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Posted Date: 19 July 2024

doi: 10.20944/preprints202407.1562.v1

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p-adic Equiangular Lines and p-adic van Lint-Seidel Relative Bound

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Abstract: We introduce the notion of p-adic equiangular lines and derive the first fundamental relation between common angle, dimension of the space and the number of lines. More precisely, we show that if $\{\tau_j\}_{j=1}^n$ is p-adic γ-equiangular lines in $\mathbb{Q}_p^d$, then $|n|^2 \leq |d| \max\{|n|, \gamma^2\}$. We call Inequality (1) as the p-adic van Lint-Seidel relative bound. We believe that this complements fundamental van Lint-Seidel *[Indag. Math., 1966]* relative bound for equiangular lines in the p-adic case.

Keywords: equiangular lines; p-adic hilbert space

MSC: 12J25; 46S10; 47S10; 11D88

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1. Introduction

Let $d \in \mathbb{N}$ and $\gamma \in [0,1]$. Recall that a collection $\{\tau_j\}_{j=1}^n$ of unit vectors in $\mathbb{R}^d$ is said to be **γ-equiangular lines** [1,2] if

$$|\langle \tau_j, \tau_k \rangle| = \gamma, \quad \forall 1 \leq j,k \leq n, j \neq k.$$

A fundamental problem associated with equiangular lines is the following.

Problem 1.1. *Given $d \in \mathbb{N}$ and $\gamma \in [0,1]$, what is the upper bound on n such that there exists a collection $\{\tau_j\}_{j=1}^n$ of γ-equiangular lines in $\mathbb{R}^d$?*

An answer to Problem 1.1 which is fundamental driving force in the study of equiangular lines is the following result of van Lint and Seidel [2,3].

Theorem 1.2. *[2,3] (**van Lint-Seidel Relative Bound**) Let $\{\tau_j\}_{j=1}^n$ be γ-equiangular lines in $\mathbb{R}^d$. Then*

$$n(1-d\gamma^2) \leq d(1-\gamma^2).$$

In particular, if

$$\gamma < \frac{1}{\sqrt{d}},$$

then

$$n \leq \frac{d(1-\gamma^2)}{1-d\gamma^2}.$$

While deriving p-adic Welch bounds, the notion of p-adic equiangular lines is hinted in [4]. In this paper, we make it more rigorous and derive a fundamental relation which complements Theorem 1.2.

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2. p-adic Equiangular Lines

We begin by recalling the notion of p-adic Hilbert space. We refer [5–9] for more on p-adic Hilbert spaces.

Definition 2.1. *[6,7] Let $\mathbb{K}$ be a non-Archimedean valued field (with valuation $|\cdot|$) and $\mathcal{X}$ be a non-Archimedean Banach space (with norm $\|\cdot\|$) over $\mathbb{K}$. We say that $\mathcal{X}$ is a* ***p-adic Hilbert space*** *if there is a map (called as p-adic inner product) $\langle\cdot,\cdot\rangle : \mathcal{X}\times\mathcal{X}\to\mathbb{K}$ satisfying following.* (i)(i)

1. If $x\in\mathcal{X}$ is such that $\langle x,y\rangle = 0$ for all $y\in\mathcal{X}$, then $x=0$.
2. $\langle x,y\rangle = \langle y,x\rangle$ for all $x,y\in\mathcal{X}$.
3. $\langle \alpha x, y+z\rangle = \alpha\langle x,y\rangle + \langle x,z\rangle$ for all $\alpha\in\mathbb{K}$, for all $x,y,z\in\mathcal{X}$.
4. $|\langle x,y\rangle| \leq \|x\|\|y\|$ for all $x,y\in\mathcal{X}$.

Following is the standard example which we consider in the paper.

Example 2.2. *[5] Let p be a prime. For $d\in\mathbb{N}$, let $\mathbb{Q}_p^d$ be the standard p-adic Hilbert space equipped with the inner product*

$$\langle (a_j)_{j=1}^d, (b_j)_{j=1}^d\rangle := \sum_{j=1}^d a_j b_j, \quad \forall (a_j)_{j=1}^d, (b_j)_{j=1}^d \in \mathbb{Q}_p^d$$

and norm

$$\|(x_j)_{j=1}^d\| := \max_{1\leq j\leq d} |x_j|, \quad \forall (x_j)_{j=1}^d \in \mathbb{Q}_p^d.$$

Through various trials, we believe following is the correct definition of equiangular lines in the p-adic setting.

Definition 2.3. *Let p be a prime, $d\in\mathbb{N}$ and $\gamma\geq 0$. A collection $\{\tau_j\}_{j=1}^n$ in $\mathbb{Q}_p^d$ is said to be* ***p-adic γ-equiangular lines*** *if the following conditions hold:* (i)(i)

1. $\langle \tau_j, \tau_j\rangle = 1, \quad \forall 1\leq j\leq n.$
2. $|\langle \tau_j,\tau_k\rangle| = \gamma, \forall 1\leq j,k\leq n, j\neq k.$
3. The operator

$$S_\tau : \mathbb{Q}_p^d \ni x \mapsto \sum_{j=1}^n \langle x,\tau_j\rangle\tau_j \in \mathbb{Q}_p^d$$

is similar (through invertible operator) to a diagonal operator over $\mathbb{Q}_p$ with eigenvalues $\lambda_1,\dots,\lambda_d\in\mathbb{Q}_p$ satisfying

$$\left|\sum_{j=1}^d \lambda_j\right|^2 \leq |d| \left|\sum_{j=1}^d \lambda_j^2\right|.$$

The result of this paper is the following p-adic version of Theorem 1.2.

Theorem 2.4. ***(p-adic van Lint-Seidel Relative Bound)*** *Let p be a prime, $d\in\mathbb{N}$ and $\gamma\geq 0$. If $\{\tau_j\}_{j=1}^n$ is p-adic γ-equiangular lines in $\mathbb{Q}_p^d$, then*

$$|n|^2 \leq |d|\max\{|n|,\gamma^2\}.$$

In particular, we have following. (i)(i)

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1. If $|n| \leq \gamma^2$, then
$$|n|^2 \leq |d|\gamma^2.$$
2. If $|n| \geq \gamma^2$, then
$$|n| \leq |d|.$$

Proof. We see that

$$\begin{aligned}
\operatorname{Tra}(S_\tau) &= \sum_{j=1}^{n} \langle \tau_j, \tau_j \rangle, \\
\operatorname{Tra}(S_\tau^2) &= \sum_{j=1}^{n}\sum_{k=1}^{n} \langle \tau_j, \tau_k \rangle \langle \tau_k, \tau_j \rangle = \sum_{j=1}^{n}\sum_{k=1}^{n} \langle \tau_j, \tau_k \rangle^2.
\end{aligned}$$

Using Definition 2.3,

$$\begin{aligned}
|n|^2 &= \left|\sum_{j=1}^{n} \langle \tau_j, \tau_j \rangle\right|^2 = |\operatorname{Tra}(S_\tau)|^2 = \left|\sum_{j=1}^{d} \lambda_j\right|^2 \leq |d| \left|\sum_{j=1}^{d} \lambda_j^2\right| \\
&= |d| \left|\sum_{j=1}^{n}\sum_{k=1}^{n} \langle \tau_j, \tau_k \rangle^2\right| = |d| \left|\sum_{j=1}^{n} \langle \tau_j, \tau_j \rangle^2 + \sum_{1\leq j,k \leq n, j\neq k} \langle \tau_j, \tau_k \rangle^2\right| \\
&= |d| \left|\sum_{j=1}^{n} 1 + \sum_{1\leq j,k \leq n, j\neq k} \langle \tau_j, \tau_k \rangle^2\right| = |d| \left| n + \sum_{1\leq j,k \leq n, j\neq k} \langle \tau_j, \tau_k \rangle^2\right| \\
&\leq |d| \max\left\{ |n|, \left|\sum_{1\leq j,k \leq n, j\neq k} \langle \tau_j, \tau_k \rangle^2\right| \right\} \\
&\leq |d| \max\left\{ |n|, \max_{1\leq j,k \leq n, j\neq k} |\langle \tau_j, \tau_k \rangle|^2 \right\} = |d| \max\{|n|, \gamma^2\}.
\end{aligned}$$

□

Corollary 2.5. *Let $\{\tau_j\}_{j=1}^n$ be a collection in $\mathbb{Q}_p^d$ satisfying following.* (i)(i)

1. $\langle \tau_j, \tau_j \rangle = 1, \quad \forall 1 \leq j \leq n.$
2. There exists a nonzero element $b \in \mathbb{Q}_p$ such that
$$bx = \sum_{j=1}^{n} \langle x, \tau_j \rangle \tau_j, \quad \forall x \in \mathbb{Q}_p^d.$$

   Then
$$|n|^2 \leq |d| \max\{|n|, \gamma^2\}.$$

A careful observation of proof of Theorem 2.4 gives following general p-adic Welch bound.

Theorem 2.6. *(**General p-adic Welch Bound**) Let $\{\tau_j\}_{j=1}^n$ be a collection in $\mathbb{Q}_p^d$ satisfying following.* (i)(i)

1. $\langle \tau_j, \tau_j \rangle = 1, \quad \forall 1 \leq j \leq n.$
2. The operator
$$S_\tau : \mathbb{Q}_p^d \ni x \mapsto \sum_{j=1}^{n} \langle x, \tau_j \rangle \tau_j \in \mathbb{Q}_p^d$$

 doi:10.20944/preprints202407.1562.v1

is similar to a diagonal operator over $\mathbb{Q}_p$ with eigenvalues $\lambda_1, \ldots, \lambda_d \in \mathbb{Q}_p$ satisfying

$$\left|\sum_{j=1}^{d} \lambda_j\right|^2 \leq |d| \left|\sum_{j=1}^{d} \lambda_j^2\right|.$$

Then

$$|n|^2 \leq |d| \max_{1 \leq j,k \leq n, j \neq k} \left\{|n|, |\langle \tau_j, \tau_k \rangle|^2\right\}.$$

We can generalize Definition 2.3 in the following way.

Definition 2.7. *Let p be a prime, $d \in \mathbb{N}$, $\gamma \geq 0$ and $a \in \mathbb{Q}_p$ be nonzero. A collection $\{\tau_j\}_{j=1}^n$ in $\mathbb{Q}_p^d$ is said to be* ***p-adic** (γ, a)**-equiangular lines** if the following conditions hold:* (i)(i)

1. $\langle \tau_j, \tau_j \rangle = a, \quad \forall 1 \leq j \leq n.$
2. $|\langle \tau_j, \tau_k \rangle| = \gamma, \forall 1 \leq j, k \leq n, j \neq k.$
3. The operator

$$S_\tau : \mathbb{Q}_p^d \ni x \mapsto \sum_{j=1}^{n} \langle x, \tau_j \rangle \tau_j \in \mathbb{Q}_p^d$$

is similar to a diagonal operator over $\mathbb{Q}_p$ with eigenvalues $\lambda_1, \ldots, \lambda_d \in \mathbb{Q}_p$ satisfying

$$\left|\sum_{j=1}^{d} \lambda_j\right|^2 \leq |d| \left|\sum_{j=1}^{d} \lambda_j^2\right|.$$

Note that division by norm of an element is not allowed in a p-adic Hilbert space. Thus we can not reduce Definition 2.7 to Definition 2.3 (unlike the real case). By modifying earlier proofs, we easily get following theorems.

Theorem 2.8. *If $\{\tau_j\}_{j=1}^n$ is p-adic (γ, a)-equiangular lines in $\mathbb{Q}_p^d$, then*

$$|n|^2 \leq |d| \max\left\{|n|, \frac{\gamma^2}{|a^2|}\right\}.$$

In particular, we have following. (i)(i)

1. If $|a^2 n| \leq \gamma^2$, then

$$|n|^2 \leq |d| \frac{\gamma^2}{|a^2|}.$$

2. If $|a^2 n| \geq \gamma^2$, then

$$|n| \leq |d|.$$

Corollary 2.9. *Let $\{\tau_j\}_{j=1}^n$ be a collection in $\mathbb{Q}_p^d$ satisfying following.* (i)(i)

1. There exists a nonzero element $a \in \mathbb{Q}_p$ such that $\langle \tau_j, \tau_j \rangle = a, \forall 1 \leq j \leq n.$
2. There exists a nonzero element $b \in \mathbb{Q}_p$ such that

$$bx = \sum_{j=1}^{n} \langle x, \tau_j \rangle \tau_j, \quad \forall x \in \mathbb{Q}_p^d.$$

Then

$$|n|^2 \leq |d| \max\left\{|n|, \frac{\gamma^2}{|a^2|}\right\}.$$

Theorem 2.10. *Let $\{\tau_j\}_{j=1}^n$ be a collection in $\mathbb{Q}_p^d$ satisfying following.* (i)(i)

1. There exists a nonzero element $a \in \mathbb{Q}_p$ such that $\langle \tau_j, \tau_j \rangle = a, \forall 1 \leq j \leq n$.
2. The operator

$$S_\tau : \mathbb{Q}_p^d \ni x \mapsto \sum_{j=1}^{n} \langle x, \tau_j \rangle \tau_j \in \mathbb{Q}_p^d$$

is similar to a diagonal operator over $\mathbb{Q}_p$ with eigenvalues $\lambda_1, \dots, \lambda_d \in \mathbb{Q}_p$ satisfying

$$\left|\sum_{j=1}^{d} \lambda_j\right|^2 \leq |d| \left|\sum_{j=1}^{d} \lambda_j^2\right|.$$

Then

$$|n|^2 \leq |d| \max_{1\leq j,k\leq n, j\neq k} \left\{|n|, \frac{|\langle \tau_j, \tau_k\rangle|^2}{|a^2|}\right\}.$$

Note that there is a universal bound for equiangular lines known as Gerzon bound.

Theorem 2.11. *[10] (**Gerzon Universal Bound**) Let $\{\tau_j\}_{j=1}^n$ be γ-equiangular lines in $\mathbb{R}^d$. Then*

$$n \leq \frac{d(d+1)}{2}.$$

We are unable to derive p-adic version of Theorem 2.11. It is clear that, in the paper, we can replace $\mathbb{Q}_p$ by any non-Archimedean field.

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 doi:10.20944/preprints202407.1562.v1