

Article

Not peer-reviewed version

Lie Symmetries and Classification of Plane Symmetric Static Space-Times

[Jing-Li Fu](#)^{*}, Yan-Min Li, [Hui-Dong Cheng](#), [Xiao-Fan Sun](#)^{*}

Posted Date: 15 July 2024

doi: 10.20944/preprints202407.1172.v1

Keywords: static space-times; plane symmetric; Lie symmetry; conserved quantity; classification



Preprints.org is a free multidiscipline platform providing preprint service that is dedicated to making early versions of research outputs permanently available and citable. Preprints posted at Preprints.org appear in Web of Science, Crossref, Google Scholar, Scilit, Europe PMC.

Copyright: This is an open access article distributed under the Creative Commons Attribution License which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Article

Lie Symmetries and Classification of Plane Symmetric Static Space-Times

Jing-Li Fu ^{1,2,*}, Yian-Min Li ³, Hui-Dong Cheng ¹ and Sun Xiao-Fan ^{1,*}

¹ School of Artificial Intelligence, Shandong Vocational University of Foreign Affairs, Weihai 264504, China

² Department of Physics, Zhejiang Sci-Tech University, Hangzhou 310018, China

³ Department of Physics, Shangqiu normal university, Shangqiu 476000, China

* Correspondence: sqfujingli@163.com; sunxiaofan@sdws.edu.cn

Abstract: In this study, we provide the Lie symmetries and a classification of plane symmetric static space-times. Based on the invariance of the Lagrange equations of plane symmetries static space-time systems under the transformation Lie group, we provide the Lie symmetry determination equation, Lie symmetry theorem, and the conserved quantity of the systems; by utilizing the Lie symmetry method to solve the system, we give complete classification of the plane symmetric static space-times systems. The research results indicate that using the Lie symmetry method to study plane symmetric static spacetime can identify a series of conserved quantities that exist in the system; Discovered 5 basic Lie symmetries in the system; When the metric parameters of the system are appropriately selected, the plane symmetric static spacetime can have 6, 7, 8, 9, 11, and 17 Lie symmetric symmetries and conserved quantities.

Keywords: static space-times; plane symmetric; Lie symmetry; conserved quantity; classification

1. Introduction

It is generally known that space-time symmetries play a significant role in the motion of particles, specifically in gravity theories. The classification of space-time symmetry has become a hot topic in general relativity [1–7]. These classifications not only classify space-times according to such space-time symmetries, but also provide new solutions to the Einstein field equations, which are given in standard gravitational units $c = G = 1$ as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = k T_{\mu\nu}$$

The classification of Einstein field equations (EFE) constitutes a significant part of general relativity research. It is practically impossible to find general solutions for the EFE in a closed analytic form. These are non-linear partial differential equations, and it is not easy to obtain the exact solutions of these equations.

In 1974, Lie introduced infinitesimal transformations into differential equations and proposed a symmetric solution for solving differential equations [8]. The symmetry method is a fundamental method for solving differential equations [9–15]. By using the symmetry-solving method, we can solve the dynamical equations, reduce the order of the differential equations, and linearize nonlinear the dynamical equations. These methods are used to reduce the number of variables in partial differential equations. There are two basic symmetry methods under the transformation of the Lie group. One is based on the invariance of the Hamiltonian action of the dynamic system under the transformation Lie group, which is called the Noether symmetry method [16]; Other is based on the invariance of dynamical equations of system under the transformation Lie group, which is called the Lie symmetry method [8,16].

Symmetries help to find solutions of the 4 dimensional space-time. Various approaches have been used to classify space-times and to find solutions to Einstein field equations [17–23]. However, in previous studies, the classification and exact solutions of four-dimensional space-time were mostly based on the invariance of the Lagrange function describing the system under the transformation Lie

group; that is, the Noether symmetry of a four-dimensional space-time system to give the conservation of the system's existence and classify the system.

In recent years, the Lie symmetry method has been successfully applied to solve problems in conservative and non conservative, holonomic and non holonomic constrained mechanical systems, as well as in phase space constrained mechanical systems [24–38]. Scholars have applied the Lie symmetry method to solve electromechanical coupled dynamic systems and flexible robot systems [39–46].

In this study, the plane symmetric static space-times were classified using Lie symmetries. We introduce the concepts of generalized coordinates and generalized momentum in four-dimensional space-time, and provide the corresponding Lagrange equations. Based on the invariance of the Lagrange equations in the transformation Lie group of the system, we derive the Lie symmetry determination equation for the system, and further solve the transformation Lie group corresponding to the plane symmetric static space-time. We also propose the Lie symmetry theorem and conserved quantity (first integral) for plane symmetry static space-time. Lie symmetry theorem proved that for every symmetry there is a conservation law (conserved quantity). Classification of the plane symmetry space-times by Lie symmetries provides those symmetries of space-times corresponding to which there are conserved quantities. Isometries and homotheties also correspond to conserved quantities and they form subset(s) of the set of Lie symmetries. For the classification, examples of the Lie symmetries are symmetry under spatial translations implies conservation of linear momentums, the symmetry under time translation implies conservation of energy, and the symmetry under rotation implies conservation of angular momentum.

2. Motion Equations of Plane Symmetry Static Space-Time System

In four-dimensional space, generally the line element of n th dimensional space-time takes the form

$$ds^2 = g_{ij} dx^i dx^j \quad i, j = 1, 2, \dots, n \quad (1)$$

The Lagrangian L for the metric (1) is given by [23–26]

$$L = g_{ij} \dot{x}^i \dot{x}^j. \quad (2)$$

The spherically symmetric space-time is an important exact solution of the Einstein field equation. It is the conformally flat solution of the Einstein-Maxwell field equations for a nonnull electromagnetic field. We take the Lagrangian of general plane symmetric static space-time [7]

$$L = \frac{1}{2} \left[e^{\nu(x)} \dot{t}^2 - \dot{x}^2 - e^{\mu(x)} (\dot{y}^2 + \dot{z}^2) \right] \quad (3)$$

where s is the independent variable, x^i are the dependent variables on t, x, y, z and $\dot{x}^i$ are their derivatives with respect to curve s ; and here the metric coefficients $\nu(x)$ and $\mu(x)$ are functions of coordinate x .

The static space-time of the plane symmetries can be considered as a holonomic dynamical system. We can prove that one satisfies Lagrange equations in the following form:

$$\frac{d}{ds} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = Q_k \quad q_k = q_k(s, t, x, y, z). \quad (4)$$

The Lagrange equations (4) are different from the Lagrange equations in analytical mechanics, as it is the Lagrange equations in four-dimensional space-time.

Substituting Eq. (3) into Eqs. (4), we obtain the Lagrange equations for the plane symmetries static spacetime as follows

$$\ddot{t} = -\frac{d\nu}{dx} \dot{x} \dot{t} = \alpha_0, \quad (5)$$

$$\ddot{x} = \frac{1}{2} e^{\nu(x)} \frac{d\nu}{dx} \dot{t}^2 - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} (\dot{y}^2 + \dot{z}^2) = \alpha_1, \quad (6)$$

$$\ddot{y} = -\frac{d\mu}{dx} \dot{x}\dot{y} = \alpha_2, \quad (7)$$

$$\ddot{z} = -\frac{d\mu}{dx} \dot{x}\dot{z} = \alpha_3. \quad (8)$$

3. Lie Symmetries and Conserved Quantities of the Plane Symmetries Static Space-Time

In this section, we perform infinitesimal transformations on the independent and dependent variables of the plane symmetries static space-time based on the invariance of the Lagrange equation of the plane symmetries static space-time under infinitesimal transformations. We provide the Lie symmetry determination equation, Lie symmetry theorem, and this system possesses the conserved quantity (first integral).

3.1. Lie Symmetries and Its Determining Equations of the Plane Symmetries Static Space-Time

Introducing infinitesimal transformations on independent variable s and generalized coordinates,

$$s^* = s + \varepsilon \xi(s, q_0, q_1, \dots, q_k), \quad q_k^* = q_k + \varepsilon \eta^k(s, q_0, q_1, \dots, q_k) \quad k = 0, 1, \dots, n \quad (9)$$

where ε is an infinitesimal parameter, ξ and η^k are infinitesimal generators, and $n=4N-l$. Introducing the following infinitesimal generator vector,

$$X^{[0]} = \xi \frac{\partial}{\partial s} + \sum_{k=0}^n \eta^k \frac{\partial}{\partial q_k}, \quad (10)$$

and, one of its extensions,

$$X^{[1]} = X^{[0]} + \sum_{k=0}^n \left(\dot{\eta}^k - \dot{q}_k \dot{\xi} \right) \frac{\partial}{\partial \dot{q}_k}, \quad (11)$$

and its secondary expansion,

$$X^{[2]} = X^{[1]} + \sum_{k=0}^n \left(\ddot{\eta}^k - 2\alpha_k \dot{\xi} - \dot{q}_k \ddot{\xi} \right) \frac{\partial}{\partial \ddot{q}_k}. \quad (12)$$

In this work, we take $K=t, x, y, z$.

The invariance of differential equation (5)-(8) under infinitesimal transformation (9) is reduced to the following equations:

$$\ddot{\eta}^0 + 2 \frac{d\nu}{dx} \dot{x} \dot{\xi} - \dot{t} \ddot{\xi} = -X^1 \left(\frac{d\nu}{dx} \dot{x} \dot{t} \right), \quad (13)$$

$$\begin{aligned} \ddot{\eta}^1 - \left(e^{\nu(x)} \frac{d\nu}{dx} \dot{t}^2 - e^{\mu(x)} \frac{d\mu}{dx} (\dot{y}^2 + \dot{z}^2) \right) \dot{\xi} - \dot{x} \ddot{\xi} \\ = -X^1 \left(\frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} (\dot{y}^2 + \dot{z}^2) - \frac{1}{2} e^{\nu(x)} \frac{d\nu}{dx} \dot{t}^2 \right), \end{aligned} \quad (14)$$

$$\ddot{\eta}^2 + 2 \frac{d\mu}{dx} \dot{x} \dot{\xi} - \dot{y} \ddot{\xi} = -X^1 \left(\frac{d\mu}{dx} \dot{x} \dot{y} \right), \quad (15)$$

$$\ddot{\eta}^3 + 2 \frac{d\mu}{dx} \dot{x}\dot{\xi} - \dot{z}\ddot{\xi} = -X^1 \left(\frac{d\mu}{dx} \dot{x}\dot{z} \right). \quad (16)$$

Equations (13)-(16) are called the Lie symmetry determination equations for the plane symmetry static space-time in rectangular coordinate system.

By expanding Eqs. (13)-(16), we obtain the four partial differential equations of the plane symmetries static space-time as follows:

$$\begin{aligned} & \frac{\partial^2 \eta^0}{\partial s^2} + \left(2 \frac{\partial^2 \eta^0}{\partial t \partial s} - \frac{\partial^2 \xi}{\partial s^2} + \frac{\partial \eta^1}{\partial s} \frac{dv}{dx} \right) \dot{t} + \left(2 \frac{\partial^2 \eta^0}{\partial x \partial s} + \frac{\partial \eta^0}{\partial s} \frac{dv}{dx} \right) \dot{x} \\ & + 2 \frac{\partial^2 \eta^0}{\partial y \partial s} \dot{y} + 2 \frac{\partial^2 \eta^0}{\partial z \partial s} \dot{z} \\ & + \left(2 \frac{\partial^2 \eta^0}{\partial t \partial y} - 2 \frac{\partial^2 \xi}{\partial y \partial s} + \frac{\partial \eta^1}{\partial y} \frac{dv}{dx} \right) \dot{t}\dot{y} + \left(2 \frac{\partial^2 \eta^0}{\partial t \partial z} - 2 \frac{\partial^2 \xi}{\partial z \partial s} + \frac{\partial \eta^1}{\partial z} \frac{dv}{dx} \right) \dot{t}\dot{z} \\ & + \left(2 \frac{\partial^2 \eta^0}{\partial x \partial y} - \frac{d\mu}{dx} \frac{\partial \eta^0}{\partial y} + \frac{\partial \eta^0}{\partial y} \frac{dv}{dx} \right) \dot{x}\dot{y} + \left(2 \frac{\partial^2 \eta^0}{\partial x \partial z} - \frac{d\mu}{dx} \frac{\partial \eta^0}{\partial z} + \frac{\partial \eta^0}{\partial z} \frac{dv}{dx} \right) \dot{x}\dot{z} \\ & + 2 \frac{\partial^2 \eta^0}{\partial z \partial y} \dot{y}\dot{z} \\ & + \left(2 \frac{dv}{dx} \frac{\partial \xi}{\partial s} - \frac{dv}{dx} \frac{\partial \eta^0}{\partial t} - 2 \frac{\partial^2 \xi}{\partial x \partial s} + 2 \frac{\partial^2 \eta^0}{\partial t \partial x} + \eta^1 \frac{d^2 v}{dx^2} + \frac{\partial \eta^0}{\partial t} \frac{dv}{dx} + \frac{\partial \eta^1}{\partial x} \frac{dv}{dx} \right. \\ & \left. - 2 \frac{dv}{dx} \frac{\partial \xi}{\partial s} \right) \dot{x}\dot{t} \\ & + \left(\frac{1}{2} e^{\nu(x)} \frac{\partial \eta^0}{\partial x} \frac{dv}{dx} + \frac{\partial^2 \eta^0}{\partial t \partial t} - 2 \frac{\partial^2 \xi}{\partial t \partial s} + \frac{\partial \eta^1}{\partial t} \frac{dv}{dx} \right) \dot{t}^2 + \left(\frac{\partial \eta^0}{\partial x} \frac{dv}{dx} + \frac{\partial^2 \eta^0}{\partial x \partial x} \right) \dot{x}^2 \\ & + \left(\frac{\partial^2 \eta^0}{\partial y \partial y} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^0}{\partial x} \right) \dot{y}^2 + \left(\frac{\partial^2 \eta^0}{\partial z \partial z} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^0}{\partial x} \right) \dot{z}^2 \\ & + \left(-\frac{\partial^2 \xi}{\partial t \partial t} \dot{t}^3 - \frac{1}{2} e^{\nu(x)} \frac{dv}{dx} \frac{\partial \xi}{\partial x} \right) \dot{t}^3 + \left(2 \frac{dv}{dx} \frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial x \partial x} - 2 \frac{dv}{dx} \frac{\partial \xi}{\partial x} \right) \dot{t}\dot{x}^2 \\ & + \left(\frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial y \partial y} \right) \dot{t}\dot{y}^2 + \left(\frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial z \partial z} \right) \dot{t}\dot{z}^2 \\ & + \left(\frac{dv}{dx} \frac{\partial \xi}{\partial t} - 2 \frac{\partial^2 \xi}{\partial t \partial x} - 2 \frac{dv}{dx} \frac{\partial \xi}{\partial t} + 2 \frac{dv}{dx} \frac{\partial \xi}{\partial t} \right) \dot{x}\dot{t}^2 - 2 \frac{\partial^2 \xi}{\partial t \partial y} \dot{y}\dot{t}^2 - 2 \frac{\partial^2 \xi}{\partial t \partial z} \dot{z}\dot{t}^2 \\ & + \left(\frac{d\mu}{dx} \frac{\partial \xi}{\partial y} - 2 \frac{\partial^2 \xi}{\partial x \partial y} + 2 \frac{dv}{dx} \frac{\partial \xi}{\partial y} - 2 \frac{dv}{dx} \frac{\partial \xi}{\partial y} \right) \dot{t}\dot{y}\dot{x} \\ & + \left(2 \frac{dv}{dx} \frac{\partial \xi}{\partial z} + \frac{d\mu}{dx} \frac{\partial \xi}{\partial z} - 2 \frac{\partial^2 \xi}{\partial x \partial z} - 2 \frac{dv}{dx} \frac{\partial \xi}{\partial z} \right) \dot{t}\dot{z}\dot{x} - 2 \frac{\partial^2 \xi}{\partial z \partial y} \dot{t}\dot{y}\dot{z} \\ & = 0 \end{aligned} \quad (17)$$

$$\begin{aligned}
& \frac{\partial^2 \eta^0}{\partial s^2} = 0, 2 \frac{\partial^2 \eta^0}{\partial t \partial s} - \frac{\partial^2 \xi}{\partial s^2} + \frac{\partial \eta^1}{\partial s} \frac{dv}{dx} = 0, +2 \frac{\partial^2 \eta^0}{\partial x \partial s} + \frac{\partial \eta^0}{\partial s} \frac{dv}{dx} = 0, \\
& +2 \frac{\partial^2 \eta^0}{\partial y \partial s} = 0, +2 \frac{\partial^2 \eta^0}{\partial z \partial s} = 0, +2 \frac{\partial^2 \eta^0}{\partial t \partial y} - 2 \frac{\partial^2 \xi}{\partial y \partial s} + \frac{\partial \eta^1}{\partial y} \frac{dv}{dx} = 0, \\
& +2 \frac{\partial^2 \eta^0}{\partial t \partial z} - 2 \frac{\partial^2 \xi}{\partial z \partial s} + \frac{\partial \eta^1}{\partial z} \frac{dv}{dx} = 0, +2 \frac{\partial^2 \eta^0}{\partial x \partial y} - \frac{d\mu}{dx} \frac{\partial \eta^0}{\partial y} + \frac{\partial \eta^0}{\partial y} \frac{dv}{dx} = 0, \\
& +2 \frac{\partial^2 \eta^0}{\partial x \partial z} - \frac{d\mu}{dx} \frac{\partial \eta^0}{\partial z} + \frac{\partial \eta^0}{\partial z} \frac{dv}{dx} = 0, +2 \frac{\partial^2 \eta^0}{\partial z \partial y} = 0 \\
& +2 \frac{dv}{dx} \frac{\partial \xi}{\partial s} - \frac{dv}{dx} \frac{\partial \eta^0}{\partial t} - 2 \frac{\partial^2 \xi}{\partial x \partial s} + 2 \frac{\partial^2 \eta^0}{\partial t \partial x} + \eta^1 \frac{d^2 v}{dx^2} + \frac{\partial \eta^0}{\partial t} \frac{dv}{dx} \\
& + \frac{\partial \eta^1}{\partial x} \frac{dv}{dx} - 2 \frac{dv}{dx} \frac{\partial \xi}{\partial s} = 0, \frac{\partial \eta^0}{\partial x} \frac{dv}{dx} + \frac{\partial^2 \eta^0}{\partial x \partial x} = 0, \\
& + \frac{1}{2} e^{\nu(x)} \frac{\partial \eta^0}{\partial x} \frac{dv}{dx} + \frac{\partial^2 \eta^0}{\partial t \partial t} - 2 \frac{\partial^2 \xi}{\partial t \partial s} + \frac{\partial \eta^1}{\partial t} \frac{dv}{dx} = 0, \\
& \frac{\partial^2 \eta^0}{\partial y \partial y} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^0}{\partial x} = 0, \frac{\partial^2 \eta^0}{\partial z \partial z} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^0}{\partial x} = 0, \\
& - \frac{\partial^2 \xi}{\partial t \partial t} \dot{t}^3 - \frac{1}{2} e^{\nu(x)} \frac{dv}{dx} \frac{\partial \xi}{\partial x} = 0, 2 \frac{dv}{dx} \frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial x \partial x} - 2 \frac{dv}{dx} \frac{\partial \xi}{\partial x} = 0, \\
& \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial y \partial y} = 0, \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial z \partial z} = 0, \\
& \frac{dv}{dx} \frac{\partial \xi}{\partial t} - 2 \frac{\partial^2 \xi}{\partial t \partial x} - 2 \frac{dv}{dx} \frac{\partial \xi}{\partial t} + 2 \frac{dv}{dx} \frac{\partial \xi}{\partial t} = 0, -2 \frac{\partial^2 \xi}{\partial t \partial y} = 0, \\
& -2 \frac{\partial^2 \xi}{\partial t \partial z} = 0, \frac{d\mu}{dx} \frac{\partial \xi}{\partial y} - 2 \frac{\partial^2 \xi}{\partial x \partial y} + 2 \frac{dv}{dx} \frac{\partial \xi}{\partial y} - 2 \frac{dv}{dx} \frac{\partial \xi}{\partial y} = 0, \\
& 2 \frac{dv}{dx} \frac{\partial \xi}{\partial z} + \frac{d\mu}{dx} \frac{\partial \xi}{\partial z} - 2 \frac{\partial^2 \xi}{\partial x \partial z} - 2 \frac{dv}{dx} \frac{\partial \xi}{\partial z} = 0, -2 \frac{\partial^2 \xi}{\partial z \partial y} = 0.
\end{aligned}
\tag{17}$$

$$\begin{aligned}
& \frac{\partial^2 \eta^1}{\partial s^2} + 2 \frac{\partial^2 \eta^1}{\partial t \partial s} \dot{t} + \left(2 \frac{\partial^2 \eta^1}{\partial x \partial s} - \frac{\partial^2 \xi}{\partial s^2} \right) \dot{x} + 2 \frac{\partial^2 \eta^1}{\partial y \partial s} \dot{y} + 2 \frac{\partial^2 \eta^1}{\partial z \partial s} \dot{z} \\
& + \left(\frac{\partial^2 \eta^1}{\partial t \partial t} + \frac{1}{2} e^{\nu(x)} \frac{d\nu}{dx} \frac{\partial \eta^1}{\partial x} - e^{\nu(x)} \frac{d\nu}{dx} \frac{\partial \xi}{\partial s} \right) \dot{t}^2 + \left(\frac{\partial^2 \eta^1}{\partial x \partial x} - 2 \frac{\partial^2 \xi}{\partial x \partial s} \right) \dot{x}^2 \\
& + \left(\frac{\partial^2 \eta^1}{\partial y \partial y} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^1}{\partial x} + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial s} \right) \dot{y}^2 \\
& + \left(\frac{\partial^2 \eta^1}{\partial z \partial z} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^1}{\partial x} + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial s} \right) \dot{z}^2 \\
& + \left(2 \frac{\partial^2 \eta^1}{\partial t \partial x} - \frac{d\nu}{dx} \frac{\partial \eta^1}{\partial t} - 2 \frac{\partial^2 \xi}{\partial t \partial s} \right) \dot{x} \dot{t} + 2 \frac{\partial^2 \eta^1}{\partial t \partial y} \dot{y} \dot{t} + 2 \frac{\partial^2 \eta^1}{\partial t \partial z} \dot{z} \dot{t} \\
& - 2 \frac{\partial^2 \xi}{\partial y \partial s} \dot{x} \dot{y} - 2 \frac{\partial^2 \xi}{\partial z \partial s} \dot{x} \dot{z} + \left(2 \frac{\partial^2 \eta^1}{\partial x \partial y} - \frac{d\mu}{dx} \frac{\partial \eta^1}{\partial y} \right) \dot{x} \dot{y} + \left(2 \frac{\partial^2 \eta^1}{\partial x \partial z} - \frac{d\mu}{dx} \frac{\partial \eta^1}{\partial z} \right) \dot{x} \dot{z} \\
& + 2 \frac{\partial^2 \eta^1}{\partial z \partial y} \dot{y} \dot{z} - e^{\nu(x)} \frac{d\nu}{dx} \frac{\partial \xi}{\partial t} \dot{t}^3 - \frac{\partial^2 \xi}{\partial x \partial x} \dot{x}^3 + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial y} \dot{y}^3 + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial z} \dot{z}^3 \\
& + \left(- \frac{\partial^2 \xi}{\partial t \partial t} - e^{\nu(x)} \frac{d\nu}{dx} \frac{\partial \xi}{\partial x} \right) \dot{t}^2 \dot{x} - e^{\nu(x)} \frac{d\nu}{dx} \frac{\partial \xi}{\partial y} \dot{t}^2 \dot{y} - e^{\nu(x)} \frac{d\nu}{dx} \frac{\partial \xi}{\partial z} \dot{t}^2 \dot{z} \\
& + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial t} \dot{t} \dot{y}^2 + \left(e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial y \partial y} + \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} \right) \dot{x} \dot{y}^2 + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial z} \dot{y}^2 \dot{z} \\
& + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial t} \dot{t} \dot{z}^2 + \left(e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial z \partial z} + \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} \right) \dot{x} \dot{z}^2 + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial y} \dot{y} \dot{z}^2 \\
& + \left(\frac{d\nu}{dx} \frac{\partial \xi}{\partial t} - 2 \frac{\partial^2 \xi}{\partial t \partial x} \right) \dot{x}^2 \dot{t} - \frac{1}{2} e^{\nu(x)} \frac{d\nu}{dx} \frac{\partial \xi}{\partial x} \dot{x} \dot{t}^2 \\
& + \left(\frac{d\mu}{dx} \frac{\partial \xi}{\partial z} - 2 \frac{\partial^2 \xi}{\partial x \partial z} \right) \dot{z} \dot{x}^2 + \left(\frac{d\mu}{dx} \frac{\partial \xi}{\partial y} - 2 \frac{\partial^2 \xi}{\partial x \partial y} \right) \dot{x}^2 \dot{y} \\
& - 2 \frac{\partial^2 \xi}{\partial t \partial y} \dot{x} \dot{y} \dot{t} - 2 \frac{\partial^2 \xi}{\partial t \partial z} \dot{x} \dot{z} \dot{t} - 2 \frac{\partial^2 \xi}{\partial z \partial y} \dot{x} \dot{y} \dot{z}
\end{aligned}$$

$$\begin{aligned}
& \frac{\partial^2 \eta^1}{\partial s^2} + \left(2 \frac{\partial^2 \eta^1}{\partial t \partial s} - e^{\nu(x)} \frac{dv}{dx} \frac{\partial \eta^0}{\partial s} \right) \dot{t} + \left(2 \frac{\partial^2 \eta^1}{\partial x \partial s} - \frac{\partial^2 \xi}{\partial s^2} \right) \dot{x} \\
& + \left(2 \frac{\partial^2 \eta^1}{\partial y \partial s} + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial s} \right) \dot{y} + \left(2 \frac{\partial^2 \eta^1}{\partial z \partial s} + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^3}{\partial s} \right) \dot{z} \\
& + \left(\frac{\partial^2 \eta^1}{\partial t \partial t} + \frac{1}{2} e^{\nu(x)} \frac{dv}{dx} \frac{\partial \eta^1}{\partial x} - e^{\nu(x)} \frac{dv}{dx} \frac{\partial \xi}{\partial s} - \frac{1}{2} e^{\nu(x)} \left(\frac{dv}{dx} \right)^2 \eta^1 \right. \\
& \quad \left. - e^{\nu(x)} \frac{dv}{dx} \frac{\partial \eta^0}{\partial t} + e^{\nu(x)} \frac{dv}{dx} \frac{\partial \xi}{\partial s} \right) \dot{t}^2 \\
& + \left(\frac{\partial^2 \eta^1}{\partial x \partial x} - 2 \frac{\partial^2 \xi}{\partial x \partial s} \right) \dot{x}^2 \\
& + \left(\frac{\partial^2 \eta^1}{\partial y \partial y} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^1}{\partial x} + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial s} + \frac{1}{2} e^{\mu(x)} \left(\frac{d\mu}{dx} \right)^2 \eta^1 \right. \\
& \quad \left. + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial y} - e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial s} \right) \dot{y}^2 \\
& + \left(\frac{\partial^2 \eta^1}{\partial z \partial z} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^1}{\partial x} + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial s} + \frac{1}{2} e^{\mu(x)} \left(\frac{d\mu}{dx} \right)^2 \eta^1 \right. \\
& \quad \left. + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^3}{\partial z} - e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial s} \right) \dot{z}^2
\end{aligned} \tag{18}$$

$$\begin{aligned}
& \frac{\partial^2 \eta^1}{\partial s^2} = 0, 2 \frac{\partial^2 \eta^1}{\partial t \partial s} - e^{\nu(x)} \frac{dv}{dx} \frac{\partial \eta^0}{\partial s} = 0, 2 \frac{\partial^2 \eta^1}{\partial x \partial s} - \frac{\partial^2 \xi}{\partial s^2} = 0, \\
& 2 \frac{\partial^2 \eta^1}{\partial y \partial s} + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial s} = 0, 2 \frac{\partial^2 \eta^1}{\partial z \partial s} + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^3}{\partial s} = 0, \\
& \frac{\partial^2 \eta^1}{\partial t \partial t} + \frac{1}{2} e^{\nu(x)} \frac{dv}{dx} \frac{\partial \eta^1}{\partial x} - \frac{1}{2} e^{\nu(x)} \left(\frac{dv}{dx} \right)^2 \eta^1 - e^{\nu(x)} \frac{dv}{dx} \frac{\partial \eta^0}{\partial t} = 0, \\
& \frac{\partial^2 \eta^1}{\partial y \partial y} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^1}{\partial x} + \frac{1}{2} e^{\mu(x)} \left(\frac{d\mu}{dx} \right)^2 \eta^1 + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial y} = 0, \\
& \frac{\partial^2 \eta^1}{\partial z \partial z} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^1}{\partial x} + \frac{1}{2} e^{\mu(x)} \left(\frac{d\mu}{dx} \right)^2 \eta^1 + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^3}{\partial z} = 0, \\
& 2 \frac{\partial^2 \eta^1}{\partial t \partial x} - \frac{dv}{dx} \frac{\partial \eta^1}{\partial t} - 2 \frac{\partial^2 \xi}{\partial t \partial s} - e^{\nu(x)} \frac{dv}{dx} \frac{\partial \eta^0}{\partial x} = 0, \\
& e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial x} - 2 \frac{\partial^2 \xi}{\partial y \partial s} + 2 \frac{\partial^2 \eta^1}{\partial x \partial y} - \frac{d\mu}{dx} \frac{\partial \eta^1}{\partial y} = 0, \\
& e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^3}{\partial x} - 2 \frac{\partial^2 \xi}{\partial z \partial s} + 2 \frac{\partial^2 \eta^1}{\partial x \partial z} - \frac{d\mu}{dx} \frac{\partial \eta^1}{\partial z} = 0, \\
& e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^3}{\partial t} - e^{\nu(x)} \frac{dv}{dx} \frac{\partial \eta^0}{\partial z} + 2 \frac{\partial^2 \eta^1}{\partial t \partial z} = 0, \\
& e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial t} - e^{\nu(x)} \frac{dv}{dx} \frac{\partial \eta^0}{\partial y} + 2 \frac{\partial^2 \eta^1}{\partial t \partial y} = 0, \\
& e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial z} + e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^3}{\partial y} + 2 \frac{\partial^2 \eta^1}{\partial z \partial y} = 0, \\
& e^{\nu(x)} \frac{dv}{dx} \frac{\partial \xi}{\partial t} - e^{\nu(x)} \frac{dv}{dx} \frac{\partial \xi}{\partial t} = 0, -\frac{\partial^2 \xi}{\partial x \partial x} = 0, \frac{\partial^2 \eta^1}{\partial x \partial x} - 2 \frac{\partial^2 \xi}{\partial x \partial s} = 0, \\
& -\frac{\partial^2 \xi}{\partial t \partial t} - e^{\nu(x)} \frac{dv}{dx} \frac{\partial \xi}{\partial x} + \frac{1}{2} e^{\nu(x)} \frac{dv}{dx} \frac{\partial \xi}{\partial x} = 0, \frac{dv}{dx} \frac{\partial \xi}{\partial t} - 2 \frac{\partial^2 \xi}{\partial t \partial x} = 0, \\
& \frac{d\mu}{dx} \frac{\partial \xi}{\partial z} - 2 \frac{\partial^2 \xi}{\partial x \partial z} = 0, e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial t} - e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial t} = 0, \frac{d\mu}{dx} \frac{\partial \xi}{\partial y} - 2 \frac{\partial^2 \xi}{\partial x \partial y} = 0, \\
& e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial y \partial y} + \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} - e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} = 0, \\
& e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial z} - e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial z} = 0, e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial y} - e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial y} = 0, \\
& -\frac{\partial^2 \xi}{\partial z \partial z} + \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} = 0, -2 \frac{\partial^2 \xi}{\partial t \partial y} = 0, 2 \frac{\partial^2 \xi}{\partial t \partial z} = 0, -2 \frac{\partial^2 \xi}{\partial z \partial y} = 0.
\end{aligned} \tag{18}$$

$$\begin{aligned}
& \frac{\partial^2 \eta^2}{\partial s \partial s} + 2 \frac{\partial^2 \eta^2}{\partial s \partial t} \dot{t} + 2 \frac{\partial^2 \eta^2}{\partial s \partial x} \dot{x} + \left(2 \frac{\partial^2 \eta^2}{\partial s \partial y} - \frac{\partial^2 \xi}{\partial s^2} + \frac{\partial \eta^1}{\partial s} \frac{d\mu}{dx} + \frac{\partial \eta^2}{\partial s} \frac{d\mu}{dx} \right) \dot{y} + 2 \frac{\partial^2 \eta^2}{\partial s \partial z} \dot{z} \\
& + \left(\frac{\partial^2 \eta^2}{\partial t \partial t} + \frac{1}{2} e^{\mu(x)} \frac{dv}{dx} \frac{\partial \eta^2}{\partial x} - \frac{1}{2} e^{\mu(x)} \frac{dv}{dx} \frac{\partial \xi}{\partial x} \right) \dot{t}^2 + \frac{\partial^2 \eta^2}{\partial x \partial x} \dot{x}^2 \\
& + \left(\frac{\partial^2 \eta^2}{\partial x \partial y} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial x} - 2 \frac{\partial^2 \xi}{\partial y \partial s} + \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} + \frac{\partial \eta^1}{\partial y} \frac{d\mu}{dx} + \frac{\partial \eta^2}{\partial y} \frac{d\mu}{dx} \right) \dot{y}^2 \\
& + \left(\frac{\partial^2 \eta^2}{\partial z \partial z} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial x} + \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} \right) \dot{z}^2 + \left(2 \frac{\partial^2 \eta^2}{\partial t \partial x} - \frac{dv}{dx} \frac{\partial \eta^2}{\partial t} + \frac{dv}{dx} \frac{\partial \xi}{\partial t} \right) \dot{x} \dot{t} \\
& + \left(2 \frac{\partial^2 \eta^2}{\partial t \partial y} - 2 \frac{\partial^2 \xi}{\partial t \partial s} + \frac{\partial \eta^1}{\partial t} \frac{d\mu}{dx} + \frac{\partial \eta^2}{\partial t} \frac{d\mu}{dx} \right) \dot{t} \dot{y} + 2 \frac{\partial^2 \eta^2}{\partial t \partial z} \dot{t} \dot{z} \\
& + \left(2 \frac{\partial^2 \eta^2}{\partial x \partial y} - \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial y} + 2 \frac{\partial \xi}{\partial s} \frac{d\mu}{dx} - 2 \frac{\partial^2 \xi}{\partial x \partial s} + \frac{d\mu}{dx} \frac{\partial \xi}{\partial y} - 2 \frac{\partial \xi}{\partial s} \frac{d\mu}{dx} + \eta^1 \frac{d^2 \mu}{dx^2} \right) \dot{x} \dot{y} \\
& + \left(\frac{\partial \eta^1}{\partial x} \frac{d\mu}{dx} + \frac{\partial \eta^2}{\partial x} \frac{d\mu}{dx} \right) \dot{x} \\
& + \left(2 \frac{\partial^2 \eta^2}{\partial x \partial z} - \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial z} + \frac{d\mu}{dx} \frac{\partial \xi}{\partial z} \right) \dot{x} \dot{z} + \left(2 \frac{\partial^2 \eta^2}{\partial y \partial z} - 2 \frac{\partial^2 \xi}{\partial z \partial s} + \frac{\partial \eta^1}{\partial z} \frac{d\mu}{dx} + \frac{\partial \eta^2}{\partial z} \frac{d\mu}{dx} \right) \dot{y} \dot{z} \\
& + \left(2 \frac{\partial \xi}{\partial t} \frac{d\mu}{dx} - 2 \frac{\partial^2 \xi}{\partial t \partial x} - 2 \frac{\partial \xi}{\partial t} \frac{d\mu}{dx} \right) \dot{x} \dot{t} \dot{y} + \left(2 \frac{\partial \xi}{\partial x} \frac{d\mu}{dx} - \frac{\partial^2 \xi}{\partial x \partial x} - 2 \frac{\partial \xi}{\partial x} \frac{d\mu}{dx} \right) \dot{x}^2 \dot{y} \\
& + \left(2 \frac{\partial \xi}{\partial y} \frac{d\mu}{dx} - 2 \frac{\partial^2 \xi}{\partial x \partial y} - 2 \frac{\partial \xi}{\partial y} \frac{d\mu}{dx} \right) \dot{y}^2 \dot{x} \\
& + \left(2 \frac{\partial \xi}{\partial z} \frac{d\mu}{dx} - 2 \frac{\partial^2 \xi}{\partial x \partial z} - 2 \frac{\partial \xi}{\partial z} \frac{d\mu}{dx} \right) \dot{z} \dot{x} \dot{y} - \frac{\partial^2 \xi}{\partial t \partial t} \dot{t}^2 \dot{y} - \frac{\partial^2 \xi}{\partial z \partial z} \dot{z}^2 \dot{y} \\
& - 2 \frac{\partial^2 \xi}{\partial t \partial y} \dot{y}^2 \dot{t} - 2 \frac{\partial^2 \xi}{\partial t \partial z} \dot{y} \dot{z} \dot{t} - \frac{\partial^2 \xi}{\partial y \partial y} \dot{y}^3 - 2 \frac{\partial^2 \xi}{\partial z \partial y} \dot{y}^2 \dot{z} = 0
\end{aligned} \tag{19}$$

$$\begin{aligned}
& \frac{\partial^2 \eta^2}{\partial s \partial s} = 0, +2 \frac{\partial^2 \eta^2}{\partial s \partial t} = 0, 2 \frac{\partial^2 \eta^2}{\partial s \partial x} = 0, 2 \frac{\partial^2 \eta^2}{\partial s \partial y} - \frac{\partial^2 \xi}{\partial s^2} + \frac{\partial \eta^1}{\partial s} \frac{d\mu}{dx} + \frac{\partial \eta^2}{\partial s} \frac{d\mu}{dx} = 0, \\
& \frac{\partial^2 \eta^2}{\partial s \partial z} = 0, \frac{\partial^2 \eta^2}{\partial t \partial t} + \frac{1}{2} e^{v(x)} \frac{dv}{dx} \frac{\partial \eta^2}{\partial x} - \frac{1}{2} e^{v(x)} \frac{dv}{dx} \frac{\partial \xi}{\partial x} = 0, + \frac{\partial^2 \eta^2}{\partial x \partial x} = 0, \\
& \frac{\partial^2 \eta^2}{\partial x \partial y} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial x} - 2 \frac{\partial^2 \xi}{\partial y \partial s} + \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} + \frac{\partial \eta^1}{\partial y} \frac{d\mu}{dx} + \frac{\partial \eta^2}{\partial y} \frac{d\mu}{dx} = 0, \\
& \frac{\partial^2 \eta^2}{\partial z \partial z} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial x} + \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} = 0, 2 \frac{\partial^2 \eta^2}{\partial t \partial x} - \frac{dv}{dx} \frac{\partial \eta^2}{\partial t} + \frac{dv}{dx} \frac{\partial \xi}{\partial t} = 0, \\
& 2 \frac{\partial^2 \eta^2}{\partial t \partial y} - 2 \frac{\partial^2 \xi}{\partial t \partial s} + \frac{\partial \eta^1}{\partial t} \frac{d\mu}{dx} + \frac{\partial \eta^2}{\partial t} \frac{d\mu}{dx} = 0, +2 \frac{\partial^2 \eta^2}{\partial t \partial z} = 0, \\
& 2 \frac{\partial^2 \eta^2}{\partial x \partial y} - \frac{d\mu}{dx} \frac{\partial \eta^2}{\partial y} - 2 \frac{\partial^2 \xi}{\partial x \partial s} + \frac{d\mu}{dx} \frac{\partial \xi}{\partial y} + \eta^1 \frac{d^2 \mu}{dx^2} + \frac{\partial \eta^1}{\partial x} \frac{d\mu}{dx} + \frac{\partial \eta^2}{\partial x} \frac{d\mu}{dx} = 0, \\
& 2 \frac{\partial^2 \eta^2}{\partial x \partial z} + \frac{d\mu}{dx} \frac{\partial \xi}{\partial z} = 0, 2 \frac{\partial^2 \eta^2}{\partial y \partial z} - 2 \frac{\partial^2 \xi}{\partial z \partial s} + \frac{\partial \eta^1}{\partial z} \frac{d\mu}{dx} = 0, \\
& -2 \frac{\partial^2 \xi}{\partial t \partial x} = 0, \frac{\partial^2 \xi}{\partial x \partial x} = 0, 2 \frac{\partial^2 \xi}{\partial x \partial y} = 0, \\
& 2 \frac{\partial \xi}{\partial z} \frac{d\mu}{dx} - 2 \frac{\partial^2 \xi}{\partial x \partial z} - 2 \frac{\partial \xi}{\partial z} \frac{d\mu}{dx} = 0, \frac{\partial^2 \xi}{\partial t \partial t} = 0, \frac{\partial^2 \xi}{\partial z \partial z} = 0, \\
& \frac{\partial^2 \xi}{\partial t \partial y} = 0, \frac{\partial^2 \xi}{\partial t \partial z} = 0, \frac{\partial^2 \xi}{\partial y \partial y} = 0, \frac{\partial^2 \xi}{\partial z \partial y} = 0
\end{aligned} \tag{19}$$

$$\begin{aligned}
& \frac{\partial^2 \eta^3}{\partial s^2} + 2 \frac{\partial^2 \eta^3}{\partial t \partial s} \dot{t} + \left(2 \frac{\partial^2 \eta^3}{\partial x \partial s} + \frac{\partial \eta^3}{\partial s} \frac{d\mu}{dx} \right) \dot{x} + 2 \frac{\partial^2 \eta^3}{\partial y \partial s} \dot{y} + \left(2 \frac{\partial^2 \eta^3}{\partial z \partial s} - \frac{\partial^2 \xi}{\partial s^2} + \frac{\partial \eta^1}{\partial s} \frac{d\mu}{dx} \right) \dot{z} \\
& + \left(\frac{\partial^2 \eta^3}{\partial t \partial t} + \frac{1}{2} e^{\nu(x)} \frac{dv}{dx} \frac{\partial \eta^3}{\partial x} \right) \dot{t}^2 + \left(\frac{\partial^2 \eta^3}{\partial x \partial x} + \frac{\partial \eta^3}{\partial x} \frac{d\mu}{dx} \right) \dot{x}^2 \\
& + \left(\frac{\partial^2 \eta^3}{\partial y \partial y} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^3}{\partial x} \right) \dot{y}^2 \\
& + \left(-\frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^3}{\partial x} - 2 \frac{\partial^2 \xi}{\partial z \partial s} + \frac{\partial \eta^1}{\partial z} \frac{d\mu}{dx} + \frac{\partial^2 \eta^3}{\partial z \partial z} \right) \dot{z}^2 \\
& + \left(2 \frac{\partial^2 \eta^3}{\partial t \partial x} - \frac{dv}{dx} \frac{\partial \eta^3}{\partial t} + \frac{\partial \eta^3}{\partial t} \frac{d\mu}{dx} \right) \dot{x} \dot{t} + 2 \frac{\partial^2 \eta^3}{\partial t \partial y} \dot{y} \dot{t} + \left(2 \frac{\partial^2 \eta^3}{\partial t \partial z} - 2 \frac{\partial^2 \xi}{\partial t \partial s} + \frac{\partial \eta^1}{\partial t} \frac{d\mu}{dx} \right) \dot{t} \dot{z} \\
& + \left(2 \frac{\partial^2 \eta^3}{\partial x \partial y} - \frac{d\mu}{dx} \frac{\partial \eta^3}{\partial y} + \frac{\partial \eta^3}{\partial y} \frac{d\mu}{dx} \right) \dot{x} \dot{y} \\
& + \left(2 \frac{\partial^2 \eta^3}{\partial x \partial z} - \frac{d\mu}{dx} \frac{\partial \eta^3}{\partial z} + 2 \frac{\partial \xi}{\partial s} \frac{d\mu}{dx} + \eta^1 \frac{d^2 \mu}{dx^2} - 2 \frac{\partial \xi}{\partial s} \frac{d\mu}{dx} \right. \\
& \left. - 2 \frac{\partial^2 \xi}{\partial x \partial s} + \frac{\partial \eta^1}{\partial x} \frac{d\mu}{dx} + \frac{\partial \eta^3}{\partial z} \frac{d\mu}{dx} \right) \dot{x} \dot{z} \\
& + \left(2 \frac{\partial^2 \eta^3}{\partial z \partial y} - 2 \frac{\partial^2 \xi}{\partial y \partial s} + \frac{\partial \eta^1}{\partial y} \frac{d\mu}{dx} \right) \dot{y} \dot{z} \\
& + \left(2 \frac{\partial \xi}{\partial t} \frac{d\mu}{dx} - 2 \frac{\partial^2 \xi}{\partial t \partial x} + \frac{dv}{dx} \frac{\partial \xi}{\partial t} - 2 \frac{\partial \xi}{\partial t} \frac{d\mu}{dx} \right) \dot{x} \dot{t} \dot{z} - 2 \frac{\partial^2 \xi}{\partial t \partial y} \dot{y} \dot{t} \dot{z} \\
& + \left(2 \frac{\partial \xi}{\partial x} \frac{d\mu}{dx} - \frac{\partial^2 \xi}{\partial x \partial x} - 2 \frac{\partial \xi}{\partial x} \frac{d\mu}{dx} \right) \dot{x}^2 \dot{z} \\
& + \left(2 \frac{\partial \xi}{\partial y} \frac{d\mu}{dx} - 2 \frac{\partial^2 \xi}{\partial x \partial y} + \frac{d\mu}{dx} \frac{\partial \xi}{\partial y} - 2 \frac{\partial \xi}{\partial y} \frac{d\mu}{dx} + \frac{d\mu}{dx} \frac{\partial \xi}{\partial z} \right) \dot{y} \dot{x} \dot{z} \\
& + \left(\frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial y \partial y} \right) \dot{y}^2 \dot{z} - 2 \frac{\partial^2 \xi}{\partial z \partial y} \dot{y} \dot{z}^2 - 2 \frac{\partial^2 \xi}{\partial t \partial z} \dot{z}^2 \dot{t} \\
& + \left(-\frac{\partial^2 \xi}{\partial t \partial t} - \frac{1}{2} e^{\nu(x)} \frac{dv}{dx} \frac{\partial \xi}{\partial x} \right) \dot{z} \dot{t}^2 + \left(2 \frac{\partial \xi}{\partial z} \frac{d\mu}{dx} - 2 \frac{\partial^2 \xi}{\partial x \partial z} - 2 \frac{\partial \xi}{\partial z} \frac{d\mu}{dx} \right) \dot{z}^2 \dot{x} \\
& + \left(\frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial z \partial z} \right) \dot{z}^3 = 0
\end{aligned} \tag{20}$$

$$\begin{aligned}
\frac{\partial^2 \eta^3}{\partial s^2} &= 0, 2 \frac{\partial^2 \eta^3}{\partial t \partial s} = 0, 2 \frac{\partial^2 \eta^3}{\partial x \partial s} + \frac{\partial \eta^3}{\partial s} \frac{d\mu}{dx} = 0, 2 \frac{\partial^2 \eta^3}{\partial y \partial s} = 0, \\
2 \frac{\partial^2 \eta^3}{\partial z \partial s} - \frac{\partial^2 \xi}{\partial s^2} + \frac{\partial \eta^1}{\partial s} \frac{d\mu}{dx} &= 0, \frac{\partial^2 \eta^3}{\partial t \partial t} + \frac{1}{2} e^{\nu(x)} \frac{dv}{dx} \frac{\partial \eta^3}{\partial x} = 0, \\
\frac{\partial^2 \eta^3}{\partial x \partial x} + \frac{\partial \eta^3}{\partial x} \frac{d\mu}{dx} &= 0, \frac{\partial^2 \eta^3}{\partial y \partial y} - \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^3}{\partial x} = 0, \\
-\frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \eta^3}{\partial x} - 2 \frac{\partial^2 \xi}{\partial z \partial s} + \frac{\partial \eta^1}{\partial z} \frac{d\mu}{dx} + \frac{\partial^2 \eta^3}{\partial z \partial z} &= 0, \\
2 \frac{\partial^2 \eta^3}{\partial t \partial x} - \frac{dv}{dx} \frac{\partial \eta^3}{\partial t} + \frac{\partial \eta^3}{\partial t} \frac{d\mu}{dx} &= 0, 2 \frac{\partial^2 \eta^3}{\partial t \partial y} = 0, \\
2 \frac{\partial^2 \eta^3}{\partial t \partial z} - 2 \frac{\partial^2 \xi}{\partial t \partial s} + \frac{\partial \eta^1}{\partial t} \frac{d\mu}{dx} &= 0, 2 \frac{\partial^2 \eta^3}{\partial x \partial y} = 0, \\
2 \frac{\partial^2 \eta^3}{\partial x \partial z} + \eta^1 \frac{d^2 \mu}{dx^2} - 2 \frac{\partial^2 \xi}{\partial x \partial s} + \frac{\partial \eta^1}{\partial x} \frac{d\mu}{dx} &= 0, \\
2 \frac{\partial^2 \eta^3}{\partial z \partial y} - 2 \frac{\partial^2 \xi}{\partial y \partial s} + \frac{\partial \eta^1}{\partial y} \frac{d\mu}{dx} &= 0, -2 \frac{\partial^2 \xi}{\partial t \partial x} + \frac{dv}{dx} \frac{\partial \xi}{\partial t} = 0, \\
-2 \frac{\partial^2 \xi}{\partial t \partial y} &= 0, 3 \frac{\partial \xi}{\partial y} \frac{d\mu}{dx} - 2 \frac{\partial^2 \xi}{\partial x \partial y} - 2 \frac{\partial \xi}{\partial y} \frac{d\mu}{dx} + \frac{d\mu}{dx} \frac{\partial \xi}{\partial z} = 0, \\
\frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial y \partial y} &= 0, \frac{\partial^2 \xi}{\partial z \partial y} = 0, \frac{\partial^2 \xi}{\partial t \partial z} = 0, \frac{\partial^2 \xi}{\partial x \partial x} = 0, \\
-\frac{\partial^2 \xi}{\partial t \partial t} - \frac{1}{2} e^{\nu(x)} \frac{dv}{dx} \frac{\partial \xi}{\partial x} &= 0, 2 \frac{\partial^2 \xi}{\partial x \partial z} = 0, \frac{1}{2} e^{\mu(x)} \frac{d\mu}{dx} \frac{\partial \xi}{\partial x} - \frac{\partial^2 \xi}{\partial z \partial z} = 0.
\end{aligned} \tag{20}$$

3.2. Five Lie Symmetry Generators and Vector Fields of Plane Symmetry Static Space-Time

This system consists of nine unknowns ξ and η^i ($i = 0, 1, 2, 3$), ν, μ, λ , and G . Solutions of this system give the Lagrange equations along with the Lie symmetry corresponding to these Lagrange equations. One can easily write plane symmetries space-time, which are the exact solutions of the EFE. The Eqs.(17)-(20) yield the following solutions:

$$\xi = 1, \eta^0 = \eta^1 = \eta^2 = \eta^3 = 0, \quad Y_0 = \frac{\partial}{\partial s}; \tag{21}$$

$$\eta^0 = 1, \xi = \eta^1 = \eta^2 = \eta^3 = 0, \quad X_0 = \frac{\partial}{\partial t}; \tag{22}$$

$$\eta^2 = 1, \xi = \eta^0 = \eta^1 = \eta^3 = 0, \quad X_2 = \frac{\partial}{\partial y}; \tag{23}$$

$$\eta^3 = 1, \xi = \eta^0 = \eta^1 = \eta^2 = 0, \quad X_3 = \frac{\partial}{\partial z}; \tag{24}$$

$$\eta^3 = y, \eta^2 = -z, \xi = \eta^0 = \eta^1 = 0, \quad X_3 = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}. \tag{25}$$

In other words, when the metric parameter ν, μ, λ take any value, the plane symmetry static space-times possess at least five basic Lie symmetries.

3.3. Lie symmetry Theorem of Plane Symmetry Static Space-Time

Theorem: for the generators $\xi, \eta^0, \eta^1, \eta^2, \eta^3$ of infinitesimal transformation for plane symmetry static space-time that satisfies determining equations (13)-(16), if there is a gauge function $G(s, t, x, y, z)$, satisfies the following equation:

$$L\dot{\xi} + X^{(1)}(L) + \sum_{k=0}^3 Q_k (\dot{\eta}^k - \dot{q}_k \dot{\xi}) + \dot{G} = 0, \quad (26)$$

then the plane static space time possesses the conserved quantity:

$$I = L\xi + \sum_{k=0}^3 \frac{\partial L}{\partial \dot{q}_k} (\eta^k - \dot{q}_k \xi) + G = \text{const}, \quad (27)$$

where the Q_k are non-potential generalized forces in a plane symmetry static space-time. In this study, we take $Q_k=0$.

Proof:

$$\frac{dI}{ds} = \dot{L}\xi + L\dot{\xi} + \sum_{k=0}^3 \frac{d}{ds} \frac{\partial L}{\partial \dot{q}_k} (\eta^k - \dot{q}_k \xi) + \sum_{k=0}^3 \frac{\partial L}{\partial \dot{q}_k} (\dot{\eta}^k - \ddot{q}_k \xi - \dot{q}_k \dot{\xi}) + \dot{G},$$

Using Eq.(25) in this equation and making further simplification, we obtain

$$\frac{dI}{ds} = \sum_{k=1}^4 (\xi^k - \dot{q}_k \xi_0) \left(\frac{dL}{ds} \frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} - Q_k \right) = 0.$$

Equation (26) is called the structural equation of Lie symmetries of a plane symmetry static space-time.

3.4. Conserved Quantities of the Plane Symmetry Static Space-Time

The set Eqs.(20) –(24) of Lie symmetries for plane symmetry static space-time with a constant value of the gauge function. This is the minimal set of Lie symmetries for plane symmetric static space-times. If the metric parameters ν, μ, λ of this system are obtain in the following form:

$$(a) \quad \nu(x) = \ln \left(\frac{x}{a} \right)^2, \quad \mu(x) = \left(\frac{x}{b} \right);$$

$$(b) \quad \nu(x) = \left(\frac{x}{b} \right), \quad \mu(x) = \ln \left(\frac{x}{a} \right)^2;$$

$$(c) \quad \nu(x) = \left(\frac{x}{a} \right)^2, \quad \mu(x) = \ln \cosh^2 \left(\frac{x}{b} \right);$$

$$(d) \quad \nu(x) = \left(\frac{x}{a} \right)^2, \quad \mu(x) = \ln \cos^2 \left(\frac{x}{b} \right);$$

$$(e) \quad \nu(x) = \ln \cosh^2 \left(\frac{x}{b} \right), \quad \mu(x) = \left(\frac{x}{a} \right)^2;$$

$$(f) \quad \nu(x) = \ln \cos^2\left(\frac{x}{b}\right), \quad \mu(x) = \left(\frac{x}{a}\right)^2;$$

$$(g) \quad \nu(x) = \ln \cosh^2\left(\frac{x}{b}\right), \quad \mu(x) = \left(\frac{x}{a}\right);$$

$$(h) \quad \nu(x) = \ln \cos^2\left(\frac{x}{b}\right), \quad \mu(x) = \left(\frac{x}{a}\right);$$

$$(i) \quad \nu'(x) \neq 0, \nu(x) \neq \mu(x), \quad \mu(x) = \ln \cosh^2\left(\frac{x}{a}\right), \ln \cos^2\left(\frac{x}{a}\right);$$

$$(j) \quad \mu(x) = \ln \cosh^2\left(\frac{x}{a}\right), \ln \cos^2\left(\frac{x}{a}\right), \quad \nu(x) \neq \mu(x), \mu'(x) \neq 0;$$

$$(k) \quad \nu''(x) \neq 0, \quad \mu(x) = a \ln \frac{x}{a};$$

$$(l) \quad \nu(x) = 2 \ln\left(\frac{x}{a}\right), \quad \mu''(x) \neq 0, \mu(x) \neq a \ln \frac{x}{a}$$

$$(m) \quad \nu(x) \neq \mu(x), \nu'(x) \neq 0, \quad \mu''(x) \neq 0, \mu(x) \neq a \ln \frac{x}{a};$$

$$(n) \quad \nu''(x) \neq 0, \nu(x) \neq a \ln \frac{x}{a}, \quad \nu(x) \neq \mu(x), \mu'(x) \neq 0.$$

By using Lie's theorem, we can provide conserved quantities for the existence of a plane symmetry static space-time and provide an accurate solution for the system.

By substituting generators (21)-(25) into (26) and (27) respectively, we obtain that the plane symmetry static space-time possesses conserved quantities in the following forms:

$$\begin{aligned} X_0: \quad I_0 &= -e^{\nu(x)} \dot{t} = \text{const}; \\ X_1: \quad I_2 &= e^{\mu(x)} \dot{y} = \text{const}; \\ X_2: \quad I_3 &= e^{\mu(x)} \dot{z} = \text{const}; \\ X_3: \quad I_4 &= e^{\mu(x)} (z\dot{y} - y\dot{z}) = \text{const}; \\ Y_0: \quad I_4 &= e^{\nu(x)} \dot{t}^2 - \dot{x}^2 - e^{\mu(x)} (\dot{y}^2 + \dot{z}^2) = \text{const}. \end{aligned} \tag{28}$$

It has been shown that in these Lie symmetries X_0 , X_1 , X_2 , and X_3 are isometries and correspond to the conservation of energy, linear momentum in y -direction, linear momentum in z -direction, and angular momentum, and Y_0 is the symmetry corresponding to the Lagrange equation. It is important to note that all the isometries are independent of parameter s .

4. Lie Symmetry Classification of a Plane Symmetry Static Space-Time

4.1. The Plane Symmetry Static Space-Time With Six Lie Symmetries

From the five basic Lie symmetry groups of a plane symmetry static space-time, it can be observed that the metric parameters $\nu(x)$ and $\mu(x)$ can take any value; therefore, there may be infinitely many classes for five Lie symmetries. It can be seen that a plane symmetry static space-time

consists of an infinite number of classifications. However, we provide some examples of metrics with six Lie symmetries of a plane symmetry static space-time in section.

For the $\nu(x)$ and $\mu(x)$ of the plane symmetry static space-time with six Lie symmetries, we can give the following six forms:

a. If the metric parameters $\nu(x) = \frac{x}{a}, \mu(x) = \frac{x}{b}; a \neq b$, five symmetries **X0, X1, X2, X3, and Y0** are the same as given by Eqs.(21)-(25), and we take sixth symmetry as

$$\eta^0 = -\frac{t}{2a}, \eta^1 = 1, \eta^2 = -\frac{y}{zb}, \eta^3 = -\frac{z}{2b}, \xi = 0, X_4 = -\frac{t}{2a} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} - \frac{y}{2b} \frac{\partial}{\partial y} - \frac{z}{2b} \frac{\partial}{\partial z}. \quad (29)$$

The sixth conserved quantity corresponding to this system is written as

$$I_6 = \frac{t\dot{t}}{a} e^{\frac{x}{a}} - 2\dot{x} - \frac{e^{x/b}}{b} (y\dot{y} - z\dot{z}) = \text{const} \quad (30)$$

b. If the metric parameters $\nu(x) = 0, \mu(x) = 2 \ln \cosh^2 \frac{x}{a}, a \neq 0$; The symmetries **X0, X1, X2, X3, and Y0** of the system are the same as given in Eqs.(21)-(25), and the sixth symmetry along with the gauge function are

$$\eta^0 = 0, \xi = \eta^1 = \eta^2 = \eta^3 = 0; Y_1 = s \frac{\partial}{\partial t}. \quad (31)$$

This system possesses the sixth conserved quantity as

$$I_6 = 2(t - st) = \text{const} \quad (32)$$

c. If the metric parameters $\nu(x) = 0, \mu(x) = 2 \ln \cos^2 \frac{x}{a}, a \neq 0$; The symmetries **X0, X1, X2, X3, and Y0** of the system are the same as given in Eqs.(21)-(25) and the sixth symmetry along with the gauge function and the sixth conserved quantity also are Eqs.(30) and (31)

d. If the metric parameters $\nu(x) = 2 \ln \left(\frac{x}{a}\right)^2, \mu(x) = 2 \ln \left(\frac{x}{b}\right)^\alpha, \alpha \neq 0, \alpha \neq 2$, and we take the sixth symmetry as

$$\begin{aligned} \xi = s, \eta^1 = \frac{x}{2}, \eta^2 = \frac{2-\alpha}{4} y, \eta^3 = \frac{2-\alpha}{4} z, \eta^0 = 0; \\ Y_1 = s \frac{\partial}{\partial s} + \frac{x}{2} \frac{\partial}{\partial x} + \frac{2-\alpha}{4} y \frac{\partial}{\partial y} + \frac{2-\alpha}{4} z \frac{\partial}{\partial z}. \end{aligned} \quad (33)$$

This system possesses the sixth conserved quantity

$$I_6 = s \left[\left(\frac{x}{a} \right)^2 \dot{t}^2 - \dot{x}^2 - \left(\frac{x}{b} \right)^\alpha (\dot{y}^2 + \dot{z}^2) \right] + x\dot{x} + \frac{2-\alpha}{2} \left(\frac{x}{b} \right)^\alpha (y\dot{y} + z\dot{z}) = \text{const} \quad (34) \quad \text{e. If}$$

the metric parameters $\nu(x) = 2 \ln \left(\frac{x}{b}\right)^\alpha, \mu(x) = 2 \ln \left(\frac{x}{a}\right)^2, \alpha \neq 0, \alpha \neq 2$, the metric admits minimal set of Lie symmetries and following symmetry (homothety)

$$Y_1 = s \frac{\partial}{\partial s} + \frac{2-\alpha}{4} t \frac{\partial}{\partial t} + \frac{x}{2} \frac{\partial}{\partial x}. \quad (35)$$

The plane symmetry static space-time possesses the sixth conserved quantity

$$I_6 = s \left[\left(\frac{x}{b} \right)^2 \dot{t}^2 - \dot{x}^2 - \left(\frac{x}{a} \right)^\alpha (\dot{y}^2 + \dot{z}^2) \right] + x\dot{x} + \frac{2-\alpha}{2} \left(\frac{x}{a} \right)^\alpha t\dot{t} = \text{const} \quad (36)$$

f. If the metric parameters $\nu(x) = 2 \ln \left(\frac{x}{a} \right)^\beta$, $\mu(x) = 2 \ln \left(\frac{x}{b} \right)^\alpha$, $2 \neq \alpha \neq \beta \neq 2$, which it admits a scaling symmetry (homothety)

$$Y_1 = s \frac{\partial}{\partial s} + \frac{2-\beta}{4} t \frac{\partial}{\partial t} + \frac{x}{2} \frac{\partial}{\partial x} + \frac{2-\alpha}{4} y \frac{\partial}{\partial y} + \frac{2-\alpha}{4} z \frac{\partial}{\partial z}, \quad (37)$$

In addition to the five symmetries given in Eqs.(20)-(24). This system possesses the sixth conserved quantity

$$I_6 = s \left[\left(\frac{x}{a} \right)^\beta \dot{t}^2 - \dot{x}^2 - \left(\frac{x}{b} \right)^\alpha (\dot{y}^2 + \dot{z}^2) \right] + x\dot{x} + \frac{2-\beta}{2} \left(\frac{x}{a} \right)^\beta t\dot{t} + \frac{2-\alpha}{2} \left(\frac{x}{b} \right)^\alpha = \text{const} \quad (38)$$

4.2. The Plane Symmetry Static Space-Time with Seven Lie Symmetries

In this section, the classes for seven Lie symmetries are given. There are two classes of the plane symmetry static space-time that admit seven Lie symmetries

Class 1 if $\nu(x) = 2 \ln \cosh^2 \frac{x}{a}$, $\mu(x) = 2 \ln \cosh^2 \frac{x}{a}$, the Lagrange equations of plane symmetry static space-time are driven as

$$\begin{aligned} \ddot{t} &= -\frac{4}{a} \frac{1}{\cosh \frac{x}{a}} \left(1 - \cosh^2 \frac{x}{a} \right) \dot{x} \dot{t} = \alpha_0, \\ \ddot{x} &= \frac{4}{a} \cosh \frac{x}{a} \left(1 - \cosh^2 \frac{x}{a} \right) \dot{t}^2 - \frac{4}{a} \cosh \frac{x}{a} \left(1 - \cosh^2 \frac{x}{a} \right) (\dot{y}^2 + \dot{z}^2) = \alpha_1, \\ \ddot{y} &= -\frac{4}{a} \frac{1}{\cosh \frac{x}{a}} \left(1 - \cosh^2 \frac{x}{a} \right) \dot{x} \dot{y} = \alpha_2, \quad \ddot{z} = -\frac{4}{a} \frac{1}{\cosh \frac{x}{a}} \left(1 - \cosh^2 \frac{x}{a} \right) \dot{x} \dot{z} = \alpha_3. \end{aligned} \quad (39)$$

Equations (39) admits seven Lie symmetries, four of which are given in Eqs.(21)-(25) and the Eqs.(17)-(20) which also possess two additional Lie symmetries in the following forms:

$$X_5 = y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}, \quad X_6 = z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z}. \quad (40)$$

Conserved quantities corresponding to these symmetries are given as

$$I_5 = 2 \ln \cosh^2 \frac{x}{a} (t\dot{y} - y\dot{t}) = \text{const}, \quad I_6 = 2 \ln \cosh^2 \frac{x}{a} (t\dot{z} - z\dot{t}) = \text{const}. \quad (41)$$

Class 2 if $\nu(x) = 2 \ln \cos^2 \frac{x}{a}$, $\mu(x) = 2 \ln \cos^2 \frac{x}{a}$, the Lagrange equations of plane symmetry static space-time are driven as

$$\begin{aligned} \ddot{t} &= \frac{4}{a} \tan \frac{x}{a} \dot{x} \dot{t} = \alpha_0, \quad \ddot{x} = -\frac{2}{a} \sin \frac{2x}{a} \dot{t}^2 + \frac{2}{a} \sin \frac{2x}{a} (\dot{y}^2 + \dot{z}^2) = \alpha_1, \\ \ddot{y} &= \frac{4}{a} \tan \frac{x}{a} \dot{x} \dot{y} = \alpha_2, \quad \ddot{z} = \frac{4}{a} \tan \frac{x}{a} \dot{x} \dot{z} = \alpha_3 \end{aligned} \quad (42)$$

Equations (42) admits seven Lie symmetries, four of which are given in Eqs.(21)-(25) and the Eqs.(17)-(20) which also possess two additional Lie symmetries in the following forms:

$$X_5 = y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}, X_6 = z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z}. \quad (43)$$

Conserved quantities corresponding to these symmetries are given as

$$I_5 = 2 \ln \cos^2 \frac{x}{a} (ty - yt) = \text{const}, I_6 = 2 \ln \cos^2 \frac{x}{a} (tz - zt) = \text{const}. \quad (44)$$

Class 3 if $\nu(x) = 0, \mu(x) = 2 \ln \left(\frac{x}{b} \right)^\alpha$, $2 \neq \alpha \neq 0$, the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = 0 = \alpha_0, \ddot{x} = -\frac{2\alpha}{b} \left(\frac{x}{b} \right)^{\alpha-1} (\dot{y}^2 + \dot{z}^2) = \alpha_1, \ddot{y} = -\frac{2\alpha}{x} \dot{x}\dot{y} = \alpha_2, \ddot{z} = -\frac{2\alpha}{x} \dot{x}\dot{z} = \alpha_3. \quad (45)$$

Equations (45) admits seven Lie symmetries in clouding five basic Lie symmetries which are given in Eqs.(21)-(25) and two additional Lie symmetries are given as

$$Y_1 = s \frac{\partial}{\partial s} + \frac{t}{2} \frac{\partial}{\partial t} + \frac{x}{2} \frac{\partial}{\partial x} + \frac{2-\alpha}{4} y \frac{\partial}{\partial y} + \frac{2-\alpha}{4} z \frac{\partial}{\partial z}, Y_2 = s \frac{\partial}{\partial s}, G_2 = 2t. \quad (46)$$

Conserved quantities corresponding to two symmetries are given as

$$I_5 = s \left[\left(\frac{x}{b} \right)^\alpha \dot{t}^2 - \dot{x}^2 - \left(\frac{x}{b} \right)^\alpha (\dot{y}^2 + \dot{z}^2) \right] + x\dot{x} - t\dot{t} + \frac{2-\alpha}{2} \left(\frac{x}{b} \right)^\alpha (y\dot{y} + z\dot{z}) = \text{const}, \quad (47)$$

$$I_6 = t - s\dot{t} = \text{const}.$$

Class 4 if $\nu(x)$ is arbitrary, and $\nu(x) \neq a \ln \frac{x}{b}, a \neq 0, \nu''(x) \neq 0, \mu(x) = 0$, the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = -\frac{d\nu}{dx} \dot{x}\dot{t} = \alpha_0, \ddot{x} = \frac{1}{2} e^{\nu(x)} \frac{d\nu}{dx} \dot{t}^2 = \alpha_1, \ddot{y} = 0 = \alpha_2, \ddot{z} = 0 = \alpha_3, \quad (48)$$

Equations (48) admits the five basic Lie symmetries which are given in Eqs.(21)-(25) and two additional Lie symmetries are given as

$$Y_1 = s \frac{\partial}{\partial y}, G_1 = -2y, Y_2 = s \frac{\partial}{\partial z}, G_2 = -2z. \quad (49)$$

Conserved quantities corresponding to two additional Lie symmetries are

$$I_5 = s\dot{y} - y = \text{const}, I_6 = s\dot{z} - z = \text{const}. \quad (50)$$

4.3. The Plane Symmetry Static Space-Time with Eight Lie Symmetries

In this section, the classes of the eight Lie symmetries presented. There are three classes of plane symmetry static space-time that admit eight Lie symmetries

Class 1 If the metric parameters $\nu(x) = 2 \ln \left(\frac{x}{b} \right)^\alpha, \mu(x) = 2 \ln \left(\frac{x}{b} \right)^\alpha$, the Lagrange equations of plane symmetry static space-time are driven as

$$\begin{aligned}\ddot{t} &= -\frac{2\alpha}{x} \dot{x} \dot{t} = \alpha_0, \ddot{x} = \frac{2\alpha}{b} \left(\frac{x}{b}\right)^{\alpha-1} \dot{t}^2 - \frac{2\alpha}{b} \left(\frac{x}{b}\right)^{\alpha-1} (\dot{y}^2 + \dot{z}^2) = \alpha_1, \\ \ddot{y} &= -\frac{2\alpha}{x} \dot{x} \dot{y} = \alpha_2, \ddot{z} = -\frac{2\alpha}{x} \dot{x} \dot{z} = \alpha_3.\end{aligned}\quad (51)$$

This system possess eight Lie symmetries which including five basic forms of Lie symmetries are given by Eqs.(21)-(25) and three additional Lie symmetries are

$$\begin{aligned}X_5 &= y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}, X_6 = z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z}, \\ Y_1 &= s \frac{\partial}{\partial s} + \frac{2-\alpha}{4} t \frac{\partial}{\partial t} + \frac{x}{2} \frac{\partial}{\partial x} + \frac{2-\alpha}{4} y \frac{\partial}{\partial y} + \frac{2-\alpha}{4} z \frac{\partial}{\partial z}.\end{aligned}\quad (52)$$

The plane symmetries static space-time possess the three additional conserved quantities

$$\begin{aligned}I_5 &= 2 \left(\frac{x}{b}\right)^\alpha (\dot{y}t - yt\dot{y}) = \text{const}, \quad I_6 = 2 \left(\frac{x}{b}\right)^\alpha (\dot{z}t - zt\dot{z}) = \text{const}, \\ I_7 &= s \left[\left(\frac{x}{b}\right)^\alpha \dot{t}^2 - \dot{x}^2 - \left(\frac{x}{b}\right)^\alpha (\dot{y}^2 + \dot{z}^2) \right] - \frac{2-\alpha}{2} \left(\frac{x}{b}\right)^\alpha t\dot{t} + x\dot{x} \\ &+ \frac{2-\alpha}{2} \left(\frac{x}{b}\right)^\alpha (y\dot{y} + z\dot{z}) = \text{const}.\end{aligned}\quad (53)$$

Class 2 If the metric parameters $\nu(x)=0, \mu(x)=2\ln\left(\frac{x}{a}\right)^2$, the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = 0 = \alpha_0, \ddot{x} = -\frac{4}{a^2} x (\dot{y}^2 + \dot{z}^2) = \alpha_1, \ddot{y} = -\frac{4}{x} \dot{x} \dot{y} = \alpha_2, \ddot{z} = -\frac{4}{x} \dot{x} \dot{z} = \alpha_3 \quad (54)$$

This system possess eight Lie symmetries which including five basic forms of Lie symmetries are given by Eqs.(21)-(25) and three additional Lie symmetries are given as

$$Y_1 = s \frac{\partial}{\partial t}, Y_2 = s \frac{\partial}{\partial s} + \frac{t}{2} \frac{\partial}{\partial t} + \frac{x}{2} \frac{\partial}{\partial x}, Y_3 = s^2 \frac{\partial}{\partial s} + st \frac{\partial}{\partial t} + sx \frac{\partial}{\partial x}.\quad (55)$$

The plane symmetry static space-time possesses the three additional conserved quantities as

$$\begin{aligned}I_5 &= 2 \left(\frac{x}{b}\right)^\alpha (\dot{y}t - yt\dot{y}) = \text{const}, \quad I_6 = 2 \left(\frac{x}{b}\right)^\alpha (\dot{z}t - zt\dot{z}) = \text{const}, \\ I_7 &= s \left[\left(\frac{x}{b}\right)^\alpha \dot{t}^2 - \dot{x}^2 - \left(\frac{x}{b}\right)^\alpha (\dot{y}^2 + \dot{z}^2) \right] - \frac{2-\alpha}{2} \left(\frac{x}{b}\right)^\alpha t\dot{t} + x\dot{x} \\ &+ \frac{2-\alpha}{2} \left(\frac{x}{b}\right)^\alpha (y\dot{y} + z\dot{z}) = \text{const}.\end{aligned}\quad (56)$$

Class 3 If the metric parameters $\nu(x) = 2 \ln\left(\frac{x}{b}\right)^\alpha$, $\mu(x) = 0$, the Lagrange equations of the plane symmetry static space-time are driven as

$$\ddot{t} = -\frac{2\alpha}{x} \dot{x} \dot{t} = \alpha_0, \ddot{x} = \frac{2\alpha}{b} \left(\frac{x}{b}\right)^{\alpha-1} \dot{t}^2 = \alpha_1, \ddot{y} = 0 = \alpha_2, \ddot{z} = 0 = \alpha_3, \quad (57)$$

This system possesses eight Lie symmetries including five basic forms, which are given by Eqs.(21)-(25) and the three additional Lie symmetries are

$$Y_1 = s \frac{\partial}{\partial s} + \frac{2-\alpha}{4} t \frac{\partial}{\partial t} + \frac{x}{2} \frac{\partial}{\partial x} + \frac{y}{2} \frac{\partial}{\partial y} + \frac{z}{2} \frac{\partial}{\partial z}, Y_2 = s \frac{\partial}{\partial y}, G_2 = -2y, G_3 = -2z, Y_3 = s \frac{\partial}{\partial z}. \quad (58)$$

The plane symmetries static space-time possesses the three additional conserved quantities respectively as

$$I_5 = s \left[\left(\frac{x}{b}\right)^\alpha \dot{t}^2 - \dot{x}^2 - (\dot{y}^2 + \dot{z}^2) \right] + x \dot{x} - \frac{2-\alpha}{2} \left(\frac{x}{b}\right)^\alpha t \dot{t} + (y \dot{y} + z \dot{z}) = \text{const}, \quad (58)$$

$$I_6 = y - s \dot{y} = \text{const}, I_7 = z - s \dot{z} = \text{const}.$$

4.4. The Plane Symmetry Static Space-Time with Nine Lie Symmetries

In this section, the classes for nine Lie symmetries are given. There are five classes of the plane symmetry static space-time that admit nine Lie symmetries

Class 1 If the metric parameters $\nu(x) = 0$, $\mu(x) = \frac{x}{a}$, the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = 0 = \alpha_0, \ddot{x} = -\frac{1}{a} e^{\frac{x}{a}} (\dot{y}^2 + \dot{z}^2) = \alpha_1, \ddot{y} = -\frac{1}{a} \dot{x} \dot{y} = \alpha_2, \ddot{z} = -\frac{1}{a} \dot{x} \dot{z} = \alpha_3, \quad (59)$$

This system possesses nine Lie symmetries which including five basic forms of Lie symmetries are given by Eqs.(21)-(25) and four additional Lie symmetries are written as

$$X_5 = \frac{\partial}{\partial x} - \frac{y}{2a} \frac{\partial}{\partial y} - \frac{z}{2a} \frac{\partial}{\partial z}, X_7 = y \frac{\partial}{\partial x} + \left(-\frac{y^2}{4a} + \frac{z^2}{4a} + a e^{-\frac{x}{a}} \right) \frac{\partial}{\partial y} - \frac{yz}{2a} \frac{\partial}{\partial z}, \quad (60)$$

$$X_8 = z \frac{\partial}{\partial x} - \left(\frac{y^2}{4a} - \frac{z^2}{4a} + a e^{-\frac{x}{a}} \right) \frac{\partial}{\partial z} - \frac{yz}{2a} \frac{\partial}{\partial y}, Y_1 = s \frac{\partial}{\partial y}, G_1 = 2t.$$

The plane symmetries static space-time possesses the four additional conserved quantities respectively as

$$I_5 = 2\dot{x} - \frac{e^{x/a}}{a} (y\dot{y} - z\dot{z}) = \text{const};$$

$$I_6 = 2\dot{x}y + \frac{e^{x/a}}{2a} \left[\dot{y} (z^2 - y^2 + 4a^2 e^{-x/a}) - yz\dot{z} \right] = \text{const}; \quad (61)$$

$$I_7 = 2\dot{x}z + \frac{e^{x/a}}{2a} \left[\dot{z} (-z^2 + y^2 + 4a^2 e^{-x/a}) - yz\dot{y} \right] = \text{const}; I_8 = t - s\dot{t} = \text{const}.$$

Class 2 If the metric parameters $\nu(x) = \frac{x}{a}, \mu(x) = 0$, the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = -\frac{1}{a}\dot{x}\dot{t} = \alpha_0, \ddot{x} = \frac{1}{a}e^{x/a}\dot{t}^2 = \alpha_1, \ddot{y} = 0 = \alpha_2, \ddot{z} = 0 = \alpha_3. \quad (62)$$

which admits the following four symmetries, in which X_5 and X_6 are isometries, along with the minimal set

$$X_5 = \frac{\partial}{\partial x} - \frac{t}{2a} \frac{\partial}{\partial t}, X_6 = t \frac{\partial}{\partial x} - \left(\frac{t^2}{4a} + ae^{x/a} \right) \frac{\partial}{\partial t}, \quad (63)$$

$$Y_1 = s \frac{\partial}{\partial y}, G_1 = -2y, Y_2 = s \frac{\partial}{\partial z}, G_2 = -2z.$$

This plane symmetry static space-time possesses the four additional conserved quantities are

$$I_5 = \frac{t\dot{t}e^{x/a}}{a} - 2\dot{x} = \text{const}; I_6 = 2\dot{x}t + \left(t^2 e^{x/a} + 4a^2 \right) \frac{\dot{t}}{2a} = \text{const}; \quad (64)$$

$$I_7 = s\dot{y} - y = \text{const}; I_8 = s\dot{z} - z = \text{const}.$$

Class 3 If the metric parameters $\nu(x) = 2 \ln \cosh^2 \frac{x}{a}, \mu(x) = 0$, the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = -\frac{4}{a} \tanh \frac{x}{a} \dot{x}\dot{t} = \alpha_0; \ddot{x} = \frac{2}{a} \left(1 + 2 \cosh^2 \frac{x}{a} \right) \dot{t}^2 = \alpha_1; \quad (65)$$

$$\ddot{y} = 0 = \alpha_2; \ddot{z} = 0 = \alpha_3.$$

For this metric the plane symmetry static space-time possesses the four additional symmetries are given by

$$X_5 = -\tanh \frac{x}{a} \sin \frac{t}{a} \frac{\partial}{\partial t} + \cos \frac{t}{a} \frac{\partial}{\partial x}; X_6 = \tanh \frac{x}{a} \cos \frac{t}{a} + \sin \frac{t}{a} \frac{\partial}{\partial x}; \quad (66)$$

$$Y_1 = s \frac{\partial}{\partial y}, G_1 = -2y, Y_2 = s \frac{\partial}{\partial z}, G_2 = -2z.$$

This plane symmetry static space-time possesses the four additional conserved quantities are given respectively as

$$I_5 = \dot{t} \sinh \frac{x}{a} \sin \frac{t}{a} \cosh \frac{x}{a} + \dot{x} \cos \frac{t}{a} = \text{const}; I_7 = s\dot{y} - y = \text{const}; \quad (67)$$

$$I_6 = -\dot{t} \sinh \frac{x}{a} \cos \frac{t}{a} \cosh \frac{x}{a} + \dot{x} \sin \frac{t}{a} = \text{const}; I_8 = s\dot{z} - z = \text{const}.$$

Class 4 If the metric parameters $\nu(x) = 2 \ln \cos^2 \frac{x}{a}, \mu(x) = 0$, the Lagrange equations of plane symmetry static space-time are driven as

$$\ddot{t} = 4 \tan \frac{x}{a} \dot{x}\dot{t} = \alpha_0, \ddot{x} = -\frac{2}{a} \sin \frac{2x}{a} \dot{t}^2 = \alpha_1, \ddot{y} = 0 = \alpha_2, \ddot{z} = 0 = \alpha_3. \quad (68)$$

For this metric the plane symmetry static space-time possesses the four additional symmetries are given by

$$\begin{aligned}
 X_5 &= -\tanh \frac{x}{a} \sin \frac{t}{a} \frac{\partial}{\partial t} + \cos \frac{t}{a} \frac{\partial}{\partial x}; X_6 = \tanh \frac{x}{a} \cos \frac{t}{a} \frac{\partial}{\partial t} + \sin \frac{t}{a} \frac{\partial}{\partial x}; \\
 Y_1 &= s \frac{\partial}{\partial y}, G_1 = -2y, Y_2 = s \frac{\partial}{\partial z}, G_2 = -2z.
 \end{aligned}
 \tag{69}$$

This plane symmetry static space-time possesses the four additional conserved quantities are

$$\begin{aligned}
 I_5 &= \dot{t} \sin \frac{x}{a} \sin \frac{t}{a} \cos \frac{x}{a} + \dot{x} \cos \frac{t}{a} = \text{const}; I_7 = s\dot{y} - y = \text{const}; \\
 I_6 &= -\dot{t} \sin \frac{x}{a} \cos \frac{t}{a} \cos \frac{x}{a} + \dot{x} \sin \frac{t}{a} = \text{const}; I_8 = s\dot{z} - z = \text{const}.
 \end{aligned}
 \tag{70}$$

Class 5 If the metric parameters $\nu(x) = 2 \ln \left(\frac{x}{a} \right)^2$, $\mu(x) = 2 \ln \left(\frac{x}{b} \right)^2$, the Lagrange equations of

plane symmetry static space-time are driven as

$$\ddot{t} = -\frac{4}{x} \dot{x} \dot{t} = \alpha_0, \ddot{x} = \frac{4}{a^2} x \dot{t}^2 - \frac{4}{a^2} x (\dot{y}^2 + \dot{z}^2) = \alpha_1, \ddot{y} = -\frac{4}{x} \dot{x} \dot{y} = \alpha_2, \ddot{z} = -\frac{4}{x} \dot{x} \dot{z} = \alpha_3. \tag{71}$$

The symmetries **X0**, **X1**, **X2**, **X3**, and **Y0** of (71) are the same as given in Eqs.(21)-(25) and the four additional symmetries are given by

$$\begin{aligned}
 X_5 &= y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}; X_6 = z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z}; \\
 Y_1 &= s \frac{\partial}{\partial s} + \frac{x}{2} \frac{\partial}{\partial x}; Y_2 = s^2 \frac{\partial}{\partial s} + sx \frac{\partial}{\partial x}, G_2 = -2x^2.
 \end{aligned}
 \tag{72}$$

This plane symmetry static space-time possesses the four additional conserved quantities are written as

$$\begin{aligned}
 I_5 &= 2 \left(\frac{x}{b} \right)^2 (\dot{y}t - y\dot{t}) = \text{const}; I_6 = 2 \left(\frac{x}{b} \right)^2 (\dot{z}t - z\dot{t}) = \text{const}; \\
 I_7 &= s \left[\left(\frac{x}{b} \right)^2 \dot{t}^2 - \dot{x}^2 - \left(\frac{x}{b} \right)^2 (\dot{y}^2 + \dot{z}^2) \right] + x\dot{x} = \text{const}; \\
 I_8 &= s^2 \left[\left(\frac{x}{b} \right)^2 \dot{t}^2 - \dot{x}^2 - \left(\frac{x}{b} \right)^2 (\dot{y}^2 + \dot{z}^2) \right] + 2sx\dot{x} - 2x^2 = \text{const}.
 \end{aligned}
 \tag{73}$$

4.5. The Plane Symmetry Static Space-Time with Eleven Lie Symmetries

In this section, the classes for eleven Lie symmetries are given.

If the metric parameters $\nu(x) = \frac{x}{a}$, $\mu(x) = \frac{x}{a}$, the Lagrange equations of plane symmetry static space-time are driven by

$$\ddot{t} = -\frac{1}{a} \dot{x} \dot{t} = \alpha_0, \ddot{x} = \frac{1}{a} e^{\frac{x}{a}} \dot{t}^2 - \frac{1}{a} e^{\frac{x}{a}} (\dot{y}^2 + \dot{z}^2) = \alpha_1, \ddot{y} = -\frac{1}{a} \dot{x} \dot{y} = \alpha_2, \ddot{z} = -\frac{1}{a} \dot{x} \dot{z} = \alpha_3. \tag{74}$$

The symmetries **X0**, **X1**, **X2**, **X3**, and **Y0** of (74) are the same as given in Eqs.(21)-(25) and the six additional symmetries are given by

$$\begin{aligned}
X_5 &= -\frac{t}{2a} \frac{\partial}{\partial t} + \frac{\partial}{\partial x} - \frac{y}{2a} \frac{\partial}{\partial y} - \frac{z}{2a} \frac{\partial}{\partial z} t; X_6 = z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z}; X_7 = y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}; \\
X_8 &= y \frac{\partial}{\partial x} - \frac{yt}{2a} \frac{\partial}{\partial t} - \frac{yz}{2a} \frac{\partial}{\partial z} - \left(\frac{t^2}{4a} + \frac{y^2}{4a} - \frac{z^2}{4a} - ae^{-\frac{x}{a}} \right) \frac{\partial}{\partial y}; \\
X_9 &= z \frac{\partial}{\partial x} - \frac{zt}{2a} \frac{\partial}{\partial t} - \frac{zy}{2a} \frac{\partial}{\partial z} - \left(\frac{t^2}{4a} - \frac{y^2}{4a} + \frac{z^2}{4a} - ae^{-\frac{x}{a}} \right) \frac{\partial}{\partial z}; \\
X_{10} &= t \frac{\partial}{\partial x} - \frac{ty}{2a} \frac{\partial}{\partial y} - \frac{tz}{2a} \frac{\partial}{\partial z} - \left(\frac{t^2}{4a} + \frac{y^2}{4a} + \frac{z^2}{4a} + ae^{-\frac{x}{a}} \right) \frac{\partial}{\partial t}.
\end{aligned} \tag{75}$$

The six additional conserved quantities of the system are

$$\begin{aligned}
I_5 &= 2\dot{x} + \frac{e^{x/a}}{a} (t\dot{t} - y\dot{y} - z\dot{z}) = \text{const}; I_6 = 2e^{x/a} (t\dot{z} - z\dot{t}) = \text{const}; \\
I_7 &= 2e^{x/a} (t\dot{y} - y\dot{t}) = \text{const}; \\
I_8 &= 2\dot{x}y + \frac{e^{x/a}}{2a} \left[2y\dot{t}t - 2yz\dot{z} + \left(z^2 - y^2 - t^2 + 4a^2 e^{-\frac{x}{a}} \right) \dot{y} \right] = \text{const}; \\
I_9 &= 2\dot{x}z + \frac{e^{x/a}}{2a} \left[2z\dot{t}t - 2yz\dot{y} + \left(z^2 - y^2 - t^2 + 4a^2 e^{-\frac{x}{a}} \right) \dot{z} \right] = \text{const}; \\
I_{10} &= 2\dot{x}t + \frac{e^{x/a}}{2a} \left[-2y\dot{y}t - 2z\dot{z}t + \left(z^2 + y^2 + t^2 + 4a^2 e^{-\frac{x}{a}} \right) \dot{t} \right] = \text{const};
\end{aligned} \tag{76}$$

4.6. The Plane Symmetry Static Space-Time with Seventeen Lie Symmetries

If the metric parameters $\nu(x) = 0, \mu(x) = 0$, the plane symmetry static space-time is a Minkowski space-time, the Lagrange equations of the system are driven by

$$\ddot{t} = 0 = \alpha_0, \ddot{x} = 0 = \alpha_1, \ddot{y} = 0 = \alpha_2, \ddot{z} = 0 = \alpha_3. \tag{77}$$

The symmetries **X0**, **X1**, **X2**, **X3**, and **Y0** of Eqs. (77) are the same as those given in Eqs.(21)-(25) and the 12 additional symmetries are written in the following forms:

$$\begin{aligned}
X_5 &= y \frac{\partial}{\partial t} + t \frac{\partial}{\partial y}; X_6 = z \frac{\partial}{\partial t} + t \frac{\partial}{\partial z}; X_7 = x \frac{\partial}{\partial t} + t \frac{\partial}{\partial x}; X_8 = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}; \\
X_9 &= x \frac{\partial}{\partial z} - z \frac{\partial}{\partial x}; X_{10} = \frac{\partial}{\partial x}; Y_1 = s \frac{\partial}{\partial t}, G_1 = t; \\
Y_2 &= 2s \frac{\partial}{\partial s} + t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}; Y_3 = -s \frac{\partial}{\partial y}, G_3 = y; Y_4 = -s \frac{\partial}{\partial z}, G_4 = z; \\
Y_5 &= s^2 \frac{\partial}{\partial s} + st \frac{\partial}{\partial t} + sx \frac{\partial}{\partial x} + sy \frac{\partial}{\partial y} + sz \frac{\partial}{\partial z}, G_5 = t^2 - x^2 - y^2 - z^2; \\
Y_6 &= -s \frac{\partial}{\partial x}, G_6 = x,
\end{aligned} \tag{78}$$

where, **X5**, **X6**, **X7**, **X8**, **X9**, and **X10** are isometries and **Y2** is homothety. The twelve additional conserved quantities of the system are given as

$$\begin{aligned}
I_5 &= t\dot{y} - y\dot{t} = \text{const}; I_6 = t\dot{z} - z\dot{t} = \text{const}; I_7 = t\dot{x} - x\dot{t} = \text{const}; \\
I_8 &= x\dot{y} - \dot{x}y = \text{const}; I_9 = x\dot{z} - \dot{x}z = \text{const}; I_{10} = \dot{x} = \text{const}; I_{11} = t - s\dot{t} = \text{const}; \\
I_{12} &= s\left(\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2\right) - t\dot{t} + x\dot{x} + y\dot{y} + z\dot{z} = \text{const}; \\
I_{13} &= s\dot{y} + y = \text{const}; I_{14} = s\dot{z} - zy = \text{const}; \\
I_{15} &= s^2\left(\dot{t}^2 - \dot{x}^2 - \dot{y}^2 - \dot{z}^2\right) + 2s\left(-t\dot{t} + x\dot{x} + y\dot{y} + z\dot{z}\right) + t^2 - x^2 - y^2 - z^2 = \text{const}; \\
I_{16} &= s\dot{x} + x = \text{const}.
\end{aligned} \tag{79}$$

5. Conclusion

This article uses the method of analytical mechanics to study the gravitational field problem in four-dimensional space-time, and provides several important results: firstly, by introducing four-dimensional space-time coordinates as generalized coordinates and taking curve coordinates as independent variables, the Lagrangian equations of the plan symmetry static space-time are established; the secondly is to introduce transformation of the Lie groups and corresponding vector fields related to curve coordinates and four-dimensional space-time; thirdly, based on the invariance of the Lagrangian equations of the static space-time of plane symmetry under the transformation of the Lie group, the Lie symmetry determination equations (13) - (16) and a series of symmetry killing equations (17)-(20) for the four-dimensional static space-time are given; fourthly proposed and proved the Lie symmetry theorem for the plane symmetry static space-time, and provided the structural equation (26) for the existence of the gravitational fields and the form of conserved quantities (27).

We have classified four-dimensional plane symmetry static space-time using the Lagrange equations and Lie symmetry theorem of the plane symmetric gravitational field, and obtained some useful conclusions; 1. There are five basic Lie symmetries (21) - (25) and conserved quantities in (27) static four-dimensional space-time of the planar symmetry; 2. When the metric coefficients take six different forms, the static four-dimensional space-time of the planar symmetry has five basic Lie symmetries and an additional symmetry and conservation quantity; 3. When there are four different forms of metric coefficient regions, the planar symmetric four-dimensional space-time has five basic symmetries and two additional Lie symmetries and conserved quantities; 4. When the metric coefficients take three different forms, a plane symmetric gravitational field not only has five basic Lie symmetries, but also three additional Lie symmetries and conserved quantities; 5. When the metric coefficients take five different forms, the plane symmetric gravitational field has nine Lie symmetries and conserved quantities, including five basic Lie symmetries and four additional Lie symmetries and conserved quantities; 6. There is only one case where a static space-time of the plane symmetry has 11 Lie symmetries and conserved quantities, including 6 additional Lie symmetries and conserved quantities; 7. If the metric coefficients are all set to zero, a planar symmetric static space-time has 12 additional Lie symmetries and conserved quantities, as well as 5 basic Lie symmetries and conserved quantities in (28).

It should be noted that there are two basic methods to solve the Lie symmetry of differential equations. Firstly, the characteristic equation is given using the Lie symmetry of the equation, and then the first integral of the system is obtained by integration, as shown in reference [27]. The second is to solve the generator from the Killing equation of Lie symmetry, and use the Lie symmetry theorem provide the conserved quantity of the system's existence. The first method can provide non Noether conservation quantities and Noether form conservation quantities for the existence of the system; the second method provides the Noether form conservation for the existence of the system. This article adopts the second method to study the Lie symmetry properties of plane symmetric static spacetime, and uses the Lie symmetry theorem to find the Noether type conserved quantity that exists in the system. The system is classified using the Lie symmetry method and a series of results are obtained. The conclusion presented in this article is consistent with existing findings.

Acknowledgments: This work was supported by the National Natural Science Foundation of China (11872335), and supported by the Natural Science Foundation of Shandong Province of China (ZR2023MA070)

References

1. Kramer, D., Stephani, H., MacCallum, M. A. H., and Herlt, E., Exact solution of Einstein field equations, Cambridge University Press, 1980.
2. Ayesha Mahmood, Azad A. Siddiqui, Tooba Feroze, Non-static spherically symmetric exact solution of the Einstein-Maxwell field equations, Journal of the Korean Physical Society, 2017, 71:396–403
3. Feroze, T. Mahomed, F.M. and Qadir, A., The connection between isometries and symmetries of geodesic equations of the underlying spaces., Nonlinear Dynamics, 2006, 64: 65-74
4. Qadir, A. Ziad, M. The classification of spherically symmetric space-time. IL Nuovo Cimento B 1995, 110(3):1971-1996
5. Qadir, A. Ziad, M. Classification of cylindrically symmetric static spacetimes according to their Ricci collineations. General Relativity and Gravitation, 2003, 35(11):1927-1975
6. Foyster, G.M and McIntosh, C.B.G. The classification of some spherically symmetric spacetime metrics, Bulletin of the Australian Mathematical Society, 1973, 8(2):187-190
7. Bokhari, A.H., Kashif, A.R. and Qadir, A., Classification of curvature collineation of plane symmetric static space-times. J. Math. Phys. 2000, 41(4):2167-2172
8. Lie S., Theorie der transformationsgruppen, Teubner, Leipzig, 1893.
9. Ovisiannikov L V. Group analysis of difference equations. New York: Academic, 1982
10. Ibragimov, N.H. CRC Handbook of Lie group analysis of differential equations, Volume 1:symmetries, exact solutions and conservation laws; CRC Press: Boca Raton, FL, USA, 1994.
11. Ibragimov, N.H., Kovalev, V.F., Pustovalov V.V., Symmetries of integro- differential equations: A survey of methods illustrated by the Benny equations. Nonlinear Dyn. 2002, 28, 135–153.
12. Olver P.J., Applications of Lie group to differential equations, New York, Springer-Verlag, 1986
13. Bluman G.W., Kumei S., Symmetries and differential equations, New York, Springer-Verlag, 1989
14. Stephani, H., Differential equations their solution using symmetries, Cambridge University Press, 1990.
15. Ibragimov, N.H. Elementary Lie group analysis and ordinary differential equations, Wiley, Chichester, 1999.
16. E. Noether, Invariant variation problems, Transport Theory and Statistical Physics, 1971,1(3): 186207
17. Mei F X The application of Lie groups and Lie algebras to constrained mechanical systems (Beijing: Science Press) (1999) (*in Chinese*)
18. [17]. Qadir, A.; Ziad, M. The classification of spherically symmetric space-times. IL Nuovo Cimento B **1995**, 110,317–334.
19. [18]. Bokhari, A.H.; Kashif, A.R.; Qadir, A. Classification of curvature collineation of plane symmetric staticspace-times. J. Math. Phys. **2000**, 41, 2167–2172.
20. [19]. Feroze, T.; Qadir, A.; Zaid, M. The classification of plane symmetric static space time by isometries.J. Math. Phys. **2001**, 42, 4947–4955.
21. [20]. Bokhari, A.H.; Kashif, A.R.; Qadir, A. A complete classification of curvature collineations of cylindrically symmetric static metrics. Gen. Relativ. Gravit. **2003**, 35, 1059–1076.
22. [21]. Foyster, G.M.; McIntosh, C.B.G. The classification of some spherically symmetric spacetime metrics.Bull. Aust. Math. Soc. **1973**, 8, 187–190.
23. [22]. Tupper, B.O.; Keane, A.J.; Carot, J. A classification of spherically symmetric spacetimes. Class. Quantum Gravity **2012**, 29, 145016.
24. [23]. Qadir, A.; Ziad, M. The classification of static cylindrically symmetric spacetime. IL Nuovo Cimento **1995**, 110,277–290.
25. Lutzky M., Dynamical symmetries and conserved quantities, J. Phys., A: Math. Gen., 1979,12(7):973-981
26. Djukić Dj. D. VujanovićB., Noether's theory in classical nonconservative mechanics, Acta Mechanics, 1975,23,17-27
27. Fu J L, Chen L Q. Non-Noether symmetries and conserved quantities of nonconservative dynamical systems. Phys Lett A, 2003, 317: 255–259
28. Guo Y X, Jiang L Y, Yu Y. Symmetries of mechanical systems with nonlinear nonholonomic constraints. Chin Phys, 2001, 10: 181–185
29. Liu R W, Chen L Q. Lie symmetries and invariants of constrained Hamiltonian systems. Chin Phys, 2004, 13: 1615–1619
30. Chen X W, Li Y M. Perturbation to symmetries and adiabatic invariants of a type of nonholonomic singular system. Chin Phys, 2003, 12:1349–1353

31. Zhou S., Fu H., Fu J. L., Symmetry theories of Hamiltonian systems with fractional derivatives, *Science China: Physics, Mechanics & Astronomy*, 2011, 54 ,(10): 1847- 1853
32. Mei F. X. Lie symmetries and conserved quantities of holonomic systems with remainder coordinates , *J.BJT*,1998,7(1):26-31
33. Fu J. L., Sallalor J. and Tang Y. F. and Luis V., Construction of exact invariants of time-dependent linear nonholonomic dynamical systems, *Physics Letters A*, 2008,372,1555-1561
34. Fu J. L., Fu L.P., Chen B.Y., Sun Y. Lie symmetries and their inverse problems of nonholonomic Hamilton systems with fractional derivatives. *Physics Letters A* (2016)380, 1-2: 15–21
35. Fu J. L., Zhang L.J., Cao S., Xiang C. and Zao W.J.. A symplectic algorithm for constrained Hamiltonian systems, *Axioms*,2022,11,217
36. Cao S., Fu J. L., Symmetry theories for canonicalized equations of constrained Hamiltonian system, *Nonlinear Dynamics*, 2018,92 (4) 1947-1954
37. Cai P. P., Fu J. L., Guo Y. X., Lie symmetries and conserved quantities of the constraint mechanical systems on time scale, *Report on Mathematical Physics*, 2017,79(3): 279-298
38. Zhou S., Fu H., Fu J. L., Symmetry theories of Hamiltonian systems with fractional derivatives, *Science China: Physics, Mechanics & Astronomy*, 2011, 54 (10): 1847- 1853
39. Fu J. L., Zhang L. J., Khalique C. M., and Guo M. L., Motion equations and non-Noether symmetries of Lagrangian systems with the conformable fractional derivatives. *Thermal Science*, 2021, 25, (2B): 1365-1372
40. Zhou S., Fu H., Fu J. L., Symmetry theories of Hamiltonian systems with fractional derivatives, *Science China: Physics, Mechanics & Astronomy*, 2011, 54 (10): 1847- 1853
41. Fu J. L., Fu H., Liu R. W., Hojman conserved quantities of discrete mechanico– electrical systems constructed by continuous symmetries. *Physics Letters A* 2010, 374: 1812–1818
42. Fu J. L., Dai G. D., Jimenes Salvaolor and Tang Y. F., Discrete variational principle and first integrals for Lagrange-Maxwell mechanico-electrical systems, *Chinese Physics*, 2007, 16(3), 570-577
43. Fu J. L., Chen B. Y., Fu H., Zhao G. L. Liu R. W., and Zhu. Z. Y., Velocity-dependent symmetries and non-Noether conserved quantities of electromechanical systems, *Science China: Physics, Mechanics & Astronomy*, 2011,54 ,(2): 288–295.
44. Fu J. L., Lu X. D, Xiang C. and Guo Y. X., Lie group analysis method for wall climbing robot systems, *Indian J Phys.* 2022, 96: 4231–4243
45. Fu J. L., Xiang C., and Meng L., Algebraic structure and Poisson integral method of snake-like robot systems, *Frontiers of Physics*, 2021,9: 643016
46. Wang L L., Fu J. L, and Li L L. Fractional Hamilton’s canonical equations and Poisson theorem of mechanical systems with fractional factor. *Mathematics MDPI*, **2023**, 11, 1803
47. Xu H. and Fu J. Li. Lie group analysis for torsional vibration of serve motor driven feeder drive system, *Chinese Journal of Theoretical and Applied Mechanics*, 2023,59(9):2000-2009

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.