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*Article*

# Global Existence and Boundedness of Solutions to a Chemotaxis-Haptotaxis Model with Nonlinear Diffusion and Signal Production ,†

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**Abstract:** In this paper, we investigate the following chemotaxis-haptotaxis model

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(H(u)\nabla v) - \nabla\cdot(I(u)\nabla w) + u(a-\mu u^{k-1}-\lambda w), \\ v_t = \triangle v - v + u^{\gamma}, \\ w_t = -vw \end{cases} \qquad (*)$$

under homogenous Neumann boundary condition and for a bounded domain $\Omega\subset\mathbb{R}^n(n\ge 2)$, with $\lambda,\mu,\gamma>0$, $k>1$, $a\in\mathbb{R}$, and $D(u)\ge K_D(u+1)^{m-1}$, $0\le H(u)\le \chi u(u+1)^{-\alpha}$, $0\le I(u)\le \xi u(u+1)^{-\beta}$ for $K_D,\chi,\xi>0,m,\alpha,\beta\in\mathbb{R}$. It has been demonstrated that
(i) For $0<\gamma\le\frac{2}{n}$, if $\alpha>\gamma-k+1$ and $\beta>1-k$, problem $(*)$ admits a classical solution $(u,v,w)$ which is globally bounded.
(ii) For $\frac{2}{n}<\gamma\le 1$, if $\alpha>\gamma-k+\frac{1}{e}+1$ and $\beta>\max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n}-k+1,\frac{(n\gamma-2)(\gamma+\frac{1}{e})}{n}-k+1\}$ or $\alpha>\gamma-k+1$ and $\beta>\max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n}-k+1,\frac{(n\gamma-2)(\alpha+k-1)}{n}-k+1\}$, problem $(*)$ admits a classical solution $(u,v,w)$ which is globally bounded.

**Keywords:** Boundedness; Chemotaxis-haptotaxis; Nonlinear diffusion; Signal production

**MSC:** 35K55; 35K65; 35A07; 35B35

## 1. Introduction

In the present work, we consider the following chemotaxis-haptotaxis system with nonlinear diffusion and signal production

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(H(u)\nabla v) - \nabla\cdot(I(u)\nabla w) + u(a-\mu u^{k-1}-\lambda w), x\in\Omega, t>0, \\ v_t = \triangle v - v + g(u), & x\in\Omega,\ t>0, \\ w_t = -vw, & x\in\Omega,\ t>0, \\ D(u)\frac{\partial u}{\partial \nu} - H(u)\frac{\partial v}{\partial \nu} - I(u)\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega, t>0, \\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & x\in\Omega, \end{cases} \qquad (1.1)$$

where $\Omega\subset\mathbb{R}^n(n\ge 2)$ is a bounded domain with smooth boundary, the function $u=u(x,t)$ denotes the cancer cell density, $v=v(x,t)$ represents the concentration of matrix degrading enzymes, and $w=w(x,t)$ represents the density of extracellular matrix. We assume that $D,H,I\in C^2([0,\infty))$ fulfill for all $s\ge 0$,

$$D(s)\ge K_D(s+1)^{m-1}, \qquad (1.2)$$

$$0\le H(s)\le \chi s(s+1)^{-\alpha}\ \text{ and } H(0)=0, \qquad (1.3)$$

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$$0 \le I(s) \le \xi s(s+1)^{-\beta} \text{ and } I(0) = 0, \tag{1.4}$$

with $K_D, \chi, \xi > 0$ and $\alpha, \beta, m \in \mathbb{R}$. Moreover, we assume $g \in C^1([0,\infty))$ such that

$$0 \le g(s) \le K_g s^\gamma \text{ for all } s \ge 0, \tag{1.5}$$

where $K_g, \gamma > 0$. To this end, we assume that the initial data satisfy

$$\begin{cases} u_0, v_0 \in W^{1,\infty}(\Omega),\ u_0 \ge 0,\ \ v_0 \ge 0 \text{ in } \Omega, \\ w_0 \in C^{2,\alpha}(\overline{\Omega}) \text{ with } \alpha \in (0,1), w_0 \ge 0 \text{ in } \Omega,\ \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases} \tag{1.6}$$

When $w \equiv 0$, (1.1) is reduced to the Keller-Segel system which has been widely studied by many authors during the past four decades (see[1-20]). When $f(u) = a - \mu u^k$ and $D(u), H(u)$ satisfy (1.2), (1.3), Zheng [5] proved that all solutions are global and uniformly bounded if $0 < 2 - \alpha - m < \max\{k - m, \frac{2}{N}\}$ or $2 - \alpha = k$ and $\mu$ is large enough. In the case of $g(u) = u^\gamma$, Tao et.al [6] considered problem (1.1) , it is shown that if $1 + \gamma - \alpha < k$ or $1 + \gamma - \alpha = k$ and $\mu$ is large enough, then the solutions of (1.1) are globally bounded. When $f \equiv 0$ and $1 - \alpha - m + \gamma < \frac{2}{N}$, they also proved that system (1.1) possesses a nonnegative classical solution $(u, v)$ which is globally bounded. Later, Ding et. al [7] provided a boundedness result under $1 - \alpha - m + \gamma < \frac{2}{n}$ and proved the asymptotic stability when damping effects of logistic source are strong enough. Nowadays, there are more and more mathematical models used to describe complex natural phenomena, and the results are also very impressive (see [21]-[31]).

In 2016, Chaplain and Lolas [32] presented the chemotaxis-haptotaxis model which can be represented by the following equation

$$\begin{cases} u_t = \triangle u - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla w) + u(a - \mu u^{k-1} - \lambda w), \\ v_t = \triangle v - v + u, \\ w_t = -vw. \end{cases} \tag{1.7}$$

When $k = 2, a = \lambda = \mu$, Tao, Wang [33,34,35] proved the global solvability and uniform boundedness for $n = 1, 2$. For the case $n = 3$, the global existence and boundedness was proved for $\frac{\mu}{\chi}$ is sufficiently large (see [33,36]). Zheng and Ke [37] proved that model (1.7) possesses a global classical solution which is bounded for $k > 2$ or $k = 2$, with $\mu$ is sufficiently large. And they demonstrated that if $\mu$ is large enough, the corresponding solution of (1.7) exponentially decays to $((\frac{a}{\mu})^{\frac{1}{k-1}}, (\frac{a}{\mu})^{\frac{1}{k-1}}, 0)$.

In recent year, many authors have begun to studied the chemotaxis-haptotaxis model with non-linear diffusion (see [38]-[46]). For problem (1.1) with $g(u) = u, k = 2, a = \lambda = \mu$. Liu et. al. [44] demanstrated the global boundedness of solutions for $n = 2$ if $\max\{1 - \alpha, 1 - \beta\} < m + \frac{2}{n} - 1$ or for $n \ge 3$ if $\max\{1 - \alpha, 1 - \beta\} < m + \frac{2}{n} - 1$ with either $m > 2 - \frac{2}{n}$ or $m \le 1$. Subsequently, Xu et. al [45] proved that if $m > 0, \alpha > 0, \beta \ge 0$ for $n = 3$, problem (1.1) possesses a global bounded weak solution, And they investigated the large time behavior of solutions and showed that when $0 < m \le 1$, for appropriately large $\mu$, $(u, v, w) \to (1, 1, 0)$ as $t \to \infty$. Later, Jia et al [46] extends the boundedness result of [45], which deals with the global boundedness of solutions with $\alpha > 0, \beta > -\frac{1}{6}$. This paper is devoted to research the global existence and boundedness for (1.1) with nonlinear diffusion and signal production in the case of $n \ge 2$.

Now, we present the primary result of this paper.

**Theorem 1.1.** Let $\Omega \subset \mathbb{R}^n (n \ge 2)$ be a bounded domain with smooth boundary and $(u_0, v_0, w_0)$ satisfy (1.6). Suppose that $D, H, I$ and $g$ fulfill (1.2)-(1.5). Then

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(i) For $0 < \gamma \le \frac{2}{n}$, if $\alpha > \gamma - k + 1$ and $\beta > 1 - k$, problem (1.1) possesses a classical solution $(u, v, w)$ which is globally bounded.

(ii) For $\frac{2}{n} < \gamma \le 1$, if $\alpha > \gamma - k + \frac{1}{e} + 1$ and $\beta > \max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1, \frac{(n\gamma-2)(\gamma+\frac{1}{e})}{n} - k + 1\}$ or $\alpha > \gamma - k + 1$ and $\beta > \max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1, \frac{(n\gamma-2)(\alpha+k-1)}{n} - k + 1\}$, problem (1.1) possesses a classical solution $(u, v, w)$ which is globally bounded.

The rest of this paper is organized as follows. In section 2, we present the local existence of classical solutions to system (1.1) and recall some preliminaries. Finally, we establish the global existence and boundedness of solutions to system (1.1) in section 3.

## 2. Preliminaries

We first prove the local existence of classical solutions, which proceeds along the idea of the arguments of [38], [43].

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^n (n \ge 2)$ be a bounded domain with smooth boundary. Suppose that $D, H, I$ and $g$ fulfill (1.2)-(1.5). Then for any initial data $(u_0, v_0, w_0)$ fulfilling (1.6), there exists $T_{\max} \in (0, \infty]$ and a local-in-time classical solution $(u, v, w)$ satisfies

$$\begin{cases} u, v \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times [0, T_{\max}), \\ w \in C^{2,1}(\overline{\Omega} \times [0, T_{\max}) \end{cases} \tag{2.1}$$

with $u, v \ge 0$ in $\Omega \times (0, T_{\max})$ and $0 \le w \le \|w\|_{L^\infty(\Omega)}$. Moreover, if $T_{\max} < \infty$, then

$$\lim_{t \to T_{\max}} \sup \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty.$$

Then, we will give a useful lemma referred to as a variation of maximal Sobolev regularity, as obtained in [47, 48].

**Lemma 2.2.** Let $s_0 \in W^{2,p}(\Omega)$ and $f \in L^p(0, T; L^p(\Omega))$. Then the following problem

$$\begin{cases} s_t = \triangle s - s + f, \\ \frac{\partial s}{\partial \nu} = 0, \\ v(x, 0) = v_0(x), \end{cases} \tag{2.2}$$

possesses a unique solution $s \in L^p_{loc}((0, +\infty); W^{2,p}(\Omega))$ and $s_t \in L^p_{loc}((0, +\infty); L^p(\Omega))$, and if $t_0 \in (0, T)$, then

$$\int_{t_0}^T \int_\Omega e^{pt} |\triangle s|^p dx dt \le C_p \int_{t_0}^T \int_\Omega e^{pt} |f|^p dx dt + C_p \|s(\cdot, t_0)\|^p_{W^{2,p}(\Omega)}, \tag{2.3}$$

where $C_p$ is a constant independent of $t_0$.

By [35], we have the following lemma.

**Lemma 2.3.** Assume $(u, v, w)$ be the solution of model (1.1). Then

$$-\triangle w(x, t) \le \|w_0\|_{L^\infty(\Omega)} v(x, t) + C \ \text{ for } (x, t) \in \Omega \times (0, T_{\max}), \tag{2.4}$$

where

$$C := \|w_0\|_{L^\infty(\Omega)} + 4\|\nabla \sqrt{w_0}\|^2_{L^\infty(\Omega)} + \frac{\|w_0\|_{L^\infty(\Omega)}}{e}. \tag{2.5}$$

The following lemma is important to prove the Theorem 1.1. The main ideas comes from [39].

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**Lemma 2.4.** Assume that $D, H, I$ and $g$ fulfill (1.2)-(1.5) with $0 < \gamma \le 1$, then we have

(i) There exists $K > 0$ such that for all $t \in (0, T_{\max})$

$$\|u(\cdot,t)\|_{L^1(\Omega)} \le K\mu^{-\frac{1}{k-1}}. \tag{2.6}$$

(ii)For $s \in [1, \frac{n}{(n\gamma-2)_+})$, there exists $K_s > 0$ such that for all $t \in (0, T_{\max})$

$$\|v(\cdot,t)\|_{L^s(\Omega)} \le K_s. \tag{2.7}$$

where $(n\gamma - 2)_+ := \max\{n\gamma - 2, 0\}$.

(iii) Assume that $p > \max\{\frac{n\gamma}{2}, \gamma\}$ and $\|u(\cdot,t)\|_{L^p(\Omega)} \le K$. Then there exists $K_p > 0$ such that for all $t \in (0, T_{\max})$

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le K_p. \tag{2.8}$$

(iv) Assume that $q > nr$ and $\|u(\cdot,t)\|_{L^q(\Omega)} \le K$. Then there exists a positive constant $K_q$ such that for all $t \in (0, T_{\max})$

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le K_q. \tag{2.9}$$

## 3. Proof of Theorem 1.1

In this section, we deal with global existence and boundedness, we firstly give the following lemma, which is important to prove the main theorem. For convenience, we denote $T = T_{\max}$.

**Lemma 3.1.** Assume that $D, H, I$ and $g$ fulfill (1.2)-(1.5) with $\beta > 1 - k$. Then

(i) Let $\alpha > \gamma - k + \frac{1}{e} + 1$ and $p > \max\{1, \beta, \frac{n\gamma}{2} + 1 - k, \gamma - k + \frac{1}{e} + 1\}$. If there exists $K_0 > 0$ fulfills for all $t \in (0, T)$

$$\|v(\cdot,t)\|_{L^{\frac{p+k-1}{\beta+k-1}}(\Omega)} \le K_0, \tag{3.1}$$

then

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le K \ \text{ for } t \in (0, T), \tag{3.2}$$

where $K > 0$ depends on $K_0, \mu$.

(ii) Let $\alpha > \gamma - k + 1$ and $p > \max\{1, \alpha, \beta\}$. If there exists $K_0 > 0$ fulfills for all $t \in (0, T)$

$$\|v(\cdot,t)\|_{L^{\frac{p+k-1}{\beta+k-1}}(\Omega)} \le K_0, \tag{3.3}$$

then

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le K \ \text{ for } t \in (0, T), \tag{3.4}$$

where $K > 0$ depends on $K_0, \mu$.

**Proof.** Multiplying the first equation in (1.1) with $p(1+u)^{p-1}$, and integrating by parts yields that

$$\begin{aligned} &\tfrac{d}{dt}\int_\Omega (u+1)^p dx \\ &\le -p(p-1)K_D \int_\Omega (u+1)^{m+p-3}|\nabla u|^2 dx + p(p-1)\int_\Omega (u+1)^{p-2}H(u)\nabla u\cdot\nabla v dx \\ &\quad + p(p-1)\int_\Omega (u+1)^{p-2}I(u)\nabla u\cdot\nabla w dx + p\int_\Omega (u+1)^{p-1}u(a-\mu u^{k-1}-\lambda w)dx. \end{aligned} \tag{3.5}$$

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Since $(u+1)^k \le 2^{k-1}(u^k+1)$, we have

$$\begin{aligned} & p\int_\Omega (u+1)^{p-1}u(a-\mu u^{k-1}-\lambda w)dx \\ & \le |a|p\int_\Omega (u+1)^p dx - \frac{\mu p}{2^{k-1}}\int_\Omega (u+1)^{k+p-1}dx + \mu p\int_\Omega (u+1)^{p-1}dx \\ & \le -\frac{5\mu}{6\cdot 2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_1, \end{aligned} \tag{3.6}$$

where $C_1$ ia a positive constant. It follows from (3.5) and (3.6) that

$$\begin{aligned} & \frac{d}{dt}\int_\Omega (u+1)^p dx \\ & \le p(p-1)\int_\Omega (u+1)^{p-2}H(u)\nabla u\cdot\nabla v dx + p(p-1)\int_\Omega (u+1)^{p-2}I(u)\nabla u\cdot\nabla w dx \\ & -\frac{5\mu}{6\cdot 2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_1. \end{aligned} \tag{3.7}$$

Define

$$\varphi(s) := \int_0^s (1+\sigma)^{p-2}H(\sigma)d\sigma \quad \text{for } s \ge 0.$$

We infer from (1.3) that

$$0 \le \varphi(s) \le \chi\int_0^s (1+\sigma)^{p-\alpha-1}d\sigma.$$

This implies

$$\varphi(s) \le \begin{cases} \frac{2\chi}{|p-\alpha|}, & \text{for } p<\alpha \\ \chi\ln(1+s), & \text{for } p=\alpha \\ \frac{\chi}{p-\alpha}(1+s)^{p-\alpha}, & \text{for } p>\alpha \end{cases} \tag{3.8}$$

for $z \ge 0$. Integrating by parts, we obtain that

$$\begin{aligned} & p(p-1)\int_\Omega (1+u)^{p-2}H(u)\nabla u\cdot\nabla v dx \\ & = p(p-1)\int_\Omega \nabla\varphi(u)\cdot\nabla v dx \\ & \le p(p-1)\int_\Omega \varphi(u)|\triangle v|dx. \end{aligned} \tag{3.9}$$

**Case (i).** Combining (3.8) with (3.9) yields that, for $\gamma - k + \frac{1}{e} + 1 < p < \alpha$ and $n \ge 2$, we have

$$\begin{aligned} & p(p-1)\int_\Omega (1+u)^{p-2}H(u)\nabla u\cdot\nabla v dx \le \frac{2\chi p(p-1)}{|p-\alpha|}\int_\Omega |\triangle v|dx \\ & \le \frac{2\chi p(p-1)}{|p-\alpha|}\int_\Omega |\triangle v|^{\frac{n}{2}}dx + C_2, \end{aligned} \tag{3.10}$$

where $C_2$ is a positive constant. For $p > \alpha$, we obtain

$$\begin{aligned} & p(p-1)\int_\Omega (1+u)^{p-2}H(u)\nabla u\cdot\nabla v dx \\ & \le \frac{\chi p(p-1)}{p-\alpha}\int_\Omega (1+u)^{p-\alpha}|\triangle v|dx, \\ & \le \frac{\mu}{3\cdot 2^k}\int_\Omega (1+u)^{p+k-1} + C_3\int_\Omega |\triangle v|^{\frac{p+k-1}{k+\alpha-1}}, \end{aligned} \tag{3.11}$$

where $C_3$ is a positive constant. For $p = \alpha$, we get

$$\begin{aligned} & p(p-1)\int_\Omega (1+u)^{p-2}H(u)\nabla u\cdot\nabla v dx \\ & \le p(p-1)\chi\int_\Omega \ln(1+u)|\triangle v|dx, \\ & \le p(p-1)\chi\int_\Omega (1+u)^{\frac{1}{e}}|\triangle v|dx, \\ & \le \frac{\mu}{3\cdot 2^k}\int_\Omega (1+u)^{p+k-1} + C_4\int_\Omega |\triangle v|^{\frac{e(p+k-1)}{e(p+k-1)-1}}, \end{aligned} \tag{3.12}$$

where $C_4$ is a positive constant.

Denote $\psi(s) = \int_0^s (1+\sigma)^{p-2}I(\sigma)d\sigma$ for all $s \ge 0$. We infer from (1.4) and $p > \beta$ that

$$0 \le \psi(s) \le \frac{\xi}{p-\beta}(1+s)^{p-\beta}, \tag{3.13}$$

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for $s \geq 0$. This together with Lemma 2.3 and $\beta > 1-k$ entail that

$$\begin{aligned}
&p(p-1)\int_\Omega (1+u)^{p-2} I(u)\nabla u\cdot\nabla w dx\\
=&-p(p-1)\int_\Omega \psi(u)\cdot\triangle w dx\\
\leq& p(p-1)\|w_0\|_{L^\infty(\Omega)}\int_\Omega v\psi(u)dx + p(p-1)C\int_\Omega \psi(u)dx\\
\leq& \frac{\xi p(p-1)}{p-\beta}\|w_0\|_{L^\infty(\Omega)}\int_\Omega (1+u)^{p-\beta} v dx + \frac{\xi M p(p-1)}{p-\beta}\int_\Omega (1+u)^{p-\beta}dx\\
\leq& \frac{\mu}{3\cdot 2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_5\int_\Omega v^{\frac{p+k-1}{\beta+k-1}}dx + C_6,
\end{aligned} \tag{3.14}$$

where $C_5, C_6$ are positive constants.

For $\gamma - k + \frac{1}{e} + 1 < p < \alpha$, we infer from (3.1), (3.7), (3.10) and (3.14) that

$$\frac{d}{dt}\int_\Omega (1+u)^p \leq -\frac{\mu}{2^k}\int_\Omega (1+u)^{p+k-1}dx + \frac{2\chi p(p-1)}{|p-\alpha|}\int_\Omega |\triangle v|^{\frac{n}{2}}dx + C_7, \tag{3.15}$$

Since

$$\frac{n}{2}\int_\Omega (1+u)^p dx \leq \frac{\mu}{3\cdot 2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_8, \tag{3.16}$$

Combining (3.15) with (3.16), we get

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (1+u)^p dx + \frac{n}{2}\int_\Omega (1+u)^p dx\\
\leq& -\frac{\mu}{3\cdot 2^k}\int_\Omega (1+u)^{p+k-1}dx + \frac{2\chi p(p-1)}{|p-\alpha|}\int_\Omega |\triangle v|^{\frac{n}{2}}dx + C_9,
\end{aligned} \tag{3.17}$$

where $C_9$ is a positive constant. This together with the variation-of-constants formula shows that

$$\begin{aligned}
&\int_\Omega (1+u)^p dx\\
\leq& -\frac{\mu}{3\cdot 2^k}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}(1+u)^{p+k-1}dxds + \frac{2\chi p(p-1)}{|p-\alpha|}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}|\triangle v|^{\frac{n}{2}}dxds\\
&+e^{-\frac{n}{2}(t-t_0)}\int_\Omega (1+u(\cdot,t_0))^p dx + C_9\int_{t_0}^t e^{-\frac{n}{2}(t-s)}ds\\
\leq& -\frac{\mu}{3\cdot 2^k}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}(1+u)^{p+k-1}dxds + \frac{2\chi p(p-1)}{|p-\alpha|}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}|\triangle v|^{\frac{n}{2}}dxds + C_{10}.
\end{aligned} \tag{3.18}$$

Since $p > \frac{n\gamma}{2} + 1 - k$, we have from Lemma 2.2 and (1.5) that

$$\begin{aligned}
&\frac{2\chi p(p-1)}{|p-\alpha|}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}|\triangle v|^{\frac{n}{2}}dxds\\
\leq& \frac{2\chi p(p-1)C_n}{|p-\alpha|}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}u^{\frac{n\gamma}{2}}dxds + \frac{2\chi p(p-1)C_n}{|p-\alpha|}\|v(\cdot,t_0)\|^{\frac{n}{2}}_{W^{2,\frac{n}{2}}(\Omega)}\\
\leq& \frac{\mu}{3\cdot 2^k}\int_{t_0}^t\int_\Omega e^{-\frac{n}{2}(t-s)}(u+1)^{p+k-1}dxds + C_{11},
\end{aligned} \tag{3.19}$$

where $C_{11}$ is a positive constant. The combination of (3.18)-(3.19), we conclude that

$$\int_\Omega (u+1)^p dx \leq C_{12}, \tag{3.20}$$

where $C_{12}$ is a positive constant.

For $p > \alpha$, we infer from (3.1), (3.7), (3.11) and (3.14) that

$$\frac{d}{dt}\int_\Omega (1+u)^p dx \leq -\frac{\mu}{3\cdot 2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_3\int_\Omega |\triangle v|^{\frac{p+k-1}{\alpha+k-1}}dx + C_7. \tag{3.21}$$

Define

$$m := \frac{p+k-1}{\alpha+k-1},$$

we known from (3.21) that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega (1+u)^p dx + m\int_\Omega (1+u)^p dx\\
\leq& -\frac{\mu}{3\cdot 2^k}\int_\Omega (1+u)^{p+k-1}dx + C_3\int_\Omega |\triangle v|^m dx + C_{13},
\end{aligned} \tag{3.22}$$

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where $C_{13}$ is a positive constant. Recalling Lemma 2.2, it can be obtained from (3.22) that

$$\begin{aligned}
&\int_\Omega (1+u)^p dx \\
\leq& -\frac{\mu}{3\cdot 2^k}\int_{t_0}^t\int_\Omega e^{-m(t-s)}(1+u)^{p+k-1}dxds + C_3\int_{t_0}^t\int_\Omega e^{-m(t-s)}|\triangle v|^m dxds \\
&+e^{-m(t-t_0)}\int_\Omega (1+u(\cdot,t_0))^p dx + C_{13}\int_{t_0}^t e^{-m(t-s)}ds \\
\leq& -\frac{\mu}{3\cdot 2^k}\int_{t_0}^t\int_\Omega e^{-m(t-s)}(1+u)^{p+k-1}dxds + C_3C_m\int_{t_0}^t\int_\Omega e^{-m(t-s)}(1+u)^{m\gamma}dxds \\
&+C_3C_m\|v(\cdot,t_0)\|^m_{W^{2,m}(\Omega)} + C_{14},
\end{aligned} \tag{3.23}$$

where $C_{14}$ is a positive constant. Since $\alpha > r-k+\frac{1}{e}+1$, then $m\gamma < p+k-1$, we have from Young's inequality that

$$\begin{aligned}
&C_3C_m\int_{t_0}^t\int_\Omega e^{-m(t-s)}(1+u)^{m\gamma}dxds \\
\leq& \frac{\mu}{3\cdot 2^k}\int_{t_0}^t\int_\Omega e^{-m(t-s)}(1+u)^{p+k-1}dxds + C_{15},
\end{aligned} \tag{3.24}$$

where $C_{15}$ is a positive constant. Inserting (3.24) into (3.23), we have

$$\int_\Omega (u+1)^p dx \leq C_{16}, \tag{3.25}$$

where $C_{16}$ is a positive constant.

For $p=\alpha$, we infer from (3.1), (3.7), (3.12) and (3.14) that

$$\frac{d}{dt}\int_\Omega (1+u)^p dx \leq -\frac{\mu}{3\cdot 2^{k-1}}\int_\Omega (1+u)^{p+k-1}dx + C_4\int_\Omega |\triangle v|^{\frac{e(p+k-1)}{e(p+k-1)-1}}dx + C_{17}. \tag{3.26}$$

Define

$$\widetilde{m} := \frac{e(p+k-1)}{e(p+k-1)-1},$$

Similar to (3.23), we have

$$\begin{aligned}
&\int_\Omega (1+u)^p dx \\
\leq& -\frac{\mu}{3\cdot 2^k}\int_{t_0}^t\int_\Omega e^{-\widetilde{m}(t-s)}(1+u)^{p+k-1}dxds + C_4C_{\widetilde{m}}\int_{t_0}^t\int_\Omega e^{-\widetilde{m}(t-s)}(1+u)^{\widetilde{m}\gamma}dxds \\
&+C_4C_{\widetilde{m}}\|v(\cdot,t_0)\|^{\widetilde{m}}_{W^{2,\widetilde{m}}(\Omega)} + C_{18},
\end{aligned} \tag{3.27}$$

where $C_{18}$ is a positive constant.

Since $\alpha = p > r-k+\frac{1}{e}+1$, then $\widetilde{m}\gamma < p+k-1$, we have from Young's inequality that

$$\widetilde{C}_2C_{\widetilde{m}}\int_{t_0}^t\int_\Omega e^{-\widetilde{m}(t-s)}(1+u)^{\widetilde{m}\gamma}dxds \leq \frac{\mu}{3\cdot 2^k}\int_{t_0}^t\int_\Omega e^{-\widetilde{m}(t-s)}(1+u)^{p+k-1}dxds + C_{19}, \tag{3.28}$$

where $C_{19}$ is a positive constant. Inserting (3.28) into (3.27), we have

$$\int_\Omega (u+1)^p dx \leq C_{20}, \tag{3.29}$$

where $C_{20}$ is a positive constant.

**Case (ii).** For $p > \alpha$ and $\alpha > \gamma-k+1$, define

$$m := \frac{p+k-1}{\alpha+k-1},$$

we have from (3.23) that

$$\begin{aligned}
&\int_\Omega (1+u)^p dx \\
\leq& -\frac{\mu}{3\cdot 2^k}\int_{t_0}^t\int_\Omega e^{-m(t-s)}(1+u)^{p+k-1}dxds + C_3C_m\int_{t_0}^t\int_\Omega e^{-m(t-s)}(1+u)^{m\gamma}dxds \\
&+C_3C_m\|v(\cdot,t_0)\|^m_{W^{2,m}(\Omega)} + C_{21}.
\end{aligned} \tag{3.30}$$

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Since $\alpha > r - k + 1$, then $m\gamma < p + k - 1$. We infer from Young's inequality and (3.30) that

$$\int_\Omega (u+1)^p dx \le C_{22}, \tag{3.31}$$

where $C_{22}$ is a positive constant. This complete the proof of Lemma 3.1. □

**Lemma 3.2.** Assume that $D, H, I$ and $g$ fulfill (1.2)-(1.5). Then

(i) For $0 < \gamma \le \frac{2}{n}$, if $\alpha > \gamma - k + 1$ and $\beta > 1 - k$, there exists a constant $C > 0$ such that $\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C$.

(ii) For $\frac{2}{n} < \gamma \le 1$, if $\alpha > \gamma - k + \frac{1}{e} + 1$ and $\beta > \max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1, \frac{(n\gamma-2)(\gamma+\frac{1}{e})}{n} - k + 1\}$ or $\alpha > \gamma - k + 1$ and $\beta > \max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1, \frac{(n\gamma-2)(\alpha+k-1)}{n} - k + 1\}$, there exists a constant $C > 0$ such that $\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le C$.

**Proof. Case (i)** Since $0 < \gamma \le \frac{2}{n}$, we have

$$n\gamma - 2 \le 0. \tag{3.32}$$

Lemma 2.4(ii) yields that

$$\|v(\cdot,t)\|_{L^s(\Omega)} \le C_{23} \quad \text{for all } t \in (0,T), \tag{3.33}$$

for any $s \ge 1$. Taking $p_1 > \max\{\frac{n\gamma}{2}, 1, \alpha, \beta\}$, this implies $\frac{p_1+k-1}{\beta+k-1} > 1$, and so we get

$$\|v(\cdot,t)\|_{L^{\frac{p_1+k-1}{\beta+k-1}}(\Omega)} \le C_{24} \quad \text{for all } t \in (0,T), \tag{3.34}$$

which, along with Lemma 3.1 (ii), we have for all $t \in (0,T)$

$$\|u(\cdot,t)\|_{L^{p_1}(\Omega)} \le C_{25}. \tag{3.35}$$

Since $p_1 > \frac{n\gamma}{2}$, we infer from Lemma 2.4(iii) that

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \le C_{26} \quad \text{for all } t \in (0,T). \tag{3.36}$$

By Lemma 3.1 (ii) again and let $p_2 > \max\{n\gamma, 1, \alpha, \beta\}$, one can find

$$\|u(\cdot,t)\|_{L^{p_2}(\Omega)} \le C_{27} \quad \text{for all } t \in (0,T). \tag{3.37}$$

This together with Lemma 2.4(iv), we obtain

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \le C_{28} \quad \text{for all } t \in (0,T). \tag{3.38}$$

This complete the proof of Case (i).

**Case (ii)** For $\frac{2}{n} < \gamma \le 1$. Since $\beta > \frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1$ and $\beta > \frac{(n\gamma-2)(\gamma+\frac{1}{e})}{n} - k + 1$, we have

$$\begin{aligned} \frac{n\gamma}{2} &< \frac{n}{nr-2}(\beta+k-1) + 1 - k, \\ \gamma - k + \frac{1}{e} + 1 &< \frac{n}{n\gamma-2}(\beta+k-1) + 1 - k, \\ \beta &< \frac{n}{nr-2}(\beta+k-1) + 1 - k. \end{aligned} \tag{3.39}$$

Taking $\max\{\frac{n\gamma}{2}, \gamma - k + \frac{1}{e} + 1, \beta\} < p_3 < \frac{n}{nr-2}(\beta+k-1) + 1 - k$, then

$$\frac{p_3+k-1}{\beta+k-1} \in (1, \frac{n}{n\gamma-2}). \tag{3.40}$$

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By Lemma 2.4(ii), we get

$$\|v(\cdot,t)\|_{L^{\frac{p_3+k-1}{\beta+k-1}}(\Omega)} \leq C_{29} \quad \text{for all } t \in (0,T), \tag{3.41}$$

which, along with Lemma 3.1 (i), we have

$$\|u(\cdot,t)\|_{L^{p_3}(\Omega)} \leq C_{30} \quad \text{for all } t \in (0,T). \tag{3.42}$$

Since $p_3 > \frac{n\gamma}{2}$, applying Lemma 2.4(iii), we obtain

$$\|v(\cdot,t)\|_{L^\infty(\Omega)} \leq C_{31} \quad \text{for all } t \in (0,T). \tag{3.43}$$

By Lemma 3.1(i) again and let $p_4 > \max\{n\gamma, \gamma - k + \frac{1}{e} + 1, \beta\}$, one can find

$$\|u(\cdot,t)\|_{L^{p_4}(\Omega)} \leq C_{32} \quad \text{for all } t \in (0,T). \tag{3.44}$$

This together with Lemma 2.4(iv), we obtain for all $t \in (0,T)$

$$\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \leq C_{33}. \tag{3.45}$$

Similarly, we infer from $\beta > \max\{\frac{(n\gamma-2)(n\gamma+2k-2)}{2n} - k + 1, \frac{(n\gamma-2)(\alpha+k-1)}{n} - k + 1\}$ that there exists a positive constant $p_5$ such that

$$\max\{\frac{n\gamma}{2}, \alpha, \beta\} < p_5 < \frac{n}{nr-2}(\beta + k - 1) + 1 - k, \tag{3.46}$$

thus $\frac{p_5+k-1}{\beta+k-1} \in (1, \frac{n}{n\gamma-2})$. Combining Lemma 2.4(iii) and Lemma 3.1(ii), we have $\|u(\cdot,t)\|_{L^{p_5}(\Omega)} \leq C_{34}$ and $\|v(\cdot,t)\|_{L^\infty(\Omega)} \leq C_{35}$. Using Lemma 3.1(ii) and Lemma 2.4(iv), we deduce that $\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)} \leq C_{36}$. This complete the proof of Case (ii). □

**The proof of Theorem 1.1.** From Lemma 3.2 and the well-known Moser-Alikakos iteration [39, 40, 45], we obtain the boundedness of $\|u\|_{L^\infty(\Omega)}$. The proof of Theorem 1.1 is complete by Lemma 2.1.

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