \ldots,\, V_k;\, \theta\rangle = \\ =&\langle \mu;\, U_1,\, \ldots,\, U_n,\, V_1,\, \ldots,\, V_k;\, \theta\rangle \in \mathcal{B}(\mu). \end{aligned}$$

Proposition 1 is proved. □

Remark 2. For an open set U sets

$$\langle U;\, \varepsilon\rangle_- = \{\xi \in \mathsf{I}(X) \colon\, \xi(U) < \varepsilon\} \text{ and } \langle U;\, \varepsilon\rangle_+ = \{\xi \in I(X) \colon\, \xi(U) > \varepsilon\}$$

are open. Indeed, we have $\mu(U) = a < \varepsilon$ for every $\mu \in \langle U;\, \varepsilon\rangle_-$. Then for each $\nu \in \langle \mu;\, U;\, \varepsilon - a\rangle_-$ one has $-(\varepsilon - a) + \mu(U) < \nu(U) < \mu(U) + (\varepsilon - a)$, $2a - \varepsilon < \nu(U) < \varepsilon$. Hence, $\nu \in \langle U;\, \varepsilon\rangle_-$. The openness of the second set will be established like to the openness of the first set.

Remark 3. For a closed set F sets

$$[F;\, \varepsilon]_- = \{\xi \in \mathsf{I}(X) \colon\, \xi(F) \le \varepsilon\} \text{ and } [F;\, \varepsilon]_+ = \{\xi \in I(X) \colon\, \xi(F) \ge \varepsilon\}$$

are closed. Indeed, for each $\{\xi_\alpha\} \subset [F;\, \varepsilon]_- \subset \mathsf{I}(X)$ there exists $\xi_0 \in \mathsf{I}(X)$ such that $\xi_0 = \lim\limits_{\alpha} \xi_\alpha$ (i. e, $\xi(A) = \lim\limits_{\alpha} \xi_\alpha(A)$ for each $A \in \mathfrak{B}(X)$). Hence, $\xi_0(F) \le \varepsilon$. Thus, $\xi_0 \in [F;\, \varepsilon]_-$. In the same way one can show that $[F;\, \varepsilon]_+$ is also closed.

3. Idempotent probability measures on compact Hausdorff spaces

The construction of $I(X)$ gives that $\mathsf{I}(X) \subset \mathbb{R}^{\mathfrak{B}(X)}$. Equip the set $\mathsf{I}(X)$ with the topology generated by the above neighbourhoods system. Obviously, this topology coincides with the induced topology from the product topology of $\mathbb{R}^{\mathfrak{B}(X)}$ to $\mathsf{I}(X)$.

Theorem 1. *For a compact Hausdorff space X the topological space $\mathsf{I}(X)$ is also a compact Hausdorff space.*

Proof. At first we show that $\mathsf{I}|(X)$ is a Hausdorff space. Let $\mu,\, \nu \in \mathsf{I}(X)$ be different idempotent measures. Then there exists a set $U \in \mathcal{B}$ such that $\mu(U) \ne \nu(U)$. Put $|\mu(U) - \nu(U)| = a$. One has $a \ne 0$, $|\mu(U) - \nu(U)| > \frac{a}{2}$. Assume that there exists an idempotent probability measure ξ belonging to both sets $\langle \mu;\, U;\, \frac{a}{4}\rangle$ and $\langle \nu;\, U;\, \frac{a}{4}\rangle$. Then

$$\begin{aligned} |\nu(U) - \mu(U)| &= |\nu(U) - \xi(U) + \xi(U) - \mu(U)| \le \\ &\le |\nu(U) - \xi(U)| + |\xi(U) - \mu(U)| < \tfrac{a}{4} + \tfrac{a}{4} = \tfrac{a}{2}. \end{aligned}$$

The obtained contradiction shows that $\langle \mu;\, U;\, \frac{a}{4}\rangle \cap \langle \nu;\, U;\, \frac{a}{4}\rangle = \varnothing$.

Now we will prove the compactness of $I(X)$ with respect to the topology generated by the neighbourhoods system $\mathcal{B}_{\mathsf{I}(X)}$.

Clearly, $\mathsf{I}(X)$ is bounded in $\mathbb{R}^{\mathfrak{B}(X)}$. It remains to show its closedness. Let $\mu \in \mathbb{R}^{\mathfrak{B}(X)} \setminus I(X)$. Then the following cases are possible.

Case 1. $\mu(\varnothing) = a \ne 0$. In this case we have $\langle \mu;\, \varnothing;\, \frac{|a|}{2}\rangle \cap \mathsf{I}(X) = \varnothing$.

Case 2. $\mu(X) = a \ne 1$. Then it os obvious that $\langle \mu;\, X;\, \frac{|a+1|}{2}\rangle \cap \mathsf{I}(X) = \varnothing$.

Case 3. $\mu(G \cup \Gamma) = a \ne b = \max\{\mu(G),\, \mu(\Gamma)\}$ for some open sets G and Γ. Whereupon, $\langle \mu;\, G,\, \Gamma,\, G \cup \Gamma;\, \frac{|b-a|}{2}\rangle \cap I(X) = \varnothing$.

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Case 4. $\mu\left(\bigcup\limits_{\alpha\in D}\Gamma_\alpha\right) = a \neq b = \sup\limits_{\alpha\in D}\{\mu(\Gamma_\alpha)\}$ for a net $\{\Gamma_\alpha\}_{\alpha\in D}$ of open sets.

The fourth case has the following subcases.

Subcase 4.1. Assume $0 \leq a < b \leq 1$. By the definition there exist such an index α_0 that $\mu(\Gamma_{\alpha_0}) > b - \frac{b-a}{4} = \frac{a+3b}{4}$. We claim that $\langle \mu; \Gamma_{\alpha_0}, \bigcup\limits_{\alpha\in D}\Gamma_\alpha; \frac{b-a}{4}\rangle \cap \mathsf{I}(X) = \emptyset$. Presume, it is not so. Then there exists ν belonging to this intersection. From here we get

$$\left|\nu\left(\bigcup_{\alpha\in D}\Gamma_\alpha\right) - \mu\left(\bigcup_{\alpha\in D}\Gamma_\alpha\right)\right| < \tfrac{b-a}{4},$$
$$|\nu(\Gamma_{\alpha_0}) - \mu(\Gamma_{\alpha_0})| < \tfrac{b-a}{4}.$$

These inequalities provide, correspondingly,

$$\tfrac{3a-b}{4} < \nu\left(\bigcup_{\alpha\in D}\Gamma_\alpha\right) < \tfrac{3a+b}{4},$$
$$\tfrac{a+b}{2} < \nu(\Gamma_{\alpha_0}) < \tfrac{5b-a}{4}.$$

In the issue one obtains

$$\sup_{\alpha\in D}\{\nu(\Gamma_\alpha)\} \geq \nu(\Gamma_{\alpha_0}) > \tfrac{a+b}{2} > \tfrac{3a+b}{4} > \nu\left(\bigcup_{\alpha\in D}\Gamma_\alpha\right),$$

i.e. $\sup\limits_{\alpha\in D}\{\nu(\Gamma_\alpha)\} > \nu\left(\bigcup\limits_{\alpha\in D}\Gamma_\alpha\right)$. The last inequality goes against $\nu \in \mathsf{I}(X)$. The resulting contradiction shows that our assumption is wrong, and our statement about the emptiness of the intersection is true.

Subcase 4.2. Assume $0 \leq b < a \leq 1$. Take an any $\xi \in \langle \mu; \bigcup\limits_{\alpha\in D}\Gamma_\alpha; \frac{a-b}{4}\rangle$. Then

$$\sup_{\alpha\in D}\{\xi(\Gamma_\alpha)\} < b + \tfrac{a-b}{4} < a - \tfrac{a-b}{4} = \mu\left(\bigcup_{\alpha\in D}\Gamma_\alpha\right) - \tfrac{a-b}{4} < \xi\left(\bigcup_{\alpha\in D}\Gamma_\alpha\right),$$

i. e. $\sup\limits_{\alpha\in D}\{\xi(\Gamma_\alpha)\} < \xi\left(\bigcup\limits_{\alpha\in D}\Gamma_\alpha\right)$. Hence, $\xi \notin I(X)$. Thus, $\langle \mu; \bigcup\limits_{\alpha\in D}\Gamma_\alpha; \frac{a-b}{4}\rangle \cap \mathsf{I}(X) = \emptyset$.

Subcase 4.3. Assume $b < 0$. Then $\mu(\Gamma_\alpha) \leq b$ for every $\alpha \in D$. For $\frac{|b|}{2}$ there exists an index $\alpha_0 \in D$ such that $\frac{3b}{2} = b - \frac{|b|}{2} < \mu(\Gamma_{\alpha_0}) < b + \frac{|b|}{2} = \frac{b}{2}$. Evidently, $\langle \mu; \Gamma_{\alpha_0}; \frac{|b|}{2}\rangle \cap \mathsf{I}(X) = \emptyset$.

Subcase 4.4. Assume $a < 0$. Then $\langle \mu; \bigcup\limits_{\alpha\in D}\Gamma_\alpha; \frac{|a|}{2}\rangle \cap \mathsf{I}(X) = \emptyset$.

Subcase 4.5. Assume $b > 1$. Then there exists an index $\alpha_0 \in D$ such that $\frac{b+1}{2} = b - \frac{b-1}{2} < \mu(\Gamma_{\alpha_0}) < b + \frac{b-1}{2} = \frac{3b-1}{2}$. Hence, $\langle \mu; \Gamma_{\alpha_0}; \frac{b-1}{2}\rangle \cap \mathsf{I}|(X) = \emptyset$.

Subcase 4.6. Assume $a > 1$. Then $\langle \mu; \bigcup\limits_{\alpha\in D}\Gamma_\alpha; \frac{a-1}{2}\rangle \cap \mathsf{I}(X) = \emptyset$.

On this, all possible cases have been exhausted. Consequently, μ is an interior point of $\mathbb{R}^{\mathfrak{B}(X)} \setminus \mathsf{I}(X)$ which immediately implies that $\mathsf{I}(X)$ is closed in $\mathbb{R}^{\mathfrak{B}(X)}$ which means its compactness. The proof of Theorem 1 is completed. □

For a mapping $f\colon X \to Y$ of compact Hausdorff spaces X and Y we define a mapping

$$\mathsf{I}(f)\colon \mathsf{I}(X) \to \mathsf{I}(Y)$$

by the rule

$$\mathsf{I}(f)(\mu)(B) = \mu(f^{\leftarrow}(B)), \qquad B \in \mathfrak{B}(Y).$$

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Proposition 2. *For every pair of compact Hausdorff spaces X and Y and any continuous mapping $f\colon X \to Y$ the mapping $\mathsf{I}(f)\colon \mathsf{I}(X) \to \mathsf{I}(Y)$ is continuous.*

Proof. Take any $\mu \in \mathsf{I}(X)$, and let $\nu = \mathsf{I}(f)(\mu) \in \mathsf{I}(Y)$. Consider an arbitrary neighbourhood $\langle \nu; V_1, \dots, V_k; \delta\rangle$ of ν. Then for any $\xi \in \langle \mu; f^{\leftarrow}(V_1), \dots, f^{\leftarrow}(V_k); \delta\rangle$ we have

$$|\mathsf{I}(f)(\xi)(V_i)) - \nu(V_i))| = |\xi(f^{\leftarrow}(V_i)) - \mu(f^{\leftarrow}(V_i))| < \delta,$$

which yields $\mathsf{I}(f)(\xi) \in \langle \nu; V_1, \dots, V_k; \delta\rangle$. By virtue of arbitrariness of ξ we obtain that $\mathsf{I}(f)(\langle \mu; f^{\leftarrow}(V_1), \dots, f^{\leftarrow}(V_k); \delta\rangle) \subset \langle \nu; V_1, \dots, V_k; \delta\rangle$. □

4. τ-smooth idempotent probability measures

To continue our investigation we need the notion of an outer idempotent measure.
An *outer idempotent measure* of an arbitrary subset $A \subset X$ defines as

$$\mu^*(A) = \inf\{\mu(B) : B \supset A,\, B \in \mathfrak{B}(X)\}.$$

So, we got an extension μ^* of μ. Now Remark 1 may be improved as follows

Remark 4. For every idempotent measure μ its extension μ^* is increasing, i. e. if $A \subset B$, then $\mu^*(A) \leq \mu^*(B)$.

Lemma 2. *For any pair of $A, B \subset X$ and every idempotent measure we have*

$$\mu^*(A \cup B) = \max\{\mu^*(A),\, \mu^*(B)\}.$$

Proof. It is easy to see that $\mu^*(A \cup B) \geq \max\{\mu^*(A), \mu^*(B)\}$. Let us show the inverse inclusion, i. e. $\mu^*(A \cup B) \leq \max\{\mu^*(A), \mu^*(B)\}$. By the definition of the outer idempotent measure one has

$$\mu^*(A \cup B) = \inf\{\mu(E)\colon E \in \mathfrak{B}(X),\, E \supset A \cup B\}.$$

Clearly,

$$\{E\colon E \in \mathfrak{B}(X), E \supset A \cup B\} \supset \{C \cup D\colon C, D \in \mathfrak{B}(X), C \supset A \text{ and } D \supset B\}.$$

Then

$$\begin{aligned}\mu^*(A \cup B) &= \inf\{\mu(E)\colon E \in \mathfrak{B}(X), E \supset A \cup B\} \leq \\ &\leq \inf\{\mu(C \cup D)\colon C, D \in \mathfrak{B}(X), C \supset A \text{ and } D \supset B\} = \\ &= \inf\{\max\{\mu(C), \mu(D)\}\colon C, D \in \mathfrak{B}(X), C \supset A \text{ and } D \supset B\} = \\ &= \max\{\inf\{\mu(C)\colon C \in \mathfrak{B}(X), C \supset A\}, \inf\{\mu(D)\colon D \in \mathfrak{B}(X), D \supset B\}\} = \\ &= \max\{\mu^*(A), \mu^*(B)\}\end{aligned}$$

Thus, $\mu^*(A \cup B) = \max\{\mu^*(A), \mu^*(B)\}$. The proof of Lemma 2 is completed. □

Let X be a Tychonoff space, βX the Stone-Čech compactification of X. We determine the following set:

$$\mathsf{I}_{\emptyset}(X) = \{\mu \in I(\beta X)\colon \mu(F) = 0 \text{ for every } F \in \mathfrak{B}(\beta X), F \subset \beta X \setminus X\}. \tag{3}$$

It is easy to see that $\mu \in I_\tau(X)$ implies $\mu^*(X) = 1$.

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From the definition we have

$$\mathsf{I}_{\emptyset}(\emptyset) = \{\mu \in \mathsf{I}(\beta X) \colon \mu(F) = 0, \forall F \in \mathfrak{B}(\beta X), F \subset \beta X\} =$$
$$= \{\mu \in \mathsf{I}(\beta X) \colon (\mu(\beta X) = 1) \wedge (\mu(\beta X) = 0)\} = \emptyset,$$

i. e.

$$\mathsf{I}_{\emptyset}(\emptyset) = \emptyset. \tag{4}$$

Elements of $\mathsf{I}_{\emptyset}(X)$ are said to be *τ-smooth idempotent probability measures*.

For each $\mu \in \mathsf{I}_{\emptyset}(X)$ we define a set function $\tilde{\mu} \colon \mathfrak{B}(X) \to \mathbb{R}$ on the family $\mathfrak{B}(X)$ of all Borel subsets of X by the formula

$$\tilde{\mu}(A) = \mu^*(A) = \inf\{\mu(B) \colon B \in \mathfrak{B}(\beta X), B \supset A\}, \qquad A \in \mathfrak{B}(X).$$

Lemma 3. *$\tilde{\mu}$ is an idempotent probability measure on X.*

Proof. Evidently, that $\tilde{\mu}(\emptyset) = 0$. Equality 2) in Definition 1 holds because of Lemma 2. We should show that equality 3) in Definition 1 is also true.

Let $\{A_\alpha, \alpha \in D\} \subset \mathfrak{B}(X)$ be an increasing net such that $\bigcup\limits_{\alpha\in D} A_\alpha \in \mathfrak{B}(X)$. For every α there exists $M_\alpha \in \mathfrak{B}(\beta X)$ such that $M_\alpha \cap X = A_\alpha$ and $\bigcup\limits_{\alpha\in D} M_\alpha \in \mathfrak{B}(\beta X)$. One has

$$\Big\{M \in \mathfrak{B}(\beta X) \colon M \supset \bigcup_{\alpha\in D} A_\alpha\Big\} \supset \Big\{\bigcup_{\alpha\in D} M_\alpha \in \mathfrak{B}(\beta X) \colon \mathfrak{B}(\beta X) \ni M_\alpha \supset A_\alpha, \alpha \in D\Big\}.$$

Then

$$\tilde{\mu}\left(\bigcup_{\alpha\in D} A_\alpha\right) = \inf\Big\{\mu(M) \colon M \in \mathfrak{B}(\beta X), M \supset \bigcup_{\alpha\in D} A_\alpha\Big\} \le$$
$$\le \inf\left\{\mu\left(\bigcup_{\alpha\in D} M_\alpha\right) \colon \bigcup_{\alpha\in D} M_\alpha \in \mathfrak{B}(\beta X), \mathfrak{B}(\beta X) \ni M_\alpha \supset A_\alpha, \alpha \in D\right\} =$$
$$= \inf_{\substack{\bigcup\limits_{\alpha\in D} M_\alpha \in \mathfrak{B}(\beta X), \\ M_\alpha \supset A_\alpha, \alpha\in D}} \left\{\mu\left(\bigcup_{\alpha\in D} M_\alpha\right)\right\} = \inf_{\substack{M_\alpha \in \mathfrak{B}(\beta X), \\ M_\alpha \supset A_\alpha, \alpha\in D}} \left\{\sup_{\alpha\in D}\{\mu(M_\alpha)\}\right\} =$$
$$= \sup_{\alpha\in D}\left\{\inf_{\substack{M_\alpha \in \mathfrak{B}(\beta X), \\ M_\alpha \supset A_\alpha}} \{\mu(M_\alpha)\}\right\} = \sup_{\alpha\in D}\{\tilde{\mu}(A_\alpha)\},$$

i. e. $\tilde{\mu}(\bigcup\limits_{\alpha\in D} A_\alpha) \le \sup\limits_{\alpha\in D}\{\tilde{\mu}(A_\alpha)\}$. Remark 4 implies the inverse inequality. The proof of Lemma 3 is finished. □

It is easy to see that the idempotent measure $\tilde{\mu}$ is τ-smooth. Conversely, each τ-smooth probability measure $\tilde{\mu}$ on X defines a measure $\mu \in \mathsf{I}_{\emptyset}(X)$, by means of the formula $\mu(A) = \tilde{\mu}(A \cap X)$, $A \in \mathfrak{B}(\beta X)$.

Lemma 4. *Let X be a Tychonoff space. If $\mu \in \mathsf{I}_{\emptyset}(X)$, then $\mu(A) = \mu(B)$ for any two Borel subsets $A, B \subset \beta X$ such that $A \cap X = B \cap X$.*

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Proof. An arbitrary set $A \subset \beta X$ can be expressed in the form $A = (A \cap X) \cup (A \setminus X)$. We will show that

$$\mu(A) = \mu^*((A \cap X) \cup (A \setminus X)) = \max\{\mu^*(A \cap X), \mu^*(A \setminus X)\} = \mu^*(A \cap X).$$

Outer measures of sets $A \cap X$ and $A \setminus X$ are

$$\mu^*(A \cap X) = \inf\{\mu(A \cap C) \colon C \in \mathfrak{B}(\beta X), C \supset X\}$$

and

$$\mu^*(A \setminus X) = \inf\{\mu(D) \colon D \in \mathfrak{B}(\beta X), D \supset A \setminus X\}.$$

From the relations $D \supset A \setminus X$, $A \supset A \setminus X$ the outer measure of the set $A \setminus X$ can be rewritten as

$$\mu^*(A \setminus X) = \inf\{\mu(A \cap D) : D \in \mathfrak{B}(\beta X), D \supset A \setminus X\}.$$

Since $\mu \in \mathsf{I}_\varnothing(X)$, for any sets $D \in \mathfrak{B}(\beta X)$, $D \supset A \setminus X$ and $C \in \mathfrak{B}(\beta X)$, $C \supset X$, we have $\mu(C) = 1 = \mu(\beta X) \geq \mu(D)$. Then $\mu(A \cap C) \geq \mu(A \cap D)$ for each $A \in \mathfrak{B}(\beta X)$. Since the sets C and D are arbitrary, we have the inequality $\mu^*(A \cap X) \geq \mu^*(A \setminus X)$. So, $\mu(A) = \mu^*(A \cap X)$.

Like this, we can obtain the equality $\mu(B) = \mu^*(B \cap X)$ for a set B as well. By the data $A \cap X = B \cap X$, which implies $\mu(A) = \mu(B)$. □

Let X and Y be Tychonoff spaces, $f \colon X \to Y$ a continuous mapping, and $\beta f \colon \beta X \to \beta Y$ the Stone-Čech compactification of f.

Theorem 2. *For Tychonoff spaces X, Y and a continuous mapping $f \colon X \to Y$ we have $I(\beta f)(\mathsf{I}_\varnothing(X)) \subset \mathsf{I}_\varnothing(Y)$.*

Proof. Take an arbitrary $\mu \in \mathsf{I}(\beta X)$ and suppose $\mathsf{I}(\beta f)(\mu) \notin \mathsf{I}_\tau(Y)$. Then we have $\mathsf{I}(\beta f)(\mu)(\Gamma) = a > 0$ for some $\Gamma \in \mathfrak{B}(\beta Y)$, $\Gamma \cap Y = \varnothing$. From the equality $(\beta f)^{\leftarrow}(\beta Y \setminus Y) = \beta X \setminus X$, we get $(\beta f)^{\leftarrow}(\Gamma) \cap X = \varnothing$, and on the other side, $(\beta f)^{\leftarrow}(\Gamma) \in \mathfrak{B}(\beta X)$. Whereupon, $\mu((\beta f)^{\leftarrow}(\Gamma)) = a > 0$. Hence, $\mu \notin \mathsf{I}_\varnothing(X)$. Thus, the required inclusion, and Theorem 2 are established. □

Just proved Theorem 2 gives us an opportunity to determine the following mapping

$$\mathsf{I}_\varnothing(f) = \mathsf{I}(\beta f)|_{\mathsf{I}_\varnothing(X)} \colon \mathsf{I}_\varnothing(X) \to \mathsf{I}_\varnothing(Y).$$

Let us recall that a mapping $f \colon X \to Y$ between topological spaces is called perfect, if it is closed and the preimage $f^{\leftarrow}(y)$ of every point $y \in Y$ is compact.

Theorem 3. *The operation $\mathsf{I}_\varnothing$ putting the mapping $\mathsf{I}_\varnothing(f) \colon \mathsf{I}_\varnothing(X) \to \mathsf{I}_\varnothing(Y)$ in correspondence with the mapping $f \colon X \to Y$, preserves the class of perfect mappings.*

Proof. Let $f \colon X \to Y$ be a perfect mapping of Tychonoff spaces. Then for the extension $\beta f \colon \beta X \to \beta Y$ of f one has $\beta f(\beta X \setminus X) \subset \beta Y \setminus Y$ [3]. We consider a mapping $I(\beta f) \colon I(\beta X) \to I(\beta Y)$, and prove that the relation $\mathsf{I}(\beta f)(\mathsf{I}(\beta X) \setminus \mathsf{I}_\varnothing(X)) \subset \mathsf{I}(\beta Y) \setminus \mathsf{I}_\varnothing(Y)$ holds as well.

Let us take an arbitrary element $\mu \in \mathsf{I}(\beta Y) \setminus \mathsf{I}_\varnothing(Y)$. We should show that $\mathsf{I}(\beta f)^{\leftarrow}(\mu) \subset \mathsf{I}(\beta X) \setminus \mathsf{I}_\varnothing(X)$. Since $\mu \notin \mathsf{I}_\varnothing(Y)$, we have $\mu(\Gamma) = a > 0$ for some $\Gamma \in \mathfrak{B}(\beta Y)$, $\Gamma \cap Y = \varnothing$. By definition, for every $\nu \in I(\beta f)^{\leftarrow}(\mu)$ we have $I(\beta f)(\nu) = \mu$. From the equality $(\beta f)^{\leftarrow}(\beta Y \setminus Y) = \beta X \setminus X$, we get $(\beta f)^{\leftarrow}(\Gamma) \cap X = \varnothing$, and on the other side, $(\beta f)^{\leftarrow}(\Gamma) \in \mathfrak{B}(\beta X)$. Whereupon, $\mu(\Gamma) = \nu((\beta f)^{\leftarrow}(\Gamma)) = a > 0$. Hence, $\nu \in \mathsf{I}(\beta X) \setminus \mathsf{I}_\varnothing(X)$. By virtue of arbitrariness of $\nu \in I(\beta f)^{\leftarrow}(\mu)$, we conclude that $\mathsf{I}(\beta f)^{\leftarrow}(\mu) \subset \mathsf{I}(\beta X) \setminus \mathsf{I}_\varnothing(X)$.

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Thus, $\mathsf{I}(\beta f)(\mathsf{I}(\beta X) \setminus \mathsf{I}_\varnothing(X)) \subset \mathsf{I}(\beta Y) \setminus \mathsf{I}_\varnothing(Y)$. Since $\mathsf{I}(\beta f)\colon \mathsf{I}(\beta X) \to \mathsf{I}(\beta Y)$ is a mapping between compact Hausdorff spaces, the mapping $\mathsf{I}_\varnothing(f) = \mathsf{I}(\beta f)|_{\mathsf{I}_\varnothing(X)}\colon \mathsf{I}_\varnothing(X) \to \mathsf{I}_\varnothing(Y)$ is perfect. Theorem 3 is proved. □

Theorem 4. *The operation $\mathsf{I}_\varnothing$ putting the mapping $\mathsf{I}_\varnothing(f)\colon \mathsf{I}_\varnothing(X) \to \mathsf{I}_\varnothing(Y)$ in correspondence with the mapping $f\colon X \to Y$, preserves the class of embeddings.*

Proof. Let $f\colon X \to Y$ be an embedding of topological spaces and $\beta f\colon \beta X \to \beta Y$ its Stone-Čech compact extension.

As we reminded above, $\beta f(\beta X \setminus X) \subset \beta Y \setminus f(X)$. Put $A = \mathsf{I}_\varnothing(f(X))$. One can prove the relation $\mathsf{I}(\beta f)(\mathsf{I}(\beta X) \setminus \mathsf{I}_\varnothing(X)) \subset \mathsf{I}(\beta Y) \setminus A$ similarly to the proof of Theorem 3.

Theorem 2 gives $\mathsf{I}(\beta f)(\mathsf{I}_\varnothing(X)) \subset A$. Therefore, $\mathsf{I}_\varnothing(f) = \mathsf{I}(\beta f)|_{\mathsf{I}_\varnothing(X)}\colon \mathsf{I}_\varnothing(X) \to A$ is a proper mapping.

We will show that $\mathsf{I}_\varnothing(f)\colon \mathsf{I}_\varnothing(X) \to \mathsf{I}_\varnothing(Y)$ is an injective mapping. Then it follows that $\mathsf{I}_\varnothing(f)$ is an embedding. Let $\mu, \eta \in \mathsf{I}_\varnothing(X)$ are two different measures. Then there exists a Borel set $Z \subset \beta X$ such that $\mu(Z) \neq \eta(Z)$. We should prove that $\mathsf{I}_\varnothing(f)(\mu)(\beta f(Z)) \neq \mathsf{I}_\varnothing(f)(\eta)(\beta f(Z))$.

Put $Z' = (\beta f)^{\leftarrow}(\beta f(Z))$. By definition of $\mathsf{I}_\varnothing(f)(\mu)$ we have $\mathsf{I}_\varnothing(f)(\mu)(\beta f(Z)) = \mu(Z')$ and $\mathsf{I}_\varnothing(f)(\eta)(\beta f(Z)) = \eta(Z')$. Since, f is an embedding, $Z' \cap X = Z \cap X$ holds. Really, since $Z = (Z \cap X) \cup (Z \setminus X)$ one has

$$\begin{aligned} Z' \cap X &= ((\beta f)^{\leftarrow}(\beta f)((Z \cap X) \cup (Z \setminus X))) \cap X = \\ &= ((\beta f)^{\leftarrow}(\beta f(Z \cap X) \cup \beta f(Z \setminus X))) \cap X = \\ &= (((\beta f)^{\leftarrow}(\beta f(Z \cap X))) \cup ((\beta f)^{\leftarrow}(\beta f(Z \setminus X)))) \cap X = \\ &= (\beta f)^{\leftarrow}(\beta f(Z \cap X)) = Z \cap X. \end{aligned}$$

Then, by Lemma 4, we have

$$\mathsf{I}_\varnothing(f)(\mu)(\beta f(Z)) = \mu(Z') = \mu(Z) \neq \eta(Z) = \eta(Z') = \mathsf{I}_\varnothing(f)(\eta)(\beta f(Z)).$$

Theorem 4 is proved. □

Theorem 5. *The operation $\mathsf{I}_\varnothing$ preserves preimages of Borel sets, i. e. for every continuous mapping $f\colon X \to Y$ of Tychonoff spaces and any Borel subset $A \subset Y$ the equality $\mathsf{I}_\varnothing(f)^{\leftarrow}(\mathsf{I}_\varnothing(A)) = \mathsf{I}_\varnothing(f^{\leftarrow}(A))$ holds.*

Proof. We state that $\mathsf{I}_\varnothing(f^{\leftarrow}(A)) \subset \mathsf{I}_\varnothing(f)^{\leftarrow}(\mathsf{I}_\varnothing(A))$ for any (not necessary Borel) subset $A \subset Y$. Take an arbitrary $\mu \in \mathsf{I}(\beta X)$ and suppose $\mathsf{I}_\varnothing(f)(\mu) \notin \mathsf{I}_\varnothing(A)$. Then we have $\mathsf{I}_\varnothing(f)(\mu)(\Gamma) = a > 0$ for some $\Gamma \in \mathfrak{B}(\beta Y)$, $\Gamma \cap A = \varnothing$. Obviously, $f^{\leftarrow}(\Gamma) \cap f^{\leftarrow}(A) = f^{\leftarrow}(\Gamma \cap A) = \varnothing$, and on the other side, $f^{\leftarrow}(\Gamma) \in \mathfrak{B}(\beta X)$. Whereupon, $\mu(f^{\leftarrow}(\Gamma)) = \mathsf{I}_\varnothing(f)(\mu)(\Gamma) = a > 0$. Hence, $\mu \notin \mathsf{I}_\varnothing(f^{\leftarrow}(A))$.

Let us show that the inverse inclusion $\mathsf{I}_\varnothing(f)^{\leftarrow}(\mathsf{I}_\varnothing(A)) \subset \mathsf{I}_\varnothing(f^{\leftarrow}(A))$ holds for any Borel subset $A \subset Y$. Take an arbitrary element $\mu \in \mathsf{I}_\varnothing(f)^{\leftarrow}(\mathsf{I}_\varnothing(A))$. Then $\mathsf{I}_\varnothing(f)(\mu) \in \mathsf{I}_\varnothing(A)$, and we have

$$\mu(X \setminus f^{\leftarrow}(A)) = \mu(f^{\leftarrow}(Y \setminus A)) = \mathsf{I}_\varnothing(f)(\mu)(Y \setminus A) = 0.$$

Consequently, $\mu \in \mathsf{I}_\varnothing(f^{\leftarrow}(A))$. □

The authors do not know the answer to the next question.

Question 1. *Let $f\colon X \to Y$ be a continuous mapping of Tychonoff spaces, $A \subset Y$ an arbitrary set, $B \subset X$ a Borel set containing $f^{\leftarrow}(A)$. Does there exist a Borel set C in Y such that $f^{\leftarrow}(A) \subset f^{\leftarrow}(C) \subset B$?*

An affirmative answer to Question **??** makes it possible to obtain a more rigorous result in Theorem 5 by getting rid of the condition on one of the sets to be Borel.

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Theorem 6. *Let X be a Tychonoff space and $A, B \subset X$ any subsets such that at least one of them is Borel. Then the equality $\mathsf{I}_\varnothing(A \cap B) = \mathsf{I}_\varnothing(A) \cap \mathsf{I}_\varnothing(B)$ holds.*

Proof. Assume $A \cap B = \varnothing$. Then (4) implies $\mathsf{I}_\varnothing(A \cap B) = \varnothing$.

Now we will show that $\mathsf{I}_\varnothing(A) \cap \mathsf{I}_\varnothing(B) = \varnothing$. Suppose A is a Borel set. For an arbitrary $\mu \in \mathsf{I}_\varnothing(A)$ we have $\mu(\beta X \setminus A) = 0$. Consequently, $\mu^*(B) = 0$ which provides $\mu \notin \mathsf{I}_\varnothing(B)$. On the other side, for every $\mu \in \mathsf{I}_\varnothing(B)$ we have $\mu(A) = 0$ and $\mu \notin \mathsf{I}_\varnothing(A)$. So, $I_\tau(A) \cap \mathsf{I}_\varnothing(B) = \varnothing$. Hence, we obtain $I_\tau(A \cap B) = \mathsf{I}_\varnothing(A) \cap \mathsf{I}_\varnothing(B)$ in this case.

Consider the case $A \cap B \neq \varnothing$. The inclusion $\mathsf{I}_\varnothing(A \cap B) \subset \mathsf{I}_\varnothing(A) \cap \mathsf{I}_\varnothing(B)$ directly follows from Theorem 4.

Let us show the inverse inclusion, i. e. $\mathsf{I}_\varnothing(A \cap B) \supset \mathsf{I}_\varnothing(A) \cap \mathsf{I}_\varnothing(B)$. Suppose, $\mu \in \mathsf{I}_\varnothing(A) \cap \mathsf{I}_\varnothing(B) \subset \mathsf{I}_\varnothing(X)$, and A is a Borel set. Since $\mu \in \mathsf{I}_\varnothing(A)$, we get $\mu(\beta X \setminus A) = 0$. Then the inclusion $\mu \in I_\tau(B)$ implies $\mu^*(B \setminus A) = \mu^*(B \cap (\beta X \setminus A)) = 0$ and $\mu^*(B \cap A) = 1$. We should proof $\mu(\Phi) = 0$ for every $\Phi \in \mathfrak{B}(\beta X)$ with $\Phi \cap (A \cap B) = \varnothing$. We decompose $\Phi = \Phi_A \cup \Phi_B$ setting $\Phi_A = \Phi \setminus A$ and $\Phi_B = \Phi \cap A$. Then $\mu(\Phi_A) = 0$ by definition. And $\mu(\Phi_B) = 0$ because $\Phi_B \cap B = \varnothing$. Consequently, $\mu(\Phi) = 0$. So, $\mu \in \mathsf{I}_\varnothing(A \cap B)$.

Thus, Theorem 6 is completely proved. □

Remark 5. The assertion of Theorem 6 cannot be improved, i. e. the requirement that at least one of the given sets be Borel set cannot be omitted. Really, consider segment $[0; 1]$. We will say $x \sim y$ if and only if $y - x \in \mathbb{Q}$, here $\mathbb{Q}$ is the set of all rational numbers. The relation $\sim$ is an equivalence relation, (i. e. reflexive, symmetric and transitive). Denote by $[x]$ an equivalence class containing $x \in [0; 1]$. Whereupon, we got a partition of $[0; 1]$ into disjoint equivalence classes. Each class $[x]$ has countable many elements. By $\mathcal{A}$ we denote a system of all mutually distinct equivalence classes. Since $[0; 1] = \bigcup\limits_{[x] \in \mathcal{A}} [x]$, the system $\mathcal{A}$ has a power of the continuum. On the other side, one can see that each class is everywhere dense in $[0; 1]$. From each class (using the Choice Axiom) we take one point so that the resulting set A is everywhere dense in $[0; 1]$. Then from each class (using the Choice Axiom again) we choose a point that differs from the one taken before, and construct a set B which is also everywhere dense in $[0; 1]$. Neither A nor B are Borel sets, and moreover $A \cap B = \varnothing$ by virtue of the construction. It is clear, there exists no couple of Borel sets U and V in $[0; 1]$ such that $A \subset U$, $B \subset V$ and $U \cap V = \varnothing$. Note, Borel sets in $[0; 1]$ which can contain A and B are only $(0; 1)$, $[0; 1)$, $(0; 1]$ or $[0; 1]$.

Suppose the only Borel set containing the set A is $[0; 1)$. Two cases are possible: $B \subset (0; 1)$ or $B \subset (0; 1]$. Consequently, $\mathsf{I}_\varnothing(A) = \mathsf{I}_\varnothing([0; 1))$, and $\mathsf{I}_\varnothing(B) = \mathsf{I}_\varnothing((0; 1))$ or $\mathsf{I}_\varnothing(B) = \mathsf{I}_\varnothing((0; 1])$. Since $[0; 1)$, $(0; 1)$ and $(0; 1]$ are Borel sets we have

$$\mathsf{I}_\varnothing(A) \cap \mathsf{I}_\varnothing(B) = \mathsf{I}_\varnothing((0; 1)) \neq \mathsf{I}_\varnothing(\varnothing) = \mathsf{I}_\varnothing(A \cap B).$$

In the other cases ($A \subset (0; 1)$, $A \subset (0; 1]$), in exactly the same way as the above one can establish that the conclusion of Theorem 6 is not true for the sets A and B.

One can select sets A and B such that $A \cap B \subset \left[\frac{1}{3}; \frac{2}{3}\right]$ and the intersection is dense in $\left(\frac{1}{3}; \frac{2}{3}\right)$. Then the only Borel set containing the intersection is $\left(\frac{1}{3}; \frac{2}{3}\right)$. Evidently,

$$\mathsf{I}_\varnothing(A) \cap \mathsf{I}_\varnothing(B) = \mathsf{I}_\varnothing\Big((0; 1)\big) \neq \mathsf{I}_\varnothing\big(\left(\tfrac{1}{3}; \tfrac{2}{3}\right)\Big) = \mathsf{I}_\varnothing(A \cap B).$$

The set of type A (or B) considered in Remark 5 was found by Giuseppe Vitali in 1905. Therefore, such a set is called the Vitali set.

5. Idempotent probability measures spaces on Čech-complete Tychonoff spaces

We begin our investigation of spaces of idempotent probability measures on a Čech-complete Tychonoff space by stating the following result.

Theorem 7. *Let X be a Tychonoff space and $\{X_\alpha : \alpha \in D\}$ any family of Borel subsets of X such that $\bigcap\limits_{\alpha\in D} X_\alpha \in \mathfrak{B}(\beta X)$. Then the equality $I_\varnothing\left(\bigcap\limits_{\alpha\in D} X_\alpha\right) = \bigcap\limits_{\alpha\in D} I_\varnothing(X_\alpha)$ holds.*

Proof. It is easy to see that $I_\varnothing\left(\bigcap\limits_{\alpha\in D} X_\alpha\right) \subset \bigcap\limits_{\alpha\in D} I_\varnothing(X_\alpha)$. Really, it is clear that $\bigcap\limits_{\alpha\in D} X_\alpha \subset X_\alpha$ for every $\alpha \in D$. Since $\mu \in I_\varnothing\left(\bigcap\limits_{\alpha\in D} X_\alpha\right)$, we have $\mu\left(\beta X \setminus \left(\bigcap\limits_{\alpha\in D} X_\alpha\right)\right) = 0$. Hence $\mu(\beta X \setminus X_\alpha) = 0$ for arbitrary $\alpha \in D$, i. e. $\mu \in \bigcap\limits_{\alpha\in D} I_\varnothing(X_\alpha)$.

Let us show the inverse inclusion, i. e. $I_\varnothing\left(\bigcap\limits_{\alpha\in D} X_\alpha\right) \supset \bigcap\limits_{\alpha\in D} I_\varnothing(X_\alpha)$. Suppose, $\mu \in \bigcap\limits_{\alpha\in D} I_\varnothing(X_\alpha)$. Then $\mu \in I_\varnothing(X_\alpha)$ for every $\alpha \in D$. Hence for any $\alpha \in D$ one has $\mu(\beta X \setminus X_\alpha) = 0$. Consequently,

$$\mu\left(\beta X \setminus \left(\bigcap_{\alpha\in D} X_\alpha\right)\right) = \mu\left(\bigcup_{\alpha\in D} (\beta X \setminus X_\alpha)\right) = \sup_{\alpha\in D}\{\mu(\beta X \setminus X_\alpha)\} = 0.$$

Thus, $\mu \in I_\varnothing\left(\bigcap\limits_{\alpha\in D} X_\alpha\right)$. Theorem 7 is proved. □

Corollary 1. *Let X be a Tychonoff space and $\{X_\alpha : \alpha \in D\}$ any family of closed subsets of X. Then $I_\varnothing\left(\bigcap\limits_{\alpha\in D} X_\alpha\right) = \bigcap\limits_{\alpha\in D} I_\varnothing(X_\alpha)$.*

Corollary 2. *Let X be a Tychonoff space and $\{X_\alpha : \alpha \in D\}$ any family of open subsets of X such that $\bigcap\limits_{\alpha\in D} X_\alpha \in \mathfrak{B}(\beta X)$. Then $I_\varnothing\left(\bigcap\limits_{\alpha\in D} X_\alpha\right) = \bigcap\limits_{\alpha\in D} I_\varnothing(X_\alpha)$.*

Note that the following assertion ensures that every X_α is closed in Corollary 1.

Proposition 3. *For any closed subset $F \subset X$ of a Tychonoff space X the set $I_\varnothing(F)$ is closed in $I_\varnothing(X)$.*

Proof. Take any $\mu \in I_\varnothing(X) \setminus I_\varnothing(F)$. Then there exists an open subset $V \subset \beta X$ such that $F \cap V = \varnothing$ and $\mu(V) = a > 0$. Consider an open neighbourhood $\langle \mu;\, V;\, \frac{a}{2}\rangle$ of μ and claim that $\langle \mu;\, V;\, \frac{a}{2}\rangle \cap I_\varnothing(F) = \varnothing$. Really, for every $\xi \in I_\tau(F)$ one has $|\xi(V) - \mu(V)| = |0 - a| = a > \frac{a}{2}$, and $\xi \notin \langle \mu;\, V;\, \frac{a}{2}\rangle$. Or, for each $\zeta \in \langle \mu;\, V;\, \frac{a}{2}\rangle$ one has $|\zeta(V) - \mu(V)| = |\zeta(V) - a| < \frac{a}{2}$, which gives $\zeta(V) > \frac{a}{2} \neq 0$. Hence, $\zeta \notin I_\varnothing(F)$. Thus, $I_\varnothing(F)$ is closed in $I_\varnothing(X)$. □

Now we are ready to formulate the following result. Note that a linear (in the classical sense) version of this result was established in [9]

Theorem 8. *The operation $I_\varnothing$ preserves the Čech-completeness of Tychonoff spaces, in the other words, if X is a Čech-complete Tychonoff space, then $I_\varnothing(X)$ is also a Čech-complete Tychonoff space.*

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Proof. Let X be a Čech-complete Tychonoff space. Then X is a G_δ-set in βX, i. e. there are countable many open sets $U_n \subset \beta X$, such that $X = \bigcap_{n=1}^{\infty} U_n$. Then

$$\mathsf{I}_{\emptyset}(X) = \mathsf{I}_{\emptyset}\left(\bigcap_{n=1}^{\infty} U_n\right) = \left\{\mu \in \mathsf{I}(\beta X) \colon \mu\left(\beta X \setminus \bigcap_{n=1}^{\infty} U_n\right) = 0\right\} =$$
$$= \bigcap_{n=1}^{\infty}\left\{\mu \in I(\beta X) \colon \mu(\beta X \setminus U_n) < \tfrac{1}{n}\right\}.$$

Applying Remark 2, by [23] we get that $\mathsf{I}_{\emptyset}(X)$ is a Čech-complete Tychonoff space. Theorem 8 is proved. □

Since Čech-completeness is hereditary with respect to G_δ-subsets (Theorem 3.9.6, [23]), one obtains the following result.

Corollary 3. *For a Tychonoff space X the space $\mathsf{I}_{\emptyset}(X)$ Čech-complete if and only if X is Čech-complete.*

From Theorem 8 and Corollary 1 one can immediately extract that, in contrast to closed sets, for an open set U, the set $\mathsf{I}_{\emptyset}(U)$ is a G_δ-set in $I(\beta X \setminus X)$ for a Čech-complete Tychonoff space X.

Proposition 4. *For an open set $U \subset X$ of a Čech-complete Tychonoff space X the set $\mathsf{I}_{\emptyset}(U)$ is a G_δ-set.*

Proof. Since X is a Čech-complete Tychonoff space, it is a G_δ-set in βX, and let $X = \bigcap_{n=1}^{\infty} U_n$, where U_n are open subsets of βX. Then $X \in \mathfrak{B}(\beta X)$, and $U \in \mathfrak{B}(\beta X)$ for each open $U \subset X$. An open set $U \subset X$ admits a representation $U = \bigcap_{n=1}^{\infty}(U \cap U_n)$. Finally,

$$\mathsf{I}_{\emptyset}(U) = \mathsf{I}_{\emptyset}\left(\bigcap_{n=1}^{\infty}(U \cap U_n)\right) = \left\{\mu \in \mathsf{I}(\beta X) \colon \mu\left(\beta X \setminus \bigcap_{n=1}^{\infty}(U \cap U_n)\right) = 0\right\} =$$
$$= \bigcap_{n=1}^{\infty}\left\{\mu \in \mathsf{I}(\beta X) \colon \mu(\beta X \setminus (U \cap U_n)) < \tfrac{1}{n}\right\}.$$

Proposition 4 is proved. □

6. Conclusions

We provided a manner to define topology T on the set I(X)R of probability measures on a compact Hausdorff space X and described some closed and some open sets in that topology. It is shown that the space $(I(X), T)$ is compact Hausdorff if XR is so. The topology T (on βX) is used to define topology on the set $\mathsf{I}_{\emptyset}$(X) of τ-smooth idempotent probability measures on a Tychonoff space X. Some properties of I_τ are established and it is shown that the space $\mathsf{I}_\tau(X)$ is Čech-complete if and only if X is Čech-complete. It is natural to continue investigation of properties of $\mathsf{I}_{[\tau}(X)$ for some other topological properties which are not studied in the literature.

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