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Lagrange Duality and Saddle Point Optimality Conditions for Nonsmooth Interval-Valued Multiobjective Semi-infinite Programming Problems with Vanishing Constraints

Balendu Bhooshan Upadhyay [1,†], Shivani Sain [1,†] and Ioan Stancu-Minasian [2,†,*]

[1] Department of Mathematics, Indian Institute of Technology Patna, Patna 801103, Bihar, India; bhooshan@iitp.ac.in (B.B.U.); shivani_2321ma04@iitp.ac.in (S.S);
[2] "Gheorghe Mihoc-Caius Iacob" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, 050711, Bucharest, Romania; stancu_minasian@yahoo.com
* Correspondence: stancu_minasian@yahoo.com;
† These authors contributed equally to this work.

Abstract: In this article, we consider a class of nonsmooth interval-valued multiobjective semi-infinite programming problems with vanishing constraints (in short, NIMSIPVC). We introduce the VC-Abadie constraint qualification (in short, VC-ACQ) for NIMSIPVC and employ it to establish Karush-Kuhn-Tucker (in short, KKT)-type necessary optimality conditions. Related to NIMSIPVC, we formulate interval-valued vector Lagrange type dual and scalarized Lagrange type dual problems. Subsequently, we establish weak, strong, and converse duality results relating NIMSIPVC and corresponding dual problems. In addition, we introduce the notions of saddle points for interval-valued vector Lagrangian and scalarized Lagrangian of NIMSIPVC. Moreover, we establish the saddle point optimality criteria for NIMSIPVC. Various non-trivial examples are provided to demonstrate the validity of established results. To the best of our knowledge, optimality conditions, Lagrange type duality, and saddle point optimality criteria for NIMSIPVC have not been investigated yet via Clarke subdifferentials.

Keywords: Interval-valued multiobjective programming; vanishing constraints; Clarke subdifferential; Lagrange duality; saddle point

MSC: 49J52; 65G40; 90C29; 90C34; 90C46

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1. Introduction

In the realm of mathematical programming, any constrained optimization problem involving vanishing constraints is referred to as a mathematical programming problem with vanishing constraints (in short, MPVC). The formulation of MPVC has been presented by Achtziger and Kanzow [1]. It is imperative to note that the term vanishing constraints refers to the fact that in various applications of MPVC, some of the constraints are often seen to vanish or become redundant at some points of the feasible set. One of the primary challenges encountered in the investigation of MPVC is the fact that the feasible set of MPVC may be non-convex and non-connected, despite the presence of convex constraint functions (see, for instance, [1,2]). Furthermore, in general, the majority of standard constraint qualifications, such as Mangasarian-Fromovitz constraint qualification (in short, MFCQ) and linear independence constraint qualification (in short, LICQ), are violated at every feasible point of MPVC (see, [1,2]). For a more comprehensive study of MPVC in various settings, we refer the readers to [3–7] and the references cited therein. If the feasible set of MPVC is defined by an infinite number of inequality constraints, then MPVC is termed a semi-infinite programming problem with vanishing constraints (in short, SIPVC). Tung [8] has studied optimality conditions as well as duality results for SIPVC involving continuously differentiable functions.

Multiobjective optimization problems involve the simultaneous maximization or minimization of two or more conflicting objectives subject to some set of constraints. Due to their diverse applications in several real-world problems, including science and engineering (see, for instance, [9,10]),

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multiobjective optimization problems have been extensively studied by numerous researchers in various settings (see, [7,11–13] and the references cited therein). Maeda [14] has studied constraint qualifications for multiobjective optimization problems. Further, Li [15] has established KKT-type necessary optimality conditions for nonsmooth multiobjective optimization problems. Guu et al. [16] have derived strong KKT-type sufficient optimality criteria for multiobjective SIPVC under generalized convexity assumptions. Antczak [17] has studied necessary as well as sufficient optimality conditions for multiobjective SIPVC involving invex functions. Further, Tung [18,19] have derived KKT-type necessary optimality conditions, as well as duality results for multiobjective SIPVC involving smooth and nonsmooth functions under convexity assumptions.

In general, optimization problems involve deterministic values of the coefficients of objective and constraint functions, leading to precise solutions. However, it is significant to observe that many real-life optimization problems often involve uncertain or imprecise data due to measurement errors or variations due to market fluctuations. Therefore, several techniques have been developed in the literature for addressing optimization problems that involve uncertainty in data within various frameworks (see, for instance, [20–28] and the references cited therein). It is worthwhile to note that interval-valued optimization can effectively handle uncertain data even in those situations when it is difficult to determine an exact probability distribution or fuzzy membership function. Therefore, interval-valued optimization is the preferred method to address optimization problems involving uncertain data rather than stochastic and fuzzy optimization (see, [29,30]). Wu [31] has studied KKT-type optimality conditions for multiobjective interval-valued optimization problems. Further, Singh et al. [32] have derived KKT-optimality conditions for interval-valued multiobjective programming problems involving generalized differentiable functions. Tung [33] has established KKT-type optimality conditions for semi-infinite programming problems involving multiple interval-valued objective functions under convexity assumptions. Further, optimality conditions and duality results for an interval-valued SIPVC have been developed by Su and Dinh [34]. Recently, Yadav and Gupta [35] have investigated optimality criteria as well as duality results for multiobjective interval-valued semi-infinite programming problems with vanishing constraints.

Over the past few decades, Lagrange duality and saddle point optimality criteria have gained significant attention, see, for instance, [36–39]. Sawaragi et al. [40] have studied Lagrange duality theory for multiobjective optimization problems under convexity and regularity assumptions. Further, Luc [11] has widely discussed Lagrange duality and saddle point optimality conditions for multiobjective optimization problems involving set-valued data. Wang et al. [41] have further extended the results obtained by Sawargi et al. [40] for cone-subconvexlike functions. Jayswal et al. [42] have studied saddle point optimality conditions for interval-valued optimization problems involving nonsmooth functions. Further, Dar et al. [43] have studied optimality and saddle point optimality conditions for interval-valued nondifferentiable multiobjective fractional programming problems. Recently, Tung et al. [44] have studied Lagrange duality and saddle point optimality conditions for multiobjective SIPVC with vanishing constraints. However, it is worth noting that Lagrange duality and saddle point optimality conditions for a broader class of optimization problems, namely, NIMSIPVC, have not yet been studied in terms of Clarke subdifferentials.

It is worthwhile to note that several researchers have studied necessary optimality conditions for single objective as well as multiobjective optimization problems, see, for instance, [3,5,16,19, 45]. Moreover, Lagrange type duality and saddle point optimality criteria for various nonlinear programming problems have been studied by numerous researchers (see, for instance, [44,46,47] and the references cited therein). However, KKT-type necessary optimality conditions for NIMSIPVC have not been investigated yet via Clarke subdifferentials. Furthermore, Lagrange type duality and saddle point optimality criteria for interval-valued multiobjective SIPVC involving nonsmooth locally Lipschitz functions have not been explored before. In this paper, we aim to address this research gap by considering a class of interval-valued multiobjective semi-infinite programming problems with vanishing constraints involving nonsmooth locally Lipschitz functions. We introduce the notions

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of VC-stationary points and VC-ACQ for NIMSIPVC. Subsequently, we derive KKT-type necessary optimality conditions for NIMSIPVC by employing VC-ACQ. We formulate several Lagrange type dual problems corresponding to NIMSIPVC, in particular, interval-valued weak vector, interval-valued vector, and scalarized Lagrange type dual problems. We derive various weak, converse, and strong duality results related to NIMSIPVC and the corresponding dual problems. In addition, the notions of saddle points for the interval-valued vector Lagrangian and scalarized Lagrangian of NIMSIPVC are introduced in the present article. Further, we have established the saddle point optimality criteria for NIMSIPVC by establishing the relationships between saddle points and LU-optimal solutions of Lagrangians of NIMSIPVC and the primal problem NIMSIPVC, respectively.

The novelty and contributions of this paper are fourfold: In the first fold, we extend the corresponding results derived in [19,44,45] from smooth SIPVC to a nonsmooth category of optimization problems, namely, NIMSIPVC. In the second fold, we extend the corresponding results established by Joshi et al. [45] from single objective interval-valued SIPVC to multiobjective SIPVC involving nonsmooth interval-valued objective function. In the third fold, several well-known results, see, for instance, [18,19,44] are extended from SIPVC with real-valued objective functions to multiobjective SIPVC with interval-valued objective functions. In view of the fact that NIMSIPVC belongs to a more general class of optimization problems, we extend the corresponding results derived by Kanzi [48] from single objective semi-infinite programming problems to NIMSIPVC in the fourth fold. In addition, we extend several well-known results derived in [1,14,30] for a broader category of optimization problems, in particular, NIMSIPVC.

The rest of the article is organized in the following manner: Some basic mathematical preliminaries and fundamental concepts used in the sequel are discussed in Section 2. We establish KKT-type necessary optimality conditions for NIMSIPVC by employing VC-ACQ in Section 3. In Section 4, we formulate interval-valued vector Lagrange type dual problems corresponding to NIMSIPVC, followed by the weak, converse, and strong duality results. Further, we establish saddle point optimality criteria for NIMSIPVC by utilizing the saddle points of an interval-valued vector Lagrangian of NIMSIPVC. In addition, we have formulated the scalarized Lagrange type dual problems corresponding to NIMSIPVC and derived weak, converse, and strong duality results in Section 5. Moreover, in Section 5, we derive the saddle point optimality criteria for NIMSIPVC. Section 6 concludes the work presented in this paper and provides various future research avenues.

2. Mathematical Preliminaries

In this article, $\mathbb{N}$ and $\mathbb{R}^n$ are used to symbolize the set of natural numbers and Euclidean space of dimension n, respectively, and $\mathbb{R}^n_+$ represents the non-negative orthant of $\mathbb{R}^n$. The standard inner product is denoted by the symbol $\langle \cdot, \cdot \rangle$. Let $\mathcal{L} \subseteq \mathbb{R}$ be an infinite set. Then $\mathbb{R}^{|\mathcal{L}|}$ signify the linear space, which is defined as follows:

$$\mathbb{R}^{|\mathcal{L}|} := \{\mu = (\mu_k)_{k\in\mathcal{L}} \mid \mu_k = 0, \ \forall k \in \mathcal{L}, \ \text{except } \mu_k \neq 0, \ \text{for finitely many } k \in \mathcal{L}\}.$$

The symbol $\mathbb{R}^{|\mathcal{L}|}_+$ is used to denote the positive cone of $\mathbb{R}^{|\mathcal{L}|}$, which is defined as follows:

$$\mathbb{R}^{|\mathcal{L}|}_+ := \{\mu = (\mu_k)_{k\in\mathcal{L}} \in \mathbb{R}^{|\mathcal{L}|} \mid \mu_k \geq 0, \ \forall k \in \mathcal{L}\}.$$

Let $\mathcal{C} \subseteq \mathbb{R}^n$. The symbols $\text{cl}(\mathcal{C}), \ \text{span}(\mathcal{C}), \text{co}(\mathcal{C}), \text{pos}(\mathcal{C})$ are used to denote the closure, span, convex hull, and positive conic hull of $\mathcal{C}$, respectively. The following sets will be used in the subsequent part of this article:

$$\mathcal{C}^- := \{z \in \mathbb{R}^n : \langle z, \mu \rangle \leq 0, \forall \mu \in \mathcal{C}\},$$
$$\mathcal{C}^\ominus := \{z \in \mathbb{R}^n : \langle z, \mu \rangle < 0, \forall \mu \in \mathcal{C}\},$$
$$\mathcal{C}^0 := \{z \in \mathbb{R}^n : \langle z, \mu \rangle = 0, \forall \mu \in \mathcal{C}\}.$$

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Let us consider $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{R}^n$. The following relations are from [18,19].

$$\text{pos}(C_1 \cup C_2) = \text{pos}(C_1) + \text{pos}(C_2),$$
$$\text{span}(C_1 \cup C_2) = \text{span}(C_1) + \text{span}(C_2).$$

Consider any $\mu, \nu \in \mathbb{R}^n$. We define the following notations that will be used in the sequel of this paper.

$$\mu \prec \nu \Leftrightarrow \mu_j < \nu_j, \forall j = 1, 2, \ldots, l.$$
$$\mu \preceq \nu \Leftrightarrow \begin{cases} \mu_j \leq \nu_j, & \forall j = 1, 2, \ldots, l, \\ \mu_k < \nu_k, & \text{for at least one } k \in \{1, 2, \ldots, l\}. \end{cases}$$
$$\mu \precsim \nu \Leftrightarrow \mu_j \leq \nu_j, \forall j \in \{1, 2, \ldots, l\}.$$

A function $\Psi : H \subseteq \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function around $\mu \in H$, if there exists a neighbourhood V of μ and a constant $K > 0$ such that $|\Psi(u) - \Psi(v)| \leq K||u - v||, \forall u, v \in V$.

In the following definition, we recall the notion of a contingent cone of a non-empty subset of $\mathbb{R}^n$ as given in [49].

Definition 1. *Let $\emptyset \neq H \subseteq \mathbb{R}^n$ and $\overline{\mu} \in \text{cl}(H)$. The contingent cone to H at $\overline{\mu}$ is symbolized by $\mathcal{T}(\overline{\mu}, H)$, and is given by:*

$$\mathcal{T}(\overline{\mu}, H) := \{d \in \mathbb{R}^n |\ d_k \in \mathbb{R}^n,\ t_k \downarrow 0,\ \overline{\mu} + t_k d_k \in H,\ \forall k \in \mathbb{N}\}.$$

The following definition of a convex subset of $\mathbb{R}^n$ is from [50].

Definition 2. *Let $H \subseteq \mathbb{R}^n$ and μ, w be any two arbitrary distinct elements of H. H is said to be a convex set if the following condition holds:*

$$(1 - \tau)\mu + \tau w \in H,\ \forall \tau \in [0, 1],\ .$$

In the following definition, we recall the notions of Clarke's directional derivative and Clarke's subdifferential for a real-valued locally Lipschitz function (see [51]).

Definition 3. *Let $\Psi : H \subseteq \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function around $\mu \in H$. Then*

(i) the Clarke's directional derivative of Ψ at μ in direction ν is defined as follows:

$$\Psi^\circ(\mu; \nu) = \limsup_{y \to \mu, t \downarrow 0} \frac{\Psi(y + t\nu) - \Psi(\mu)}{t}.$$

(ii) the Clarke's subdifferential of Ψ at μ is given by:

$$\partial_c \Psi(\mu) := \{\eta \in \mathbb{R}^n | \Psi^\circ(\mu; \nu) \geq \langle \eta, \nu \rangle,\ \forall \nu \in \mathbb{R}^n\}.$$

The following Lemma from [51] presents various properties of Clarke's directional derivative and Clarke's subdifferential of a real-valued locally Lipschitz function, which will be used in the sequel.

Lemma 1. *Let Ψ and $\phi : H \subseteq \mathbb{R}^n \to \mathbb{R}$ be two locally Lipschitz functions around $\mu \in H$. Then the following statements hold:*

(i) $\partial_c \Psi(\mu)$ is a non-empty, convex, and compact subset of $\mathbb{R}^n$.
(ii) The Clarke directional derivative of Ψ at μ for every $\nu \in \mathbb{R}^n$, satisfy the following:

$$\Psi^\circ(\mu; \nu) = max\{\langle \eta, \nu \rangle |\ \eta \in \partial_c \Psi(\mu)\}.$$

(iii) The set-valued map $\mu \rightrightarrows \partial_c \Psi(\mu)$ is an upper semicontinuous set-valued function, provided Ψ is locally Lipschitz on $\mathbb{R}^n$.

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(iv) *For any $\sigma \in \mathbb{R}$, we have $\partial_c(\sigma\Psi(\mu)) = \sigma\partial_c\Psi(\mu)$. Furthermore, $\partial_c(\Psi + \phi)(\mu) \subseteq \partial_c\Psi(\mu) + \partial_c\phi(\mu)$.*
(v) *If Ψ is locally Lipschitz on an open set containing $[\mu, v]$, then*

$$\Psi(\mu) - \Psi(v) = \langle \xi^*, v - \mu \rangle,$$

for some $z \in [\mu, v)$ and $\xi^* \in \partial_c\Psi(z)$.

In the following, we recall the definition of a convex function defined on a convex set $H \subseteq \mathbb{R}^n$ (see, [52,53]).

Definition 4. *Let $\Psi : H \subseteq \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. Ψ is said to be;*

(i) *convex at $\overline{\mu} \in H$, provided the following condition holds:*

$$\Psi(\mu) - \Psi(\overline{\mu}) \geq \langle \xi, \mu - \overline{\mu} \rangle, \ \forall \xi \in \partial_c\Psi(\mu), \forall \mu \in H.$$

(ii) *strictly convex at $\overline{\mu} \in F$, provided the following condition holds:*

$$\Psi(\mu) - \Psi(\overline{\mu}) > \langle \xi, \mu - \overline{\mu} \rangle, \ \forall \xi \in \partial_c\Psi(\mu), \ \forall \mu \in H \setminus \{\overline{\mu}\}.$$

Let us discuss the interval analysis presented in [54].
Let $\mathcal{I}$ be the collection of all closed intervals in $\mathbb{R}$, defined as follows:

$$\mathcal{I} := \{[m^L, m^U] | m^L \leq m^U\}.$$

For any two intervals $\mathcal{M} = [m^L, m^U]$ and $\mathcal{N} = [n^L, n^U] \in \mathcal{I}$, we define

(a$_1$) $\mathcal{M} + \mathcal{N} = \{m + n : m \in \mathcal{M} \text{ and } n \in \mathcal{N}\} = [m^L + n^L, m^U + n^U]$.
(b$_1$) $-\mathcal{M} = \{-m : m \in \mathcal{M}\} = [-m^U, -m^L]$.

It is worth noting that any real number m can be represented as a closed interval, since $\mathcal{M}_m = [m, m]$. Let $\mathcal{M} = [m^L, m^U]$ and $\mathcal{N} = [n^L, n^U] \in \mathcal{I}$. We define the following relations:

(a$_2$) $\mathcal{M} \preceq_{LU} \mathcal{N} \iff m^L \leq n^L$ and $m^U \leq n^U$,
(b$_2$) $\mathcal{M} \prec_{LU} D \iff \mathcal{M} \preceq_{LU} \mathcal{N}$ and $\mathcal{M} \neq \mathcal{N}$, that is, one of the following condition is satisfied:

$$m^L < n^L \text{ and } m^U < n^U \text{ or } m^L \leq n^L \text{ and } m^U < n^U, \text{ or } m^L < n^L \text{ and } m^U \leq n^U.$$

(c$_2$) $\mathcal{M} \prec^s_{LU} \mathcal{N} \iff m^L < n^L$ and $m^U < n^U$.

Let $\mathcal{I}^p$ be the collection of all interval-valued vectors where each element $\mathbf{M} \in \mathcal{I}^p$ can be defined as:

$\mathbf{M} = (\mathcal{M}_1, \ldots, \mathcal{M}_p)$ such that for every $i = 1, 2, \ldots, p$, $\mathcal{M}_i = [m_i^L, m_i^U]$ is a closed interval. Consider two arbitrary interval-valued vectors, $\mathbf{M}$ and $\mathbf{N}$. Then

(a$_3$) $\mathbf{M} \preceq_{LU} \mathbf{N} \Leftrightarrow \mathcal{M}_i \preceq_{LU} \mathcal{N}_i, \ \forall i = 1, 2, \ldots, p$.
(b$_3$) $\mathbf{M} \prec_{LU} \mathbf{N} \Leftrightarrow \mathcal{M}_i \preceq_{LU} \mathcal{N}_i, \ \forall i = 1, 2, \ldots, p$, and $\mathcal{M}_k \prec_{LU} \mathcal{N}_k$ for some $i \neq k$.
(c$_3$) $\mathbf{M} \prec^s_{LU} \mathbf{N} \Leftrightarrow \mathcal{M}_i \prec^s_{LU} \mathcal{N}_i, \ \forall i = 1, 2, \ldots, p$.

Remark 1. *1. If $\mathbf{M} \nprec_{LU} \mathbf{N}$, then from (a$_3$), (b$_3$), and (c$_3$) we have*

$$\begin{aligned} &(\mathcal{M}^L - \mathcal{N}^L) \notin -\mathbb{R}^l_+ \setminus \{0\}, \ (\mathcal{M}^U - \mathcal{N}^U) \notin -\mathbb{R}^l_+ \setminus \{0\}, \\ \text{or } &(\mathcal{M}^L - \mathcal{N}^L) \notin -\mathbb{R}^l_+ \setminus \{0\}, (\mathcal{M}^U - \mathcal{N}^U) \notin -\mathbb{R}^l_+, \\ \text{or } &(\mathcal{M}^L - \mathcal{N}^L) \notin -\mathbb{R}^l_+, \ (\mathcal{M}^U - \mathcal{N}^U) \notin -\mathbb{R}^l_+ \setminus \{0\}, \end{aligned}$$

where $\mathcal{M}^L = (\mathcal{M}^L_1, \mathcal{M}^L_2, \ldots, \mathcal{M}^L_l) \in \mathbb{R}^l$, $\mathcal{M}^U = (\mathcal{M}^U_1, \mathcal{M}^U_2, \ldots, \mathcal{M}^U_l) \in \mathbb{R}^l$.

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2. *If* $\mathbf{M} \not\preceq^{s}_{LU} \mathbf{N}$, *then from* ($a_3$), ($b_3$), *and* ($c_3$), *we have*

$$(\mathcal{M}^L - \mathcal{N}^L) \notin -\operatorname{int}\mathbb{R}^+_l,\ (\mathcal{M}^U - \mathcal{N}^U) \notin -\operatorname{int}\mathbb{R}^+_l.$$

A function $\Psi : \mathbb{R}^n \to \mathcal{I}$ is termed as an interval-valued function if $\Psi(\mu) = \left[\Psi^L(\mu), \Psi^U(\mu)\right]$ where $\Psi^L, \Psi^U : \mathbb{R}^n \to \mathbb{R}$ are real-valued functions such that $\Psi^L(\mu) \leq \Psi^U(\mu), \forall \mu \in \mathbb{R}^n$. An interval-valued function $\Psi : H \to \mathcal{I}$ is known as locally Lipschitz function on H if Ψ^L, Ψ^U are locally Lipschitz on H.

Let us define the following sets for a nonempty subset $\mathcal{A} \subset \mathcal{I}^p$ as follows:

$$\begin{aligned}
\text{WMin}\mathcal{A} &:= \{M \in \mathcal{A} | (I^L - M^L) \cap -\operatorname{int}\mathbb{R}^l_+ = \emptyset,\ (I^U - M^U) \cap -\operatorname{int}\mathbb{R}^l_+ = \emptyset, \forall I \in \mathcal{A}\},\\
\text{Min}\mathcal{A} &:= \{M \in \mathcal{A} | (I^L - M^L) \cap -\mathbb{R}^l_+ \setminus \{0\} = \emptyset,\ (I^U - M^U) \cap -\mathbb{R}^l_+ \setminus \{0\} = \emptyset,\ \forall I \in \mathcal{A},\\
&\quad \text{or} (I^L - M^L) \cap -\mathbb{R}^l_+ = \emptyset,\ (I^U - M^U) \cap -\mathbb{R}^l_+ \setminus \{0\} = \emptyset,\ \forall I \in \mathcal{A},\\
&\quad \text{or } (I^L - M^L) \cap -\mathbb{R}^l_+ \setminus \{0\} = \emptyset,\ (I^U - M^U) \cap -\mathbb{R}^l_+ = \emptyset,\ \forall I \in \mathcal{A}\}.
\end{aligned}$$

The notion of LU-convexity of an interval-valued function defined on a convex subset is presented in the following definition (see, for instance, [30]).

Definition 5. *Let* $\Psi : H \subseteq \mathbb{R}^n \to \mathcal{I}$ *be any interval-valued function on a convex set* H. Ψ *is said to be LU-convex at* $\mu \in H$, *if* Ψ^L *and* Ψ^U *are convex at* μ.

The following lemmas from [50,55,56] will be instrumental in establishing KKT-type necessary optimality conditions for NIMSIPVC.

Lemma 2. *Let* $\{\mathcal{D}_i | i \in \mathcal{L}\}$ *be any arbitrary collection of non-empty convex sets in* $\mathbb{R}^n$. *Further, let*

$$\mathcal{B} = \text{pos}\left(\bigcup_{i \in \mathcal{L}} \mathcal{D}_i\right).$$

Then, any non-zero vector lying in set $\mathcal{B}$ *can be expressed as a non-negative combination of at most n linearly independent vectors, each belonging to some different set* $\mathcal{D}_i$.

Lemma 3. *Let* $\mathcal{D}$, $\mathcal{E}$, *and* $\mathcal{M}$ *be any arbitrary (need not be finite) index sets. Consider the maps* $d_i : \mathcal{D} \to \mathbb{R}^n$, $e_j : \mathcal{E} \to \mathbb{R}^n$, *and* $f_m : \mathcal{M} \to \mathbb{R}^n$ *as follows:*

$$\begin{aligned}
d_i &= d(i) = (d_1(i), \ldots, d_n(i)),\\
e_j &= e(j) = (e_1(j), \ldots, e_n(j)),\\
f_m &= f(m) = (f_1(m), \ldots, f_n(m)).
\end{aligned}$$

Further, suppose that the set $\text{co}\{d_i | i \in \mathcal{D}\} + \text{pos}\{e_j | j \in \mathcal{E}\} + \text{span}\{f_m | m \in \mathcal{M}\}$ *is a closed set. Then the following statements are equivalent:*
Statement I. *The following system of inequalities*

$$\begin{aligned}
\langle d_i, \nu \rangle &< 0, i \in \mathcal{D}, \mathcal{D} \neq \emptyset,\\
\langle e_j, \nu \rangle &\leq 0, j \in \mathcal{E},\\
\langle f_m, \nu \rangle &= 0, m \in \mathcal{M},
\end{aligned}$$

has no solution $\nu \in \mathbb{R}^n$.
Statement II. *The following relation holds:*

$$0 \in \text{co}\{d_i | i \in \mathcal{D}\} + \text{pos}\{e_j | j \in \mathcal{E}\} + \text{span}\{f_m | m \in \mathcal{M}\}.$$

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Lemma 4. *Suppose that $\mathcal{D}$ is any non-empty and compact subset of $\mathbb{R}^n$. Then the following statements hold:*

(a) The convex hull of $\mathcal{D}$ is a compact set.
(b) The $\operatorname{pos} \mathcal{D}$ is a closed cone, provided $0 \notin \operatorname{co} \mathcal{D}$.

3. Optimality Conditions for NIMSIPVC

In this section, we introduce the notion of a VC-stationary point and VC-ACQ for NIMSIPVC. By employing VC-ACQ, we derive KKT-type necessary optimality conditions for NIMSIPVC in terms of Clarke subdifferentials.

Consider the following nonsmooth multiobjective interval-valued semi-infinite programming problem with vanishing constraints on $\mathbb{R}^n$, as follows:

$$\begin{aligned} \text{NIMSIPVC Minimize } \mathcal{F}(\mu) &= (\mathcal{F}_1(\mu), \mathcal{F}_2(\mu), \ldots, \mathcal{F}_l(\mu)), \\ &= ([\mathcal{F}_1^L(\mu), \mathcal{F}_1^U(\mu)], [\mathcal{F}_2^L(\mu), \mathcal{F}_2^U(\mu)] \ldots, [\mathcal{F}_l^L(\mu), \mathcal{F}_l^U(\mu]), \\ \text{subject to } \Psi_k(\mu) &\leq 0, \ \forall k \in \mathcal{L}, \\ \zeta_i(\mu) &= 0, \ \forall i \in \mathcal{B} = \{1, 2, \ldots, r\}, \\ \mathcal{Q}_i(\mu) &\geq 0, \ \forall i \in \mathcal{C} = \{1, 2, \ldots, s\}, \\ \mathcal{Q}_i(\mu)\mathcal{R}_i(\mu) &\leq 0, \ \forall i \in \mathcal{C} = \{1, 2, \ldots, s\}, \end{aligned}$$

where $\mathcal{F}_i^L$, $\mathcal{F}_i^U : \mathbb{R}^n \to \mathbb{R}$ $(i \in \mathcal{J}^{\mathcal{F}} = \{1, 2, \ldots, l\})$, $\Psi_k : \mathbb{R}^n \to \mathbb{R}$ $(k \in \mathcal{L})$, $\zeta_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in \mathcal{B})$, $\mathcal{Q}_i$, $\mathcal{R}_i : \mathbb{R}^n \to \mathbb{R}$ $(i \in \mathcal{C})$ are locally Lipschitz functions on $\mathbb{R}^n$. Notably, $\mathcal{L}$ need not be a finite set.

Remark 2.
1. *If $\mathcal{L}$ is a finite set, $\mathcal{F}_i^L(\mu) = \mathcal{F}_i^U(\mu)$, $\forall i \in \mathcal{J}^{\mathcal{F}}$, $\forall \mu \in \mathbb{R}^n$, and if $\mathcal{J}^{\mathcal{F}} = \{1\}$, then NIMSIPVC reduces to the problem MPVC as considered by Achtziger and Kanzow [1].*
2. *If $\mathcal{J}^{\mathcal{F}} = \{1\}$, then NIMSIPVC reduces to a semi-infinite interval-valued optimization problem as considered by Joshi et al. [45].*
3. *If $\mathcal{J}^{\mathcal{F}} = \{1\}$, $\mathcal{F}_1^L(\mu) = \mathcal{F}_1^U(\mu)$, $\forall \mu \in \mathbb{R}^n$, and if $\mathcal{B} = \emptyset = \mathcal{C}$ then, NIMSIPVC reduces to the semi-infinite programming problem which was considered by Kanzi [48].*
4. *If $\mathcal{F}_i^L(\mu) = \mathcal{F}_i^U(\mu)$, $\forall i \in \mathcal{J}^{\mathcal{F}}$, $\forall \mu \in \mathbb{R}^n$ and $\mathcal{L}$ is a finite set and if $\mathcal{B} = \emptyset = \mathcal{C}$, then NIMSIPVC reduces to the multiobjective constrained optimization problem (P) considered by Maeda [14].*
5. *If $\mathcal{B} = \emptyset = \mathcal{C}$ and if $\mathcal{F}_i^L$, $\mathcal{F}_i^U$, $\Psi_\tau : \mathbb{R}^n \to \mathbb{R}$ are real-valued functions for every $i \in \mathcal{J}^{\mathcal{F}}$, and if $\tau \in \mathcal{L}$, respectively, then NIMSIPVC reduces to the problem (P), considered by Tung [33].*
6. *If $\mathcal{F}_i^L(\mu) = \mathcal{F}_i^U(\mu)$, $\forall i \in \mathcal{J}^{\mathcal{F}}$, $\forall \mu \in \mathbb{R}^n$, then NIMSIPVC reduces to the problem (P) as considered by Tung [19].*
7. *If $\mathcal{F}_i^L(\mu) = \mathcal{F}_i^U(\mu)$, $\forall \mu \in \mathbb{R}^n$, and if for every $\mu \in \mathbb{R}^n$, $\partial_c \mathcal{F}_i^L(\mu) = \{\nabla \mathcal{F}_i^L(\mu)\}$, $\partial_c \mathcal{F}_i^U(\mu) = \{\nabla \mathcal{F}_i^U(\mu)\}$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\partial_c \Psi_k(\mu) = \{\nabla \Psi_k(\mu)\}$ $(k \in \mathcal{L})$, $\partial_c \zeta_i(\mu) = \{\nabla \zeta_i(\mu)\}$ $(i \in \mathcal{B})$, $\partial_c \mathcal{Q}_i(\mu) = \{\nabla \mathcal{Q}_i(\mu)\}$ $(i \in \mathcal{C})$, $\partial_c \mathcal{R}_i(\mu) = \{\nabla \mathcal{R}_i(\mu)\}$ $(i \in \mathcal{C})$, then NIMSIPVC reduces to the problem (P) as considered by Tung [18] and Tung et al. [44].*

The feasible set for NIMSIPVC is given by:

$$\mathcal{G} := \{\mu \in \mathbb{R}^n | \Psi_k(\mu) \leq 0, \ k \in \mathcal{L}, \ \zeta_i(\mu) = 0, \ i \in \mathcal{B}, \ \mathcal{Q}_i(\mu) \geq 0, i \in \mathcal{C}, \ \mathcal{Q}_i(\mu)\mathcal{R}_i(\mu) \leq 0, \ i \in \mathcal{C}\}. \quad (1)$$

Let $\overline{\mu} \in \mathcal{G}$. Then, the following sets will be used in the sequel:

$$\mathbb{P}(\overline{\mu}) := \{k \in \mathcal{L} | \Psi_k(\mu) = 0\}, \ \mathbb{P}^{\Psi}(\overline{\mu}) := \{\sigma^{\Psi} \in \mathbb{R}_+^{|\mathcal{L}|} | \sigma_k^{\Psi} \Psi_k(\overline{\mu}) = 0, \ \forall k \in \mathcal{L}\},$$

where $\mathbb{P}(\overline{\mu})$ signify the index set of all active inequality constraints and $\mathbb{P}^{\Psi}(\overline{\mu})$ contains all active constraint multipliers at $\overline{\mu}$, respectively.

In the following definition, we recall the notions of LU-efficient solutions for NIMSIPVC (see, [19,44]).

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Definition 6. *Let $\overline{\mu} \in \mathcal{G}$. Then $\overline{\mu}$ is a*

(i) *locally LU-efficient solution of NIMSIPVC, if there exists a neighborhood V of $\overline{\mu}$ such that for every $\mu \in V \cap \mathcal{G}$, the following conditions hold:*

$$\mathcal{F}_i(\mu) \npreceq_{LU} \mathcal{F}_i(\overline{\mu}),\ \forall i \in \mathcal{J}^{\mathcal{F}},$$
$$\mathcal{F}_j(\mu) \nprec_{LU} \mathcal{F}_j(\overline{\mu}),\ \text{for at least one } j \in \mathcal{J}^{\mathcal{F}}.$$

The symbol Eff_{loc} is used to denote the set of all locally LU-efficient solutions of NIMSIPVC.

(ii) *locally weakly LU-efficient solution of NIMSIPVC, if there exists a neighborhood V of $\overline{\mu}$ such that for any $\mu \in \mathcal{G} \cap V$, the following condition holds:*

$$\mathcal{F}_i(\mu) \nprec^{s}_{LU} \mathcal{F}_i(\overline{\mu}),\ \forall i \in \mathcal{J}^{\mathcal{F}}.$$

The set of all locally weakly LU-efficient solutions of NIMSIPVC is denoted by $WEff_{loc}$.

Remark 3. 1. *If $V = \mathbb{R}^n$ in Definition 6, then $\overline{\mu} \in \mathcal{G}$ is known as an LU-efficient and weakly LU-efficient solution of NIMSIPVC, respectively.*

2. *The symbols Eff and WEff are used to denote the sets of all LU-efficient and weakly LU-efficient solutions of NIMSIPVC, respectively.*

Consider an arbitrary feasible element $\overline{\mu}$. Then the following index sets are defined as follows:

$$\begin{aligned}
\mathcal{H}_{+}(\overline{\mu}) &:= \{i \in \mathcal{J} | \mathcal{Q}_i(\overline{\mu}) > 0\},\\
\mathcal{H}_{0}(\overline{\mu}) &:= \{i \in \mathcal{C} | \mathcal{Q}_i(\overline{\mu}) = 0\},\\
\mathcal{H}_{+0}(\overline{\mu}) &:= \{i \in \mathcal{C} | \mathcal{Q}_i(\overline{\mu}) > 0,\ \mathcal{R}_i(\overline{\mu}) = 0\},\\
\mathcal{H}_{+-}(\overline{\mu}) &:= \{i \in \mathcal{C} | \mathcal{Q}_i(\overline{\mu}) > 0,\ \mathcal{R}_i(\overline{\mu}) < 0\},\\
\mathcal{H}_{0+}(\overline{\mu}) &:= \{i \in \mathcal{C} | \mathcal{Q}_i(\overline{\mu}) = 0,\ \mathcal{R}_i(\overline{\mu}) > 0\},\\
\mathcal{H}_{00}(\overline{\mu}) &:= \{i \in \mathcal{C} | \mathcal{Q}_i(\overline{\mu}) = 0,\ \mathcal{R}_i(\overline{\mu}) = 0\},\\
\mathcal{H}_{0-}(\overline{\mu}) &:= \{i \in \mathcal{C} | \mathcal{Q}_i(\overline{\mu}) = 0,\ \mathcal{R}_i(\overline{\mu}) < 0\}.
\end{aligned}$$

The following definition extends the notion of a VC-stationary point for NIMSIPVC from [18].

Definition 7. *Let $\overline{\mu}$ be an arbitrary feasible element. Then $\overline{\mu}$ is known as a VC-stationary point of NIMSIPVC if there exists $(\lambda^L, \lambda^U, \sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}^l \times \mathbb{R}^l \times \mathbb{P}^{\Psi}(\overline{\mu}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that the following condition holds:*

$$\begin{aligned}
0 \in \sum_{i \in \mathcal{J}^{\mathcal{F}}} \Big(\lambda_i^L \partial_c \mathcal{F}_i^L(\overline{\mu}) + \lambda_i^U \partial_c \mathcal{F}_i^U(\overline{\mu})\Big) + \sum_{k \in \mathcal{L}} \sigma_k^{\Psi} \partial_c \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \sigma_i^{\zeta} \partial_c \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{Q}} \partial_c \mathcal{Q}_i(\overline{\mu})\\
+ \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{R}} \partial_c \mathcal{R}_i(\overline{\mu}),
\end{aligned} \tag{2}$$

where $\sum_{i \in \mathcal{J}^{\mathcal{F}}} (\lambda_i^L + \lambda_i^U) = 1,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(\overline{\mu})} = 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{00}(\overline{\mu}) \cup \mathcal{H}_{0-}(\overline{\mu})} \geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{+0}(\overline{\mu}) \cup \mathcal{H}_{00}(\overline{\mu})} \geq 0,$ *and* $\sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(\overline{\mu}) \cup \mathcal{H}_{0-}(\overline{\mu}) \cup \mathcal{H}_{0+}(\overline{\mu})} = 0.$

The symbol VC_{SP} is used to denote the set of all VC-stationary points of NIMSIPVC.

Remark 4. *If $\mathcal{F}_i^L(\mu) = \mathcal{F}_i^U(\mu),\ \forall i \in \mathcal{J}^{\mathcal{F}},\ \forall \mu \in \mathbb{R}^n$ then, Definition 7 reduces to Definition 2.2 presented by Hoheisel and Kanzow [2].*

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For any element $\overline{\mu} \in \mathcal{G}$, define the following sets:

$$
\begin{aligned}
&\mathbb{P}_+^{\Psi}(\overline{\mu}) := \{k \in \mathbb{P}(\overline{\mu}) | \sigma_k^{\Psi} > 0\}, \\
&\mathcal{B}_+^{\zeta}(\overline{\mu}) := \{i \in \mathcal{B} | \sigma_i^{\zeta} > 0\}, \qquad \mathcal{B}_-^{\zeta}(\overline{\mu}) := \{i \in \mathcal{B} | \sigma_i^{\zeta} < 0\}, \\
&\overline{\mathbb{H}}_0^{+}(\overline{\mu}) := \{i \in \mathcal{H}_0(\overline{\mu}) | \sigma_i^{\mathcal{Q}} > 0\}, \qquad \overline{\mathbb{H}}_0^{-}(\overline{\mu}) := \{i \in \mathcal{H}_0(\overline{\mu}) | \sigma_i^{\mathcal{Q}} < 0\}, \\
&\overline{\mathbb{H}}_{0+}^{+}(\overline{\mu}) := \{i \in \mathcal{H}_{0+}(\overline{\mu}) | \sigma_i^{\mathcal{Q}} > 0\}, \qquad \overline{\mathbb{H}}_{0+}^{-}(\overline{\mu}) := \{i \in \mathcal{H}_{0+}(\overline{\mu}) | \sigma_i^{\mathcal{Q}} < 0\}, \\
&\overline{\mathbb{H}}_{0-}^{+}(\overline{\mu}) := \{i \in \mathcal{H}_0(\overline{\mu}) | \sigma_i^{\mathcal{Q}} > 0\}, \\
&\mathbb{H}_{+0}^{+}(\overline{\mu}) := \{i \in \mathcal{H}_{+0}(\overline{\mu}) | \sigma_i^{\mathcal{R}} > 0\}, \qquad \mathbb{H}_{+0}^{-}(\overline{\mu}) := \{i \in \mathcal{H}_{+0}(\overline{\mu}) | \sigma_i^{\mathcal{R}} < 0\}, \\
&\mathbb{H}_{+-}^{+}(\overline{\mu}) := \{i \in \mathcal{H}_{+-}(\overline{\mu}) | \sigma_i^{\mathcal{R}} < 0\}, \qquad \mathbb{H}_{0+}^{+}(\overline{\mu}) := \{i \in \mathcal{H}_{0+}(\overline{\mu}) | \sigma_i^{\mathcal{R}} > 0\}, \\
&\mathbb{H}_{0+}^{-}(\overline{\mu}) := \{i \in \mathcal{H}_{0+}(\overline{\mu}) | \sigma_i^{\mathcal{R}} < 0\}, \qquad \mathbb{H}_{00}^{+}(\overline{\mu}) := \{i \in \mathcal{H}_{00}(\overline{\mu}) | \sigma_i^{\mathcal{R}} > 0\}, \\
&\mathbb{H}_{00}^{-}(\overline{\mu}) := \{i \in \mathcal{H}_{00}(\overline{\mu}) | \sigma_i^{\mathcal{R}} < 0\}, \qquad \mathbb{H}_{0-}^{+}(\overline{\mu}) := \{i \in \mathcal{H}_{0-}(\overline{\mu}) | \sigma_i^{\mathcal{R}} > 0\}.
\end{aligned}
$$

Now, we extend the definition of a VC-linearized cone given by Tung [33] from smooth MSIP to a broader class of optimization problems, namely, NIMSIPVC.

Definition 8. *Let $\overline{\mu} \in \mathcal{G}$. The VC-linearized cone at $\overline{\mu}$ is given by:*

$$
\begin{aligned}
\mathcal{L}_{VC}(\overline{\mu}) := \{\nu \in \mathbb{R}^n | &\langle \eta_k^{\Psi}, \nu \rangle \leq 0,\ \eta_k^{\Psi} \in \partial_c \Psi_k(\overline{\mu}),\ \forall k \in \mathbb{P}(\overline{\mu}), \\
&\langle \eta_i^{\zeta}, \nu \rangle = 0,\ \eta_i^{\zeta} \in \partial_c \zeta_i(\overline{\mu}),\ \forall i \in \mathcal{B}, \\
&\langle \eta_i^{\mathcal{Q}}, \nu \rangle = 0,\ \eta_i^{\mathcal{Q}} \in \partial_c \mathcal{Q}_i(\overline{\mu}),\ \forall i \in \mathcal{H}_{0+}(\overline{\mu}), \\
&\langle \eta_i^{\mathcal{Q}}, \nu \rangle \geq 0,\ \eta_i^{\mathcal{Q}} \in \partial_c \mathcal{Q}_i(\overline{\mu}),\ \forall i \in \mathcal{H}_{00}(\overline{\mu}) \cup \mathcal{H}_{0-}(\overline{\mu}), \\
&\langle \eta_i^{\mathcal{R}}, \nu \rangle \leq 0,\ \eta_i^{\mathcal{R}} \in \partial_c \mathcal{R}_i(\overline{\mu}),\ \forall i \in \mathcal{H}_{+0}(\overline{\mu}), \\
&\langle \eta_i^{\mathcal{R}}, \nu \rangle \leq 0,\ \eta_i^{\mathcal{R}} \in \partial_c \mathcal{R}_i(\overline{\mu}),\ \forall i \in \mathcal{H}_{00}(\overline{\mu})\}.
\end{aligned}
$$

For an arbitrary element $\overline{\mu} \in \mathcal{G}$, we define the following sets that will be used in the subsequent part of this article:

$$
\begin{aligned}
&\mathcal{E}_{\Psi} := \bigcup_{k \in \mathbb{P}(\overline{\mu})} \partial_c \Psi_k(\overline{\mu}), \quad \mathcal{E}_{\zeta} := \bigcup_{i \in \mathcal{B}} \partial_c \zeta_i(\overline{\mu}), \quad \mathcal{E}_{\mathcal{Q}_1} := \bigcup_{i \in \mathcal{H}_{0+}(\overline{\mu})} \partial_c \mathcal{Q}_i(\overline{\mu}), \\
&\mathcal{E}_{\mathcal{Q}_2} := \bigcup_{i \in \mathcal{H}_{00}(\overline{\mu}) \cup \mathcal{H}_{0-}(\overline{\mu})} -\partial_c \mathcal{Q}_i(\overline{\mu}), \quad \mathcal{E}_{\mathcal{R}_1} := \bigcup_{i \in \mathcal{H}_{+0}(\overline{\mu})} \partial_c \mathcal{R}_i(\overline{\mu}), \\
&\mathcal{E}_{\mathcal{R}_2} := \bigcup_{i \in \mathcal{H}_{+0}(\overline{\mu}) \cup \mathcal{H}_{00}(\overline{\mu})} \partial_c \mathcal{R}_i(\overline{\mu}),
\end{aligned}
$$

Remark 5. *In view of the Definition 8, it is worth noting that*

$$
\mathcal{L}_{VC}(\overline{\mu}) = (\mathcal{E}_{\Psi})^{-} \cap (\mathcal{E}_{\zeta})^{0} \cap (\mathcal{E}_{\mathcal{Q}_1})^{0} \cap (\mathcal{E}_{\mathcal{Q}_2})^{-} \cap (\mathcal{E}_{\mathcal{R}_2})^{-}.
$$

Now, we present VC-ACQ for NIMSIPVC.

Definition 9. *Let $\overline{\mu} \in \mathcal{G}$. Then VC-ACQ for NIMSIPVC is satisfied at $\overline{\mu}$, if*

$$
\mathcal{L}_{VC}(\overline{\mu}) \subseteq \mathcal{T}(\overline{\mu}, \mathcal{G}).
$$

Remark 6. *1. In view of the Remark 1, Definition 9 extends Definitions of VC-ACQ presented by Tung (see [18,19,33]) for a broader class of optimization problems, namely NIMSIPVC.*

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In the forthcoming theorem, we derive KKT-type necessary optimality conditions for NIMSIPVC by employing VC-ACQ.

Theorem 1. *Let $\overline{\mu} \in WEff_{loc}$ and $\mathcal{K}_2 = \text{pos}(\mathcal{E}_\Psi \cup \mathcal{E}_{\mathcal{Q}_2} \cup \mathcal{E}_{\mathcal{R}_2}) + \text{span}(\mathcal{E}_\zeta \cup \mathcal{E}_{\mathcal{Q}_1})$ be a closed set. Further, suppose that VC-ACQ is satisfied at $\overline{\mu}$. Then $\overline{\mu} \in VC_{SP}$.*

Proof. Since $\overline{\mu} \in \text{WEff}_{loc}$, implies that there exists a neighborhood V of $\overline{\mu}$ such that there does not exists any $\mu \in V \cap \mathcal{G}$, satisfying:

$$\mathcal{F}_i(\mu) \prec^s_{LU} \mathcal{F}_i(\overline{\mu}),\ \forall i \in \mathcal{J}^{\mathcal{F}}.$$

Let us first verify the following condition:

$$\left(\bigcup_{i \in \mathcal{J}^{\mathcal{F}}} \partial_c \mathcal{F}_i^L(\overline{\mu}) \cup \partial_c \mathcal{F}_i^U(\overline{\mu}) \right)^{\ominus} \cap \mathcal{T}(\overline{\mu}, \mathcal{G}) = \emptyset. \tag{3}$$

This verification involves two cases:
Case I. If $0 \in \partial_c \mathcal{F}_j^L(\overline{\mu})$ or $0 \in \partial_c \mathcal{F}_j^U(\overline{\mu})$ for at least one $i \in \mathcal{J}^{\mathcal{F}}$, then, we are done. Since

$$\left(\bigcup_{i \in \mathcal{J}^{\mathcal{F}}} \partial_c \mathcal{F}_i^L(\overline{\mu}) \cup \partial_c \mathcal{F}_i^U(\overline{\mu}) \right)^{\ominus} = \emptyset.$$

Therefore,

$$\left(\bigcup_{i \in \mathcal{J}^{\mathcal{F}}} \partial_c \mathcal{F}_i^L(\overline{\mu}) \cup \partial_c \mathcal{F}_i^U(\overline{\mu}) \right)^{\ominus} \cap \mathcal{T}(\overline{\mu}, \mathcal{G}) = \emptyset.$$

Case II. Assume $0 \notin \partial_c \mathcal{F}_i^L(\overline{\mu})$ and $0 \notin \partial_c \mathcal{F}_i^U(\overline{\mu})$ for any $i \in \mathcal{J}^{\mathcal{F}}$. On the contrary, we suppose that there exists $\nu \in \mathbb{R}^n$ such that $\nu \in \left(\bigcup_{i \in \mathcal{J}^{\mathcal{F}}} \partial_c \mathcal{F}_i^L(\overline{\mu}) \cup \partial_c \mathcal{F}_i^U(\overline{\mu}) \right)^{\ominus} \cap \mathcal{T}(\overline{\mu}, \mathcal{G})$. It follows that

$$\begin{aligned} \langle \xi_i^L, \nu \rangle < 0,\ \forall i \in \mathcal{J}^{\mathcal{F}},\ \forall \xi_i^L \in \partial_c \mathcal{F}_i^L(\overline{\mu}), \\ \langle \xi_i^U, \nu \rangle < 0,\ \forall i \in \mathcal{J}^{\mathcal{F}},\ \forall \xi_i^U \in \partial_c \mathcal{F}_i^U(\overline{\mu}). \end{aligned} \tag{4}$$

Moreover, $\nu \in \mathcal{T}(\overline{\mu}, \mathcal{G})$. This implies that there exist real sequences $t_m \downarrow 0, \nu_m \to \nu$ as $m \to \infty$ and $\nu_m \in \mathbb{R}^n$ such that $\overline{\mu} + t_m \nu_m \in \mathcal{G}$ for all $m \in \mathbb{N}$. Utilizing the mean value theorem from Lemma 1(v), for every $m \in \mathbb{N}$, there exist $y_m \in (\overline{\mu}, \overline{\mu} + t_m \nu_m)$ and $\xi_m^L \in \partial_c \mathcal{F}_1^L(y_m)$, satisfy the following condition:

$$\mathcal{F}_1^L(\overline{\mu} + t_m \nu_m) - \mathcal{F}_1^L(\overline{\mu}) = t_m \langle \xi_m^L, \nu_m \rangle, \tag{5}$$

In view of the fact that $\partial_c \Psi_1^L(y_m)$ is a compact set in $\mathbb{R}^n$, this implies that $\{\xi_m^L\}_{m=1}^{\infty} \subset \partial_c \Psi_1^L(y_m)$ is a bounded sequence in $\mathbb{R}^n$. By utilizing the upper semicontinuity of map $\mu \mapsto \partial_c \mathcal{F}_1^L(\mu)$, we get some subsequence $\xi_{m_k}^L$ of sequence ξ_m^L such that $\xi_{m_k}^L \to \overline{\xi}_1^L \in \partial_c \mathcal{F}_1^L(\overline{\mu})$. In view of (4), we infer that

$$\langle \overline{\xi}_1^L, \nu \rangle < 0.$$

From (5),

$$\frac{\mathcal{F}_1^L(\overline{\mu} + t_{m_k} \nu_{m_k}) - \mathcal{F}_1^L(\overline{\mu})}{t_{m_k}} = \langle \xi_{m_k}^L, \nu_{m_k} \rangle \to \langle \overline{\xi}_1^L, \nu \rangle < 0.$$

Therefore, there exists a natural number M_1 such that

$$\mathcal{F}_1^L(\overline{\mu} + t_{m_k} \nu_{m_k})) < \mathcal{F}_1^L(\overline{\mu}),\ \forall k > M_1.$$

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Hence, there exists a subsequence $\{\overline{\mu} + t_m^1 \nu_m^1\}_{m=1}^{\infty}$ of sequence $\{\overline{\mu} + t_m \nu_m\}_{m=1}^{\infty}$ such that

$$\mathcal{F}_1^L(\overline{\mu} + t_m^1 \nu_m^1)) < \mathcal{F}_1^L(\overline{\mu}).$$

On following the similar steps as above, there exists a subsequence $\{\overline{\mu} + t_m^2 \nu_m^2\}_{m=1}^{\infty}$ of the sequence $\{\overline{\mu} + t_m^1 \nu_m^1\}_{m=1}^{\infty}$ such that

$$\mathcal{F}_1^L(\overline{\mu} + t_m^2 \nu_m^2)) < \mathcal{F}_1^L(\overline{\mu}),$$
$$\mathcal{F}_2^L(\overline{\mu} + t_m^2 \nu_m^2)) < \mathcal{F}_2^L(\overline{\mu}).$$

Similarly, we can get a subsequence $\{\overline{\mu} + t_m^l \nu_m^l\}_{m=1}^{\infty}$ of the sequence $\{\overline{\mu} + t_m \nu_m\}_{m=1}^{\infty}$ such that

$$\begin{aligned} \mathcal{F}_1^L(\overline{\mu} + t_m^l \nu_m^l)) &< \mathcal{F}_1^L(\overline{\mu}), \\ \mathcal{F}_2^L(\overline{\mu} + t_m^l \nu_m^l)) &< \mathcal{F}_2^L(\overline{\mu}), \\ \cdots \;\cdots \;\cdots \;\cdots \;\cdots&\cdots\cdots \\ \mathcal{F}_l^L(\overline{\mu} + t_m^l \nu_m^l) &< \mathcal{F}_l^L(\overline{\mu}). \end{aligned}$$

In view of the Definition 1, we have $\overline{\mu} + t_m^l \nu_m^l \in \mathcal{G}$ for sufficiently large $m \in \mathbb{N}$ such that $\overline{\mu} + t_m^l \nu_m^l \in V$, contradicting the fact that $\overline{\mu} \in \mathrm{WEff}_{loc}$. Hence,

$$\left(\bigcup_{i \in \mathcal{J}^{\mathcal{F}}} \partial_c \mathcal{F}_i^L(\overline{\mu}) \cup \partial_c \mathcal{F}_i^U(\overline{\mu}) \right)^{\ominus} \cap \mathcal{T}(\overline{\mu}, \mathcal{G}) = \emptyset.$$

From the given hypothesis, VC-ACQ holds at $\overline{\mu}$. This implies that there does not exist any $\nu \in \mathbb{R}^n$ such that the following system of inequalities have any solution. That is,

$$\begin{aligned} &\langle \xi_i^L, \nu \rangle < 0, \; \xi_i^L \in \partial_c \mathcal{F}_i^L(\overline{\mu}), \; \forall i \in \mathcal{J}^{\mathcal{F}}, \\ &\langle \xi_i^U, \nu \rangle < 0, \; \xi_i^U \in \partial_c \mathcal{F}_i^U(\overline{\mu}), \; \forall i \in \mathcal{J}^{\mathcal{F}}, \\ &\langle \eta_k^{\Psi}, \; \nu \rangle \le 0, \; \eta_k^{\Psi} \in \partial_c \Psi_k(\overline{\mu}), \; \forall k \in \mathbb{P}(\overline{\mu}), \\ &\langle \eta_i^{\zeta}, \nu \rangle = 0, \; \eta_i^{\zeta} \in \partial_c \zeta_i(\overline{\mu}), \; \forall i \in \mathcal{B}, \\ &\langle \eta_i^{\mathcal{Q}}, \nu \rangle = 0, \; \eta_i^{\mathcal{Q}} \in \partial_c \mathcal{Q}_i(\overline{\mu}), \; \forall i \in \mathcal{H}_{0+}(\overline{\mu}), \\ &\langle \eta_i^{\mathcal{Q}}, \nu \rangle \ge 0, \; \eta_i^{\mathcal{Q}} \in \partial_c \mathcal{Q}_i(\overline{\mu}), \; \forall i \in \mathcal{H}_{00}(\overline{\mu}) \cup \mathcal{H}_{0-}(\overline{\mu}), \\ &\langle \eta_i^{\mathcal{R}}, \nu \rangle \le 0, \; \eta_i^{\mathcal{H}} \in \partial_c \mathcal{R}_i(\overline{\mu}), \; \forall i \in \mathcal{H}_{+0}(\overline{\mu}) \cup \mathcal{H}_{00}(\overline{\mu}). \end{aligned}$$

Moreover, from Lemma 1, $\mathrm{co}\{\bigcup_{i \in \mathcal{J}^{\mathcal{F}}} (\partial_c \mathcal{F}_i^L(\overline{\mu}) \cup \partial_c \mathcal{F}_i^U(\overline{\mu}))\}$ is a compact set. This implies that $\mathrm{co}\{\bigcup_{i \in \mathcal{J}^{\mathcal{F}}} (\partial_c \mathcal{F}_i^L(\overline{\mu}) \cup \partial_c \mathcal{F}_i^U(\overline{\mu}))\} + \mathcal{K}_2$ is a closed set. From Lemma 3 it follows that

$$0 \in \mathrm{co} \left\{ \bigcup_{i \in \mathcal{J}^{\mathcal{F}}} (\partial_c \mathcal{F}_i^L(\overline{\mu}) \cup \partial_c \mathcal{F}_i^U(\overline{\mu})) \right\} + \mathrm{pos}(\mathcal{E}_{\Psi} \cup \mathcal{E}_{\mathcal{Q}_2} \cup \mathcal{E}_{\mathcal{R}_2}) + \mathrm{span}(\mathcal{E}_{\zeta} \cup \mathcal{E}_{\mathcal{Q}_1}).$$

Equivalently,

$$\begin{aligned} 0 \in \mathrm{co} \left\{ \bigcup_{i \in \mathcal{J}^{\mathcal{F}}} (\partial_c \mathcal{F}_i^L(\overline{\mu}) \cup \partial_c \mathcal{F}_i^U(\overline{\mu})) \right\} + \mathrm{pos}(\mathcal{E}_{\Psi}) + \mathrm{pos}(\mathcal{E}_{\mathcal{Q}_2}) + \mathrm{pos}(\mathcal{E}_{\mathcal{R}_2}) + \mathrm{span}(\mathcal{E}_{\zeta}) \\ + \mathrm{span}(\mathcal{E}_{\mathcal{Q}_1}). \end{aligned}$$

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Therefore, there exists $(\lambda^L, \lambda^U, \sigma^\Psi, \sigma^\zeta, \sigma^\mathcal{Q}, \sigma^\mathcal{R}) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+ \times \mathbb{P}(\overline{\mu}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that the following condition holds:

$$0 \in \sum_{i\in\mathcal{J}^\mathcal{F}} \Big(\lambda^L_i \partial_c \mathcal{F}^L_i(\overline{\mu}) + \lambda^U_i \partial_c \mathcal{F}^U_i(\overline{\mu})\Big) + \sum_{k\in\mathbb{P}(\overline{\mu})} \sigma^\Psi_k \partial_c \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma^\zeta_i \partial_c \zeta_i(\overline{\mu})$$
$$- \sum_{i\in\mathcal{C}} \sigma^\mathcal{Q}_i \partial_c \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma^\mathcal{R}_i \partial_c \mathcal{R}_i(\overline{\mu}),$$

with $\sum_{i\in\mathcal{J}^\mathcal{F}} (\lambda^L_i + \lambda^U_i) = 1, \sigma^\mathcal{Q}_{\mathcal{H}_+(\overline{\mu})} = 0, \sigma^\mathcal{Q}_{\mathcal{H}_{00}(\overline{\mu})\cup\mathcal{H}_{0-}(\overline{\mu})} \geq 0, \sigma^\mathcal{R}_{\mathcal{H}_{+0}(\overline{\mu})\cup\mathcal{H}_{00}(\overline{\mu})} \geq 0$, and $\sigma^\mathcal{R}_{\mathcal{H}_{+-}(\overline{\mu})\cup\mathcal{H}_{0-}(\overline{\mu})\cup\mathcal{H}_{0+}(\overline{\mu})} = 0$. □

Remark 7.
1. *If $\mathcal{J}^\mathcal{F} = \{1\}$ and $\mathcal{L}$ is a finite set and if $\mathcal{F}^L_i(\mu) = \mathcal{F}^U_i(\mu)$, $\forall i \in \mathcal{J}^\mathcal{F}$, $\forall \mu \in \mathbb{R}^n$ then, in view of the Remark 2, Theorem 1 reduces to Theorem 1 derived by Achtziger and Kanzow [1].*
2. *If $\mathcal{F}^L_i(\mu) = \mathcal{F}^U_i(\mu)$, $\forall i \in \mathcal{J}^\mathcal{F}$, $\forall \mu \in \mathbb{R}^n$, in view of the Remark 2, Theorem 1 reduces to Proposition 3.1(ii) from [19].*
3. *If $\mathcal{F}^L_i(\mu) = \mathcal{F}^U_i(\mu)$, $\forall \mu \in \mathbb{R}^n$, $\forall i \in \mathcal{J}^\mathcal{F}$ and if for every $\mu \in \mathbb{R}^n$, $\partial_c \mathcal{F}^L_i(\mu) = \{\nabla \mathcal{F}^L_i(\mu)\}$, $\partial_c \mathcal{F}^U_i(\mu) = \{\nabla \mathcal{F}^U_i(\mu)\}$ $(i \in \mathcal{J}^\mathcal{F})$, $\partial_c \Psi_k(\mu) = \{\nabla \Psi_k(\mu)\}$ $(k \in \mathcal{L})$, $\partial_c \zeta_i(\mu) = \{\nabla \zeta_i(\mu)\}$ $(i \in \mathcal{B})$, $\partial_c \mathcal{Q}_i(\mu) = \{\nabla \mathcal{Q}_i(\mu)\}$ $(i \in \mathcal{C})$, $\partial_c \mathcal{R}_i(\mu) = \{\nabla \mathcal{R}_i(\mu)\}$ $(i \in \mathcal{C})$ then, Theorem 1 reduces to Proposition 1(ii) deduced by Tung [18].*

In the following example, we illustrate the significance of Theorem 1.

Example 1. *Consider the problem $(\mathcal{P}_1)$ as follows:*

$$(\mathcal{P}_1)\ \text{Minimize}\ \mathcal{F}(\mu) = (\mathcal{F}_1(\mu), \mathcal{F}_2(\mu))$$
$$= \left(\left[|\mu_1 - 1|, |\mu_1 - 1| + \mu_2^2\right], \left[\frac{1}{2}(\mu_1 - 1)^2, \frac{1}{2}\Big((\mu_1 - 1)^2 + (\mu_2 - 1)^2\Big)\right]\right),$$
$$\text{subject to}\ \Psi_\tau(\mu) = (\tau - 1)(\mu_2 - 1) \leq 0, \tau \in \mathcal{L} = [0, 1],$$
$$\mathcal{Q}_1(\mu) = (\mu_1 - 1) \geq 0,$$
$$\mathcal{Q}_1(\mu)\mathcal{R}_1(\mu) = (\mu_1 - 1)(\mu_2 - 1) \leq 0.$$

The feasible set of the considered problem is given as follows:

$$\mathcal{G} := \{(\mu_1, \mu_2) \in \mathbb{R}^2 | \mu_1 > 1,\ \mu_2 = 1\} \cup \{(\mu_1, \mu_2) \in \mathbb{R}^2 | \mu_1 = 1,\ \mu_2 = 1\}$$
$$\cup \{(\mu_1, \mu_2) \in \mathbb{R}^2 | \mu_1 = 1,\ \mu_2 > 1\}.$$

Evidently, $\overline{\mu} = (1, 1)$ is an LU-efficient solution of $(\mathcal{P}_1)$. In particular, $\overline{\mu}$ is a weakly LU-efficient solution of $(\mathcal{P}_1)$.
The contingent cone to the set $\mathcal{G}$ at $\overline{\mu}$ is given by:

$$\mathcal{T}(\overline{\mu}, \mathcal{G}) = \{(d_1, d_2) \in \mathbb{R}^2 | d_1 > 0,\ d_2 = 0\} \cup \{(d_1, d_2) \in \mathbb{R}^2 | d_1 = 0,\ d_2 = 0\}$$
$$\cup \{(d_1, d_2) \in \mathbb{R}^2 | d_1 = 0,\ d_2 > 0\}.$$

The VC-linearized cone at $\overline{\mu}$ is given by:

$$\mathcal{L}_{VC}(\overline{\mu}) := \{(d_1, d_2) \in \mathbb{R}^2 | d_1 \geq 0,\ d_2 = 0\} \subseteq \mathcal{T}(\overline{\mu}, \mathcal{G}).$$

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This implies that VC-ACQ is satisfied at $\overline{\mu}$. Now, the Clarke subdifferentials of every function involved in the problem $(\mathcal{P}_1)$ are given by:

$$\begin{aligned} &\partial_c \mathcal{F}_1^L(\overline{\mu}) = \text{co}\{(-1,0),(1,0)\}, && \partial_c \mathcal{F}_1^U(\overline{\mu}) = \text{co}\{(-1,2),(1,2)\}, \\ &\partial_c \mathcal{F}_2^L(\overline{\mu}) = \{(2,0)\}, && \partial_c \mathcal{F}_2^U(\overline{\mu}) = \{(2,2)\}, \\ &\partial_c \Psi_\tau(\overline{\mu}) = \{(0,(\tau-1))\}, && \partial_c \mathcal{Q}_1(\overline{\mu}) = \{(1,0)\}, \\ &\partial_c \mathcal{R}_1(\overline{\mu}) = \{(0,1)\}. \end{aligned}$$

Moreover,

$$\begin{aligned} (\partial_c \Psi_\tau(\overline{\mu}))^- &= \{(d_1,d_2) \in \mathbb{R}^2 | d_2 \geq 0\}, \\ (-\partial_c \mathcal{Q}_1(\overline{\mu}))^- &= \{(d_1,d_2) \in \mathbb{R}^2 | d_1 \geq 0\}, \\ (-\partial_c \mathcal{R}_1(\overline{\mu}))^- &= \{(d_1,d_2) \in \mathbb{R}^2 | d_2 \leq 0\}. \end{aligned}$$

Therefore,

$$\mathcal{K}_2 = \text{pos}(\mathcal{E}_\Psi \cup \mathcal{E}_{\mathcal{Q}_2} \cup \mathcal{E}_{\mathcal{R}_2}) = \mathbb{R}_+ \times \mathbb{R},$$

is a closed set. All the hypotheses stated in Theorem 1 are satisfied at $\overline{\mu}$, implies that $\overline{\mu}$ is a VC-stationary point of $(\mathcal{P}_1)$. That is, there exist $\lambda_1^L = \frac{1}{4} = \lambda_2^L = \lambda_1^U = \lambda_2^U$, $\overline{\sigma} = (\overline{\sigma}_\tau^\Psi, \overline{\sigma}_1^\mathcal{Q}, \overline{\sigma}_1^\mathcal{R})$ such that

$$\overline{\sigma}_\tau^\Psi = \begin{cases} 1, & \tau = 0, \\ 0, & \textit{otherwise}, \end{cases}$$

$\overline{\sigma}_1^\mathcal{Q} = \frac{1}{2}$, $\overline{\sigma}_1^\mathcal{R} = 0$. If we choose $\xi_1^L = (-1,0)$, $\xi_1^U = (-1,2)$, $\xi_2^L = (2,0)$, $\xi_2^U = (2,2)$, $\eta_\tau^\Psi = (0,\tau - 1)$, $\eta_1^\mathcal{Q} = (1,0)$, $\eta_1^\mathcal{R} = (0,1)$, then the following condition holds:

$$0 = \lambda_1^L \xi_1^L + \lambda_2^L \xi_2^L + \lambda_1^U \xi_1^U + \lambda_2^U \xi_2^U + \sum_{\tau \in [0,1]} \overline{\sigma}_\tau^\Psi \eta_\tau - \overline{\sigma}_1^\mathcal{Q} \eta^\mathcal{Q} + \overline{\sigma}_1^\mathcal{R} \eta^\mathcal{R}.$$

Remark 8.

1. *It is worth noting that LICQ is not satisfied for NIMSIPVC at $\overline{\mu} = (1,1)$ in Example 1. Let $h_1(\mu) = (\mu_1 - 1)(\mu_2 - 1)$ in the aforementioned Example 1. Then, the Clarke subdifferentials of h_1, Ψ_τ $(\tau \in [0,1])$, and $\mathcal{Q}_1$ at $\overline{\mu}$ are given as follows:*

$$\partial_c h_1(\overline{\mu}) = \{(0,0)\}, \quad \partial_c \Psi_\tau(\overline{\mu}) = \{(0,\tau-1)\},\ \forall \tau \in [0,1],\ \partial_c \mathcal{Q}_1(\overline{\mu}) = (1,0).$$

   Notably, $\eta_1^h \in \partial_c h_1(\overline{\mu})$, $\eta_1^\Psi \in \partial_c \Psi_\tau(\overline{\mu})$ $(\tau \in [0,1])$, $\eta_1^\mathcal{Q} \in \partial_c \mathcal{Q}_1(\overline{\mu})$ are not linearly independent vectors. Hence, LICQ is not satisfied at $\overline{\mu}$.
2. *It is worthwhile to note that MFCQ is also not satisfied for NIMSIPVC at LU-efficient or weakly LU-efficient solutions. Let us consider the following example:*

$$\begin{aligned} (\mathcal{P}_2)\ \textit{Minimize}\ \mathcal{F}(\mu) &= (\mathcal{F}_1(\mu), \mathcal{F}_2(\mu)), \\ &= \Big(\big[|\mu_2 - 1|, |\mu_2 - 1| + (\mu_1 - 1)^2 \big], \big[(\mu_1 - 1)^2, (\mu_1 - 1)^2 + (\mu_2 - 1)^2 \big] \Big), \\ \textit{subject to}\ & \Psi_\tau(\mu) = -\tau(\mu_2 - 1) \leq 0,\ \tau \in [0,1], \\ & \zeta_1(\mu) = \mu_1 - 1 = 0, \\ & \mathcal{Q}_1(\mu) = (\mu_1 - \mu_2) \geq 0, \\ & h_1(\mu) = \mathcal{Q}_1(\mu)\mathcal{R}_1(\mu) = (\mu_1 - \mu_2)(\mu_2 - 1) \leq 0. \end{aligned}$$

   The feasible set of the considered problem is given as follows:

$$\mathcal{G} = \{(\mu_1,\mu_2) \in \mathbb{R}^2 | \mu_1 = 1,\ \mu_2 = 1\}.$$

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Evidently, $\overline{\mu} = (1,1)$ is an LU-efficient solution of $(\mathcal{P}_2)$. In particular, $\overline{\mu}$ is a weakly LU-efficient solution of $(\mathcal{P}_2)$.
Now, the Clarke subdifferentials of each constraint function at $\overline{\mu}$ are given as follows:

$$\partial_c \zeta_1(\overline{\mu}) = \{(1,0)\}, \qquad \partial_c \Psi_\tau(\overline{\mu}) = \{(0,-\tau)\}, \ \forall \tau \in [0,1],$$
$$\partial_c \mathcal{Q}_1(\overline{\mu}) = \{(1,-1)\}, \qquad \partial_c h_1(\overline{\mu}) = \{(0,1)\}.$$

Evidently, $\partial_c \zeta_1(\overline{\mu})$ is a linearly independent set. Suppose that there exists a vector $d = (d_1, d_2) \in \mathbb{R}^2$ such that

$$\begin{aligned} &\langle \eta_1^{\zeta}, (d_1,d_2)\rangle = 0, \ \eta_1^{\zeta} \in \partial_c \zeta_1(\overline{\mu}), \qquad \langle \eta_\tau^{\Psi}, (d_1,d_2)\rangle < 0, \ \eta_\tau^{\Psi} \in \partial_c \Psi_\tau(\overline{\mu}), \ \tau \in [0,1], \\ &\langle \eta_1^{\mathcal{Q}}, (d_1,d_2)\rangle < 0, \ \eta_1^{\mathcal{Q}} \in \partial_c \mathcal{Q}_1(\overline{\mu}), \qquad \langle \eta_1^{h}, (d_1,d_2)\rangle < 0, \ \eta_1^{h} \in \partial_c h_1(\overline{\mu}). \end{aligned} \tag{6}$$

It follows that

$$\begin{aligned} &\langle \eta_1^{\zeta}, (d_1,d_2)\rangle = d_1 = 0, \ \eta_1^{\zeta} \in \partial_c \zeta_1(\overline{\mu}), \\ &\langle \eta_\tau^{\Psi}, (d_1,d_2)\rangle = -\tau d_2 < 0, \ \eta_\tau^{\Psi} \in \partial_c \Psi_\tau(\overline{\mu}), \ \tau \in [0,1], \\ &\langle \eta_1^{\mathcal{Q}}, (d_1,d_2)\rangle = d_1 - d_2 < 0, \ \eta_1^{\mathcal{Q}} \in \partial_c \mathcal{Q}_1(\overline{\mu}), \\ &\langle \eta_1^{h}, (d_1,d_2)\rangle = 0 < 0, \ \eta_1^{h} \in \partial_c h_1(\overline{\mu}). \end{aligned} \tag{7}$$

It is evident from (7) that the system of inequalities in (6) does not have any solution $d \in \mathbb{R}^2$. This claims that MFCQ is not satisfied at $\overline{\mu}$.

4. Interval-Valued Vector Lagrange Type Duality Models and Saddle Points for NIMSIPVC

In this section, we formulate interval-valued vector Lagrange type dual problems for NIMSIPVC, namely, interval-valued weak vector and interval-valued vector Lagrange type dual problems. Further, we establish various weak, strong, and converse duality results that elucidate the relationship between the primal problem NIMSIPVC and its associated Lagrange type dual problems. Moreover, this section deals with the notions of saddle points for the interval-valued vector Lagrangian of NIMSIPVC, in particular, weakly LU-saddle point and LU-saddle point.

Let us formulate the interval-valued weak vector Lagrange type dual problem for NIMSIPVC. Consider $\sigma = (\sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}_+^{|\mathcal{L}|} \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$, and $e = (1,1,\ldots,1) \in \mathbb{R}^l$. Then the interval-valued vector Lagrangian $\mathbb{L} : \mathbb{R}^n \times \mathbb{R}_+^{|\mathcal{L}|} \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s \to \mathcal{I}^l$ is defined as follows:

$$\mathbb{L}(\mu,\sigma) = (\mathbb{L}_1(\mu,\sigma), \mathbb{L}_2(\mu,\sigma), \ldots, \mathbb{L}_l(\mu,\sigma)),$$

where $\mathbb{L}_i(\mu,\sigma) = [\mathbb{L}_i^L(\mu,\sigma), \mathbb{L}_i^U(\mu,\sigma)]$, $\forall i \in \mathcal{J}^{\mathcal{F}}$, and

$$\begin{aligned} \mathbb{L}^L(\mu,\sigma) &:= \mathcal{F}^L(\mu) + \left(\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\mu) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\mu) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\mu) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\mu)\right)e, \\ \mathbb{L}^U(\mu,\sigma) &:= \mathcal{F}^U(\mu) + \left(\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\mu) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\mu) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\mu) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\mu)\right)e. \end{aligned} \tag{8}$$

4.1. *Interval-Valued Weak Vector Lagrange Type Duality*

We define an interval-valued weak vector Lagrangian dual function $\psi : \mathbb{R}_+^{|\mathcal{L}|} \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s \rightrightarrows \mathcal{I}^l$ as follows:

$$\psi(\sigma) = \operatorname{WMin}\{\mathbb{L}(\mu,\sigma) | \mu \in \mathcal{G}\}.$$

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Let $\mu \in \mathcal{G}$. Then the interval-valued weak vector Lagrange type dual problem of NIMSIPVC is formulated as follows:

$$\begin{aligned} \text{VCD}^{\text{WVL}}(\mu)\ \text{WMax}\ \psi(\sigma) \\ \text{subject to}\ \sigma^{\Psi}_{\mathcal{L}\setminus\mathbb{P}(\mu)} \geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(\mu)\cup\mathcal{H}_{0-}(\mu)} \geq 0, \\ \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(\mu)} \leq 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(\mu)} \geq 0. \end{aligned}$$

The feasible set of $\text{VCD}^{\text{WVL}}(\mu)$ is denoted by $\mathcal{G}_{\text{WVL}(\mu)}$, and is defined as follows:

$$\begin{aligned} \mathcal{G}_{\text{WVL}(\mu)} := \{\sigma = (\sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}^{|\mathcal{L}|}_{+} \times \mathbb{R}^{r} \times \mathbb{R}^{s} \times \mathbb{R}^{s} | \sigma^{\Psi}_{\mathcal{L}\setminus\mathbb{P}(\mu)} \geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(\mu)\cup\mathcal{H}_{0-}(\mu)} \geq 0, \\ \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(\mu)} \leq 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(\mu)} \geq 0\}. \end{aligned}$$

The notion of a weakly LU-efficient point of $\text{VCD}^{\text{WVL}}(\mu)$ is presented in the following definition by extending the corresponding definition presented by Tung et al. [44] from smooth multiobjective SIPVC to a broader class of optimization problems, namely, NIMSIPVC. For more details, we refer the readers to [41].

Definition 10. *An element $\bar{I} \in \bigcup_{\sigma \in \mathcal{G}_{WVL(\mu)}} \psi(\sigma)$ is said to be a weakly LU-efficient point of $VCD^{WVL}(\mu)$, provided*

$$\bar{I} \in WMax \bigcup_{\sigma \in \mathcal{G}_{WVL(\mu)}} \psi(\sigma).$$

Equivalently, there does not exist any $I \in \bigcup_{\sigma \in \mathcal{G}_{WVL(\mu)}} \psi(\sigma)$ such that

$$\bar{I} \prec^{s}_{LU} I.$$

Remark 9. *It is worth noting that $VCD^{WVL}(\mu)$ depends on the feasible point μ.*

Now, we propose the interval-valued weak vector Lagrange type dual problem of NIMSIPVC, independent of any feasible point, as follows:

$$\begin{aligned} \text{VCD}^{\text{WVL}}\ \text{WMax}\ \psi(\sigma) \\ \text{subject to}\ \sigma \in \mathcal{G}_{\text{WVL}} = \bigcap_{\mu \in \mathcal{G}} \mathcal{G}_{\text{WVL}(\mu)}. \end{aligned}$$

Remark 10. *One can easily note that the feasible set of VCD^{WVL} is a non-empty set. That is, $\mathcal{G}_{WVL} = \bigcap_{\mu \in \mathcal{G}} \mathcal{G}_{WVL(\mu)} \neq \emptyset$.*

In the following theorem, we derive weak duality results that relate NIMSIPVC with its corresponding Lagrange type dual problem VCD^{WVL}.

Theorem 2. *Let μ be an arbitrary element of $\mathcal{G}$ and $I \in \bigcup_{\sigma \in \mathcal{G}_{WVL(\mu)}} \psi(\sigma)$. Then*

$$\mathcal{F}(\mu) \not\prec^{s}_{LU} I.$$

Proof. From the given hypothesis, there exists $\sigma \in \mathcal{G}_{\text{WVL}(\mu)}$ such that $I \in \psi(\sigma)$. Therefore, we have

$$\mathcal{F}(\mu) + \left(\sum_{k \in \mathcal{L}} \sigma^{\Psi}_{k} \Psi_{k}(\mu) + \sum_{i \in \mathcal{B}} \sigma^{\zeta}_{i} \zeta_{i}(\mu) - \sum_{i \in \mathcal{C}} \sigma^{\mathcal{Q}}_{i} \mathcal{Q}_{i}(\mu) + \sum_{i \in \mathcal{C}} \sigma^{\mathcal{R}}_{i} \mathcal{R}_{i}(\mu) \right) e \not\prec^{s}_{LU} I. \tag{9}$$

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On contrary, we suppose that

$$\mathcal{F}(\mu) \prec^{s}_{LU} I,$$

which implies that

$$\mathcal{F}^L_i(\mu) < I^L_i,\ \mathcal{F}^U_i(\mu) < I^U_i, \forall i \in \mathcal{J}^{\mathcal{F}}. \tag{10}$$

In view of the fact that $\mu \in \mathcal{G}$, we infer that

$$\Psi_k(\mu) \leq 0\ (k \in \mathcal{L}),\ \zeta_i(\mu) = 0\ (i \in \mathcal{B}),\ \mathcal{Q}_i(\mu) \geq 0\ (i \in \mathcal{C}).$$

Moreover, $\sigma \in \mathcal{G}_{\mathrm{WVL}(\mu)}$ implies that

$$\begin{aligned}
\sum_{k\in\mathbb{P}(\mu)} \sigma^{\Psi}_k \Psi_k(\mu) = 0, \quad & \sum_{k\in\mathcal{L}\setminus\mathbb{P}(\mu)} \sigma^{\Psi}_k \Psi_k(\mu) \leq 0,\\
& \sum_{i\in\mathcal{B}} \sigma^{\zeta}_i \zeta_i(\mu) = 0,\\
\sum_{i\in\mathcal{H}_0(\mu)} \sigma^{\mathcal{Q}}_i \mathcal{Q}_i(\mu) = 0, \quad & \sum_{i\in\mathcal{H}_+(\mu)} \sigma^{\mathcal{Q}}_i \mathcal{Q}_i(\mu) \geq 0,\\
\sum_{i\in\mathcal{H}_{0+}(\mu)} \sigma^{\mathcal{R}}_i \mathcal{R}_i(\mu) \leq 0, \quad & \sum_{i\in\mathcal{H}_{+-}(\mu)\cup\mathcal{H}_{0-}(\mu)} \sigma^{\mathcal{R}}_i \mathcal{R}_i(\mu) \leq 0.
\end{aligned}$$

These equations yield that

$$\sum_{k\in\mathcal{L}} \sigma^{\Psi}_k \Psi_k(\mu) + \sum_{i\in\mathcal{B}} \sigma^{\zeta}_i \zeta_i(\mu) - \sum_{i\in\mathcal{C}} \sigma^{\mathcal{Q}}_i \mathcal{Q}_i(\mu) + \sum_{i\in\mathcal{C}} \sigma^{\mathcal{R}}_i \mathcal{R}_i(\mu) \leq 0, \tag{11}$$

and hence,

$$\mathcal{F}^L_i(\mu) + \sum_{k\in\mathcal{L}} \sigma^{\Psi}_k \Psi_k(\mu) + \sum_{i\in\mathcal{B}} \sigma^{\zeta}_i \zeta_i(\mu) - \sum_{i\in\mathcal{C}} \sigma^{\mathcal{Q}}_i \mathcal{Q}_i(\mu) + \sum_{i\in\mathcal{C}} \sigma^{\mathcal{R}}_i \mathcal{R}_i(\mu) \leq \mathcal{F}^L_i(\mu),\ \forall i \in \mathcal{J}^{\mathcal{F}},$$

$$\mathcal{F}^U_i(\mu) + \sum_{k\in\mathcal{L}} \sigma^{\Psi}_k \Psi_k(\mu) + \sum_{i\in\mathcal{B}} \sigma^{\zeta}_i \zeta_i(\mu) - \sum_{i\in\mathcal{C}} \sigma^{\mathcal{Q}}_i \mathcal{Q}_i(\mu) + \sum_{i\in\mathcal{C}} \sigma^{\mathcal{R}}_i \mathcal{R}_i(\mu) \leq \mathcal{F}^U_i(\mu),\ \forall i \in \mathcal{J}^{\mathcal{F}}.$$

From (10),

$$\mathcal{F}^L_i(\mu) + \sum_{k\in\mathcal{L}} \sigma^{\Psi}_k \Psi_k(\mu) + \sum_{i\in\mathcal{B}} \sigma^{\zeta}_i \zeta_i(\mu) - \sum_{i\in\mathcal{C}} \sigma^{\mathcal{Q}}_i \mathcal{Q}_i(\mu) + \sum_{i\in\mathcal{C}} \sigma^{\mathcal{R}}_i \mathcal{R}_i(\mu) < I^L_i,\ \forall i \in \mathcal{J}^{\mathcal{F}},$$

$$\mathcal{F}^U_i(\mu) + \sum_{k\in\mathcal{L}} \sigma^{\Psi}_k \Psi_k(\mu) + \sum_{i\in\mathcal{B}} \sigma^{\zeta}_i \zeta_i(\mu) - \sum_{i\in\mathcal{C}} \sigma^{\mathcal{Q}}_i \mathcal{Q}_i(\mu) + \sum_{i\in\mathcal{C}} \sigma^{\mathcal{R}}_i \mathcal{R}_i(\mu) < I^U_i,\ \forall i \in \mathcal{J}^{\mathcal{F}},$$

which is a contradiction to (9). Hence, the proof of the theorem is complete. □

Remark 11. *If $\mathcal{F}^L_i(\mu) = \mathcal{F}^U_i(\mu)$, $\forall i \in \mathcal{J}^{\mathcal{F}}$, $\forall \mu \in \mathbb{R}^n$ and if $\partial_c \mathcal{F}^L_i(\mu) = \{\nabla \mathcal{F}^L_i(\mu)\}$, $\partial_c \mathcal{F}^U_i(\mu) = \{\nabla \mathcal{F}^U_i(\mu)\}$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\partial_c \Psi_k(\mu) = \{\nabla \Psi_k(\mu)\}$ $(k \in \mathcal{L})$, $\partial_c \zeta_i(\mu) = \{\nabla \zeta_i(\mu)\}$ $(i \in \mathcal{B})$, $\partial_c \mathcal{Q}_i(\mu) = \{\nabla \mathcal{Q}_i(\mu)\}$ $(i \in \mathcal{C})$, $\partial_c \mathcal{R}_i(\mu) = \{\nabla \mathcal{R}_i(\mu)\}$ $(i \in \mathcal{C})$ then, Theorem 2 reduces to Proposition 3.1 from [44].*

The relationship between weakly LU-efficient solution and weakly LU-efficient point of NIM-SIPVC and VCD$^{\mathrm{WVL}}$ has been derived in the following theorem.

Theorem 3. *Consider an arbitrary $\overline{\mu} \in \mathcal{G}$, $\overline{\sigma} \in \mathcal{G}_{WVL(\overline{\mu})}$, and $\mathcal{F}(\overline{\mu}) \in \psi(\overline{\sigma})$. Then $\mathcal{F}(\overline{\mu})$ is a weakly LU-efficient point of VCD$^{WVL}(\overline{\mu})$.*

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Proof. On the contrary, we suppose that $\mathcal{F}(\overline{\mu})$ is not a weakly LU-efficient point of $\mathrm{VCD}^{\mathrm{WVL}}(\overline{\mu})$. This implies that there exists $I \in \psi(\overline{\sigma})$ for some $\overline{\sigma} \in \mathcal{G}_{\mathrm{WVL}(\overline{\mu})}$ such that

$$\mathcal{F}(\overline{\mu}) \prec^{s}_{LU} I. \tag{12}$$

In view of the fact that $\overline{\mu} \in \mathcal{G}, \overline{\sigma} \in \mathcal{G}_{\mathrm{WVL}(\overline{\mu})}$, and from the proof of Theorem 2, we have

$$\mathcal{F}_i^L(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) \leq \mathcal{F}_i^L(\overline{\mu}),\ \forall i \in \mathcal{J}^{\mathcal{F}},$$

$$\mathcal{F}_i^U(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) \leq \mathcal{F}_i^U(\overline{\mu}),\ \forall i \in \mathcal{J}^{\mathcal{F}}.$$

From (12), we yield that

$$\mathcal{F}_i^L(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) < I_i^L,\ \forall i \in \mathcal{J}^{\mathcal{F}},$$

$$\mathcal{F}_i^U(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) < I_i^U,\ \forall i \in \mathcal{J}^{\mathcal{F}},$$

contradicting the fact that $I \in \psi(\overline{\sigma})$. Therefore, $\mathcal{F}(\overline{\mu})$ is a weakly LU-efficient point of $\mathrm{VCD}^{\mathrm{WVL}}(\overline{\mu})$. □

In the following theorem, we derive a converse duality result that relates our primal problem NIMSIPVC and the corresponding interval-valued weak vector Lagrange type dual problem $\mathrm{VCD}^{\mathrm{WVL}}$.

Theorem 4. *Let $\overline{\mu} \in \mathcal{G}$, $\overline{\sigma} \in \mathcal{G}_{\mathrm{WVL}}$, and $\mathcal{F}(\overline{\mu}) \in \psi(\overline{\sigma})$. Then $\overline{\mu} \in$ WEff.*

Proof. On the contrary, we suppose that $\overline{\mu} \notin$ WEff. This implies that there exists $\mu \in \mathcal{G}$ such that

$$\mathcal{F}(\mu) \prec^{s}_{LU} \mathcal{F}(\overline{\mu}). \tag{13}$$

On following the similar steps in Theorem 2 and in view of the fact that $\mu \in \mathcal{G}$, $\overline{\sigma} \in \mathcal{G}_{\mathrm{WVL}} \subset \mathcal{G}_{\mathrm{WVL}(\mu)}$, it follows that

$$\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\mu) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\mu) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\mu) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\mu) \leq 0.$$

From (13) for every $i \in \mathcal{J}^{\mathcal{F}}$ we have

$$\mathbb{L}_i^L(\mu,\overline{\sigma}) := \mathcal{F}_i^L(\mu) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\mu) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta(\mu) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\mu) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\mu) < \mathcal{F}_i^L(\overline{\mu}),$$
$$\mathbb{L}_i^U(\mu,\overline{\sigma}) := \mathcal{F}_i^U(\mu) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\mu) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\mu) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\mu) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\mu) < \mathcal{F}_i^U(\overline{\mu}),,$$

which is a contradiction to the fact that $\mathcal{F}(\overline{\mu}) \in \psi(\overline{\sigma})$. Therefore, $\overline{\mu} \in$ WEff. □

Remark 12. *Theorems 3 and 4 extend Proposition 3.2 established by Tung et al. [44] from smooth multiobjective semi-infinite programming problems with vanishing constraints to a broader class of optimization problem, in particular, NIMSIPVC.*

In the following example, we illustrate the significance of Theorems 2, 3, and 4.

Example 2. *Consider the problem $(\mathcal{P}_1)$ from Example 1.*

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The feasible set of the considered problem is given as follows:

$$\mathcal{G} := \{(\mu_1,\mu_2) \in \mathbb{R}^2 | \mu_1 > 1,\ \mu_2 = 1\} \cup \{(\mu_1,\mu_2) \in \mathbb{R}^2 | \mu_1 = 1,\ \mu_2 = 1\} \\ \cup \{(\mu_1,\mu_2) \in \mathbb{R}^2 | \mu_1 = 1,\ \mu_2 > 1\}.$$

For the sake of convenience, we break the feasible set into three disjoint sets as follows:

$$\mathcal{G}^1 := \{(\mu_1,\mu_2) \in \mathbb{R}^2 | \mu_1 > 1,\ \mu_2 = 1\},$$
$$\mathcal{G}^2 := \{(\mu_1,\mu_2) \in \mathbb{R}^2 | \mu_1 = 1,\ \mu_2 = 1\},$$
$$\mathcal{G}^3 := \{(\mu_1,\mu_2) \in \mathbb{R}^2 | \mu_1 = 1,\ \mu_2 > 1\}.$$

Formulate the interval-valued vector Lagrangian for $(\mathcal{P}_1)$ *as follows:*

$$\mathbb{L}(\mu,\sigma) = \mathcal{F}(\mu) + \left(\sum_{\tau\in[0,1]} \sigma_\tau^\Psi \Psi_\tau(\mu) - \sigma_1^{\mathcal{Q}} \mathcal{Q}(\mu) + \sigma_1^{\mathcal{R}} \mathbb{R}(\mu) \right) e.$$

Then,

$$\mathbb{L}^L(\mu,\sigma) = \begin{pmatrix} |\mu_1 - 1| + \sum_{\tau\in[0,1]} \sigma_\tau^\Psi (\tau-1)(\mu_2-1) - \sigma_1^{\mathcal{Q}}(\mu_2-1) + \sigma_1^{\mathcal{R}}(\mu_1-1) \\ \frac{1}{2}(\mu_1-1)^2 + \sum_{\tau\in[0,1]} \sigma_\tau^\Psi (\tau-1)(\mu_2-1) - \sigma_1^{\mathcal{Q}}(\mu_2-1) + \sigma_1^{\mathcal{R}}(\mu_1-1) \end{pmatrix},$$

and

$$\mathbb{L}^U(\mu,\sigma) = \begin{pmatrix} |\mu_1 - 1| + \mu_2^2 + \sum_{\tau\in[0,1]} \sigma_\tau^\Psi (\tau-1)(\mu_2-1) - \sigma_1^{\mathcal{Q}}(\mu_2-1) + \sigma_1^{\mathcal{R}}(\mu_1-1) \\ \frac{1}{2}((\mu_1-1)^2 + (\mu_2-1)^2) + \sum_{\tau\in[0,1]} \sigma_\tau^\Psi (\tau-1)(\mu_2-1) - \sigma_1^{\mathcal{Q}}(\mu_2-1) + \sigma_1^{\mathcal{R}}(\mu_1-1) \end{pmatrix}.$$

Now, we define $\psi : \mathbb{R}_+^{|[0,1]|} \times \mathbb{R} \times \mathbb{R} \rightrightarrows \mathbb{R}^2$ *as follows:*

$$\psi(\sigma) = WMin\{\mathbb{L}(\mu,\sigma) | \mu \in \mathcal{G}\}.$$

Consider an arbitrary point $\mu \in \mathcal{G}^1$. *Then* $\mathbb{P}(\mu) = [0,1]$, $\mathcal{H}_{0+}(\mu) = \{1\}$. *The interval-valued weak vector Lagrange type dual problem* $VCD^{WVL1}(\mu)$ *of* $(\mathcal{P}_1)$ *is formulated as:*

$$VCD^{WVL1}(\mu)\ WMax\ \psi(\sigma)$$
$$\text{subject to } \sigma^\Psi \in \mathbb{R}_+^{|[0,1]|},\ \sigma_1^{\mathcal{R}} \in \mathbb{R},\ \sigma_1^{\mathcal{Q}} \geq 0.$$

Similarly, for $\mu \in \mathcal{G}^2$, *we formulate the following interval-valued Lagrange type dual problem corresponding to* $(\mathcal{P}_1)$:

$$VCD^{WVL2}(\mu)\ WMax\ \psi(\sigma)$$
$$\text{subject to } \sigma^\Psi \in \mathbb{R}_+^{|\mathcal{L}|},\ \sigma_1^{\mathcal{R}} \in \mathbb{R},\ \sigma_1^{\mathcal{Q}} \in \mathbb{R},$$

and for some $\mu \in \mathcal{G}^3$, *interval-valued Lagrange type dual problem of* $(\mathcal{P}_1)$ *is given by:*

$$VCD^{WVL3}(\mu)\ WMax\ \psi(\sigma)$$
$$\text{subject to } \sigma^\Psi \in \mathbb{R}_+^{|\mathcal{L}|},\ \sigma_1^{\mathcal{R}} \leq 0,\ \sigma_1^{\mathcal{Q}} \in \mathbb{R}.$$

The interval-valued weak vector Lagrange type dual problem, which is independent of a feasible point, is defined as follows:

$$VCD^{WVL} \; WMax \; \psi(\sigma) \quad \text{subject to } \sigma^{\Psi} \in \mathbb{R}_{+}^{|[0,1]|},\; \sigma_1^{\mathcal{R}} \leq 0,\; \sigma_1^{\mathcal{Q}} \geq 0.$$

Let $\overline{\mu} = (1,1)$. *Then* $\mathcal{F}(\overline{\mu}) = ([0,1],[0,0])$. *Let* $\overline{\sigma} = (\overline{\sigma}_{\tau}^{\Psi}, \overline{\sigma}_1^{\mathcal{Q}}, \overline{\sigma}_1^{\mathcal{R}})$ *such that*

$$\overline{\sigma}_{\tau}^{\Psi} = \begin{cases} 1, & \tau = 1, \\ 0, & \textit{otherwise}, \end{cases}$$

$\overline{\sigma}_1^{\mathcal{Q}} = 0,\; \overline{\sigma}_1^{\mathcal{R}} = 1.$

Now,

$$\psi(\overline{\sigma}) = WMin \left\{ \left(\begin{matrix} [|\mu_1 - 1| + (\mu_1 - 1), |\mu_1 - 1| + (\mu_1 - 1) + \mu_2^2] \\ \left[\frac{1}{2}(\mu_1 - 1)^2 + (\mu_1 - 1), \frac{1}{2}((\mu_1 - 1)^2 + (\mu_2 - 2)^2) + (\mu_1 - 1)\right] \end{matrix} \right) | \mu \in \mathcal{G} \right\}.$$

Then one can easily verify that $\mathcal{F}(\overline{\mu}) \in \psi(\overline{\sigma})$. *Therefore, from Theorem 3 ,* $\mathcal{F}(\overline{\mu})$ *is a weakly LU-efficient point of* $VCD^{WVL}(\overline{\mu})$.

Let $\overline{\sigma} = (\overline{\sigma}_{\tau}^{\Psi}, \overline{\sigma}_1^{\mathcal{Q}}, \overline{\sigma}_1^{\mathcal{R}})$ *such that* $\overline{\sigma}_{\tau}^{\Psi} = 0, \forall \tau \in [0,1], \overline{\sigma}_1^{\mathcal{Q}} = 0, \overline{\sigma}_1^{\mathcal{R}} = 0$. *Then one can easily verify that* $\mathcal{F}(\overline{\mu}) \in \psi(\overline{\sigma})$. *Therefore, from Theorem 4 we conclude that* $(1,1)$ *is a weakly LU-efficient solution of problem* $(\mathcal{P}_1)$.

In the next theorem we derive the strong duality result, which elucidates the relationship between NIMSIPVC and interval-valued weak vector Lagrange type dual problem.

Theorem 5. *Let* $\overline{\mu} \in WEff_{loc}$ *such that VC-ACQ is satisfied at* $\overline{\mu}$ *and let* $\mathcal{K}_2$ *be a closed set. Further, assume that* $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$, Ψ_k $(k \in \mathbb{P}^{\Psi}(\overline{\mu}))$, ζ_i $(i \in \mathcal{B}_+^{\zeta})$, $-\zeta_i$ $(i \in \mathcal{B}_-^{\zeta})$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^{-}(\overline{\mu}))$, $-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^{+}(\overline{\mu}) \cup \overline{\mathbb{H}}_{00}^{+}(\overline{\mu}) \cup \overline{\mathbb{H}}_{0-}^{+}(\overline{\mu}))$, $\mathcal{R}_i$ $(i \in \mathbb{H}_{+0}^{+}(\overline{\mu}) \cup \mathbb{H}_{00}^{+}(\overline{\mu}))$ *are LU-convex and convex at* $\overline{\mu}$, *respectively. Then there exists* $\overline{\sigma} \in \mathcal{G}_{WVL(\overline{\mu})}$ *such that* $\mathcal{F}(\overline{\mu}) \in \psi(\overline{\sigma})$. *Furthermore,* $\mathcal{F}(\overline{\mu})$ *is a weakly LU-efficient point of* $VCD^{WVL}(\overline{\mu})$.

Proof. Since $\overline{\mu} \in \text{WEff}_{loc}$ and VC-ACQ is satisfied at $\overline{\mu}$ then, from Theorem 1, $\overline{\mu} \in \text{VC}_{SP}$. Therefore, there exist $\overline{\lambda}^L, \overline{\lambda}^U \in \mathbb{R}_+^l \times \mathbb{R}_+^l$, $\overline{\sigma} = (\overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{P}^{\Psi}(\overline{\mu}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that the following condition holds:

$$0 \in \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \partial_c \mathcal{F}_i^L(\overline{\mu}) + \overline{\lambda}_i^U \partial_c \mathcal{F}_i^U(\overline{\mu}) \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \partial_c \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \partial_c \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \partial_c \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \partial_c \mathcal{R}_i(\overline{\mu}),$$

where $\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L + \overline{\lambda}_i^U \right) = 1$, $\overline{\sigma}_{\mathcal{H}_+(\overline{\mu})}^{\mathcal{Q}} = 0$, $\overline{\sigma}_{\mathcal{H}_{00}(\overline{\mu}) \cup \mathcal{H}_{0-}(\overline{\mu})}^{\mathcal{Q}} \geq 0$, $\overline{\sigma}_{\mathcal{H}_{+0}(\overline{\mu}) \cup \mathcal{H}_{00}(\overline{\mu})}^{\mathcal{R}} \geq 0$, and $\overline{\sigma}_{\mathcal{H}_{+-}(\overline{\mu}) \cup \mathcal{H}_{0-}(\overline{\mu}) \cup \mathcal{H}_{0+}(\overline{\mu})}^{\mathcal{R}} = 0$. This implies that there exist $\hat{\xi}_i^L \in \partial_c \mathcal{F}_i^L(\overline{\mu})$, $\hat{\xi}_i^U \in \partial_c \mathcal{F}_i^U(\overline{\mu})$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\hat{\eta}_k^{\Psi} \in \partial_c \Psi_k(\overline{\mu})$ $(k \in \mathcal{L})$, $\hat{\eta}_i^{\zeta} \in \partial_c \zeta_i(\overline{\mu})$ $(i \in \mathcal{B})$, $\hat{\eta}_i^{\mathcal{Q}} \in \partial_c \mathcal{Q}_i(\overline{\mu})$ $(i \in \mathcal{C})$, $\hat{\eta}_i^{\mathcal{R}} \in \partial_c \mathcal{R}_i(\overline{\mu})$ $(i \in \mathcal{C})$ such that

$$\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \hat{\xi}_i^L + \overline{\lambda}_i^U \hat{\xi}_i^U \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \hat{\eta}_k^{\Psi} + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \hat{\eta}_i^{\zeta} - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \hat{\eta}_i^{\mathcal{Q}} + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \hat{\eta}_i^{\mathcal{R}} = 0. \tag{14}$$

Further, one can obtain the following:

$$\sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) = 0.$$

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Hence,
$$\mathbb{L}^L(\overline{\mu},\overline{\sigma}) = \mathcal{F}^L(\overline{\mu}),\ \mathbb{L}^U(\overline{\mu},\overline{\sigma}) = \mathcal{F}^U(\overline{\mu}). \tag{15}$$
Let us assume that there exists $\tilde{\mu} \in \mathcal{G}$ such that
$$\mathbb{L}(\tilde{\mu},\overline{\sigma}) \prec^s_{LU} \mathbb{L}(\overline{\mu},\overline{\sigma}) = \mathcal{F}(\overline{\mu}).$$
That is,
$$\mathbb{L}^L_i(\tilde{\mu},\overline{\sigma}) < \mathbb{L}^L_i(\overline{\mu},\overline{\sigma}), \quad \mathbb{L}^U_i(\tilde{\mu},\overline{\sigma}) < \mathbb{L}^U_i(\overline{\mu},\overline{\sigma}),\ \forall i \in \mathcal{J}^{\mathcal{F}}.$$
Equivalently, for every $i \in \mathcal{J}^{\mathcal{F}}$, the following inequalities hold:
$$\begin{aligned}
&\mathcal{F}^L_i(\tilde{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_k \Psi_k(\tilde{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_i \zeta_i(\tilde{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_i \mathcal{Q}_i(\tilde{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_i \mathcal{R}_i(\tilde{\mu})\\
<&\mathcal{F}^L_i(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_k \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_i \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_i \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_i \mathcal{R}_i(\overline{\mu}),\\
&\mathcal{F}^U_i(\tilde{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_k \Psi_k(\tilde{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_i \zeta_i(\tilde{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_i \mathcal{Q}_i(\tilde{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_i \mathcal{R}_i(\tilde{\mu})\\
<&\mathcal{F}^U_i(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_k \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_i \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_i \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_i \mathcal{R}_i(\overline{\mu}).
\end{aligned}$$
Multiply both equations by $\overline{\lambda}^L_i, \overline{\lambda}^U_i$, and add them, we get
$$\begin{aligned}
&\sum_{i\in\mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i \mathcal{F}^L_i(\tilde{\mu}) + \sum_{i\in\mathcal{J}^{\mathcal{F}}} \overline{\lambda}^U_i \mathcal{F}^U_i(\tilde{\mu}) \right) + \sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_k \Psi_k(\tilde{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_i \zeta_i(\tilde{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_i \mathcal{Q}_i(\tilde{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_i \mathcal{R}_i(\tilde{\mu})\\
&- \sum_{i\in\mathcal{J}^{\mathcal{F}}} \overline{\lambda}^L_i \mathcal{F}^L_i(\overline{\mu}) - \sum_{i\in\mathcal{J}^{\mathcal{F}}} \lambda^U_i \mathcal{F}^U_i(\overline{\mu}) - \left(\sum_{k\in\mathcal{L}} \overline{\sigma}^{\Psi}_k \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}^{\zeta}_i \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_i \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}^{\mathcal{R}}_i \mathcal{R}_i(\overline{\mu}) \right)\\
&< 0.
\end{aligned} \tag{16}$$
Now, from the LU-convexity of $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$ at $\overline{\mu}$ we have
$$\begin{aligned}
\mathcal{F}^L_i(\tilde{\mu}) - \mathcal{F}^L_i(\overline{\mu}) &\geq \langle \xi^L_i, \tilde{\mu} - \overline{\mu} \rangle,\ \forall \xi^L_i \in \partial_c \mathcal{F}^L_i(\overline{\mu}),\ i \in \mathcal{J}^{\mathcal{F}},\\
\mathcal{F}^U_i(\mu) - \mathcal{F}^U_i(\overline{\mu}) &\geq \langle \xi^U_i, \tilde{\mu} - \overline{\mu} \rangle,\ \forall \xi^U_i \in \partial_c \mathcal{F}^U_i(\overline{\mu}),\ i \in \mathcal{J}^{\mathcal{F}}.
\end{aligned}$$
Moreover, from the convexity assumptions of all the constraint functions at $\overline{\mu}$ we have the following inequalities:
$$\begin{aligned}
\langle \eta^{\Psi}_k, \tilde{\mu} - \overline{\mu} \rangle &\leq \Psi_k(\tilde{\mu}) - \Psi_k(\overline{\mu}),\ \forall \eta^{\Psi}_k \in \partial_c \Psi_k(\overline{\mu}),\ k \in \mathbb{P}^{\Psi}(\overline{\mu}),\\
\langle \eta^{\zeta}_i, \tilde{\mu} - \overline{\mu} \rangle &\leq \zeta_i(\tilde{\mu}) - \zeta_i(\overline{\mu}) = 0,\ \forall \eta^{\zeta}_i \in \partial_c \zeta_i(\overline{\mu}),\ i \in \mathcal{B}^{\zeta}_+(\overline{\mu}),\\
\langle -\eta^{\zeta}_i, \tilde{\mu} - \overline{\mu} \rangle &\leq (-\zeta_i)(\tilde{\mu}) - (-\zeta_i)(\overline{\mu}) = 0,\ \forall -\eta^{\zeta}_i \in \partial_c(-\zeta_i(\overline{\mu})),\ i \in \mathcal{B}^{\zeta}_-(\overline{\mu}),\\
\langle \eta^{\mathcal{Q}}_i, \tilde{\mu} - \overline{\mu} \rangle &\leq \mathcal{Q}_i(\tilde{\mu}) - \mathcal{Q}_i(\overline{\mu}),\ \forall \eta^{\mathcal{Q}}_i \in \partial_c \mathcal{Q}_i(\overline{\mu}),\ i \in \overline{\mathbb{H}}^{-}_{0+}(\overline{\mu}),\\
\langle -\eta^{\mathcal{Q}}_i, \tilde{\mu} - \overline{\mu} \rangle &\leq -\mathcal{Q}_i(\tilde{\mu}) - (-\mathcal{Q}_i(\overline{\mu})),\ \forall -\eta^{\mathcal{Q}}_i \in \partial_c(-\mathcal{Q}_i(\mu)),\ i \in \overline{\mathbb{H}}^{+}_{0+}(\overline{\mu}) \cup \overline{\mathbb{H}}^{+}_{0-}(\overline{\mu}) \cup \overline{\mathbb{H}}^{+}_{00}(\overline{\mu}),\\
\langle \eta^{\mathcal{R}}_i, \tilde{\mu} - \overline{\mu} \rangle &\leq \mathcal{R}_i(\tilde{\mu}) - \mathcal{R}_i(\overline{\mu}),\ \forall \eta^{\mathcal{R}}_i \in \partial_c \mathcal{R}_i(\overline{\mu}),\ i \in \mathbb{H}^{+}_{+0}(\overline{\mu}) \cup \mathbb{H}^{+}_{+-}(\overline{\mu}) \cup \mathbb{H}^{+}_{00}(\overline{\mu}).
\end{aligned}$$

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On multiplying the above inequalities with $\overline{\sigma}_k^{\Psi} > 0$ $(k \in \mathbb{P}^{\Psi}(\overline{\mu}))$, $\overline{\sigma}_i^{\zeta} > 0$ $(i \in \mathcal{B}_+^{\zeta}(\overline{\mu}))$, $\overline{\sigma}_i^{\zeta} < 0$ $(i \in \mathcal{B}_-^{\zeta}(\overline{\mu}))$, $\overline{\sigma}_i^{\mathcal{Q}} < 0$ $(i \in \overline{\mathbb{H}}_0^-(\overline{\mu}))$, $\overline{\sigma}_i^{\mathcal{Q}} > 0$ $(i \in \overline{\mathbb{H}}_{0+}^+(\overline{\mu}) \cup \overline{\mathbb{H}}_{0-}^+(\overline{\mu}) \cup \overline{\mathbb{H}}_{00}^+(\overline{\mu}))$, $\overline{\sigma}_i^{\mathcal{R}} > 0$ $(i \in \mathbb{H}_{+0}^+(\overline{\mu}) \cup \mathbb{H}_{+-}^+(\overline{\mu}) \cup \mathbb{H}_{00}^+(\overline{\mu}))$, respectively and add them we get,

$$\begin{aligned}
&\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\tilde{\mu})+\overline{\lambda}_i^U\mathcal{F}_i^U(\tilde{\mu})\right)+\left(\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\tilde{\mu})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\tilde{\mu})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{\mu})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{\mu})\right)\\
&-\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{\mu})+\overline{\lambda}_i^U\mathcal{F}_i^U(\overline{\mu})\right)+\left(\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu})\right)\\
&\qquad\geq\left\langle\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\xi_i^L+\overline{\lambda}_i^U\xi_i^U\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\eta_k^{\Psi}+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\eta_i^{\zeta}-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\eta_i^{\mathcal{Q}}+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\eta_i^{\mathcal{R}},\tilde{\mu}-\overline{\mu}\right\rangle
\end{aligned}$$

From (16) we get that for every $\xi_i^L \in \partial_c\mathcal{F}_i^L(\overline{\mu})$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\xi_i^U \in \partial_c\mathcal{F}_i^U(\overline{\mu})$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\eta_k^{\Psi} \in \partial_c\Psi_k(\overline{\mu})$ $(k \in \mathcal{L})$, $\eta_i^{\zeta} \in \partial_c\zeta_i(\overline{\mu})$ $(i \in \mathcal{B})$, $\eta_i^{\mathcal{Q}} \in \partial_c\mathcal{Q}_i(\overline{\mu})$ $(i \in \mathcal{C})$, $\eta_i^{\mathcal{R}} \in \partial_c\mathcal{R}_i(\overline{\mu})$ $(i \in \mathcal{C})$,

$$\left\langle\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\xi_i^L+\overline{\lambda}_i^L\xi_i^U\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\eta_k^{\Psi}+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\eta_i^{\zeta}-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\eta_i^{\mathcal{Q}}+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\eta_i^{\mathcal{R}},\tilde{\mu}-\overline{\mu}\right\rangle<0,$$

which is a contradiction to (14). Therefore, there does not exist any $\mu \in \mathcal{G}$ such that $\mathbb{L}(\mu,\overline{\sigma}) \prec_{LU}^s \mathbb{L}(\overline{\mu},\overline{\sigma})$. From (15) we get that $\mathcal{F}(\overline{\mu}) = \mathbb{L}(\overline{\mu},\overline{\sigma}) \in \psi(\overline{\sigma})$. Furthermore, from Theorem 3, $\mathcal{F}(\overline{\mu})$ is a weakly LU-efficient point of VCD$^{\mathrm{WVL}}(\overline{\mu})$. □

Remark 13. *If* $\mathcal{F}_i^L(\mu) = \mathcal{F}_i^U(\mu) = \mathcal{F}_i(\mu)$, $\forall i \in \mathcal{J}^{\mathcal{F}}$, $\mu \in \mathbb{R}^n$ *and if* $\partial_c\mathcal{F}_i^L(\mu) = \{\nabla\mathcal{F}_i^L(\mu)\}$, $\partial_c\mathcal{F}_i^U(\mu) = \{\nabla\mathcal{F}_i^U(\mu)\}$ $(i \in \mathcal{J}^{\mathcal{F}})$ $\partial_c\Psi_k(\mu) = \{\nabla\Psi_k(\mu)\}$ $(k \in \mathcal{L})$, $\partial_c\zeta_i(\mu) = \{\nabla\zeta_i(\mu)\}$ $(i \in \mathcal{B})$, $\partial_c\mathcal{Q}_i(\mu) = \{\nabla\mathcal{Q}_i(\mu)\}$ $(i \in \mathcal{C})$, $\partial_c\mathcal{R}_i(\mu) = \{\nabla\mathcal{R}_i(\mu)\}$ $(i \in \mathcal{C})$ *then, Theorem 5 reduces to Proposition 3.4 from [44].*

Now, we provide an example to demonstrate the significance of Theorem 5.

Example 3. *Consider Example 1 and let* $\overline{\mu} = (1,1)$. *From Example 1,* $\overline{\mu}$ *is a VC-stationary point of* $(\mathcal{P}_1)$. *Moreover,* $\mathcal{F}(\overline{\mu}) = ([0,1],[0,0])$ *and* $\mathcal{F}_i(i = 1,2)$, Ψ, $\mathcal{Q}_1$, $\mathcal{R}_1$ *are LU-convex and convex at* $\overline{\mu}$, *respectively. Therefore, all the hypotheses in Theorem 5 are satisfied. Hence, from Theorem 5 there exists* $\overline{\sigma} = (\overline{\sigma}_\tau^{\Psi},\overline{\sigma}_1^{\mathcal{Q}},\overline{\sigma}_1^{\mathcal{R}})$ *such that*

$$\overline{\sigma}_\tau^{\Psi} = \begin{cases}1, & \tau = 0,\\ 0, & \textit{otherwise},\end{cases}$$

$\overline{\sigma}_1^{\mathcal{Q}} = 0$, $\overline{\sigma}_1^{\mathcal{R}} = 1$ *such that* $\mathcal{F}(\overline{\mu}) \in \psi(\overline{\sigma})$ *and* $\mathcal{F}(\overline{\mu})$ *is a weakly LU-efficient point of* (VCD)$^{WVL2}(\overline{\mu})$.

4.2. Interval-Valued Vector Lagrange Type Duality

In this subsection, we formulate an interval-valued vector Lagrange type dual problem corresponding to NIMSIPVC and further elucidate the weak and strong duality results.

Define a set-valued function $\psi^V : \mathbb{R}_+^{|\mathcal{L}|} \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s \rightrightarrows \mathcal{I}^l$ as follows:

$$\psi^V(\sigma) := \mathrm{Min}_{VC}\{\mathbb{L}(\mu,\sigma)|\mu \in \mathcal{G}\}.$$

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Let us formulate the interval-valued vector Lagrange type dual problem of NIMSIPVC for a given $\mu \in \mathcal{G}$, in the following manner:

$$\begin{aligned} \text{VCD}^{\text{VL}}(\mu)\ &\text{Max } \psi^V(\sigma) \\ &\text{subject to } \sigma^{\Psi}_{\mathcal{L}\setminus\mathbb{P}(\mu)} \geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(\mu)} \leq 0, \\ &\qquad \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(\mu)\cup\mathcal{H}_{0-}(\mu)} \geq 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(\mu)} \geq 0. \end{aligned}$$

The feasible set of VCD^{VL} is symbolized by $\mathcal{G}_{\text{VL}(\mu)}$, and is given by:

$$\begin{aligned} \mathcal{G}_{\text{VL}(\mu)} := \{\sigma = (\sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}^{|\mathcal{L}|}_{+} \times \mathbb{R}^{r} \times \mathbb{R}^{s} \times \mathbb{R}^{s}) | \sigma^{\Psi}_{\mathcal{L}\setminus\mathbb{P}(\mu)} \geq 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(\mu)} \leq 0, \\ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(\mu)\cup\mathcal{H}_{0-}(\mu)} \geq 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(\mu)} \geq 0\}. \end{aligned}$$

In the next definition, we extend the definition of a weakly LU-efficient point of $\text{VCD}^{VL}(\mu)$ from Tung et al. [44]. For further details, we refer the readers to [11,41].

Definition 11. *An interval-valued vector $\bar{I} \in \bigcup_{\sigma \in \mathcal{G}_{VL(\mu)}} \psi^V(\sigma)$ is said to be an LU-efficient point of $VCD^{VL}(\mu)$, provided*

$$\bar{I} \in Max_{VC} \bigcup_{\sigma \in \mathcal{G}_{VL(\mu)}} \psi^V(\sigma).$$

Equivalently, there does not exists any $I \in \bigcup_{\sigma \in \mathcal{G}_{VL(\mu)}} \psi^V(\sigma)$ such that

$$\bar{I} \prec_{LU} I.$$

Remark 14. *It is worth noting that $VCD^{VL}(\mu)$ depends on the feasible point μ.*

Now, we propose the interval-valued vector Lagrange type dual problem for NIMSIPVC, which is independent of the choice of a feasible element, as follows:

$$\begin{aligned} \text{VCD}^{\text{VL}}\ &\text{Max } \psi^V(\sigma) \\ &\text{subject to } \sigma \in \mathcal{G}_{\text{VL}} = \bigcap_{\mu \in \mathcal{G}} \mathcal{G}_{\text{VL}(\mu)}. \end{aligned}$$

Remark 15. *One can easily note that the feasible region of VCD^{VL} is always non-empty i.e. $\mathcal{G}_{VL} = \bigcap_{\mu \in \mathcal{G}_{VL(\mu)}} \neq \emptyset$.*

In the following theorem, we establish the weak duality result that elucidates the relationship between NIMSIPVC and VCD^{VL}. The proof is analogous to the proof of Theorem 2 and we will omit it.

Theorem 6. *Let μ be an arbitrary element of $\mathcal{G}$ and $I \in \bigcup_{\sigma \in \mathcal{G}_{VL(\mu)}} \psi^V(\sigma)$. Then*

$$\mathcal{F}(\mu) \not\prec_{LU} I.$$

Remark 16. *Theorem 6 extends Proposition 3.6 derived by Tung et al. [44] from smooth multiobjective SIPVC to NIMSIPVC, which belongs to a broader category of optimization problems.*

In the following theorem, we derive the relationship between a feasible point of NIMSIPVC and a LU-efficient point of VCD^{VL}, respectively. The proof is analogous to the proof of Theorem 3 and we will omit it.

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Theorem 7. *Consider an arbitrary $\overline{\mu} \in \mathcal{G}$, $\overline{\sigma} \in \mathcal{G}_{VL(\overline{\mu})}$ and $\mathcal{F}(\overline{\mu}) \in \psi^V(\overline{\sigma})$. Then $\mathcal{F}(\overline{\mu})$ is an LU-efficient point of $VCD^{VL}(\overline{\mu})$.*

In the following theorem, we derive the converse duality result that relates our primal problem NIMSIPVC and the corresponding interval-valued vector Lagrange type dual problem VCD$^{\text{VL}}$. The proof is analogous to the proof of Theorem 4 and we will omit it.

Theorem 8. *Let $\overline{\mu} \in \mathcal{G}$, $\overline{\sigma} \in \mathcal{G}_{VL}$, and $\mathcal{F}(\overline{\mu}) \in \psi^V(\overline{\sigma})$. Then $\overline{\mu} \in Eff$.*

Remark 17. *Theorem 7 and 8 extend Proposition 3.7 deduced by Tung et al. [44] from smooth multiobjective SIPVC to NIMSIPVC, which belongs to a more general category of optimization problems.*

In the following theorem, we derive the strong duality result relating NIMSIPVC and the interval-valued vector Lagrange type dual problem of NIMSIPVC.

Theorem 9. *Let $\overline{\mu} \in WEff_{loc}$ such that VC-ACQ is satisfied at $\overline{\mu}$ and let $\mathcal{K}_2$ be a closed set. Further, assume that $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$, Ψ_k $(k \in \mathbb{P}^{\Psi}(\overline{\mu}))$, ζ_i $(i \in \mathcal{B}_+^{\zeta})$, $-\zeta_i$ $(i \in \mathcal{B}_-^{\zeta})$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^{-}(\overline{\mu}))$, $-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^{+}(\overline{\mu}) \cup \overline{\mathbb{H}}_{00}^{+}(\overline{\mu}) \cup \overline{\mathbb{H}}_{0-}^{+}(\overline{\mu}))$, $\mathcal{R}_i$ $(i \in \mathbb{H}_{+0}^{+}(\overline{\mu}) \cup \mathbb{H}_{00}^{+}(\overline{\mu}))$ are strictly LU-convex and convex at $\overline{\mu}$, respectively. Then there exists $\overline{\sigma} \in \mathcal{G}_{VL(\overline{\mu})}$ such that $\mathcal{F}(\overline{\mu}) \in \psi^V(\overline{\sigma})$. Furthermore, $\mathcal{F}(\overline{\mu})$ is an LU-efficient point of $VCD^{VL}(\overline{\mu})$.*

Proof. Following the similar steps in Theorem 5, we obtain

$$\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}) = 0.$$

Hence,

$$\mathbb{L}^L(\overline{\mu},\overline{\sigma}) = \mathcal{F}^L(\overline{\mu}), \quad \mathbb{L}^U(\overline{\mu},\overline{\sigma}) = \mathcal{F}^U(\overline{\mu}). \tag{17}$$

Let us assume that there exists some $\tilde{\mu} \in \mathcal{G}$ such that

$$\mathbb{L}(\tilde{\mu},\overline{\sigma}) \prec_{LU} \mathbb{L}(\overline{\mu},\overline{\sigma}).$$

This implies that

$$\mathbb{L}_i^L(\tilde{\mu},\overline{\sigma}) \leq \mathbb{L}_i^L(\overline{\mu},\overline{\sigma}),\ \mathbb{L}_i^U(\tilde{\mu},\overline{\sigma}) \leq \mathbb{L}_i^U(\overline{\mu},\overline{\sigma}),\ \forall i \in \mathcal{J}^{\mathcal{F}},$$

and for at least one $p \in \mathcal{J}^{\mathcal{F}}$, exactly one of the following relation holds:

$$\begin{cases}\mathbb{L}_p^L(\tilde{\mu},\overline{\sigma}) < \mathbb{L}_p^L(\overline{\mu},\overline{\sigma})\\ \mathbb{L}_p^U(\tilde{\mu},\overline{\sigma}) \leq \mathbb{L}_p^U(\overline{\mu},\overline{\sigma})\end{cases} \text{or} \begin{cases}\mathbb{L}_p^L(\tilde{\mu},\overline{\sigma}) \leq \mathbb{L}_p^L(\overline{\mu},\overline{\sigma})\\ \mathbb{L}_p^U(\tilde{\mu},\overline{\sigma}) < \mathcal{L}_p^U(\overline{\mu},\overline{\sigma})\end{cases} \text{or} \begin{cases}\mathbb{L}_p^L(\tilde{\mu},\overline{\sigma}) < \mathbb{L}_p^L(\overline{\mu},\overline{\sigma})\\ \mathbb{L}_p^U(\tilde{\mu},\overline{\sigma}) < \mathbb{L}_p^U(\overline{\mu},\overline{\sigma}).\end{cases}$$

Equivalently, for every $i \in \mathcal{J}^{\mathcal{F}}$, the following inequalities hold:

$$\begin{aligned}
&\mathcal{F}_i^L(\tilde{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{\mu})\\
\leq &\mathcal{F}_i^L(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}),\\
&\mathcal{F}_i^U(\tilde{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{\mu})\\
\leq &\mathcal{F}_i^U(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}),
\end{aligned}$$

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and for at least one $p \in \mathcal{J}^{\mathcal{F}}$, the following condition holds:

$$\begin{aligned}
&\mathcal{F}_p^L(\tilde{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{\mu}) \\
<&\mathcal{F}_p^L(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}), \\
&\mathcal{F}_p^U(\tilde{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{\mu}) \\
\leq&\mathcal{F}_p^U(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}),
\end{aligned}$$

or

$$\begin{aligned}
&\mathcal{F}_p^L(\tilde{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{\mu}) \\
\leq&\mathcal{F}_p^L(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}), \\
&\mathcal{F}_p^U(\tilde{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{\mu}) \\
<&\mathcal{F}_p^U(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}),
\end{aligned}$$

or

$$\begin{aligned}
&\mathcal{F}_p^L(\tilde{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{\mu}) \\
<&\mathcal{F}_p^L(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}), \\
&\mathcal{F}_p^U(\tilde{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{\mu}) \\
<&\mathcal{F}_p^U(\overline{\mu}) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}).
\end{aligned}$$

On multiplying with $\overline{\lambda}_i^L,\ \overline{\lambda}_i^U \geq 0\ (i \in \mathcal{J}^{\mathcal{F}})$ such that $\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L + \overline{\lambda}_i^U\right) = 1$, we have

$$\begin{aligned}
&\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\tilde{\mu}) + \overline{\lambda}_i^U\mathcal{F}_i^U(\tilde{\mu})\right) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{\mu}) \\
&- \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{\mu}) + \overline{\lambda}_i^U\mathcal{F}_i^U(\overline{\mu})\right) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}) \\
&\leq 0.
\end{aligned} \tag{18}$$

In view of the fact that $\mathcal{F}_i^L,\ \mathcal{F}_i^U, \forall i \in \mathcal{J}^{\mathcal{F}}$ are strictly LU-convex at $\overline{\mu}$, we have

$$\begin{aligned}
\mathcal{F}_i^L(\tilde{\mu}) - \mathcal{F}_i^L(\overline{\mu}) > \langle \xi_i^L, \tilde{\mu} - \overline{\mu}\rangle,\ \forall \xi_i^L \in \partial_c\mathcal{F}_i^L(\overline{\mu}),\ \forall i \in \mathcal{J}^{\mathcal{F}} \\
\mathcal{F}_i^U(\tilde{\mu}) - \mathcal{F}_i^U(\overline{\mu}) > \langle \xi_i^U, \tilde{\mu} - \overline{\mu}\rangle,\ \forall \xi_i^U \in \partial_c\mathcal{F}_i^U(\overline{\mu}),\ \forall i \in \mathcal{J}^{\mathcal{F}}.
\end{aligned}$$

Following the similar steps in Theorem 5 along with the convexity assumptions of all the constraint functions, we obtain

$$\begin{aligned}
&\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\tilde{\mu}) + \overline{\lambda}_i^U\mathcal{F}_i^U(\tilde{\mu})\right) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\tilde{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\tilde{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\tilde{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\tilde{\mu}) \\
&- \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{\mu}) + \overline{\lambda}_i^U\mathcal{F}_i^U(\overline{\mu})\right) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}) \\
&> \left\langle \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\xi_i^L + \overline{\lambda}_i^U\xi_i^U\right) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\eta_k^{\Psi} + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\eta_i^{\zeta} - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\eta_i^{\mathcal{Q}} + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\eta_i^{\mathcal{R}}, \tilde{\mu} - \overline{\mu}\right\rangle.
\end{aligned} \tag{19}$$

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From (18) and (19), we obtain that for every $\xi_i^L \in \partial_c \mathcal{F}_i^L(\overline{\mu})$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\xi_i^U \in \partial_c \mathcal{F}_i^U(\overline{\mu})$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\eta_k^\Psi \in \partial_c \Psi_k(\overline{\mu})$ $(k \in \mathcal{L})$, $\eta_i^\zeta \in \partial_c \zeta_i(\overline{\mu})$ $(i \in \mathcal{B})$, $\eta_i^{\mathcal{Q}} \in \partial_c \mathcal{Q}_i(\overline{\mu})$ $(i \in \mathcal{C})$, $\eta_i^{\mathcal{R}} \in \partial_c \mathcal{R}_i(\overline{\mu})$ $(i \in \mathcal{C})$,

$$\left\langle \sum_{i\in\mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \xi_i^L + \overline{\lambda}_i^U \xi_i^U\right) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^\Psi \eta_k^\Psi + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^\zeta \eta_i^\zeta - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \eta_i^{\mathcal{Q}} + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \eta_i^{\mathcal{R}}, \tilde{\mu} - \overline{\mu} \right\rangle < 0,$$

which is a contradiction to the fact that $\overline{\mu} \in \mathrm{VC}_{SP}$. From Theorem 7, $\mathcal{F}(\overline{\mu})$ is an LU-efficient point of $\mathrm{VCD}^{\mathrm{VL}}(\overline{\mu})$. Furthermore, from Theorem 8, we conclude that $\overline{\mu} \in \mathrm{Eff}$. This completes the proof. □

Remark 18. *Theorem 9 extends Proposition 3.8 deduced by Tung et al. [44] from the smooth case of multiobjective semi-infinite programming problems with vanishing constraints to nonsmooth semi-infinite programming problems with vanishing constraints, including multiple interval-valued objective functions.*

4.3. Interval-valued vector saddle point optimality criteria

In the following subsection, we introduce the notions of LU-saddle points for the interval-valued vector Lagrangian of NIMSIPVC, in particular, weakly LU-saddle point and LU-saddle point. Further, we establish several relationships between optimal solutions of NIMSIPVC and saddle points for interval-valued vector Lagrangian of NIMSIPVC.

In the following definition, we extend the notions of saddle points for an interval-valued vector Lagrangian of NIMSIPVC, which was presented by Tung et al. [44] for the vector Lagrangian of smooth multiobjective SIPVC.

Definition 12. *Let $\overline{\mu} \in \mathcal{G}$ and $\overline{\sigma} \in \mathcal{G}_{WVL(\overline{\mu})}$ be arbitrary elements. Then $(\overline{\mu}, \overline{\sigma})$ is known as*

(a) weakly LU-saddle point for the interval-valued vector Lagrangian of NIMSIPVC, provided the following condition holds:

$$\mathbb{L}(\mu, \overline{\sigma}) \not\prec_{LU}^{s} \mathbb{L}(\overline{\mu}, \overline{\sigma}) \not\prec_{LU}^{s} \mathbb{L}(\overline{\mu}, \sigma), \; \forall \mu \in \mathcal{G}, \forall \sigma \in \mathcal{G}_{WVL(\overline{\mu})}.$$

(b) LU-saddle point for the interval-valued vector Lagrangian of NIMSIPVC, provided the following condition holds:

$$\mathbb{L}(\mu, \overline{\sigma}) \not\prec_{LU} \mathbb{L}(\overline{\mu}, \overline{\sigma}) \not\prec_{LU} \mathbb{L}(\overline{\mu}, \sigma), \; \forall \mu \in \mathcal{G}, \forall \sigma \in \mathcal{G}_{WVL(\overline{\mu})}.$$

The symbols $WS^{\mathbb{L}}$ and $S^{\mathbb{L}}$ are used to denote the sets of all weakly LU-saddle points and LU-saddle points for the interval-valued vector Lagrangian of NIMSIPVC, respectively.

Remark 19. *It is worth noting that*

$$S^{\mathbb{L}} \subseteq WS^{\mathbb{L}}.$$

In the following theorem, we derive the relationship between a weakly LU-efficient solution and weakly LU-saddle point of NIMSIPVC and interval-valued vector Lagrangian of NIMSIPVC, respectively.

Theorem 10. *Let $\overline{\mu} \in WEff$ such that VC-ACQ is satisfied at $\overline{\mu}$ and let $\mathcal{K}_2$ be a closed set. Further, assume that $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$, Ψ_k $(k \in \mathbb{P}^\Psi(\overline{\mu}))$, ζ_i $(i \in \mathcal{B}_+^\zeta)$, $-\zeta_i$ $(i \in \mathcal{B}_-^\zeta)$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^-(\overline{\mu}))$, $-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^+(\overline{\mu}) \cup \overline{\mathbb{H}}_{00}^+(\overline{\mu}) \cup \overline{\mathbb{H}}_{0-}^+(\overline{\mu}))$, $\mathcal{R}_i$ $(i \in \mathbb{H}_{+0}^+(\overline{\mu}) \cup \mathbb{H}_{00}^+(\overline{\mu}))$ are LU-convex and convex at $\overline{\mu}$, respectively. Then there exists $\overline{\sigma} \in \mathcal{G}_{WVL(\overline{\mu})}$ such that $(\overline{\mu}, \overline{\sigma}) \in WS^{\mathbb{L}}$.*

Proof. From Theorem 5, there exists $\overline{\sigma} \in \mathcal{G}_{WVL(\overline{\mu})}$ satisfying

$$\mathcal{F}(\overline{\mu}) = \mathbb{L}(\overline{\mu}, \overline{\sigma}), \tag{20}$$

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as well as

$$\mathbb{L}(\mu,\overline{\sigma}) \not\prec^{s}_{LU} \mathbb{L}(\overline{\mu},\overline{\sigma}), \ \forall \mu \in \mathcal{G}. \tag{21}$$

We will prove that

$$\mathbb{L}(\overline{\mu},\overline{\sigma}) \not\prec^{s}_{LU} \mathbb{L}(\overline{\mu},\sigma), \ \forall \sigma \in \mathcal{G}_{WVL(\overline{\mu})}.$$

On the contrary, we suppose that there exists $\sigma \in \mathcal{G}_{WVL(\overline{\mu})}$ such that

$$\mathbb{L}(\overline{\mu},\overline{\sigma}) \prec^{s}_{LU} \mathbb{L}(\overline{\mu},\sigma), \ \forall \sigma \in \mathcal{G}_{WVL(\overline{\mu})}.$$

From (20) we obtain

$$\mathcal{F}(\overline{\mu}) \prec^{s}_{LU} \mathcal{F}(\overline{\mu}) + \left(\sum_{k\in\mathcal{L}} \sigma_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) \right) e.$$

Furthermore, we can rewrite the above inequality for every $i \in \mathcal{J}^{\mathcal{F}}$ as follows:

$$\mathcal{F}_i^L(\overline{\mu}) < \mathcal{F}_i^L(\overline{\mu}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}),$$
$$\mathcal{F}_i^U(\overline{\mu}) < \mathcal{F}_i^U(\overline{\mu}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}).$$

It follows that

$$\sum_{k\in\mathcal{L}} \sigma_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) > 0. \tag{22}$$

In view of the fact that $\overline{\mu} \in \mathcal{G}$ and $\sigma \in \mathcal{G}_{WVL(\overline{\mu})}$, we have

$$\sum_{k\in\mathcal{L}} \sigma_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) \leq 0,$$

which is a contradiction to (22). Therefore, there does not exist any $\sigma \in \mathcal{G}_{WVL(\overline{\mu})}$ such that

$$\mathbb{L}(\overline{\mu},\overline{\sigma}) \not\prec^{s}_{LU} \mathbb{L}(\overline{\mu},\sigma), \forall \sigma \in \mathcal{G}_{WVL(\overline{\mu})}. \tag{23}$$

Therefore, from (21) and (23), $(\overline{\mu},\overline{\sigma}) \in \mathrm{WS}^{\mathbb{L}}$. This completes the proof. □

Remark 20. *Theorem 10 extends Proposition 3.10(i) derived by Tung et al. [44] from smooth multiobjective semi-infinite programming problem with vanishing constraints to a broader class of optimization problems, in particular, NIMSIPVC.*

In the following theorem, we establish the relationship between a weakly LU-saddle point and weakly LU-efficient point of interval-valued vector Lagrangian for NIMSIPVC and $\mathrm{VCD}^{\mathrm{WVL}}(\overline{\mu})$, respectively.

Theorem 11. *Let $(\overline{\mu},\overline{\sigma}) \in \mathcal{G} \times \mathcal{G}_{WVL(\overline{\mu})}$ be a weakly LU-saddle point for the interval-valued vector Lagrangian of NIMSIPVC. Then $\mathcal{F}(\overline{\mu}) \in \psi(\overline{\sigma})$, where $\mathcal{F}(\overline{\mu})$ is a weakly LU-efficient point of $VCD^{WVL}(\overline{\mu})$.*

Proof. From the given hypothesis, $(\overline{\mu},\overline{\sigma}) \in \mathcal{G} \times \mathcal{G}_{\mathrm{WVL}(\overline{\mu})}$ is a weakly LU-saddle point for the interval-valued vector Lagrangian of NIMSIPVC. It follows that

$$\mathbb{L}(\overline{\mu},\overline{\sigma}) \not\prec^{s}_{LU} \mathbb{L}(\overline{\mu},\sigma), \ \forall \sigma \in \mathcal{G}_{\mathrm{WVL}(\overline{\mu})}.$$

Equivalently,

$$\begin{aligned}&\mathcal{F}(\overline{\mu})+\left(\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu})+\sum_{i\in\mathcal{C}}\overline{\sigma}^{\mathcal{R}}\mathcal{R}_i(\overline{\mu})\right)e\\ \not\prec^{s}_{LU}&\mathcal{F}(\overline{\mu})+\left(\sum_{k\in\mathcal{L}}\sigma_k^{\Psi}\Psi_k(\overline{\mu})+\sum_{i\in\mathcal{B}}\sigma_i^{\zeta}\zeta_i(\overline{\mu})-\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu})+\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu})\right)e.\end{aligned} \tag{24}$$

If we assume that $\sigma = 0$, then (24) can be rewritten as follows:

$$\mathcal{F}(\overline{\mu})+\left(\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu})\right)\not\prec^{s}_{LU}\mathcal{F}(\overline{\mu}). \tag{25}$$

Moreover, by following the similar steps in Theorem 2, we deduce that

$$\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu})\leq 0. \tag{26}$$

If

$$\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu})< 0,$$

then

$$\mathcal{F}(\overline{\mu})+\left(\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu})\right)\prec^{s}_{LU}\mathcal{F}(\overline{\mu}),$$

which is a contradiction to (25). Therefore, we have

$$\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu})= 0.$$

This implies that $\mathbb{L}(\overline{\mu},\overline{\sigma}) = \mathcal{F}(\overline{\mu})$. In view of the given hypothesis, $(\overline{\mu},\overline{\sigma})$ is a weakly LU-saddle point for the interval-valued vector Lagrangian of NIMSIPVC. It follows that

$$\mathbb{L}(\mu,\overline{\sigma})\not\prec^{s}_{LU}\mathbb{L}(\overline{\mu},\overline{\sigma}),\ \forall\mu\in\mathcal{G}.$$

This claims that $\mathcal{F}(\overline{\mu}) = \mathbb{L}(\overline{\mu},\overline{\sigma}) \in \text{WMin}\{\mathbb{L}(\mu,\overline{\sigma})|\mu\in\mathcal{G}\} = \psi(\overline{\sigma})$. In view of Theorem 3, $\mathcal{F}(\overline{\mu})$ is a weakly LU-efficient point of $\text{VCD}^{\text{WVL}}(\overline{\mu})$. □

Remark 21. *If $\mathcal{F}_i^L(\mu) = \mathcal{F}_i^U(\mu) = \mathcal{F}_i(\mu)$, $\forall i\in\mathcal{J}^{\mathcal{F}}$, $\mu\in\mathbb{R}^n$ and if $\partial_c\mathcal{F}_i^L(\mu) = \{\nabla\mathcal{F}_i^L(\mu)\}$, $\partial_c\mathcal{F}_i^U(\mu) = \{\nabla\mathcal{F}_i^U(\mu)\}$ $(i\in\mathcal{J}^{\mathcal{F}})$, $\partial_c\Psi_k(\mu) = \{\nabla\Psi_k(\mu)\}$ $(k\in\mathcal{L})$, $\partial_c\zeta_i(\mu) = \{\nabla\zeta_i(\mu)\}$ $(i\in\mathcal{B})$, $\partial_c\mathcal{Q}_i(\mu) = \{\nabla\mathcal{Q}_i(\mu)\}$ $(i\in\mathcal{C})$, $\partial_c\mathcal{R}_i(\mu) = \{\nabla\mathcal{R}_i(\mu)\}$ $(i\in\mathcal{C})$ then, Theorem 11 reduces to Proposition 3.10(ii) from [44].*

In the following theorem, we establish the relationship between a VC-stationary point and weakly LU-saddle point of NIMSIPVC and interval-valued vector Lagrangian of NIMSIPVC, respectively.

Theorem 12. *Let $\overline{\mu}\in VC_{SP}$. Further, assume that $\mathcal{F}_i$ $(i\in\mathcal{J}^{\mathcal{F}})$, Ψ_k $(k\in\mathbb{P}^{\Psi}(\overline{\mu}))$, ζ_i $(i\in\mathcal{B}^{\zeta}_{+})$, $-\zeta_i$ $(i\in\mathcal{B}^{\zeta}_{-})$, $\mathcal{Q}_i$ $(i\in\overline{\mathbb{H}}^{-}_{0+}(\overline{\mu}))$, $-\mathcal{Q}_i$ $(i\in\overline{\mathbb{H}}^{+}_{0+}(\overline{\mu})\cup\overline{\mathbb{H}}^{+}_{00}(\overline{\mu})\cup\overline{\mathbb{H}}^{+}_{0-}(\overline{\mu}))$, $\mathcal{R}_i$ $(i\in\mathbb{H}^{+}_{+0}(\overline{\mu})\cup\mathbb{H}^{+}_{00}(\overline{\mu}))$ are LU-convex and convex at $\overline{\mu}$. Then there exists $\overline{\sigma} = (\overline{\sigma}^{\Psi},\overline{\sigma}^{\zeta},\overline{\sigma}^{\mathcal{Q}},\overline{\sigma}^{\mathcal{R}})\in\mathbb{P}^{\Psi}(\overline{\mu})\times\mathbb{R}^r\times\mathbb{R}^s\times\mathbb{R}^s$ such that $(\overline{\mu},\overline{\sigma})\in WS^{\mathbb{L}}$.*

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Proof. In view of the fact that $\overline{\mu} \in \mathrm{VC_{SP}}$, there exist $(\overline{\lambda}^L, \overline{\lambda}^U) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+$, $\overline{\sigma} = (\overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{P}^{\Psi}(\overline{\mu}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that the following condition holds:

$$0 \in \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \partial_c \mathcal{F}_i^L(\overline{\mu}) + \overline{\lambda}_i^U \partial_c \mathcal{F}_i^U(\overline{\mu}) \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \partial_c \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \partial_c \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \partial_c \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \partial_c \mathcal{R}_i(\overline{\mu}),$$

where $\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L + \overline{\lambda}_i^U \right) = 1$, $\overline{\sigma}^{\mathcal{Q}}_{\mathcal{H}_+(\overline{\mu})} = 0$, $\overline{\sigma}^{\mathcal{Q}}_{\mathcal{H}_{00}(\overline{\mu}) \cup \mathcal{H}_{0-}(\overline{\mu})} \geq 0$, $\overline{\sigma}^{\mathcal{R}}_{\mathcal{H}_{+0}(\overline{\mu}) \cup \mathcal{H}_{00}(\overline{\mu})} \geq 0$, and $\overline{\sigma}^{\mathcal{R}}_{\mathcal{H}_{+-}(\overline{\mu}) \cup \mathcal{H}_{0-}(\overline{\mu}) \cup \mathcal{H}_{0+}(\overline{\mu})} = 0$. This implies that there exist $\hat{\xi}_i^L \in \partial_c \mathcal{F}_i^L(\overline{\mu})$, $\hat{\xi}_i^U \in \partial_c \mathcal{F}_i^U(\overline{\mu})$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\hat{\eta}_k^{\Psi} \in \partial_c \Psi_k(\overline{\mu})$ $(k \in \mathcal{L})$, $\hat{\eta}_i^{\zeta} \in \partial_c \zeta_i(\overline{\mu})$ $(i \in \mathcal{B})$, $\hat{\eta}_i^{\mathcal{Q}} \in \partial_c \mathcal{Q}_i(\overline{\mu})$ $(i \in \mathcal{C})$, $\hat{\eta}_i^{\mathcal{R}} \in \partial_c \mathcal{R}_i(\overline{\mu})$ $(i \in \mathcal{C})$ such that

$$\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \hat{\xi}_i^L + \overline{\lambda}_i^U \hat{\xi}_i^U \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \hat{\eta}_k^{\Psi} + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \hat{\eta}_i^{\zeta} - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \hat{\eta}_i^{\mathcal{Q}} + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \hat{\eta}_i^{\mathcal{R}} = 0.$$

We divide the main proof into two parts:

(a) We will prove that

$$\mathbb{L}(\mu, \overline{\sigma}) \nprec^s_{LU} \mathbb{L}(\overline{\mu}, \overline{\sigma}), \ \forall \mu \in \mathcal{G}, \ \overline{\sigma} \in \mathcal{G}_{\mathrm{WVL}(\overline{\mu})}.$$

On the contrary, we suppose that there exists $\tilde{\mu} \in \mathcal{G}$ such that

$$\mathbb{L}(\tilde{\mu}, \overline{\sigma}) \prec^s_{LU} \mathbb{L}(\overline{\mu}, \overline{\sigma}).$$

This implies that for every $i \in \mathcal{J}^{\mathcal{F}}$, the following inequalities hold:

$$\begin{aligned}
&\mathcal{F}_i^L(\tilde{\mu}) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\tilde{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\tilde{\mu}) - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\tilde{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\tilde{\mu}) \\
&< \mathcal{F}_i^L(\overline{\mu}) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}), \\
&\mathcal{F}_i^U(\tilde{\mu}) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\tilde{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\tilde{\mu}) - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\tilde{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\tilde{\mu}) \\
&< \mathcal{F}_i^U(\overline{\mu}) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}).
\end{aligned}$$

Multiply first and second inequalities by $\overline{\lambda}_i^L$, $\overline{\lambda}_i^U$, respectively. On adding them, we get

$$\begin{aligned}
&\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \mathcal{F}_i^L(\tilde{\mu}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\tilde{\mu}) \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\tilde{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\tilde{\mu}) - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\tilde{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\tilde{\mu}) \\
&- \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \mathcal{F}_i^L(\overline{\mu}) - \overline{\lambda}_i^U \mathcal{F}_i^U(\overline{\mu}) \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) \\
&< 0.
\end{aligned} \tag{27}$$

Following the similar steps in the proof of Theorem 5 we have

$$\begin{aligned}
&\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \mathcal{F}_i^L(\tilde{\mu}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\tilde{\mu}) \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\tilde{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\tilde{\mu}) - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\tilde{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\tilde{\mu}) \\
&- \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \mathcal{F}_i^L(\overline{\mu}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\overline{\mu}) \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) \\
&\geq \left\langle \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \xi_i^L + \overline{\lambda}_i^U \xi_i^U \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^{\Psi} \eta_k^{\Psi} + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^{\zeta} \eta_i^{\zeta} - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \eta_i^{\mathcal{Q}} + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \eta_i^{\mathcal{R}}, \tilde{\mu} - \overline{\mu} \right\rangle.
\end{aligned} \tag{28}$$

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From (27), we get that for every $\xi_i^L \in \partial_c \mathcal{F}_i^L(\overline{\mu})$, $\xi_i^U \in \partial_c \mathcal{F}_i^U(\overline{\mu})$, $\eta_k^\Psi \in \partial_c \Psi_k(\overline{\mu})$, $\eta_i^\zeta \in \partial_c \zeta_i(\overline{\mu})$, $\eta_i^\mathcal{Q} \in \partial_c \mathcal{Q}_i(\overline{\mu})$, $\eta_i^\mathcal{R} \in \partial_c \mathcal{R}_i(\overline{\mu})$,

$$\left\langle \sum_{i \in \mathcal{J}^\mathcal{F}} \left(\overline{\lambda}_i^L \xi_i^L + \overline{\lambda}_i^L \xi_i^U \right) + \sum_{k \in \mathcal{L}} \overline{\sigma}_k^\Psi \eta_k^\Psi + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^\zeta \eta_i^\zeta - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^\mathcal{Q} \eta_i^\mathcal{Q} + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^\mathcal{R} \eta_i^\mathcal{R}, \tilde{\mu} - \overline{\mu} \right\rangle < 0,$$

which is a contradiction to the fact that $\overline{\mu} \in \mathrm{VC}_{SP}$. Therefore,

$$\mathbb{L}(\mu, \overline{\sigma}) \not\prec_{LU}^s \mathbb{L}(\overline{\mu}, \overline{\sigma}), \ \forall \mu \in \mathcal{G}. \tag{29}$$

(b) In this part, we shall claim that

$$\mathbb{L}(\overline{\mu}, \overline{\sigma}) \not\prec_{LU}^s \mathbb{L}(\overline{\mu}, \sigma), \ \forall \sigma \in \mathcal{G}_{WVL(\overline{\mu})}.$$

On the contrary, we suppose that there exists $\sigma \in \mathcal{G}_{\mathrm{WVL}(\overline{\mu})}$ such that

$$\mathbb{L}(\overline{\mu}, \overline{\sigma}) \prec_{LU}^s \mathbb{L}(\overline{\mu}, \sigma).$$

It follows that

$$\begin{aligned} &\mathcal{F}(\overline{\mu}) + \left(\sum_{k \in \mathcal{L}} \overline{\sigma}_k^\Psi \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^\zeta \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^\mathcal{Q} \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^\mathcal{R} \mathcal{R}_i(\overline{\mu}) \right) e \\ \prec_{LU}^s &\mathcal{F}(\overline{\mu}) + \left(\sum_{k \in \mathcal{L}} \sigma_k^\Psi \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \sigma_i^\zeta \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \sigma_i^\mathcal{Q} \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \sigma_i^\mathcal{R} \mathcal{R}_i(\overline{\mu}) \right) e. \end{aligned} \tag{30}$$

In view of the fact that $\overline{\mu} \in \mathrm{VC}_{SP}$ we obtain

$$\sum_{k \in \mathcal{L}} \overline{\sigma}_k^\Psi \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}_i^\zeta \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \overline{\sigma}_i^\mathcal{Q} \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}_i^\mathcal{R} \mathcal{R}_i(\overline{\mu}) = 0. \tag{31}$$

Hence, from (30) and (31) we deduce that

$$\sum_{i \in \mathcal{J}} \sigma_i^\Psi \Psi_i(\overline{\mu}) + \sum_{i \in \mathcal{B}} \sigma_i^\zeta \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \sigma_i^\mathcal{Q} \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \sigma_i^\mathcal{R} \mathcal{R}_i(\overline{\mu}) > 0. \tag{32}$$

Since $\overline{\mu} \in \mathcal{G}$ and $\sigma \in \mathcal{G}_{WVL(\overline{\mu})}$ we infer that

$$\sum_{k \in \mathcal{L}} \sigma_k^\Psi \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \sigma_i^\zeta \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \sigma_i^\mathcal{Q} \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \sigma_i^\mathcal{R} \mathcal{R}_i(\overline{\mu}) \leq 0,$$

which is a contradiction to (32). Therefore,

$$\mathbb{L}(\overline{\mu}, \overline{\sigma}) \not\prec_{LU}^s \mathbb{L}(\overline{\mu}, \sigma), \ \forall \sigma \in \mathcal{G}_{WVL(\overline{\mu})}. \tag{33}$$

From (29) and (33) we can conclude that $(\overline{\mu}, \overline{\sigma}) \in \mathrm{WS}^\mathbb{L}$.

□

Remark 22. *Theorem 12 extends Proposition 3.11 from [44] for a general category of nonsmooth multiobjective optimization problems, particularly NIMSIPVC.*

Now, the following example illustrates the significance of Theorem 12.

Example 4. *Consider the Problem* $(\mathcal{P}_1)$ *in Example 1.*

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From Example 1, $\overline{\mu}$ is a VC-stationary point of $(\mathcal{P}_1)$. Furthermore, one can observe that $\mathcal{F}_i$ $(i = 1,2), \Psi_\tau$ $(\tau \in [0,1]), \mathcal{Q}, \mathbb{R}$ are LU-convex and convex at $\overline{\mu}$, respectively. Therefore, all the hypotheses in Theorem 12 are satisfied at $\overline{\mu}$ which implies that $(\overline{\mu}, \overline{\sigma})$ is a weakly LU-saddle point for the interval-valued vector Lagrangian of $(\mathcal{P}_1)$.

In the following theorem, we establish a relationship between LU-weakly local efficient solution and LU-saddle point of NIMSIPVC and interval-valued vector Lagrangian of NIMSIPVC, respectively.

Theorem 13. *Let $\overline{\mu} \in WEff_{loc}$ such that VC-ACQ is satisfied at $\overline{\mu}$ and let $\mathcal{K}_2$ be a closed set. Further, assume that $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$, Ψ_k $(k \in \mathbb{P}^{\Psi}(\overline{\mu})), \zeta_i$ $(i \in \mathcal{B}^{\zeta}_{+}), -\zeta_i$ $(i \in \mathcal{B}^{\zeta}_{-})$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}^{-}_{0+}(\overline{\mu}))$, $-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}^{+}_{0+}(\overline{\mu}) \cup \overline{\mathbb{H}}^{+}_{00}(\overline{\mu}) \cup \overline{\mathbb{H}}^{+}_{0-}(\overline{\mu}))$, $\mathcal{R}_i$ $(i \in \mathbb{H}^{+}_{+0}(\overline{\mu}) \cup \mathbb{H}^{+}_{00}(\overline{\mu}))$ are LU-convex and convex at $\overline{\mu}$, respectively. Then there exists $\overline{\sigma} = (\overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{P}^{\Psi}(\overline{\mu}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that $(\overline{\mu}, \overline{\sigma}) \in S^{\mathbb{L}}$.*

Proof. From the given hypotheses, $\overline{\mu} \in \mathrm{WEff}_{loc}$ such that VC-ACQ is satisfied at $\overline{\mu}$ and $\mathcal{K}_2$ is a closed set. Therefore, from Theorem 1, $\overline{\mu} \in VC_{SP}$. This implies that there exists $\overline{\sigma} = (\overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+ \times \mathbb{P}^{\Psi}(\overline{\mu}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that the following condition holds:

$$0 \in \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i \partial_c \mathcal{F}^L_i(\overline{\mu}) + \overline{\lambda}^U_i \partial_c \mathcal{F}^U_i(\overline{\mu})\right) + \sum_{k \in \mathcal{L}} \overline{\sigma}^{\Psi}_k \partial_c \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \overline{\sigma}^{\zeta}_i \partial_c \zeta_i(\overline{\mu})$$
$$- \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_i \partial_c \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{R}}_i \partial_c \mathcal{R}_i(\overline{\mu}),$$

where $\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i + \overline{\lambda}^U_i\right) = 1$, $\overline{\sigma}^{\mathcal{Q}}_{\mathcal{H}_+(\overline{\mu})} = 0$, $\overline{\sigma}^{\mathcal{Q}}_{\mathcal{H}_{00}(\overline{\mu}) \cup \mathcal{H}_{0-}(\overline{\mu})} \geq 0$, $\overline{\sigma}^{\mathcal{R}}_{\mathcal{H}_{+0}(\overline{\mu}) \cup \mathcal{H}_{00}(\overline{\mu})} \geq 0$, and $\overline{\sigma}^{\mathcal{R}}_{\mathcal{H}_{+-}(\overline{\mu}) \cup \mathcal{H}_{0-}(\overline{\mu}) \cup \mathcal{H}_{0+}(\overline{\mu})} = 0$. Therefore, $\overline{\sigma} \in \mathcal{G}_{VL(\overline{\mu})}$.

Moreover, there exist $\hat{\xi}^L_i \in \partial_c \mathcal{F}^L_i(\overline{\mu})$, $\hat{\xi}^U_i \in \partial_c \mathcal{F}^U_i(\overline{\mu})$ $(i \in \mathcal{J}^{\mathcal{F}}), \hat{\eta}^{\Psi}_k \in \partial_c \Psi_k(\overline{\mu})$ $(k \in \mathcal{L})$, $\hat{\eta}^{\zeta}_i \in \partial_c \zeta_i(\overline{\mu})$ $(i \in \mathcal{B})$, $\hat{\eta}^{\mathcal{Q}}_i \in \partial_c \mathcal{Q}_i(\overline{\mu})$ $(i \in \mathcal{C})$, $\hat{\eta}^{\mathcal{R}}_i \in \partial_c \mathcal{R}_i(\overline{\mu})$ $(i \in \mathcal{C})$ such that

$$\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i \hat{\xi}^L_i + \overline{\lambda}^U_i \hat{\xi}^U_i\right) + \sum_{k \in \mathcal{L}} \overline{\sigma}^{\Psi}_k \hat{\eta}^{\Psi}_k + \sum_{i \in \mathcal{B}} \overline{\sigma}^{\zeta}_i \hat{\eta}^{\zeta}_i - \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{Q}}_i \hat{\eta}^{\mathcal{Q}}_i + \sum_{i \in \mathcal{C}} \overline{\sigma}^{\mathcal{R}}_i \hat{\eta}^{\mathcal{R}}_i = 0.$$

Evidently, in view of the fact that $(\overline{\mu}, \overline{\sigma}) \in \mathcal{G} \times \mathcal{G}_{VL(\overline{\mu})}$ we have

$$\mathcal{L}(\overline{\mu}, \overline{\sigma}) = \mathcal{F}(\overline{\mu}).$$

From Theorem 9, $\mathcal{F}(\overline{\mu}) \in \psi^V(\overline{\sigma})$, which yields the following equation:

$$\mathbb{L}(\mu, \overline{\sigma}) \not\prec_{LU} \mathcal{F}(\overline{\mu}) = \mathbb{L}(\overline{\mu}, \overline{\sigma}), \ \forall \mu \in \mathcal{G}. \tag{34}$$

We are left to prove that

$$\mathbb{L}(\overline{\mu}, \overline{\sigma}) \not\prec_{LU} \mathbb{L}(\overline{\mu}, \sigma), \ \forall \sigma \in \mathcal{G}_{VL(\overline{\mu})}.$$

On the contrary, suppose that there exists $\sigma \in \mathcal{G}_{\mathrm{VL}(\overline{\mu})}$ such that

$$\mathbb{L}(\overline{\mu}, \overline{\sigma}) \prec_{LU} \mathbb{L}(\overline{\mu}, \sigma).$$

This implies that for every $i \in \mathcal{J}^{\mathcal{F}}$,

$$\mathcal{F}^L_i(\overline{\mu}) \leq \mathcal{F}^L_i(\overline{\mu}) + \sum_{k \in \mathcal{L}} \sigma^{\Psi}_k \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \sigma^{\zeta}_i \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \sigma^{\mathcal{Q}}_i \mathcal{Q}_i(\overline{\mu}) + \sigma^{\mathcal{R}}_i \sum_{i \in \mathcal{C}} \mathcal{R}_i(\overline{\mu}),$$
$$\mathcal{F}^U_i(\overline{\mu}) \leq \mathcal{F}^U_i(\overline{\mu}) + \sum_{k \in \mathcal{L}} \sigma^{\Psi}_k \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \sigma^{\zeta}_i \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \sigma^{\mathcal{Q}}_i \mathcal{Q}_i(\overline{\mu}) + \sum_{i \in \mathcal{C}} \sigma^{\mathcal{R}}_i \mathcal{R}_i(\overline{\mu}),$$

and for at least one $p \in \mathcal{J}^{\mathcal{F}}$, exactly one of the following relation holds:

$$\mathcal{F}_p^L(\overline{\mu}) < \mathcal{F}_p^L(\overline{\mu}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}),$$
$$\mathcal{F}_p^U(\overline{\mu}) \leq \mathcal{F}_p^U(\overline{\mu}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}),$$

or

$$\mathcal{F}_p^L(\overline{\mu}) \leq \mathcal{F}_p^L(\overline{\mu}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}),$$
$$\mathcal{F}_p^U(\overline{\mu}) < \mathcal{F}_p^U(\overline{\mu}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}),$$

or

$$\mathcal{F}_p^L(\overline{\mu}) < \mathcal{F}_p^L(\overline{\mu}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}),$$
$$\mathcal{F}_p^U(\overline{\mu}) < \mathcal{F}_p^U(\overline{\mu}) + \sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}).$$

Therefore,

$$\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}) > 0. \tag{35}$$

However, $(\overline{\mu}, \sigma) \in \mathcal{G} \times \mathcal{G}_{VL(\overline{\mu})}$, which gives that

$$\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}) \leq 0,$$

a contradiction to (35). Therefore,

$$\mathbb{L}(\overline{\mu}, \overline{\sigma}) \not<_{LU} \mathbb{L}(\overline{\mu}, \sigma), \ \forall \sigma \in \mathcal{G}_{VL(\overline{\mu})}. \tag{36}$$

From (34) and (36) we prove that $(\overline{\mu}, \overline{\sigma}) \in S^{\mathbb{L}}$. This completes the proof. □

Remark 23. *If $\mathcal{F}_i^L(\mu) = \mathcal{F}_i^U(\mu) = \mathcal{F}_i(\mu), \forall i \in \mathcal{J}^{\mathcal{F}}, \mu \in \mathbb{R}^n$ and if $\partial_c \mathcal{F}_i^L(\mu) = \{\nabla \mathcal{F}_i^L(\mu)\}$, $\partial_c \mathcal{F}_i^U(\mu) = \{\nabla \mathcal{F}_i^U(\mu)\}$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\partial_c \Psi_k(\mu) = \{\nabla \Psi_k(\mu)\}$ $(k \in \mathcal{L})$, $\partial_c \zeta_i(\mu) = \{\nabla \zeta_i(\mu)\}$ $(i \in \mathcal{B})$, $\partial_c \mathcal{Q}_i(\mu) = \{\nabla \mathcal{Q}_i(\mu)\}$ $(i \in \mathcal{C})$, $\partial_c \mathcal{R}_i(\mu) = \{\nabla \mathcal{R}_i(\mu)\}$ $(i \in \mathcal{C})$ then Theorem 13 reduces to Proposition 3.13(i) from [44].*

In the following theorem, we derive the necessary condition for a saddle point of the interval-valued vector Lagrangian of NIMSIPVC.

Theorem 14. *If $(\overline{\mu}, \overline{\sigma}) \in \mathcal{G} \times \mathcal{G}_{VL(\overline{\mu})}$ is a saddle point for the interval-valued vector Lagrangian of NIMSIPVC, then $\mathcal{F}(\overline{\mu}) \in \psi^V(\overline{\mu})$ such that $\mathcal{F}(\overline{\mu})$ is an LU-efficient point of $VCD^{VL}(\overline{\mu})$.*

Proof. Since $(\overline{\mu}, \overline{\sigma}) \in S^{\mathbb{L}}$. This implies that

$$\mathbb{L}(\overline{\mu}, \overline{\sigma}) \not<_{LU} \mathbb{L}(\overline{\mu}, \sigma), \ \forall \sigma \in \mathcal{G}_{\mathrm{WVL}(\overline{\mu})}.$$

It follows that

$$\begin{aligned} &\mathcal{F}(\overline{\mu}) + \left(\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}) \right) e \\ &\not<_{LU} \mathcal{F}(\overline{\mu}) + \left(\sum_{k\in\mathcal{L}} \sigma_k^{\Psi}\Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \sigma_i^{\zeta}\zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}) \right) e. \end{aligned} \tag{37}$$

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If we assume that $\sigma = 0$, then (37) can be rewritten as follows:

$$\mathcal{F}(\overline{\mu}) + \left(\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) \right) e \not\prec_{LU} \mathcal{F}(\overline{\mu}) \tag{38}$$

Moreover, by following the similar steps in Theorem 2, we deduce that

$$\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) \leq 0. \tag{39}$$

If

$$\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) < 0,$$

then

$$\mathcal{F}(\overline{\mu}) + \left(\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) \right) e \prec_{LU}^{s} \mathcal{F}(\overline{\mu}).$$

It follows that

$$\mathcal{F}(\overline{\mu}) + \left(\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) \right) e \prec_{LU} \mathcal{F}(\overline{\mu}),$$

which is a contradiction to (38). Therefore, we have

$$\sum_{k\in\mathcal{L}} \overline{\sigma}_k^{\Psi} \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^{\zeta} \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{Q}} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^{\mathcal{R}} \mathcal{R}_i(\overline{\mu}) = 0,$$

which implies that $\mathbb{L}(\overline{\mu}, \overline{\sigma}) = \mathcal{F}(\overline{\mu})$. In view of the given hypothesis, $(\overline{\mu}, \overline{\sigma}) \in \mathrm{WS}^{\mathbb{L}}$. It follows that

$$\mathbb{L}(\mu, \overline{\sigma}) \not\prec_{LU} \mathbb{L}(\overline{\mu}, \overline{\sigma}), \forall \mu \in \mathcal{G}.$$

This claims that $\mathcal{F}(\overline{\mu}) = \mathbb{L}(\overline{\mu}, \overline{\sigma}) \in \mathrm{Min}_{VC}\{\mathbb{L}(\mu, \overline{\sigma}) | \mu \in \mathcal{G}\} = \psi^V(\overline{\sigma})$. From Theorem 9, we prove that $\mathcal{F}(\overline{\mu})$ is an LU-efficient point of $\mathrm{VCD}^{\mathrm{WVL}}(\overline{\mu})$. $\square$

Remark 24. *Theorem 14 extends Proposition 3.13(ii) derived by Tung et al. [44] from smooth semi-infinite programming with vanishing constraints to a broader class of optimization problems, particularly NIMSIPVC.*

In the following theorem, we establish a relationship between the VC-stationary point and saddle point of NIMSIPVC and its corresponding interval-valued vector Lagrangian. The proof is analogous to the proof of Theorem 12 and we will omit it.

Theorem 15. *Let $\overline{\mu} \in VC_{SP}$. Further, assume that $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$, Ψ_k $(k \in \mathbb{P}^{\Psi}(\overline{\mu}))$, ζ_i $(i \in \mathcal{B}_+^{\zeta})$, $-\zeta_i$ $(i \in \mathcal{B}_-^{\zeta})$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^{-}(\overline{\mu}))$, $-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}_{0+}^{+}(\overline{\mu}) \cup \overline{\mathbb{H}}_{00}^{+}(\overline{\mu}) \cup \overline{\mathbb{H}}_{0-}^{+}(\overline{\mu}))$, $\mathcal{R}_i$ $(i \in \mathbb{H}_{+0}^{+}(\overline{\mu}) \cup \mathbb{H}_{00}^{+}(\overline{\mu}))$ are strictly LU-convex and convex at $\overline{\mu}$, respectively. Then there exists $\overline{\sigma} = (\overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{P}^{\Psi}(\overline{\mu}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ such that $(\overline{\mu}, \overline{\sigma}) \in S^{\mathbb{L}}$.*

Remark 25. *Theorem 15 extends Proposition 3.14, deduced by Tung et al. [44], from smooth multiobjective SIPVC to NIMSIPVC.*

5. Scalarized Lagrange Type Duality and Saddle Point Optimality Criteria for NIMSIPVC

In this section, we delve into the study of a scalarized Lagrange type dual problem corresponding to NIMSIPVC. Further, we establish various weak and strong duality results that relate the primal problem NIMSIPVC and the corresponding scalarized Lagrange type dual problem. In addition, we

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introduce the notion of a saddle point for the scalarized Lagrangian of NIMSIPVC, followed by the saddle point optimality criteria for NIMSIPVC.

5.1. Scalarized Lagrange Type Duality

In this subsection, we formulate the scalarized Lagrange type dual problem associated with NIMSIPVC. We derive various weak and strong duality results that elucidate the relationship between the scalarized Lagrange type dual problem and the primal problem NIMSIPVC.

Let $\overline{\lambda}_i^L, \overline{\lambda}_i^U \geq 0$, $(i \in \mathcal{J}^{\mathcal{F}})$ be fixed elements, and $\sigma = (\sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}_+^{|\mathcal{L}|} \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$. The scalarized Lagrangian of NIMSIPVC is a function $\mathbb{L}^s : \mathbb{R}^n \times \mathbb{R}_+^{|\mathcal{L}|} \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ defined as follows:

$$\mathbb{L}^s(\mu, \overline{\lambda}^L, \overline{\lambda}^U, \sigma) := \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \mathcal{F}_i^L(\mu) + \overline{\lambda}_i^U \mathcal{F}_i^U(\mu) \right) + \sum_{k \in \mathcal{L}} \sigma_k^{\Psi} \Psi_k(\mu) + \sum_{i \in \mathcal{B}} \sigma_i^{\zeta} \zeta_i(\mu) - \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{Q}} \mathcal{Q}_i(\mu) + \sum_{i \in \mathcal{C}} \sigma_i^{\mathcal{R}} \mathcal{R}_i(\mu).$$

Define the scalarized Lagrangian dual map $\Psi_0 : \mathbb{R}_+^l \times \mathbb{R}_+^l \times \mathbb{R}_+^{|\mathcal{L}|} \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}$ as follows:

$$\Psi_0(\overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) := \text{Minimize}_{\mu \in \mathcal{G}} \mathbb{L}^s(\mu, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}).$$

The scalarized Lagrange type dual problem for NIMSIPVC is given as follows:

$$\begin{aligned} \text{VCD}^{\text{SL}}(\mu, \overline{\lambda}^L, \overline{\lambda}^U) \ & \text{Maximize } \Psi_0(\overline{\lambda}^L, \overline{\lambda}^U, \sigma) \\ & \text{subject to } \sigma^{\Psi}_{\mathcal{L} \setminus \mathbb{P}(\mu)} \geq 0, \ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(\mu) \cup \mathcal{H}_{0-}(\mu)} \geq 0, \\ & \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(\mu)} \leq 0, \ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(\mu)} \geq 0. \end{aligned}$$

The feasible set of $(\text{VCD})^{SL}(\mu, \overline{\lambda}^L, \overline{\lambda}^U)$ is denoted by $\mathcal{G}_{SL(\mu, \overline{\lambda}^L, \overline{\lambda}^U)}$, and is defined as follows:

$$\mathcal{G}_{SL(\mu, \overline{\lambda}^L, \overline{\lambda}^U)} := \{\sigma = (\sigma^{\Psi}, \sigma^{\zeta}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathbb{R}_+^{|\mathcal{L}|} \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s | \sigma^{\Psi}_{\mathcal{L} \setminus \mathbb{P}(\mu)} \geq 0, \ \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(\mu)} \leq 0, \ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(\mu) \cup \mathcal{H}_{0-}(\mu)} \geq 0, \ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(\mu)} \geq 0\}.$$

Remark 26. *It is worth noting that* $VCD^{SL}(\mu, \overline{\lambda}^L, \overline{\lambda}^U)$ *depends on the feasible point* μ.

The scalarized Lagrange type dual problem, independent of an element's choice from the feasible set $\mathcal{G}$, is defined as follows:

$$\begin{aligned} \text{VCD}^{\text{SL}}(\overline{\lambda}^L, \overline{\lambda}^U) \ & \text{Maximize } \Psi_0(\overline{\lambda}^L, \overline{\lambda}^U, \sigma) \\ & \text{subject to } \sigma \in \mathcal{G}_{\text{SL}(\overline{\lambda}^L, \overline{\lambda}^U)} = \bigcap_{\mu \in \mathcal{G}} \mathcal{G}_{\text{SL}(\mu, \overline{\lambda}^L, \overline{\lambda}^U)} \neq \emptyset. \end{aligned}$$

In the following theorem, we establish weak duality results that demonstrate the relationship between NIMSIPVC and $\text{VCD}^{\text{SL}}(\mu, \overline{\lambda}^L, \overline{\lambda}^U)$.

Theorem 16. *Let* μ *and* σ *be any elements of* $\mathcal{G}$ *and* $\mathcal{G}_{SL(\mu, \overline{\lambda}^L, \overline{\lambda}^U)}$*, respectively. Then*

$$\Psi_0\left(\overline{\lambda}^L, \overline{\lambda}^U, \sigma\right) \leq \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}_i^L \mathcal{F}_i^L(\mu) + \overline{\lambda}_i^U \mathcal{F}_i^U(\mu) \right).$$

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Proof. From the definition of Ψ_0 and the given hypothesis that $(\mu,\sigma) \in \mathcal{G} \times \mathcal{G}_{\mathrm{SL}(\mu,\overline{\lambda}^L,\overline{\lambda}^U)}$, we have

$$\begin{aligned}\Psi_0(\overline{\lambda}^L,\overline{\lambda}^U,\sigma) &= \mathrm{Minimize}_{\mu\in\mathcal{G}}\mathbb{L}^s(\mu,\overline{\lambda}^L,\overline{\lambda}^U,\sigma)\\ &\le \sum_{i\in\mathcal{J}^{\mathcal{F}}}\Big(\overline{\lambda}_i^L\mathcal{F}_i^L(\mu)+\overline{\lambda}_i^U\mathcal{F}_i^U(\mu)\Big)+\sum_{k\in\mathcal{L}}\sigma_k^{\Psi}\Psi_k(\mu)+\sum_{i\in\mathcal{B}}\sigma_i^{\zeta}\zeta_i(\mu)-\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\mu)\\ &\quad+\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{R}}\mathcal{R}_i(\mu).\end{aligned} \tag{40}$$

On utilizing the feasibility of μ and σ we have

$$\sum_{k\in\mathcal{L}}\sigma_k^{\Psi}\Psi_k(\mu)+\sum_{i\in\mathcal{B}}\sigma_i^{\zeta}\zeta_i(\mu)-\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\mu)+\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{R}}\mathcal{R}_i(\mu)\le 0. \tag{41}$$

From (40) we infer that

$$\Psi_0(\overline{\lambda}^L,\overline{\lambda}^U,\sigma)\le\sum_{i\in\mathcal{J}^{\mathcal{F}}}\Big(\overline{\lambda}_i^L\mathcal{F}_i^L(\mu)+\overline{\lambda}_i^U\mathcal{F}_i^U(\mu)\Big).$$

This completes the proof. □

Remark 27. *If $\mathcal{F}_i^L(\mu)=\mathcal{F}_i^U(\mu)=\mathcal{F}_i(\mu)$, $\forall i\in\mathcal{J}^{\mathcal{F}}$, $\mu\in\mathbb{R}^n$ and if $\partial_c\mathcal{F}_i^L(\mu)=\{\nabla\mathcal{F}_i^L(\mu)\}$, $\partial_c\mathcal{F}_i^U(\mu)=\{\nabla\mathcal{F}_i^U(\mu)\}$ $(i\in\mathcal{J}^{\mathcal{F}})$, $\partial_c\Psi_k(\mu)=\{\nabla\Psi_k(\mu)\}$ $(k\in\mathcal{L})$, $\partial_c\zeta_i(\mu)=\{\nabla\zeta_i(\mu)\}$ $(i\in\mathcal{B})$, $\partial_c\mathcal{Q}_i(\mu)=\{\nabla\mathcal{Q}_i(\mu)\}$ $(i\in\mathcal{C})$, $\partial_c\mathcal{R}_i(\mu)=\{\nabla\mathcal{R}_i(\mu)\}$ $(i\in\mathcal{C})$ then Theorem 16 reduces to Proposition 4.1 derived by Tung et al. [44].*

In the following corollary, we derive the weak duality result relating NIMSIPVC and VCD$^{\mathrm{SL}}$.

Corollary 1. *Let μ and σ be any arbitrary elements of $\mathcal{G}$ and $\mathcal{G}_{SL}$, respectively. Then*

$$\Psi_0(\overline{\lambda}^L,\overline{\lambda}^U,\sigma)\le\sum_{i\in\mathcal{J}^{\mathcal{F}}}\Big(\overline{\lambda}_i^L\mathcal{F}_i^L(\mu)+\overline{\lambda}_i^U\mathcal{F}_i^U(\mu)\Big).$$

Remark 28. *Corollary 1 extends Corollary 4.2 derived by Tung et al. [44] from smooth multiobjective SIPVC to nonsmooth multiobjective SIPVC involving interval-valued objective function.*

In the following theorem, we establish the strong duality result relating NIMSIPVC and VCD$^{\mathrm{SL}}(\mu,\overline{\lambda}^L,\overline{\lambda}^U)$.

Theorem 17. *Let $\overline{\mu}\in WEff_{loc}$ such that VC-ACQ is satisfied at $\overline{\mu}$ and let $\mathcal{K}_2$ be a closed set. Furthermore, assume that $\mathcal{F}_i$ $(i\in\mathcal{J}^{\mathcal{F}})$, Ψ_k $(k\in\mathbb{P}^{\Psi}(\overline{\mu}))$, ζ_i $(i\in\mathcal{B}_+^{\zeta}(\overline{\mu}))$, $-\zeta_i$ $(i\in\mathcal{B}_-^{\zeta}(\overline{\mu}))$, $\mathcal{Q}_i$ $(i\in\overline{\mathbb{H}}_{0+}^{-}(\overline{\mu}))$, $-\mathcal{Q}_i$ $(i\in\overline{\mathbb{H}}_{0+}^{+}(\overline{\mu})\cup\overline{\mathbb{H}}_{00}^{+}(\overline{\mu})\cup\overline{\mathbb{H}}_{0-}^{+}(\overline{\mu}))$, $\mathcal{R}_i$ $(i\in\mathbb{H}_{+0}^{+}(\overline{\mu})\cup\mathbb{H}_{00}^{+}(\overline{\mu}))$ are LU-convex and convex at $\overline{\mu}$, respectively. Then there exists $(\overline{\lambda}^L,\overline{\lambda}^U)\in\mathbb{R}_+^l\times\mathbb{R}_+^l$, $\sum_{i\in\mathcal{J}^{\mathcal{F}}}\Big(\overline{\lambda}_i^L+\overline{\lambda}_i^U\Big)=1$ such that $\overline{\sigma}\in\mathcal{G}_{SL(\overline{\mu},\overline{\lambda}^l,\overline{\lambda}^U)}$ is an optimal solution of VCD$^{SL}(\overline{\mu},\overline{\lambda}^L,\overline{\lambda}^U)$ and*

$$\Psi_0(\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma})=\sum_{i\in\mathcal{J}^{\mathcal{F}}}\Big(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{\mu})+\overline{\lambda}_i^U\mathcal{F}_i^U(\overline{\mu})\Big).$$

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Proof. From the given hypothesis, $\overline{\mu} \in \text{WEff}_{loc}$ and VC-ACQ is satisfied at $\overline{\mu}$. Then, from Theorem 1, $\overline{\mu} \in \text{VC}_{SP}$, which implies that there exist $(\overline{\lambda}^L, \overline{\lambda}^U) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+$, $\overline{\sigma} = (\overline{\sigma}^\Psi, \overline{\sigma}^\zeta, \overline{\sigma}^\mathcal{Q}, \overline{\sigma}^\mathcal{R}) \in \mathbb{P}^\Psi(\overline{\mu}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$, satisfying:

$$0 \in \sum_{i\in\mathcal{J}} \Big(\overline{\lambda}_i^L \partial_c \mathcal{F}_i^L(\overline{\mu}) + \overline{\lambda}_i^U \partial_c \mathcal{F}_i^U(\overline{\mu})\Big) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^\Psi \partial_c \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^\zeta \partial_c \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{Q} \partial_c \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{R} \partial_c \mathcal{R}_i(\overline{\mu}),$$

where $\sum_{i\in\mathcal{J}^\mathcal{F}} \Big(\overline{\lambda}_i^L + \overline{\lambda}_i^U\Big) = 1$, $\overline{\sigma}^\mathcal{Q}_{\mathcal{H}_+(\overline{\mu})} = 0$, $\overline{\sigma}^\mathcal{Q}_{\mathcal{H}_{00}(\overline{\mu})\cup\mathcal{H}_{0-}(\overline{\mu})} \geq 0$, $\overline{\sigma}^\mathcal{R}_{\mathcal{H}_{+0}(\overline{\mu})\cup\mathcal{H}_{00}(\overline{\mu})} \geq 0$, and $\overline{\sigma}^\mathcal{R}_{\mathcal{H}_{+-}(\overline{\mu})\cup\mathcal{H}_{0-}(\overline{\mu})\cup\mathcal{H}_{0+}(\overline{\mu})} = 0$. This implies that there exist $\hat{\xi}_i^L \in \partial_c \mathcal{F}_i^L(\overline{\mu})$, $\hat{\xi}_i^U \in \partial_c \mathcal{F}_i^U(\overline{\mu})$ $(i \in \mathcal{J}^\mathcal{F})$, $\hat{\eta}_k^\Psi \in \partial_c \Psi_k(\overline{\mu})$ $(k \in \mathcal{L})$, $\hat{\eta}_i^\zeta \in \partial_c \zeta_i(\overline{\mu})$, $\hat{\eta}_i^\mathcal{Q} \in \partial_c \mathcal{Q}_i(\overline{\mu})$ $(i \in \mathcal{C})$, $\hat{\eta}_i^\mathcal{R} \in \partial_c \mathcal{R}_i(\overline{\mu})$ $(i \in \mathcal{C})$ such that

$$\sum_{i\in\mathcal{J}^\mathcal{F}} \Big(\overline{\lambda}_i^L \hat{\xi}_i^L + \overline{\lambda}_i^U \hat{\xi}_i^U\Big) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^\Psi \hat{\eta}_k^\Psi + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^\zeta \hat{\eta}_i^\zeta - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{Q} \hat{\eta}_i^\mathcal{Q} + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{R} \hat{\eta}_i^\mathcal{R} = 0. \tag{42}$$

Moreover, in view of the fact that $\overline{\mu} \in \text{VC}_{SP}$ and properties of $\overline{\sigma}$, one has

$$\sum_{k\in\mathcal{L}} \overline{\sigma}_k^\Psi \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^\zeta \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{Q} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{R} \mathcal{R}_i(\overline{\mu}) = 0.$$

It follows that

$$\mathbb{L}^s(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) = \sum_{i\in\mathcal{J}^\mathcal{F}} \Big(\overline{\lambda}_i^L \mathcal{F}_i^L(\overline{\mu}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\overline{\mu})\Big).$$

On following the similar steps as in the proof of Theorem 5 we have

$$\begin{aligned}
&\sum_{i\in\mathcal{J}^\mathcal{F}} \Big(\overline{\lambda}_i^L \mathcal{F}_i^L(\mu) + \overline{\lambda}_i^U \mathcal{F}_i^U(\mu)\Big) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^\Psi \Psi_k(\mu) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^\zeta \zeta_i(\mu) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{Q} \mathcal{Q}_i(\mu) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{R} \mathcal{R}_i(\mu) \\
&- \sum_{i\in\mathcal{J}^\mathcal{F}} \Big(\overline{\lambda}_i^L \mathcal{F}_i^L(\overline{\mu}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\overline{\mu})\Big) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^\Psi \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^\zeta \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{Q} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{R} \mathcal{R}_i(\overline{\mu}) \\
&\geq \Big\langle \sum_{i\in\mathcal{J}^\mathcal{F}} \Big(\overline{\lambda}_i^L \xi_i^L + \overline{\lambda}_i^U \xi_i^U\Big) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^\Psi \eta_k^\Psi + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^\zeta \eta_i^\zeta - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{Q} \eta_i^\mathcal{Q} + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{R} \eta_i^\mathcal{R}, \mu - \overline{\mu} \Big\rangle.
\end{aligned}$$

From (42) we get the following inequality:

$$\begin{aligned}
&\sum_{i\in\mathcal{J}^\mathcal{F}} \Big(\overline{\lambda}_i^L \mathcal{F}_i^L(\mu) + \overline{\lambda}_i^U \mathcal{F}_i^U(\mu)\Big) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^\Psi \Psi_k(\mu) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^\zeta \zeta_i(\mu) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{Q} \mathcal{Q}_i(\mu) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{R} \mathcal{R}_i(\mu) \\
&- \sum_{i\in\mathcal{J}^\mathcal{F}} \Big(\overline{\lambda}_i^L \mathcal{F}_i^L(\overline{\mu}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\overline{\mu})\Big) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^\Psi \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^\zeta \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{Q} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{R} \mathcal{R}_i(\overline{\mu}) \\
&\geq 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\sum_{i\in\mathcal{J}^\mathcal{F}} \Big(\overline{\lambda}_i^L \mathcal{F}_i^L(\mu) + \overline{\lambda}_i^U \mathcal{F}_i^U(\mu)\Big) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^\Psi \Psi_k(\mu) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^\zeta \zeta_i(\mu) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{Q} \mathcal{Q}_i(\mu) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{R} \mathcal{R}_i(\mu) \\
&\geq \sum_{i\in\mathcal{J}^\mathcal{F}} \Big(\overline{\lambda}_i^L \mathcal{F}_i^L(\overline{\mu}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\overline{\mu})\Big) + \sum_{k\in\mathcal{L}} \overline{\sigma}_k^\Psi \Psi_k(\overline{\mu}) + \sum_{i\in\mathcal{B}} \overline{\sigma}_i^\zeta \zeta_i(\overline{\mu}) - \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{Q} \mathcal{Q}_i(\overline{\mu}) + \sum_{i\in\mathcal{C}} \overline{\sigma}_i^\mathcal{R} \mathcal{R}_i(\overline{\mu}).
\end{aligned}$$

Hence,

$$\mathbb{L}^s(\mu, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) \geq \mathbb{L}^s(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) = \sum_{i\in\mathcal{J}^\mathcal{F}} \Big(\overline{\lambda}_i^L \mathcal{F}_i^L(\overline{\mu}) + \overline{\lambda}_i^U \mathcal{F}_i^U(\overline{\mu})\Big), \ \forall \mu \in \mathcal{G}.$$

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Now,

$$\begin{aligned}\Psi_0(\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma}) &= \text{Minimize}_{\mu\in\mathcal{G}}\mathbb{L}^s(\mu,\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma})\\ &= \mathbb{L}(\overline{\mu},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma})\\ &= \sum_{i\in\mathcal{J}^{\mathcal{F}}}\Big(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{\mu})+\overline{\lambda}_i^U\mathcal{F}_i^U(\overline{\mu})\Big).\end{aligned}$$

Therefore, from Theorem 16,

$$\Psi_0(\overline{\lambda}^L,\overline{\lambda}^U,\sigma)\le\Psi_0(\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma}),\ \forall\sigma\in\mathcal{G}_{SL(\overline{\mu},\overline{\lambda}^L,\overline{\lambda}^U)}.$$

Therefore, $\overline{\sigma}$ is an optimal solution of VCD$^{\text{SL}}(\overline{\mu},\overline{\lambda}^L,\overline{\lambda}^U)$. This completes the proof. □

Remark 29. *If $\mathcal{F}_i^L(\mu)=\mathcal{F}_i^U(\mu)=\mathcal{F}_i(\mu)$, $\forall i\in\mathcal{J}^{\mathcal{F}}$, $\mu\in\mathbb{R}^n$ and if $\partial_c\mathcal{F}_i^L(\mu)=\{\nabla\mathcal{F}_i^L(\mu)\}$, $\partial_c\mathcal{F}_i^U(\mu)=\{\nabla\mathcal{F}_i^U(\mu)\}$ $(i\in\mathcal{J}^{\mathcal{F}})$, $\partial_c\Psi_k(\mu)=\{\nabla\Psi_k(\mu)\}$ $(k\in\mathcal{L})$, $\partial_c\zeta_i(\mu)=\{\nabla\zeta_i(\mu)\}$ $(i\in\mathcal{B})$, $\partial_c\mathcal{Q}_i(\mu)=\{\nabla\mathcal{Q}_i(\mu)\}$ $(i\in\mathcal{C})$, $\partial_c\mathcal{R}_i(\mu)=\{\nabla\mathcal{R}_i(\mu)\}$ $(i\in\mathcal{C})$ then Theorem 17 reduces to Proposition 4.4 derived by Tung et al. [44].*

The following example demonstrates the significance of Theorem 13 and Theorem 14.

Example 5. *Consider the problem $(\mathcal{P}_s)$ as follows:*

$$\begin{aligned}(\mathcal{P}_s)\ \textit{Minimize}\ \mathcal{F}(\mu) &= (\mathcal{F}_1(\mu),\mathcal{F}_2(\mu))\\ &= \Big([|\mu_1-1|,|\mu_1-1|+(\mu_1-1)^2],\Big[(\mu_2-1)^2,(\mu_1-1)^2+(\mu_2-1)^2\Big]\Big),\\ \textit{subject to}\ \Psi_\tau(\mu) &= (\tau-1)(\mu_2-1)\le 0,\tau\in\mathcal{L}=[0,1],\\ \mathcal{Q}_1(\mu) &= (\mu_2-\mu_1)\ge 0,\\ \mathcal{R}_1(\mu)\mathcal{Q}_1(\mu) &= (\mu_2-1)(\mu_2-\mu_1)\le 0.\end{aligned}$$

The feasible set of $(\mathcal{P}_s)$ is given as follows:

$$\mathcal{G}=\cup_{i=1}^3\mathcal{G}^i,$$

where $\mathcal{G}_i, i=1,2,3$ are defined in the following manner:

$$\begin{aligned}\mathcal{G}^1 &:= \{(\mu_1,\mu_2)\in\mathbb{R}^2|\mu_2-1=0,\ \mu_2-\mu_1>0\}=\{(\mu_1,\mu_2)\in\mathbb{R}^2|\mu_2=1,\ \mu_1<1\},\\ \mathcal{G}^2 &:= \{(\mu_1,\mu_2)\in\mathbb{R}^2|\mu_2-1>0,\ \mu_2-\mu_1=0\}=\{(\mu_1,\mu_2)\in\mathbb{R}^2|\mu_2>1,\ \mu_1>1\},\\ \mathcal{G}^3 &:= \{(\mu_1,\mu_2)\in\mathbb{R}^2|\mu_2-1=0,\ \mu_2-\mu_1=0\}=\{(\mu_1,\mu_2)\in\mathbb{R}^2|\mu_2=1,\ \mu_1=1\}.\end{aligned}$$

Now, we formulate the scalarized Lagrangian for $(\mathcal{P}_s)$ for some fixed $\overline{\lambda}=(\overline{\lambda}_1^L,\overline{\lambda}_1^U,\overline{\lambda}_2^L,\overline{\lambda}_2^U)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}_+,\sum_{i=1}^2\Big(\overline{\lambda}_i^L+\overline{\lambda}_i^U\Big)=1$ as follows:

$$\mathbb{L}^s(\mu,\overline{\lambda},\sigma):=\sum_{i=1}^2\Big(\overline{\lambda}_i^L\mathcal{F}_i^L(\mu)+\overline{\lambda}_i^U\mathcal{F}_i^U(\mu)\Big)+\sum_{\tau\in[0,1]}\sigma_\tau^\Psi(\tau-1)(\mu_2-1)-\sigma^{\mathcal{Q}}\mathcal{Q}_1(\mu)+\sigma^{\mathcal{R}}\mathcal{R}_1(\mu).$$

Moreover, $\Psi_0:=\textit{Minimize}_{\mu\in\mathcal{G}}\mathbb{L}^s(\mu,\overline{\lambda},\sigma)$.

Formulating the scalarized Lagrange type dual problem corresponding to $(\mathcal{P}_s)$ in the following manner:

$$\begin{aligned}VCD_1^{SL}(\mu,\overline{\lambda})\ \ &\text{Maximize}\ \Psi_0(\overline{\lambda},\sigma)\\ &\textit{subject to}\ \sigma^\Psi_{\mathcal{L}\setminus\mathbb{P}(\mu)}\ge 0,\ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(\mu)\cup\mathcal{H}_{0-}(\mu)}\ge 0,\\ &\sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(\mu)}\le 0,\ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(\mu)}\ge 0.\end{aligned}$$

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The feasible region of $VCD_1^{SL}(\mu, \overline{\lambda})$ *corresponding to* $\mathcal{G}^1$ *is given by:*

$$\mathcal{G}^1_{SL(\mu,\overline{\lambda})} := \{(\sigma^{\Psi}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) | \sigma^{\Psi}_{\tau} \in \mathbb{R}^{|\mathcal{L}|}_{+}, \ \sigma^{\mathcal{Q}} \geq 0, \ \sigma^{\mathcal{R}} \in \mathbb{R}\}.$$

Formulation of scalarized Lagrange type dual problem corresponding to $\mathcal{G}^2$ *of* $(\mathcal{P}_s)$ *is:*

$$\begin{aligned} VCD_2^{SL}(\mu, \overline{\lambda}) \ \text{Maximize} \ & \Psi_0(\overline{\lambda}, \sigma) \\ \textit{subject to } & \sigma^{\Psi}_{\mathcal{L}\setminus\mathbb{P}(\mu)} \geq 0, \ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(\mu)\cup\mathcal{H}_{0-}(\mu)} \geq 0, \\ & \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(\mu)} \leq 0, \ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(\mu)} \geq 0. \end{aligned}$$

The feasible set of $VCD_2^{SL}(\mu, \overline{\lambda})$ *corresponding to* $\mathcal{G}^2$ *is given by:*

$$\mathcal{G}^2_{SL(\mu,\overline{\lambda})} := \{(\sigma^{\Psi}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) | \sigma^{\Psi}_{\tau} \in \mathbb{R}^{|\mathcal{L}|}_{+}, \ \sigma^{\mathcal{Q}} \in \mathbb{R}, \ \sigma^{\mathcal{R}} \leq 0\}.$$

Formulation of scalarized Lagrange type dual problem corresponding to $\mathcal{G}^3$ *is given by:*

$$\begin{aligned} VCD_3^{SL}(\mu, \overline{\lambda}) \ \text{Maximize} \ & \Psi_0(\overline{\lambda}, \sigma) \\ \textit{subject to } & \sigma^{\Psi}_{\mathcal{L}\setminus\mathbb{P}(\mu)} \geq 0, \ \sigma^{\mathcal{R}}_{\mathcal{H}_{+-}(\mu)\cup\mathcal{H}_{0-}(\mu)} \geq 0, \\ & \sigma^{\mathcal{R}}_{\mathcal{H}_{0+}(\mu)} \leq 0, \ \sigma^{\mathcal{Q}}_{\mathcal{H}_{+}(\mu)} \geq 0. \end{aligned}$$

The feasible set of $VCD_3^{SL}(\mu, \overline{\lambda})$ *corresponding to* $\mathcal{G}^3$ *is given by:*

$$\mathcal{G}^3_{SL(\mu,\overline{\lambda})} := \{(\sigma^{\Psi}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) | \sigma^{\Psi}_{\tau} \in \mathbb{R}^{|\mathcal{L}|}_{+}, \ \sigma^{\mathcal{Q}} \in \mathbb{R}, \ \sigma^{\mathcal{R}} \in \mathbb{R}\}.$$

Moreover, the scalarized Lagrange type dual problem, independent of the choice of a feasible element of $(\mathcal{P}_s)$ *is formulated as:*

$$\begin{aligned} VCD^{SL}(\overline{\lambda}) \ \text{Maximize} \ & \Psi_0(\overline{\lambda}, \sigma) \\ \textit{subject to } & (\sigma^{\Psi}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathcal{G}_{SL(\overline{\lambda})} = \bigcap_{i=1,2,3} \mathcal{G}^i_{(SL)(\overline{\lambda})}. \end{aligned}$$

Therefore, for any $(\sigma^{\Psi}, \sigma^{\mathcal{Q}}, \sigma^{\mathcal{R}}) \in \mathcal{G}_{SL(\overline{\lambda})}$, $\mu \in \mathcal{G}$, *the following inequality holds:*

$$\sum_{\tau \in \mathcal{J}} \sigma^{\Psi}_{\tau}(\tau - 1)(\mu_2 - 1) - \sigma^{\mathcal{Q}}\mathcal{Q}_1(\mu) + \sigma^{\mathcal{R}}\mathcal{R}_1(\mu) \leq 0.$$

This implies that

$$\Psi_0(\overline{\lambda}, \sigma) \leq \mathbb{L}^s(\mu, \overline{\lambda}, \sigma) \leq \sum_{i=1}^{2} \left(\overline{\lambda}^L_i \mathcal{F}^L_i(\mu) + \overline{\lambda}^U_i \mathcal{F}^U_i(\mu) \right).$$

Therefore, Theorem 13 and Corollary 1 are satisfied.

It is worthwhile to note that $\overline{\mu} = (1,1)$ *is a locally weakly LU-efficient solution of* $(\mathcal{P}_s)$ *such that VC-ACQ holds at* $\overline{\mu}$. *Let* $\overline{\mu} = (1,1)$. *Then* $\mathcal{F}(\overline{\mu}) = ([0,1],[0,0])$. *Choose* $\overline{\sigma} = (\overline{\sigma}^{\Psi}_{\tau}, \overline{\sigma}^{\mathcal{Q}}_1, \overline{\sigma}^{\mathcal{R}}_1)$ *such that*

$$\overline{\sigma}^{\Psi}_{\tau} = \begin{cases} 1, & \tau = 0, \\ 0, & \textit{otherwise}, \end{cases}$$

$\overline{\sigma}^{\mathcal{Q}}_1 = \overline{\lambda}^L_1 + 1$, $\overline{\sigma}^{\mathcal{R}}_1 = 0$. *Moreover,* $\mathcal{F}_1$, $\mathcal{F}_2$, Ψ_{τ} $(\tau \in \mathcal{L})$, $\mathcal{Q}$, $\mathcal{R}$ *are LU-convex and convex functions at* $\overline{\mu}$ *implies that all hypotheses in Theorem 14 are satisfied at* $\overline{\mu}$. *Therefore, Theorem 14 holds, that is,*

$$\Psi_0(\overline{\lambda}, \overline{\sigma}) = \sum_{i=1}^{2} \left(\overline{\lambda}^L_i \mathcal{F}^L_i(\overline{\mu}) + \overline{\lambda}^U_i \mathcal{F}^U_i(\overline{\mu}) \right).$$

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5.2. Saddle Point Optimality Criteria

In this subsection, we introduce the notion of a saddle point for the scalarized Lagrangian corresponding to NIMSIPVC and further explore saddle point optimality criteria for NIMSIPVC.

Definition 13. *Let* $\left(\overline{\lambda}^L, \overline{\lambda}^U\right) \in \mathbb{R}^l_+ \times \mathbb{R}^l_+$ *be a fixed element, such that* $\sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i + \overline{\lambda}^U_i\right) = 1$. *Further, assume* $\overline{\mu} \in \mathcal{G}$, *and* $\overline{\sigma} \in \mathcal{G}_{SL(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U)}$. *Then* $(\overline{\mu}, \overline{\sigma})$ *is known as a saddle point for the scalarized Lagrangian of NIMSIPVC, provided the following condition holds:*

$$\mathbb{L}^s(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U, \sigma) \leq \mathbb{L}^s(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) \leq \mathbb{L}^s(\mu, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}), \ \forall \mu \in \mathcal{G}, \ \forall \sigma \in \mathcal{G}_{SL(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U)}.$$

The relationship between a locally LU-weakly efficient solution and a saddle point of NIMSIPVC and scalarized Lagrangian of NIMSIPVC has been established in the following theorem.

Theorem 18. *Let* $\overline{\mu} \in WEff_{loc}$. *Further, assume that all the hypotheses in Theorem 17 are satisfied at* $\overline{\mu}$. *Then there exists* $\overline{\sigma} = (\overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{P}^{\Psi}(\overline{\mu}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ *such that* $(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma})$ *is a saddle point for the scalarized Lagrangian of NIMSIPVC.*

Proof. From the proof of Theorem 17, we have

$$\mathbb{L}^s(\mu, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) \geq \mathbb{L}^s(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}), \ \forall \mu \in \mathcal{G}.$$

We are left to prove that

$$\mathbb{L}^s(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U, \sigma) \leq \mathbb{L}^s(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}), \ \forall \sigma \in \mathcal{G}_{SL(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U)}.$$

Following the similar steps in the proof of Theorem 17 and Corollary 1, we get the following condition:

$$\begin{aligned}
\mathbb{L}^s(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma}) &= \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i \mathcal{F}^L_i(\overline{\mu}) + \overline{\lambda}^U_i \mathcal{F}^U_i(\overline{\mu})\right) \\
&\geq \sum_{i \in \mathcal{J}^{\mathcal{F}}} \left(\overline{\lambda}^L_i \mathcal{F}^L_i(\overline{\mu}) + \overline{\lambda}^U_i \mathcal{F}^U_i(\overline{\mu})\right) + \sum_{k \in \mathcal{L}} \sigma^{\Psi}_k \Psi_k(\overline{\mu}) + \sum_{i \in \mathcal{B}} \sigma^{\zeta}_i \zeta_i(\overline{\mu}) - \sum_{i \in \mathcal{C}} \sigma^{\mathcal{Q}}_i \mathcal{Q}_i(\overline{\mu}) \\
&\qquad + \sum_{i \in \mathcal{C}} \sigma^{\mathcal{R}}_i \mathcal{R}_i(\overline{\mu}) \\
&= \mathbb{L}^s(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U, \sigma).
\end{aligned}$$

This completes the proof. □

Remark 30. *Theorem 18 extends Proposition 4.7, deduced by Tung et al. [44] from smooth multiobjective SIPVC to nonsmooth multiobjective interval-valued mathematical programming problems with vanishing constraints.*

In the following proposition, we establish a relationship between the saddle point for scalarized Lagrangian of NIMSIPVC and the VC-stationary point of primal problem NIMSIPVC.

Theorem 19. *Let* $\overline{\mu} \in WEff_{loc}$ *such that VC-ACQ is satisfied at* $\overline{\mu}$ *and let* $\mathcal{K}_2$ *be a closed set. Furthermore, assume that* $\mathcal{F}_i$ $(i \in \mathcal{J}^{\mathcal{F}})$, Ψ_k $(k \in \mathbb{P}^{\Psi}(\overline{\mu}))$, ζ_i $(i \in \mathcal{B}^{\zeta}_+(\overline{\mu}))$, $-\zeta_i$ $(i \in \mathcal{B}^{\zeta}_-(\overline{\mu}))$, $\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}^-_{0+}(\overline{\mu}))$, $-\mathcal{Q}_i$ $(i \in \overline{\mathbb{H}}^+_{0+}(\overline{\mu}) \cup \overline{\mathbb{H}}^+_{00}(\overline{\mu}) \cup \overline{\mathbb{H}}^+_{0-}(\overline{\mu}))$, $\mathcal{R}_i$ $(i \in \mathbb{H}^+_{+0}(\overline{\mu}) \cup \mathbb{H}^+_{00}(\overline{\mu}))$ *are LU-convex and convex at* $\overline{\mu}$. *Then there exists* $\overline{\sigma} = (\overline{\sigma}^{\Psi}, \overline{\sigma}^{\zeta}, \overline{\sigma}^{\mathcal{Q}}, \overline{\sigma}^{\mathcal{R}}) \in \mathbb{P}^{\Psi}(\overline{\mu}) \times \mathbb{R}^r \times \mathbb{R}^s \times \mathbb{R}^s$ *such that* $(\overline{\mu}, \overline{\lambda}^L, \overline{\lambda}^U, \overline{\sigma})$ *is a saddle point for the scalarized Lagrangian of NIMSIPVC.*

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Proof. In view of the definition of scalarized Lagrangian of NIMSIPVC, we have

$$\mathbb{L}^s(\mu,\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma}) - \mathbb{L}^s(\overline{\mu},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma}) = \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\mu)+\overline{\lambda}_i^U\mathcal{F}_i^U(\mu)\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\mu)+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\mu)-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\mu)+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\mu)$$
$$-\sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{\mu})+\overline{\lambda}_i^U\mathcal{F}_i^U(\overline{\mu})\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}).$$

Furthermore, by employing the convexity assumptions on objective functions and constraint functions, we obtain the following condition by following the analogous steps in the proof of Theorem 5 and 17 as follows:

$$\begin{aligned}\mathbb{L}^s(\mu,\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma}) - \mathbb{L}^s(\overline{\mu},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma}) &\geq \Big\langle \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\xi_i^L+\overline{\lambda}_i^U\xi_i^U\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\eta_k^{\Psi}+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\eta_i^{\zeta} \\ &\qquad -\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\eta_i^{\mathcal{Q}}+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\eta_i^{\mathcal{R}},\mu-\overline{\mu}\Big\rangle \\ &\geq 0,\end{aligned}$$

due to the fact that $\overline{\mu}\in \mathrm{VC}_{SP}$. This implies that

$$\mathbb{L}^s(\mu,\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma}) \geq \mathbb{L}^s(\overline{\mu},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma}), \forall \mu\in\mathcal{G}. \tag{43}$$

Since $\overline{\mu}\in\mathcal{G}$ and $\sigma\in\mathcal{G}_{SL(\overline{\mu},\overline{\lambda}^L,\overline{\lambda}^U)}$, it follows that

$$\sum_{k\in\mathcal{L}}\sigma_k^{\Psi}\Psi_k(\mu)+\sum_{i\in\mathcal{B}}\sigma_i^{\zeta}\zeta_i(\mu)-\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\mu)+\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{R}}\mathcal{R}_i(\mu)\leq 0. \tag{44}$$

Now,

$$\begin{aligned}\mathbb{L}^s(\overline{\mu},\overline{\lambda}^L,\overline{\lambda}^U,\sigma) &= \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{\mu})+\overline{\lambda}_i^U\mathcal{F}_i^U(\overline{\mu})\right)+\sum_{k\in\mathcal{L}}\sigma_k^{\Psi}\Psi_k(\overline{\mu})+\sum_{i\in\mathcal{B}}\sigma_i^{\zeta}\zeta_i(\overline{\mu})-\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) \\ &\qquad +\sum_{i\in\mathcal{C}}\sigma_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}) \\ &\leq \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{\mu})+\overline{\lambda}_i^U\mathcal{F}_i^U(\overline{\mu})\right).\end{aligned}$$

Since $\overline{\mu}\in \mathrm{VC}_{SP}$, the following condition holds:

$$\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu})+\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu})=0.$$

Therefore, the last inequality can be rewritten as follows:

$$\begin{aligned}\mathbb{L}^s(\overline{\mu},\overline{\lambda}^L,\overline{\lambda}^U,\sigma) &\leq \sum_{i\in\mathcal{J}^{\mathcal{F}}}\left(\overline{\lambda}_i^L\mathcal{F}_i^L(\overline{\mu})+\overline{\lambda}_i^U\mathcal{F}_i^U(\overline{\mu})\right)+\sum_{k\in\mathcal{L}}\overline{\sigma}_k^{\Psi}\Psi_k(\overline{\mu})+\sum_{i\in\mathcal{B}}\overline{\sigma}_i^{\zeta}\zeta_i(\overline{\mu})-\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{Q}}\mathcal{Q}_i(\overline{\mu}) \\ &\qquad +\sum_{i\in\mathcal{C}}\overline{\sigma}_i^{\mathcal{R}}\mathcal{R}_i(\overline{\mu}) \\ &= \mathbb{L}^s(\overline{\mu},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma}).\end{aligned}$$

Hence, $(\overline{\mu},\overline{\lambda}^L,\overline{\lambda}^U,\overline{\sigma})$ is a saddle point for the scalarized Lagrangian of NIMSIPVC. □

Remark 31. *If $\mathcal{F}_i^L(\mu) = \mathcal{F}_i^U(\mu) = \mathcal{F}_i(\mu)$, $\forall i \in \mathcal{J}^{\mathcal{F}}$, $\mu \in \mathbb{R}^n$ and if $\partial_c\mathcal{F}_i^L(\mu) = \{\nabla\mathcal{F}_i^L(\mu)\}$, $\partial_c\mathcal{F}_i^U(\mu) = \{\nabla\mathcal{F}_i^U(\mu)\}$ $(i \in \mathcal{J}^{\mathcal{F}})$, $\partial_c\Psi_k(\mu) = \{\nabla\Psi_k(\mu)\}$ $(k \in \mathcal{L})$, $\partial_c\zeta_i(\mu) = \{\nabla\zeta_i(\mu)\}$ $(i \in \mathcal{B})$, $\partial_c\mathcal{Q}_i(\mu) = \{\nabla\mathcal{Q}_i(\mu)\}$ $(i \in \mathcal{C})$, $\partial_c\mathcal{R}_i(\mu) = \{\nabla\mathcal{R}_i(\mu)\}$ $(i \in \mathcal{C})$ then Theorem 19 reduces to Proposition 4.8 established by Tung et al. [44].*

Now, we provide a non-trivial example to demonstrate the validity of Theorem 19.

Example 6. *Consider the problem $(\mathcal{P}_s)$ from Example 5.*

It can be easily verify that $\overline{\mu} = (1,1)$ is a VC-stationary point of $(\mathcal{P}_s)$. Therefore, there exists $\overline{\lambda}_1^L = \overline{\lambda}_2^L = \frac{1}{4} = \overline{\lambda}_1^U = \overline{\lambda}_2^U$, $\overline{\sigma}^{\mathcal{Q}} = \frac{1}{2}$, $\overline{\sigma}^{\mathcal{R}} = 1$, and

$$\overline{\sigma}^{\Psi} = \begin{cases} \frac{1}{2}, & \tau = 0, \\ 0, & \text{otherwise}, \end{cases}$$

such that

$$0 \in \sum_{i=1}^{2}\left(\overline{\lambda}_i^L\partial_c\mathcal{F}_i^L(\overline{\mu}) + \overline{\lambda}_i^U\partial_c\mathcal{F}_i^U(\overline{\mu})\right) + \sum_{\tau\in\mathbb{P}(\overline{\mu})} \overline{\sigma}_\tau^{\Psi}\partial_c\Psi_\tau(\overline{\mu}) - \overline{\sigma}^{\mathcal{Q}}\partial_c\mathcal{Q}_1(\overline{\mu}) + \overline{\sigma}^{\mathcal{R}}\partial_c\mathcal{R}_1(\overline{\mu}).$$

Furthermore, $\mathcal{F}_i$ $(i = 1,2)$, Ψ_τ $(\tau \in \mathcal{J})$, $\mathcal{Q}_1$, $\mathcal{R}_1$ are LU-convex and convex at $\overline{\mu}$. Therefore, from Theorem 16, $(\overline{\mu}, \overline{\lambda}, \overline{\sigma})$ is a saddle point for the scalarized Lagrangian of $(\mathcal{P}_s)$.

Conclusions and Future Research Directions

This article is concerned with the KKT-type necessary optimality conditions, Lagrange type duality, and saddle point optimality conditions for NIMSIPVC. We have presented VC-ACQ for NIMSIPVC and employed it to derive KKT-type necessary optimality conditions. We have formulated several Lagrange type dual problems corresponding to NIMSIPVC, namely, interval-valued weak vector, interval-valued vector, and scalarized Lagrange type dual problems. Subsequently, we have derived weak, converse, and strong duality results relating NIMSIPVC and corresponding Lagrange type dual problems. Furthermore, we have introduced saddle points for interval-valued vector Lagrangian and scalarized Lagrangian of NIMSIPVC. Additionally, we have derived the saddle point optimality conditions for NIMSIPVC by establishing a relationship between an optimal solution of NIMSIPVC and a saddle point associated with the Lagrangian of NIMSIPVC.

The results presented in the paper extend several well-known results existing in the literature. For instance, KKT-type necessary optimality conditions established in this paper extend various well-known results (see, for instance, [1,8,18,19,33]) for a more general class of optimization problems, namely, NIMSIPVC. Moreover, we extend the corresponding results developed by Tung et al. [44] from the smooth case of multiobjective SIPVC to a broader range of optimization problems, specifically NIMSIPVC. Several non-trivial examples have been provided to illustrate the significance of established results.

The results established in the present paper suggest various potential avenues for future research. In view of the fact that limiting subdifferential is the smallest among all robust subdifferentials and provides a better Lagrange multiplier rule (see, for instance, [57,58]), the results established in this paper can be further sharpened by utilizing limiting subdifferentials.

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