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Article

Quantum Effects on Cosmic Scales as an Alternative to Dark Matter and Dark Energy

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Abstract: The spin-torsion theory is a gauge theory approach to gravity that expands upon Einstein's general relativity (GR) by incorporating the spin of microparticles. In this study, we further develop the spin-torsion theory to examine spherically symmetric and static gravitational systems that involve free-falling macroscopic particles. We posit that the quantum spin of macroscopic matter becomes noteworthy on cosmic scales. We further assume that the Dirac spinor and Dirac equation adequately capture all essential physical characteristics of the particles and their associated processes. A crucial aspect of our approach involves substituting the constant mass in the Dirac equation with a scale function, allowing us to establish a connection between quantum effects and the scale of gravitational systems. This mechanism ensures that the quantum effect of macroscopic matter is scale-dependent and diminishes locally, a phenomenon not observed in microparticles. For any given matter density distribution, our theory predicts an additional quantum term, the quantum potential energy (QPE), within the mass expression. QPE induces time dilation, distance contraction, and thus mimics a gravitational well. When applied to cosmology, our theory yields a static cosmological model. The QPE serves as a counterpart to the cosmological constant introduced by Einstein to balance gravity in his static cosmological model. The QPE also offers a plausible explanation for the origin of the Hubble redshift (traditionally attributed to the universe's expansion). The predicted luminosity distance-redshift relation aligns remarkably well with SNe Ia data from the cosmological sample of SNe Ia. In the context of galaxies, QPE functions as the equivalent of dark matter. The predicted circular velocities align well with the rotation curve data from the SPARC (Spitzer Photometry and Accurate Rotation Curves database) sample. Importantly, our conclusions in this paper are reached through a conventional approach, with the sole assumption of the quantum effects of macroscopic matter on large scales, without the need for additional modifications or assumptions.

Keywords: alternative theories of gravity; dark matter; dark energy; galactic rotation curves

1. Introduction

The primary challenge facing the current theory of gravity, the Einstein-Newton theory, lies in the unresolved mysteries of dark matter and dark energy. To date, there is a lack of direct observational evidence confirming the existence of these enigmatic components. Extensive efforts are being made to address such a topic. On the theoretical front, much attention is directed towards modifying or extending the Einstein-Newton theory of gravity to align with astronomical observations. For instance, one can enhance the standard Lagrangian in General Relativity by incorporating higher order curvature corrections [4,5,28,31,32,40], or formulate non-linear Lagrangians [8,16]. Other relevant examples include Modified Newtonian Dynamics (MOND) [15,38] and its relativistic version [1], as well as conformal gravity [33,34].

In this paper, we explore the concept of quantum spacetime on cosmic scales as a potential alternative to dark matter and dark energy. This research represents a significant extension of our previous study [10]. Rather than relying on previous assumptions, we explicitly propose the

scale-dependent quantum properties of spacetime. This proposition is motivated by our interest in potentially moving away from the notion of a preferred absolute spacetime for the universe, as implied by General Relativity (GR) and illustrated in the standard Λ CDM cosmology.

Before going further, it is imperative to clarify our actual motivation in order to prevent any possible misunderstandings. For ease of references, we categorize the universe into three distinct classes based on spatial scales: the cosmic scale (encompassing galaxies to the entire universe), the macroscopic scale (ranging from everyday life to the size of the solar system), and the microscopic scale (where quantum mechanics becomes significant as is well known). According to the mainstream view, the quantum effect for objects at the macroscopic and cosmic scales can be safely disregarded due to their large mass. This is known as the classical limit. According to our assumption, however, the quantum effects are dependent on the scale of the gravitational system considered, and the "classical limit" is only reached at macroscopic scale. In other words, quantum effects are significant not only at microscopic scale but also at cosmic scale, and can be ignored only at macroscopic scale. Intuitively, when we study cosmology, observers who perceive distant galaxies in the universe, would expect them to exhibit the quantum behavior resembling that of microscopic particles, due to their distant locations. As for the galactic dynamics, the quantum effect should also be taken into account for galaxies, although in the central part it can be ignored according to our assumption.

We will illustrate the integration of our assumption with the Einstein equations. It is natural for us to generalise the Dirac theory to describe the matter content. We present a direct application of the covariant Dirac equation without any modifications to study radially symmetric and static gravitational systems, such as galaxies and the entire universe, along with an explanation of the underlying mechanism that allows us to do so. We need the stress-energy tensor derived from the Dirac theory formulated as a new kind of hydrodynamics. Prior to incorporating the stress-energy tensor derived from Dirac theory into the Einstein equation, it is crucial to explicitly distinguish between its spin-free component (representing the classical ideal fluid) and the spin component (depicting the quantum potential energy). We have observed that without this separation, as demonstrated in prior research [10], the resulting Einstein equation is unable to accurately depict the pressure-free ideal fluid when the spin is locally negligible. By successfully solving the Einstein field equations, we have discovered some important conclusions that have not appeared in the literature.

There are two approaches to interpreting quantum mechanics. The first is the standard view, which asserts that micro particles cannot have continuous trajectories and that non-commuting observables (such as position and momentum) must satisfy the uncertainty principle. The second approach is known as the causal interpretation, which posits that each microparticle has a continuous trajectory and replaces the uncertainty principle with the concept of quantum potential [2,27]. Both approaches yield equivalent predictions for experimental results. However, the causal interpretation, particularly in its geometric algebra (GA) version, provides profound insights into the quantum nature of spacetime and matter [20,22–24]. In this paper, similar to our previous work, we utilize the spacetime algebra (STA) [12,19,24] as our mathematical language. STA is constructed from a Minkowskian vector space and provides a straightforward geometric understanding of Dirac theory. It bridges a natural link to classical mechanics, and for expressing Dirac theory with observables, it offers enhanced computational efficiency and capabilities [22] when compared to the tensor analysis method [43]. The fundamentals of STA are outlined in Appendix A. In the STA version of Dirac theory, known as real Dirac theory, Hestenes identified the complex number present in the matrix version of Dirac theory as the spin plane [20]. Furthermore, since the complex number is always accompanied by the Planck constant $\hbar$, a rigorous derivation of the Schrodinger theory from the Pauli or Dirac theory implies that the Schrodinger equation describes an electron in an eigenstate of spin, rather than, as commonly believed, an electron without spin [21]. Correspondingly, there are two gauge theories of gravity concerning our research work: one is the tensor analysis approach [17], and the other is gauge theory gravity (GTG) developed by the Cambridge group using STA [29]. The two theories are nearly equivalent, but the latter is conceptually clearer and technically more powerful for comprehending and calculating

the challenges encountered in quantum mechanics and gravity. Therefore, we choose to adopt GTG in this paper. The fundamentals of GTG and its applications to spin- $\frac{1}{2}$ particles are summarized in Appendix C.

It is now widely acknowledged that the quantum random motion of spin- $\frac{1}{2}$ particles can be fully described by its spin. In the presence of gravity, the spin gives rise to the torsion of spacetime [12,13,17,25,26,29]. For massive fermions, such as electrons and neutrons, the mass will always appear in the phase factor of the solutions of the wave equations. When gravity matters, this mass dependence remains. Consequently, it is generally believed that the gravity effect is not purely geometric at the quantum level. Hence the immediate question is, how can we reconcile, if possible, the contradiction between the quantum effect and the requirement of geometric description of gravity? For microparticles, reconciling this contradiction is indeed impossible. However, in this study, we propose that the scale-dependent quantum effect on macroscopic matter only becomes significant on cosmic scales. Locally, on a macroscopic scale, the large mass m of each particle, or equivalently, the small value of the number density ρ of the fluid, guarantees what is known as the classical limit, allowing for a geometric description of the gravitational effect on macroscopic matter. The remaining question is, what mechanism ensures that quantum effects are significant at cosmic scales while being negligible at macroscopic scales?

The answer to this question is not intricate, but rather subtle. The specifics will be provided later, but for now, we will outline the main concepts behind the mechanism. Our analysis commences with an examination of the Dirac theory within various inertial frames [22]. Generally, the Lorentz invariant Dirac spinor is defined in spacetime as

$$\psi(x) = \rho^{1/2} e^{i\beta(x)/2} R(x). \quad (1)$$

where $\rho(x)$ is a scalar, represents the proper probability density and $R(x)$ is a rotor (Lorentz rotation) satisfying $R\tilde{R} = 1$. In Dirac theory, the parameter $\beta(x)$ is intriguing. It is noteworthy that the states of plane-wave particle have $\beta = 0$, while those of antiparticle have $\beta = \pi$. In this paper, we aim to utilize the spinor $\psi(x)$ for describing free-falling macroscopic particles, hence it is logical to set $\beta = 0$ throughout. Therefore, we adopt [10]

$$\psi(x) = \rho(x)^{1/2} R(x), \quad (2)$$

The Dirac equation is

$$\hbar \nabla \psi i \gamma_3 = m \psi. \quad (3)$$

where $\hbar$ is Planck constant. This equation describes a single spin- $\frac{1}{2}$ free particle with a fixed mass m and the probability density ρ . If we interpret $\rho = \psi \tilde{\psi}$ as the number density of a fluid, then we refer to $\rho_m = m \psi \tilde{\psi}$ as the mass density. It is convenient to define a spin density trivector as (note that it is denoted with S_3 in Appendix B)

$$S = \frac{\hbar}{2} \psi i \gamma_3 \tilde{\psi} = \frac{\hbar}{2} \rho R i \gamma_3 \tilde{R}. \quad (4)$$

We show in Appendix B that, the stress-energy tensor for the spinor field is

$$T(a) = \rho_m a \cdot v v + [a \cdot \nabla (Sv)] \cdot v - (a \wedge \nabla) \cdot (Sv) v, \quad (5)$$

where v is the current velocity. When $S = 0$ (this is the classical limit), this equation becomes the stress-energy tensor of a pressureless ideal fluid without spin

$$T(a) = \rho_m a \cdot v v. \quad (6)$$

When applying these results to gravitational systems, we simply need to replace ∇ with the covariant derivative symbol D , while all other quantities remain gauge-invariant. As such, the classical limit (also

known as the short wave approximation) refers to the fact that the contribution of the spin to the stress-energy is negligible. Clearly, the magnitude of the spin $|S| = \frac{\hbar}{2}\rho$ is determined by the number density ρ . Therefore, an alternative interpretation for the classical limit is that the value of the number density, ρ , is ignorably low, whereas the significant quantum effect arises when ρ is sufficiently high. For any fixed mass density $\rho_m(x)$, the spin $|S|$ can be determined via $\rho(x) = \rho_m(x)/m$. This freedom strongly indicates that in order to incorporate the scale-dependent quantum effect on scales, we can replace the mass m in the Dirac equation with a mass function that depends on the spatial scale λ of the system, denoted as $m(\lambda)$. The mass function $m(\lambda)$ naturally satisfies the condition that, when $\lambda \rightarrow 0$, then $S \rightarrow 0$, corresponding to the macroscopic scale. Conversely, when $\lambda \rightarrow \infty$, then S becomes a λ dependent quantity related to the constant density of the universe. The intermediate functional form of $m(\lambda)$, which plays a crucial role in the study of galaxies, can be determined through observations. Consequently, we can employ the Dirac equation in the presence of gravity without any modifications, as there are no derivatives of $m(\lambda)$ involved in our calculations. The properties of the mass function $m(\lambda)$ only need to be discussed in the final results.

It is important to explicitly acknowledge that when applying the Dirac equation to gravitational systems on cosmic scales, we shall show in section 3 that the anisotropy of the spin density adheres to the symmetric properties of mass density $\rho_m(x)$. This fact applies only to macroscopic matter according to our assumption, where the gravitational effect is geometric throughout the spacetime manifold and, furthermore, the mass density and the spin density have already been averaged and must satisfy the condition that the quantum effect vanishes locally. On the other hand, when considering electrons, the anisotropy of spin must be taken into account in all cases, as the gravitational effect is not purely geometric. Thus the quantum effect can never disappear locally, as is commonly understood [6,7,9,42].

The subsequent sections present the examination of radially-symmetric and static gravitational systems based on our new postulation in Section 2, the applications of our findings to cosmology and galaxies in Section 3, and the summarized conclusions and discussions in Section 4. We employ natural units ($G = \hbar = c = 1$) throughout except stated otherwise.

2. radially Symmetric Gravitational Systems

We want to investigate the quantum nature of spacetime on cosmic scales, with the main subjects of concern being galactic dynamics and cosmology. Traditionally, the pressureless ideal fluid is a good model for both galactic dynamics and mater-dominated era cosmology. In both scenarios, matter experiences free-fall in a gravitational field and is characterized by the density distribution $\rho(x)$ and velocity field $v(x)$, where x represents the position vector in spacetime. This model is usually known as collapsing dust. In gravitational systems, it is widely recognized that a static state can only be sustained when there is a substantial gravitational potential well, as observed in galaxies. However, in the realm of cosmology, the homogeneous and isotropic distribution of matter across the entire universe renders static solutions non-existent.

Nevertheless, the introduction of quantum effects on cosmic scales fundamentally alters the situation.

The Dirac theory for radially symmetric gravitational systems is to be investigated using Gauge Theory Gravity (GTG). The latter is constructed such that the gravitational effects are described by a pair of gauge fields, $\bar{h}(a) = \bar{h}(a, x)$ and $\omega(a) = \omega(a, x)$, defined over a flat Minkowski background spacetime [29], where x is the STA position vector and is usually suppressed for short. Luckily, the majority of the necessary results for our present study have already been derived in prior works [13,29]. We have summarised them in Appendix C.

Let us consider a radially-symmetric and static gravitational system composed of free-falling particles with the identical mass m . We make the assumption that the Dirac spinor $\psi(r) = \rho(r)^{1/2}R(r)$ defined in (2) can fully capture all the essential physical aspects of the system if it satisfies the Dirac equation

$$D\psi i\gamma_3 = m\psi. \quad (7)$$

We interpret $\rho = \psi\tilde{\psi}$ as the number density and $\rho_m = m\psi\tilde{\psi}$ as the mass density. We adopt $S = \frac{1}{2}\psi i\gamma_3\tilde{\psi} = \frac{1}{2}\rho Ri\gamma_3\tilde{R}$ defined in (4) as the spin density trivector for the gravitational system. So there are four variables, namely m, ρ, ρ_m and S , that can be used to describe the gravitational system. However, only two of them are independent, as they must satisfy the conditions $\rho = \rho_m/m$ and $|S| = \frac{1}{2}\rho$. In this paper, we will eliminate m and retain the other three variables. Among them, the relationship between ρ_m and ρ can be further determined through observations by assuming a specific form for the mass function $m(\lambda)$. Once this is done, we are left with only one variable, which is the mass density ρ_m . Therefore, if we are given the mass density ρ_m of a gravitational system, we can predict the entire set of results, including the quantum effects.

Now we define a set of spherical coordinates. From the position vector of the flat spacetime

$$x = t\gamma_0 + r \sin \theta (\cos \phi \gamma_1 + \sin \phi \gamma_2) + r \cos \theta \gamma_3 \quad (8)$$

we obtain the basis vectors as follows

$$\begin{aligned} e_t &= \partial_t x = \gamma_0, \\ e_r &= \partial_r x = \sin \theta (\cos \phi \gamma_1 + \sin \phi \gamma_2) + \cos \theta \gamma_3, \\ e_\theta &= \partial_\theta x = r \cos \theta (\cos \phi \gamma_1 + \sin \phi \gamma_2) - r \sin \theta \gamma_3, \\ e_\phi &= \partial_\phi x = r \sin \theta (-\sin \phi \gamma_1 + \cos \phi \gamma_2). \end{aligned} \quad (9)$$

Since e_θ and e_ϕ are not unit, we define

$$\hat{\theta} \equiv e_\theta / r, \quad \hat{\phi} \equiv e_\phi / (r \sin \theta). \quad (10)$$

With these unit vectors, we further define the unit bivectors (relative basis vectors for $e_t = \gamma_0$)

$$\begin{aligned} \sigma_r &\equiv e_r e_t, \\ \sigma_\theta &\equiv \hat{\theta} e_t, \\ \sigma_\phi &\equiv \hat{\phi} e_t. \end{aligned} \quad (11)$$

These bivectors satisfy

$$\sigma_r \sigma_\theta \sigma_\phi = e_t e_r \hat{\theta} \hat{\phi} = i. \quad (12)$$

As in our previous paper [10], we attempted to analyze static systems initially. The set of $\bar{h}$ field that satisfies the spherically-symmetric and static matter distribution is assumed to take the form [12,29]

$$\begin{aligned} \bar{h}(e^t) &= f_1 e^t, \quad \bar{h}(e^r) = g_1 e^r + g_2 e^t, \\ \bar{h}(e^\theta) &= e^\theta, \quad \bar{h}(e^\phi) = e^\phi, \end{aligned} \quad (13)$$

where f_1, g_1 and g_2 are all function of r only. We could have tried a form $\bar{h}(e^t) = f_1 e^t + f_2 e^r$, but it is more reasonable to set f_2 to zero, which is referred to as the ‘Newtonian gauge’ [29].

The subsequent steps of this study can be summarized as follows:

$$\begin{aligned} \bar{h}(e^\mu) &\xrightarrow{D\wedge\bar{h}(a)=\kappa\bar{h}(a)\cdot S} \omega(a) \\ &\xrightarrow{\mathcal{R}(a\wedge b)=L_a\omega(b)-L_b\omega(a)+\omega(a)\times\omega(b)} \left\{ \begin{array}{l} \mathcal{R}(a) = \partial_b \cdot \mathcal{R}(b \wedge a) \\ \mathcal{R} = \partial_a \cdot \mathcal{R}(a) \end{array} \right\} \\ &\xrightarrow{\mathcal{G}(a)=\kappa\mathcal{T}(a)} g_{\mu\nu} = \underline{h}^{-1}(\mu) \cdot \underline{h}^{-1}(\nu), \end{aligned}$$

where the metric $g_{\mu\nu}$ is introduced to compare our results with those predicted by general relativity (GR) and to provide the conventional approaches for subsequent applications. Notably, $\omega(a)$ and $\mathcal{R}(a \wedge b)$

can be decomposed into torsion-free components and torsion components. This decomposition allows for the recovery of classical predictions of GR when the impact of torsion is insignificant. Conversely, given the extensive research on these classical predictions available in the existing literature, we can easily incorporate the torsion terms into the classical results when we deem them to be significant. This allows us to leverage the decompositions and enhance our understanding of the phenomena under consideration.

By solving Equation (C73), we can obtain the solution for $\omega(a)$ [29], the result is

$$\omega(a) = \omega'(a) + \frac{1}{2}\kappa a \cdot S, \quad (14)$$

where

$$\begin{aligned} \omega'(a) = & (a \cdot e_t G - a \cdot e_r F) e_r e_t - \left(\frac{g_2}{r}\right) a \cdot \hat{\theta} \hat{\theta} e_t \\ & - \left(\frac{g_1 - 1}{r}\right) a \cdot \hat{\theta} e_r \hat{\theta} - \left(\frac{g_2}{r}\right) a \cdot \hat{\phi} \hat{\phi} e_t \\ & - \left(\frac{g_1 - 1}{r}\right) a \cdot \hat{\phi} e_r \hat{\phi} \end{aligned} \quad (15)$$

denotes the torsion-free component of $\omega(a)$, and the new functions G and F are also all functions of r only. From the ω field, for any bivector B , its strength tensor can be obtained directly from Equation (C68) [13,29], the result is

$$\begin{aligned} \mathcal{R}(B) = & \mathcal{R}'(B) + \frac{1}{4}\kappa^2(B \cdot S) \cdot S \\ & - \frac{1}{2}\kappa(B \cdot D) \cdot S, \end{aligned} \quad (16)$$

where

$$\begin{aligned} \mathcal{R}'(B) = & \alpha_1 \sigma_r B \cdot \sigma_r + (\alpha_2 \sigma_\theta + \alpha_3 i \sigma_\phi) B \cdot \sigma_\theta \\ & + (\alpha_2 \sigma_\phi - \alpha_3 i \sigma_\theta) B \cdot \sigma_\phi + \alpha_6 \sigma_r (B \wedge \sigma_r) \\ & + (\alpha_4 \sigma_\theta - \alpha_5 i \sigma_\phi) (B \wedge \sigma_\theta) \\ & + (\alpha_4 \sigma_\phi + \alpha_5 i \sigma_\theta) (B \wedge \sigma_\phi) \end{aligned} \quad (17)$$

denotes the torsion free component of $R(B)$, with the understanding that $\sigma_r \wedge \sigma_\theta = \sigma_\phi \wedge \sigma_\theta = 0$, and $\alpha_1, \dots, \alpha_6$ are given by

$$\begin{aligned} \alpha_1 = & L_r G - L_t F + G^2 - F^2, \\ \alpha_2 = & -L_t \left(\frac{g_2}{r}\right) + \left(\frac{g_1}{r}\right) G - \left(\frac{g_2}{r}\right)^2, \\ \alpha_3 = & L_t \left(\frac{g_1}{r}\right) + \frac{g_1 g_2}{r^2} - \left(\frac{g_2}{r}\right) G, \\ \alpha_4 = & L_r \left(\frac{g_1}{r}\right) + \left(\frac{g_1}{r}\right)^2 - \left(\frac{g_2}{r}\right) F, \\ \alpha_5 = & L_r \left(\frac{g_2}{r}\right) + \frac{g_1 g_2}{r^2} - \left(\frac{g_1}{r}\right) F, \\ \alpha_6 = & (-g_2^2 + g_1^2 - 1)/r^2. \end{aligned} \quad (18)$$

From Eqs.(16) and (17), the Ricci tensor $\mathcal{R}(a)$ and Ricci scalar $\mathcal{R}$ are given by

$$\begin{aligned}\mathcal{R}(a) &= R'(a) + \frac{1}{2}\kappa^2(a \cdot S) \cdot S \\ &\quad - \frac{1}{2}\kappa a \cdot (D \cdot S), \\ \mathcal{R} &= \mathcal{R}' + \frac{3}{2}\kappa^2 S^2,\end{aligned}\tag{19}$$

where

$$\begin{aligned}\mathcal{R}'(a) &= [(\alpha_1 + 2\alpha_2)a \cdot e_t + 2\alpha_5 a \cdot e_r]e_t \\ &\quad + [2\alpha_3 a \cdot e_t - (\alpha_1 + 2\alpha_4)a \cdot e_r]e_r \\ &\quad - (\alpha_2 + \alpha_4 + \alpha_6)a \cdot \hat{\theta}\hat{\theta} \\ &\quad - (\alpha_2 + \alpha_4 + \alpha_6)a \cdot \hat{\phi}\hat{\phi},\end{aligned}\tag{20}$$

and

$$\mathcal{R}' = 2\alpha_1 + 4\alpha_2 + 4\alpha_4 + 2\alpha_6,\tag{21}$$

denote the torsion free components of $\mathcal{R}(a)$ and $\mathcal{R}$, respectively.

In order to solve Einstein equation Equation (C74), we need to know $T(a)$ given by Equation (C76). Noting that $\psi = \psi(r)$, we find that

$$\begin{aligned}\mathcal{T}(a) &= \langle a \cdot D\psi i\gamma_3 \tilde{\psi} \rangle_1 \\ &= \langle a \cdot \bar{h}(e^\mu) \partial_\mu \psi i\gamma_3 \tilde{\psi} + \frac{1}{2}\omega(a) \psi i\gamma_3 \tilde{\psi} \rangle_1 \\ &= a \cdot (g_1 e^r + g_2 e^t) \langle \partial_r \psi i\gamma_3 \tilde{\psi} \rangle_1 + \omega(a) \cdot S.\end{aligned}\tag{22}$$

The terms $\langle \partial_r \psi i\gamma_3 \tilde{\psi} \rangle_1$ should be derived from the Dirac equation (7). Nevertheless, as elaborated in our previous paper [10], the use of this stress-energy tensor prototype can be misleading when applied to gravitational systems consisting of macroscopic bodies. This is because, as S approaches zero, the tensor $T(a)$ defined in (22) also approaches zero. This contradicts our expectation that as S tends to zero, $T(a)$ should represent a tensor describing the pressureless ideal fluid.

An appropriate representation of the stress-energy tensor for our specific needs involves decomposing it into two components: one that characterizes the classical pressure-free ideal fluid and another that accounts for quantum effects. By substituting ∇ with D , we can rephrase equation (B32) and express $\mathcal{T}(a)$ as,

$$\mathcal{T}(a) = \rho_m a \cdot e_t e_t - (a \wedge D) \cdot (S \cdot e_t) e_t + [a \cdot D(S \cdot e_t)] \cdot e_t,\tag{23}$$

where we have replaced v with e_t due to the retained gauge freedom to perform arbitrary radial boosts in restricting the $\bar{h}$ function [12,29].

The Einstein equation is

$$\mathcal{G}(a) = R(a) - \frac{1}{2}a\mathcal{R} = \kappa\mathcal{T}(a).\tag{24}$$

A direct and efficient approach is to solve the Einstein equation separately for two scenarios: when $a = e_t$ and $a = e_r$. To be specific, in general, our results need to be derived from the following equations:

$$\begin{aligned} e_t \cdot \mathcal{G}(e_t) &= \kappa e_t \cdot \mathcal{T}(e_t), \\ e_r \cdot \mathcal{G}(e_t) &= \kappa e_r \cdot \mathcal{T}(e_t), \\ e_t \cdot \mathcal{G}(e_r) &= \kappa e_t \cdot \mathcal{T}(e_r), \\ e_r \cdot \mathcal{G}(e_r) &= \kappa e_r \cdot \mathcal{T}(e_r). \end{aligned} \quad (25)$$

However, we demonstrate that, for our intended purpose, solving the first two equations listed above is adequate. From (23) we get

$$e_t \cdot \mathcal{T}(e_t) = \rho_m - (e_t \wedge D) \cdot (S \cdot e_t). \quad (26)$$

Since $(e_t \wedge D) \cdot (S \cdot e_t) = (e_t \wedge \partial_a) \cdot [a \cdot D(S \cdot e_t)]$, we need to calculate $a \cdot D(S \cdot e_t)$ as follows

$$\begin{aligned} a \cdot D(S \cdot e_t) &= (a \cdot DS) \cdot e_t + S \cdot (a \cdot De_t) \\ &= [a \cdot \bar{h}(e^r) \partial_r S + \omega(a) \times S] \cdot e_t + S \cdot [\omega(a) \cdot e_t]. \end{aligned} \quad (27)$$

We thus have

$$\begin{aligned} (e_t \wedge \partial_a) \cdot [a \cdot D(S \cdot e_t)] &= (e_t \wedge \partial_a) \cdot [S \cdot (\omega(a) \cdot e_t)] \\ &= e_t \cdot \{[\partial_a \wedge (\omega(a) \cdot e_t)] \cdot S\} \\ &= e_t \cdot \{[-G\sigma_r + \kappa(S \cdot e_t)] \cdot S\} \\ &= \kappa(e_t \cdot S)^2 = \kappa S^2. \end{aligned} \quad (28)$$

Therefore,

$$e_t \cdot \mathcal{T}(e_t) = \rho_m - \kappa S^2. \quad (29)$$

Similarly, we have

$$\begin{aligned} e_r \cdot \mathcal{T}(e_t) &= [e_t \cdot D(S \cdot e_t)] \cdot (e_t \wedge e_r) \\ &= G(S \cdot e_r) \cdot (e_t \wedge e_r) \\ &= 0. \end{aligned} \quad (30)$$

On the other hand, for $\mathcal{G}(e_t)$, we find from (19), (20) and (21) that

$$\begin{aligned} \mathcal{G}(e_t) &= R(e_t) - \frac{1}{2} e_t R \\ &= 2\alpha_3 e_r - (2\alpha_4 + \alpha_6 + \frac{3}{4} \kappa^2 S^2) e_t \\ &\quad + \frac{1}{2} \kappa^2 (e_t \cdot S) \cdot S - \frac{1}{2} \kappa e_t \cdot (D \cdot S). \end{aligned} \quad (31)$$

Substituting (29), (30) and (31) into $e_t \cdot \mathcal{G}(e_t) = \kappa e_t \cdot \mathcal{T}(e_t)$ and $e_r \cdot \mathcal{G}(e_t) = \kappa e_r \cdot \mathcal{T}(e_t)$, we obtain

$$\begin{aligned} 2\alpha_4 + \alpha_6 &= -\kappa(\rho_m - \frac{3}{4} \kappa S^2) \\ \alpha_3 &= 0. \end{aligned} \quad (32)$$

Interestingly, from (20) we find

$$\partial_a \wedge \mathcal{R}'(a) = 2\alpha_5 \sigma_r - 2\alpha_3 \sigma_r = 0, \quad (33)$$

which gives

$$\alpha_5 = \alpha_3 = 0. \quad (34)$$

By substituting the expressions of α 's given by (18) into (32) and (34), our solutions to the Einstein equations can be summarized as follows:

$$\begin{aligned} 2 \left[g_1 \partial_r \left(\frac{g_1}{r} \right) + \left(\frac{g_1}{r} \right)^2 - F \left(\frac{g_2}{r} \right) \right] - \frac{g_2^2 - g_1^2 + 1}{r^2} &= -\kappa \left(\rho_m - \frac{3}{4} \kappa S^2 \right), \\ g_2 \partial_r \left(\frac{g_1}{r} \right) + \frac{g_1 g_2}{r^2} - G \frac{g_2}{r} &= 0, \\ g_1 \partial_r \left(\frac{g_2}{r} \right) + \frac{g_1 g_2}{r^2} - F \frac{g_1}{r} &= 0. \end{aligned} \quad (35)$$

Note that in deriving these equations, we employed $L_t = g_2 \partial_r$ instead of $L_t = f_1 \partial_t + g_2 \partial_r$ in (18). This choice was made because our subsequent study is focused on static gravitational systems.

The first and third equations in (35) can be combined to give

$$\partial_r [r(g_2^2 - g_1^2 + 1)] = \kappa \left(\rho_m - \frac{3}{4} \kappa S^2 \right) r^2. \quad (36)$$

Now if we define

$$M = \frac{r}{2} (g_2^2 - g_1^2 + 1), \quad (37)$$

we find (remembering $\kappa = 8\pi$):

$$M(r) = 4\pi \int_0^r \left(\rho_m(r') - \frac{3}{4} \kappa S^2(r') \right) r'^2 dr'. \quad (38)$$

The expression of M strongly implies that it can be identified as the mass (total energy) of a gravitational system within r at any given time t . Remarkably, the mass M contains the quantum effects, $-\frac{3}{4} \kappa S^2(r)$, as intended. Naturally, when $S = 0$, the conventional expression for M in general relativity is regained.

As mentioned earlier, there are three functions in the $\bar{h}$ field, namely f_1 , g_1 and g_2 , that need to be determined by solving Einstein equations. So far, we have obtained expressions for g_1 and g_2 , they are linked to the observables ρ_m and S . Thus, the determination of f_1 remains pending. To achieve this, we substitute $a = e^t$ into the torsion equation (C73), yielding the subsequent equations

$$\begin{aligned} D \wedge \bar{h}(e^t) &= \kappa \bar{h}(e^t) \cdot S, \\ \partial_a \wedge [a \cdot \bar{h}(e^\mu) \partial_\mu f_1 e^t + f_1 \omega(a) \cdot e^t] &= \kappa f_1 e_t \cdot S, \\ g_1 \partial_r f_1 &= -G f_1, \\ f_1 &= e^{-\int \frac{G}{g_1} dr}. \end{aligned} \quad (39)$$

On the other hand, from the last two equations of (35), it is easy to derive that

$$G = \partial_r g_1, \quad F = \partial_r g_2. \quad (40)$$

So we obtain immediately that

$$f_1 = 1/g_1. \quad (41)$$

Now we turn to the metric tensor $g_{\mu\nu} = g_\mu \cdot g_\nu = \underline{h}^{-1}(e_\mu) \cdot \underline{h}^{-1}(e_\nu)$. From (13) and (41), we obtain

$$\begin{aligned} \underline{h}^{-1}(e_t) &= g_1 e_t - g_2 e_r, & \underline{h}^{-1}(e_r) &= \frac{1}{g_1} e_r, \\ \underline{h}^{-1}(e_\theta) &= e_\theta, & \underline{h}^{-1}(e_\phi) &= e_\phi. \end{aligned} \quad (42)$$

So the metric tensor depends only on g_1 and g_2 . However, g_2^2 included in (37) served as the kinematic energy in gravitational systems [29]. To see this, we consider a radially free-falling particle with $v = e_t$. We have

$$\dot{x} = \frac{dx}{d\tau} = \dot{t}e_t + \dot{r}e_r = \underline{h}(v) = \underline{h}(e_t) = \frac{1}{g_1}e_t + g_2e_r. \quad (43)$$

Clearly, in this case $g_2 = \dot{r}$ represents the radial velocity of the particle. In general, we can understand the physical significance of g_2 by rewriting (37) as [29]

$$\frac{1}{2}(g_2)^2 - \frac{M}{r} = \frac{1}{2}((g_1)^2 - 1), \quad (44)$$

which is a Bernoulli equation for zero pressure and total non-relativistic energy $\frac{1}{2}((g_1)^2 - 1)$.

But for non-relativistic matter, the contribution of kinematic energy to gravity can be safely disregarded. So from (37) we have

$$g_1 = \left(1 - \frac{2M(r)}{r}\right)^{1/2}. \quad (45)$$

This paper specifically concentrates on non-relativistic matter within static gravitational systems, enabling us to express the metric as follows:

$$d\tau^2 = \left(1 - \frac{2M(r)}{r}\right)dt^2 - \left(1 - \frac{2M(r)}{r}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (46)$$

Remarkably, we have obtained a metric that bears a striking resemblance to the familiar form in general relativity. However, it is important to note that this metric was derived from Dirac theory, and the mass M incorporates the quantum effects arising from macroscopic matter.

In this paper, the term $-\frac{3}{4}\kappa S^2(r)$ in the mass-energy expression (38) is referred to as the "quantum potential energy". This concept was first introduced by Bohm [3] in 1952 and later by DeWitt [11] in the same year, with both authors commonly referring to it as the "quantum potential." [26]. Similar to how traditional mass-energy curves spacetime, quantum potential energy distorts spacetime. This phenomenon is known as the "spin-torsion" theory [17,29]. By explicitly decomposing the stress-energy tensor $\mathcal{T}(a)$ into a spin-free part and a spin part, as illustrated in (23), the gravitational strength, represented by Riemann tensor $\mathcal{R}(B)$, can be explicitly decomposed into a torsion-free component and a torsion component as shown in (16). This results in the two-component metric as outlined in (46). Naturally, when $S = 0$, the metric reduces to the one that describes a spherically-static gravitational system as suggested by GR.

It is crucial to grasp the physical significance of the quantum potential energy (QPE) in the metric. Up to date in the literature, QPE is attributed uniquely to microparticles. It has been confirmed that QPE would reduce to a time delation in spacetime. For instance, when negative muons are captured in atomic s-states their lifetimes are increased by a time dilation factor corresponding to the Bohr velocity. So we expect that, if this QPE induced time dilation is extended to the macroscopic matter on cosmic scales, it may disguise as some extra gravitational potential well. We will show that this extra time dilation can successfully explain the cosmic redshift and galactic rotation curve problem. For microparticles, the troubles encountered all arise from the fact that when QPE is significant, its gravitational effect is not geometric (i.e., mass-dependent). This results in the concept of the geodesic for test particles ambiguous. Therefore, as mentioned earlier, if we want to apply this metric to the gravitational systems composed of macroscopic particles so that QPE is significant only at cosmic scales and can be neglected locally, only if we can establish a mechanism that connects the quantum effects of macroscopic matter to the spatial scales of gravitational systems. The mechanism, however, can be clearly demonstrated in the applications of our theory to cosmology and galaxies, as illustrated in the following section.

3. Applications: Cosmology and Galaxies

To investigate the mechanism that connects the quantum effects of macroscopic matter to the spatial scales of gravitational systems, we begin by considering a radially-symmetric and static gravitational system consisting of N free-falling particles, each with an identical mass of m . We assume that the direct collisions and the close encounters due to gravity between the particles can be neglected. This allows us to approximate the gravitational effect of the particles by a smooth distribution of matter. Namely, we assume a smooth mass density function $\rho_m(r)$ and a smooth number density function $\rho(r)$ defined by

$$\rho_m(r) = m\rho(r) = \sum_{i=1}^N m\delta(r - r_i). \quad (47)$$

It is evident that the accuracy of the approximation improves as the mass m decreases and the particle number N increases. However, for a fixed mass density $\rho_m(r)$, reducing the mass m leads to an increase in the number density $\rho(r)$. On the other hand, the scale-dependence of a particle's mass in a gravitational system can be easily demonstrated. A particle with a mass of m contributes to the average mass density of a system with a size of λ through the following relation:

$$\rho(m, \lambda) \sim \frac{m}{\lambda^3}. \quad (48)$$

Clearly, from the perspective of average mass density, the "effective mass" of a particle decreases as the size of the system increases, following a trend of $\sim \lambda^{-3}$. While this may seem trivial, it becomes crucial in our understanding of quantum effects on cosmic scales. For example, in our Milky Way, $\lambda \sim 10$ kpc. The Sun ($m = M_\odot$) contributes to the average density of the Milky Way as an electron ($m = 9.1 \times 10^{-28} \text{g}$) contributes to a system of size $\lambda \sim 230$ cm, a macroscopic size. So as an assumption, we extend the quantum nature of electrons to macroscopic matter particles distributed in large enough systems.

In a gravitational system, the Dirac equation for a free-falling particle describes a particle of fixed mass m and the constant spin $\hbar/2$ with the probability density $\rho(r)$. We consider the probability density as the number density of the system. In the case of spin- $\frac{1}{2}$ microparticles, the mass m can vary from neutrinos ($m \approx 0$) to neutrons, while the spin remains constant at $\hbar/2$. Therefore, the spin of a particle is independent of its mass. This fact can be extended naturally to macroscopic particles. Typically, when dealing with large masses (macroscopic matter), the negligible quantum effect can be understood as the short wavelength approximation, commonly referred to as the classical limit. In the context of radially-symmetric gravitational systems, we can naturally interpret the negligible quantum effect as a result of the negligibly small number density of particles (since each particle has exactly the value spin $\hbar/2$). Given that reducing the mass m results in an increase in the number density $\rho(r)$, we can introduce a decreasing function $m(\lambda)$ (λ is the size of the gravitating system), which in turn leads to an increasing function $\rho(r)$. Consequently, the strength of the quantum effect would increase as the distance (or scale) to an observer increases, aligning with our postulation.

To be more precise, we define the density of QPE from (38) as (note that $S^2 = -\frac{1}{4}\rho^2$, and $\rho(r) = \rho_m(r)/m$)

$$\begin{aligned} \rho_Q(r, \lambda) &= -\frac{3}{4}\kappa S^2(r) \\ &= \frac{3}{16}\kappa \rho(r)^2 \\ &= \frac{3}{16}\kappa \left(\frac{\rho_m(r)}{m(\lambda)} \right)^2 \\ &= \frac{\rho_m^2(r)}{\rho_c} \alpha(\lambda), \end{aligned} \quad (49)$$

where $\alpha(\lambda)$ is a dimensionless parameter that depends on the scale λ . We require that the strength of QPE vanishes when $\lambda \rightarrow 0$ and increases with increasing λ . The specific properties of $\alpha(\lambda)$ need to be determined through observations. Note that in order to separate the size λ dependence of ρ_Q completely from coordinate r , we express $\rho_Q(r, \lambda)$ as proportional to $\rho_m^2(r)$, and the constant mass density ρ_c of the entire universe is introduced into the definition of ρ_Q only for dimensional consistency. This choice has nothing to do with our contemplation of the notion that the entire universe may exert a cosmological influence on local galaxies, as elaborated upon in subsequent sections. Needless to say, the functional form of $\alpha(\lambda)$ must be universal, meaning that it is independent of any particular gravitational system, although it can be different for the entire universe and galaxies.

From (38) we observe that the expression for M can be rewritten as

$$\begin{aligned} M(r) &= 4\pi \int_0^r (\rho_m(r') + \rho_Q(r', \lambda)) r'^2 dr' \\ &= 4\pi \int_0^r \rho_m(r') r'^2 dr' + \frac{4\pi\alpha(r)}{\rho_c} \int_0^r \rho_m^2(r') r'^2 dr' \\ &= M_m(r) + M_Q(r), \end{aligned} \quad (50)$$

where we have set $\lambda = r$ in the integrand to reflect the fact that, in $\rho_Q(r', \lambda)$, the scale λ is r , the size of the "system" with mass $M(r)$. And

$$M_m(r) = 4\pi \int_0^r \rho_m(r') r'^2 dr' \quad (51)$$

$$M_Q(r) = \frac{4\pi\alpha(r)}{\rho_c} \int_0^r \rho_m^2(r') r'^2 dr' \quad (52)$$

representing the conventional mass (energy) in general relativity and the QPE within r respectively. From (49) and (50), it can be seen clearly that the distribution of QPE exhibits no extra symmetries, i.e., it aligns with the mass distribution. The reason is that the scale λ characterizes only the amount of the QPE for macroscopic matter on cosmic scales. Therefore, as the scale increases, the amount of QPE also increases, causing the quantum effects to vanish locally. This is in contrast to electrons, whose QPE can never vanish at any scales, and thus result in the possible anisotropy if not averaged [6,7].

As previously mentioned, the functional form of $\alpha(\lambda)$ should be determined based on observations. Nonetheless, we have found that the following general form is a suitable choice:

$$\alpha(x) = \frac{A(e^x - 1)}{x^2} + B(e^x - 1) + C, \quad \text{with } x = H_0\lambda, \quad (53)$$

where H_0 represents the Hubble constant, which will be defined subsequently. We leave the coefficients A , B and C , possibly scale dependent, as free parameters to fit the observational data of the gravitational systems under consideration.

3.1. Cosmology

Now we are ready to study cosmology. Assuming a static universe with a homogeneous and isotropic matter distribution on large scales, characterized by a constant mass density ρ_c , the mass M in (50) then becomes

$$\begin{aligned} M(r) &= 4\pi \int_0^r \rho_c r'^2 dr' + 4\pi\alpha(r) \int_0^r \rho_c r'^2 dr' \\ &= \frac{1}{2} H_0^2 r^3 (1 + \alpha(r)), \end{aligned} \quad (54)$$

where

$$H_0^2 = \frac{8\pi\rho_c}{3} \quad (55)$$

is the Hubble constant. In cosmology, the entire universe can be regarded as an gravitational system with an arbitrary large scale. To compare the observations, we suggest setting the parameters in (53) as $A = 2$, $B = 0$ and $C = -1$. Namely, for cosmology, we assume

$$\alpha_c(\lambda) = \frac{2(e^{H_0\lambda} - 1)}{H_0^2\lambda^2} - 1, \quad (56)$$

which yields desirable results:

$$M(r) = r(e^{H_0r} - 1) = H_0r^2 + \frac{1}{2}H_0^2r^3 + \dots. \quad (57)$$

Before formulating the metric for the spacetime of the universe, it is important to highlight that the term $\frac{1}{2}H_0^2r^3$ resulting from ρ_c in $M(r)$ given in (54) should not be disregarded. One could argue that the universe is a distinct gravitational system, exhibiting a homogeneous and isotropic distribution of matter, and infinite in size. Consequently, the observation of gravitational redshift may not be possible under this circumstance. However, this conclusion is derived from Newtonian theory or general relativity, and does not hold true when quantum effects come into play. In fact, the term $\frac{1}{2}H_0^2r^3$ can be included as QPE if we choose $C = 0$ in (53). In any case, we should keep in mind that any choice of $\alpha_c(\lambda)$ must be subjected to testing against observations. Moreover, due to the presence of quantum effects on cosmic scales, we have to abandon the idea of absolute spacetime predicted by general relativity. This is quite similar to the case of special relativity, when Einstein had to relinquish the concept of Newtonian absolute space and time. As a result, the conventional notion of the geometry of the entire universe becomes meaningless. For instance, it loses significance to classify the universe as open, flat, or closed according to its mass density ρ_c . Thus for cosmology, we have from (45)

$$g_1 = \left(1 - \frac{2M(r)}{r}\right)^{1/2} = (3 - 2e^{H_0r})^{1/2}. \quad (58)$$

Hence the metric (46) for the universe becomes

$$d\tau^2 = (3 - 2e^{H_0r})dt^2 - (3 - 2e^{H_0r})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (59)$$

From this metric, it can be readily deduced that a horizon exists in the static universe, with the distance to any observer in the universe given by

$$r_h = \frac{1}{H_0} \ln(3/2) \approx \frac{0.4}{H_0}, \quad (60)$$

which is less than half a Hubble radius $1/H_0$.

As a crucial outcome of our theory, we now derive the Hubble redshift for the static universe model. Suppose a light signal emitted from the source at r and received by an observer at $r = 0$, the redshift arising from the time dilation can be readily derived as

$$1 + z = \frac{g_1(0)}{g_1(r)} = \frac{1}{\sqrt{3 - 2e^{H_0r}}}. \quad (61)$$

It is evident that the redshift approaches infinity as the light source approaches the horizon. In the scenario where the source is located near an observer, the redshift can be approximated by a linear function of the distance r when $z \ll 1$, consistent with observations. To see this, we expand the expression in (61) as follows

$$z = H_0r + 2H_0^2r^2 + \dots. \quad (62)$$

This shows that the first term, $H_0 r$, dominates the redshift at low values of z . In fact, it is precisely this favorable result that motivates our selection of the function $\alpha_c(\lambda)$ as assumed in (56).

The close sources' redshift z and distance r exhibit a well-established and fundamental observational fact of a linear relationship, one that remains independent of cosmological models. Hence, it is imperative to subject any plausible cosmological model to rigorous testing using the relevant observations. We validate our cosmological model by comparing the predicted luminosity distance d_L with the values derived from SN Ia data of the MLCS2k2 Full Sample [41].

The luminosity distance is defined such that the Euclidean inverse-square law for the diminution of light with distance from a point source is preserved. Let L denote the absolute luminosity of a source at distance r and l denote the observed apparent luminosity, the luminosity distance is defined as

$$d_L = \left(\frac{L}{4\pi l} \right)^{1/2}. \quad (63)$$

The area of a spherical surface centered on the source and passing through the observer is just $4\pi r^2$. The photons emitted by the source arrive at this surface having been redshifted by the quantum effect by a factor $1 + z$. We therefore find

$$l = \frac{L}{4\pi r^2} \frac{1}{1 + z}, \quad (64)$$

from which

$$d_L = r(1 + z)^{1/2}. \quad (65)$$

The coordinate distance r can be expressed in terms of z and H_0 from (61), we thus obtain

$$d_L = \frac{\sqrt{1+z}}{H_0} \ln \left[\frac{1}{2} \left(3 - \frac{1}{(1+z)^2} \right) \right]. \quad (66)$$

The distance modulus is defined by

$$\mu = m - M = 5 \log \left(\frac{d_L}{\text{Mpc}} \right) + 25, \quad (67)$$

where m and M are the apparent magnitude and absolute magnitude related to l and L respectively, and we have explicitly specified that d_L is measured in units of megaparsecs.

We use the full cosmological sample of SNe Ia, MLCS2k2, presented in Table 5 of Riess et al.'s paper [41]. The sample consists of 11 SNe with redshift ranges from 0.01 to 0.1, which is ideal for testing our cosmological model. The only free parameter in our model is the Hubble constant H_0 , with the best-fitted value being $H_0 = 59.0^{+1.68}_{-1.63} \text{ km/s/Mpc}$, as presented in Figure 1.

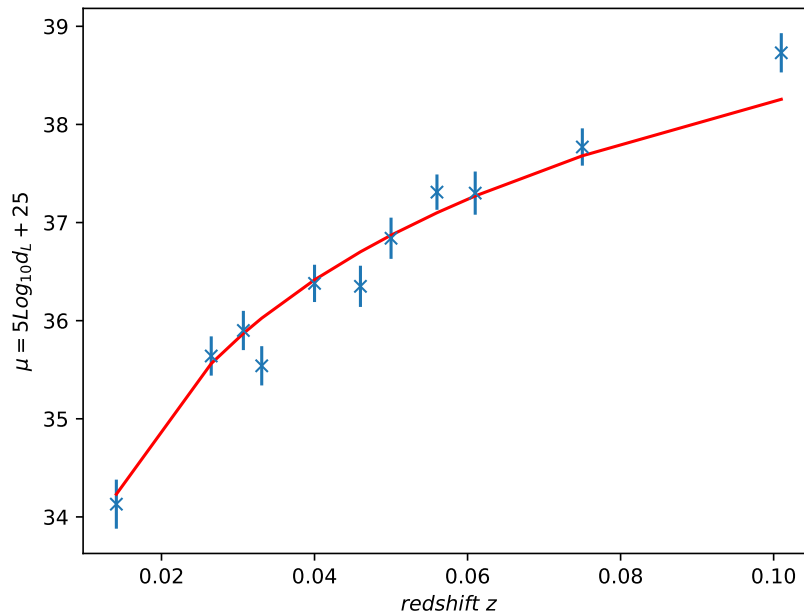


Figure 1. The fitting results of the predicted distance moduli given by (67) to the data of MLCS2k2 full sample [41]. The best fitted value of H_0 is $H_0 = 59.0^{+1.68}_{-1.63}$, and $\chi^2 = 0.0018$.

Our cosmological model fits remarkably well with the SNe Ia data, providing evidence that the cosmological redshift may be due to QPE-induced time dilation rather than the expansion of the universe. Our model is based on a static universe, which is made possible by the assumption of quantum effects on cosmic scales, described by QPE, acting as a repulsive force to balance gravity. When Einstein introduced the cosmological constant Λ to balance gravity and obtain a static universe model, he failed because Λ , regardless of its physical significance, is independent of matter distribution. Consequently, any perturbations in matter result in unstable solutions of Einstein's equations. In contrast, according to our proposal, QPE is not independent of matter distribution but is determined by the matter density, as shown in Equation (49). As a result, any perturbations in matter distribution result in corresponding perturbations in QPE to balance the extra gravity, allowing the universe to remain static.

It is worth noting from (49) that QPE depends on two independent factors: $\rho_m^2(r)$ and $\alpha(\lambda)$. As mentioned earlier, it is essential for $\alpha(\lambda)$ to satisfy the condition that $\alpha(\lambda)$ approaches zero as λ approaches zero. Thus, even though galaxies possess a high but finite central matter density, the QPE would vanish in that region. However, in the context of cosmology, the QPE that is dependent on λ displays some intricate yet justifiable characteristics. To see this, let us rephrase the function $\alpha_c(\lambda)$ in (56) as follows:

$$\alpha_c(\lambda) = \frac{2}{H_0\lambda} + \frac{1}{3}H_0\lambda + \dots \quad (68)$$

It is evident that the first term $2/H_0\lambda$ plays a crucial role in generating a linear correlation between redshift and distance at low redshift. At first sight, the inverse relationship between this term and $\alpha_c(\lambda)$ may appear to contradict the requirement that $\alpha(\lambda)$ approaches zero as λ approaches zero. However, upon further examination, this does not turn out to be the case. To grasp this paradox, it is essential to recognize that it is $M_Q(\lambda)$, not $\rho_Q(r, \lambda)$, that determines the metric and thus the geometry of spacetime for a specific observer, as shown in (59).

In cosmology, the redshift of light signals emitted from distant sources is entirely attributed to QPE, which varies with distance. However, this distance-dependent phenomenon does not imply any preferred direction in the universe, as demonstrated both theoretically and observationally.

All the conclusions reached thus far are the logical outcomes of rejecting the singular absolute spacetime concept predicted by conventional general relativity.

3.2. Our Solution to the Galactic Rotation Curve Problem

After establishing the geometry of the entire universe's spacetime, we now shift our focus to examining the galaxies. Traditionally, based on Newtonian theory, when a stellar system achieves a state of equilibrium, the kinetic energy supports the gravity, and this equilibrium state must adhere to the virial theorem. However, astronomical observations indicate that the gravitational force exerted by ordinary matter is insufficient to counterbalance the kinetic energy of the system. As a result, the presence of a mysterious substance known as dark matter is postulated to account for the discrepancy in mass, often referred to as the missing mass problem.

Based on the theory presented in this paper, we can propose an alternative explanation that challenges the need for dark matter. In fact, in the mass expression provided in equation (38), the term QPE defined by equation (49), serves as the counterpart to dark matter.

The mass expression presented in equation (50) is applicable universally to any radially-symmetric systems, making it suitable for investigating spherical galaxies. As mentioned earlier, $\alpha(\lambda)$ should also be a universal function of scale λ . We assume a functional form given by (53). In the context of cosmology, the scale can be regarded as arbitrary large up to r_h , the reduced specific form of $\alpha_c(\lambda)$ provided in equation (56) has been demonstrated to align with observational data. For galaxies, we try the parameters in (53) with $A = 0$ and $C = 0$ to obtain

$$\alpha_g(\lambda) = B(e^{H_0\lambda} - 1), \quad (69)$$

where, in this paper, we consider B as a constant, although it may be a parameter, possibly scale-dependent, to be determined by observations. When studying galaxies, the scale λ of interest is much smaller than the Hubble radius $1/H_0$, thus $\alpha_g(\lambda)$ can be approximated by $\alpha_g(\lambda) = BH_0\lambda$.

It is essential to account for the impact of the entire universe on local galaxies of finite mass, which is commonly referred to as the boundary condition when the metric of spacetime is concerned. Specifically, we mandate that the metric surrounding a local galaxy adheres to the condition that, as r approaches infinity, it converges to the metric of the entire universe (59), instead of the metric for flat spacetime, which has been conventionally used. To do so, we replace $\rho_m(r)$ in equation (50) with $\rho_m(r) + \rho_c$. Since the impact of cosmological effects on local galaxies is unclear, we make the assumption that

$$\alpha_c(\lambda) = \frac{A(\lambda)(e^{H_0\lambda} - 1)}{H_0^2\lambda^2} - 1 \quad (70)$$

when considering ρ_c , allowing us to determine the function $A(\lambda)$ based on observations of galaxies. We thus have from (50)

$$\begin{aligned} M(r) &= 4\pi \int_0^r (\rho_m(r') + \rho_c) r'^2 dr' + 4\pi \alpha_c(r) \int_0^r \rho_c r'^2 dr' + \frac{4\pi \alpha_g(r)}{\rho_c} \int_0^r \rho_m^2(r') r'^2 dr' \\ &= M_b(r) + M_m(r) + M_Q(r), \end{aligned} \quad (71)$$

where

$$\begin{aligned}
 M_b(r) &= \frac{A(r)}{2}(e^{H_0 r} - 1)r, \\
 M_m(r) &= 4\pi \int_0^r \rho_m(r')r'^2 dr', \\
 M_Q(r) &= \frac{B4\pi(e^{H_0 r} - 1)}{\rho_c} \int_0^r \rho_m^2(r')r'^2 dr' \\
 &= \frac{B32\pi^2(e^{H_0 r} - 1)}{3H_0^2} \int_0^r \rho_m^2(r')r'^2 dr'.
 \end{aligned} \tag{72}$$

Evidently, $M_b(r)$ signifies the contribution of the quantum effects of background mass density of the universe, $M_m(r)$ denotes the conventional mass and $M_Q(r)$ represents the quantum effects of a galaxy itself. We thus have

$$\begin{aligned}
 g_1 &= \left[1 - \frac{2(M_b(r) + M_m(r) + M_Q(r))}{r} \right]^{1/2} \\
 &= \left[1 - A(r)(e^{H_0 r} - 1) - \frac{2M_m(r)}{r} - \frac{2M_Q(r)}{r} \right]^{1/2}.
 \end{aligned} \tag{73}$$

We require g_1 for galaxies to satisfy the boundary condition, i.e., when $r \rightarrow \infty$, $g_1 \rightarrow (3 - 2e^{H_0 r})^{1/2}$, the value for cosmology as shown in (58). For galaxies of finite mass or finite size, the last two terms $\frac{2M_m(r)}{r} + \frac{2M_Q(r)}{r} \rightarrow 0$ when $r \rightarrow \infty$. We thus require $A(r) \rightarrow 2$ when $r \rightarrow \infty$. This condition imposes a nature constraint on the cosmological effect on local galaxies, as shown subsequently.

Our findings indicate that, from the perspective of mass-energy, the contributions of QPE to the metric exhibit no observable distinctions from conventional mass. Both QPE and conventional mass contribute to time dilation and distance contraction in precisely the identical manner, as demonstrated in Equation (46). As such, it is reasonable to employ the conventional approach in general relativity when considering the stress-energy tensor and geometry of spacetime.

Although our findings are presented in the relativistic form, the transition to the non-relativistic scenario for galaxies is straightforward. Actually, we can directly utilize the mass expression provided in equation (71) and apply Newtonian theory to investigate galaxies.

Obviously, near the center of a galaxy, the gravitational effects arising from $M_b(r)$ and $M_Q(r)$ can be neglected when compared to those of $M_m(r)$. Towards the outer regions, both $M_b(r)$ and $M_Q(r)$ become significant, while the contribution of $M_m(r)$ diminishes with the increasing radius r .

It is interesting to investigate the circular velocity of a test particle around a galaxy, which can be approximated by

$$\begin{aligned}
 v^2(r) &= \frac{M(r)}{r} \\
 &= \frac{A(r)}{2}H_0 r + \frac{4\pi}{r} \int_0^r \rho_m(r')r'^2 dr' + \frac{B32\pi^2}{3H_0} \int_0^r \rho_m^2(r')r'^2 dr',
 \end{aligned} \tag{74}$$

where we have neglected the terms with $H_0^n r^n$ when $n \geq 2$ for the scale of galaxies. For a galaxy with the finite size a , when r increases to the outer regions with $r \gg a$, we have

$$v(r) = \sqrt{\frac{A(r)}{2}H_0 r + M_Q(a)/a}. \tag{75}$$

This expression does not imply a constant value of $v(r)$ for large radii, as $H_0 r$ term in (75) becomes more and more significant with increasing r . In particular, it leads to a universal centrifugal acceleration (recall that $A(r) \rightarrow 2$ for $r \rightarrow \infty$)

$$\frac{v^2}{r} = cH_0, \quad (76)$$

where c is the speed of light. This result reflects the fact that the matter in the entire universe can have a local observable effect in galaxies, a fact first discovered in MOND theory but cannot be explained within the theory itself [37,38]. In fact, our cosmological model yields the value of $H_0 = 59.0$, resulting in $cH_0 = 5.74 \times 10^{-8} \text{cm/sec}^2$. A value which is close to, but obviously larger than, that found for the universal acceleration parameter $a_0 = 1.2 \times 10^{-8} \text{cm/sec}^2$ of the MOND theory [15,36]. This fact has a natural explanation. As an increasing function of r , $A(r) \rightarrow 2$ only at the cosmological distances, typically hundreds of Mpc. The value of a_0 is smaller than cH_0 only because it is detected at the distances in the outer regions of galaxies, the distances much smaller than the cosmological distance. Notably, conformal gravity yields comparable results when fitting the rotation curves data [34,35]. In our theory, these phenomena are not considered mysteries. The cosmological effects on local galaxies are the natural results of the boundary condition, which is determined by the requirement that the spacetime metric around an isolated galaxy, created by the mass (energy) specified in (72), should align with that of the entire universe as demonstrated in (59), when $r \rightarrow \infty$.

Despite these remarkable achievements, the most effective way to validate our theory is by leveraging the abundant observational data on the rotation curves of flattened dwarf and spiral galaxies. As our results are obtained for spherical systems, they must be converted into axisymmetric systems where cylindrical coordinates are more suitable. One could replicate the procedure for axisymmetric systems [12,14], similar to what was done for spherical systems in this paper. However, a more efficient approach to achieve our objectives is to consider the relevant terms on the right-hand side of equation (74) as the gravitational potential generated by a point mass (or a mass element). Subsequently, calculate the total potential for disk galaxies in cylindrical systems (R, ϕ, z) [34].

The first term on the right-hand side of (74) is $A(r)cH_0 r/2$. This is a linear potential originated from the quantum effect of the entire universe, and thus is independent of specific local galaxies. We simply express the contribution of this term to the total circular velocity of a test particle on the thin-disk as

$$v_L^2(R) = A(R)cH_0 R. \quad (77)$$

In this paper, we initially overlook the gradually increasing nature of the function $A(R)$ and consider it as a constant A_0 , which can be determined by rotation curve data.

The second term on the right-hand side of (74) is the usual newtonian potential. We write the corresponding potential in cylindrical coordinates as

$$\Phi_m(R, z) = -G \int_0^\infty dR' \int_0^{2\pi} d\phi' \int_{-\infty}^\infty dz' \frac{R' \rho_m(R', z')}{(R^2 + R'^2 - 2RR' \cos \phi' + (z - z')^2)^{1/2}}, \quad (78)$$

where G is the gravitational constant (we temporarily transition back from the nature units from now on in this subsection). Inserting the cylindrical coordinate Green's function Bessel function expansion

$$\frac{1}{(R^2 + R'^2 - 2RR' \cos \phi' + (z - z')^2)^{1/2}} = \sum_{-\infty}^{\infty} \int_0^\infty dk J_m(kR) J_m(kR') e^{im(\phi - \phi') - k(|z - z'|)} \quad (79)$$

into (78) yields

$$\Phi_m(R, z) = -2\pi G \int_0^\infty dk \int_0^\infty dR' \int_{-\infty}^\infty dz' R' \rho_m(R', z') J_0(kR) J_0(kR') e^{-k|z - z'|}. \quad (80)$$

For razor-thin exponential disks with $\rho_m(R, z) = \Sigma(R)\delta(z) = \Sigma_0 e^{-R/R_0}\delta(z)$, where Σ_0 is the central surface mass density and R_0 is the disk scale length, the potential is

$$\Phi_m(R) = -\pi G \Sigma_0 R_0 y [I_0(y/2)K_1(y/2) - I_1(y/2)K_0(y/2)], \quad (81)$$

where $y \equiv \frac{R}{R_0}$, the I_n and K_n are modified Bessel functions. If we differentiate equation (81) with respect to R we obtain the circular velocity contributed by the newtonian potential

$$v_m^2(y) = R \frac{\partial \Phi_m}{\partial R} = \frac{\pi c^2}{R_0} \left(\frac{GM_\odot}{c^2} \right) \left(\frac{\Sigma_0}{M_\odot} \right) R_0^2 y^2 [I_0(y/2)K_0(y/2) - I_1(y/2)K_1(y/2)]. \quad (82)$$

Another potential of the quantum effect for galaxies on the right-hand side of (74) is $(64\pi^2/3H_0) \int_0^r \rho_m^2(r') r'^2 dr'$. This corresponding to a logarithmic potential, and the circular velocity for the razor-thin exponential disk is

$$\begin{aligned} v_Q^2(R) &= \frac{B8\pi G^2}{3cH_0 R_0} \int_0^R dR' \int_0^{2\pi} d\phi' \int_{-\infty}^{\infty} dz' R' \Sigma^2(R') \delta(z') \\ &= \frac{B4\pi^2 G^2 \Sigma_0^2 R_0}{3cH_0} \left[1 - \left(1 + \frac{2R}{R_0} \right) e^{-2R/R_0} \right]. \end{aligned} \quad (83)$$

The total circular velocity used to fit data is, therefore

$$v(R) = \sqrt{v_L^2(R) + v_m^2(R) + v_Q^2(R)}. \quad (84)$$

We fit the predicted circular velocity to the data provided in Spitzer Photometry and Accurate Rotation Curves (SPARC) database [30]. SPARC is the largest sample of rotation curves to date for every galaxy. It is a sample of 175 nearby galaxies with new surface photometry at 3.6 μ m, and high-quality rotation curves from previous HI/H studies. As a first try, we consider each of the 175 galaxies in the sample as a razor-thin exponential disk characterised by the central surface mass density Σ_0 and the disk scale length R_0 . In addition to Σ_0 and R_0 , we treat $A(r) = A_0$ in (77) and B in (83) as free parameters in our fitting. Figure 2 displays 12 of them, including 6 Low surface brightness (LSB) galaxies (6 upper panels) and 6 high surface brightness (HSB) galaxies (6 lower panels).

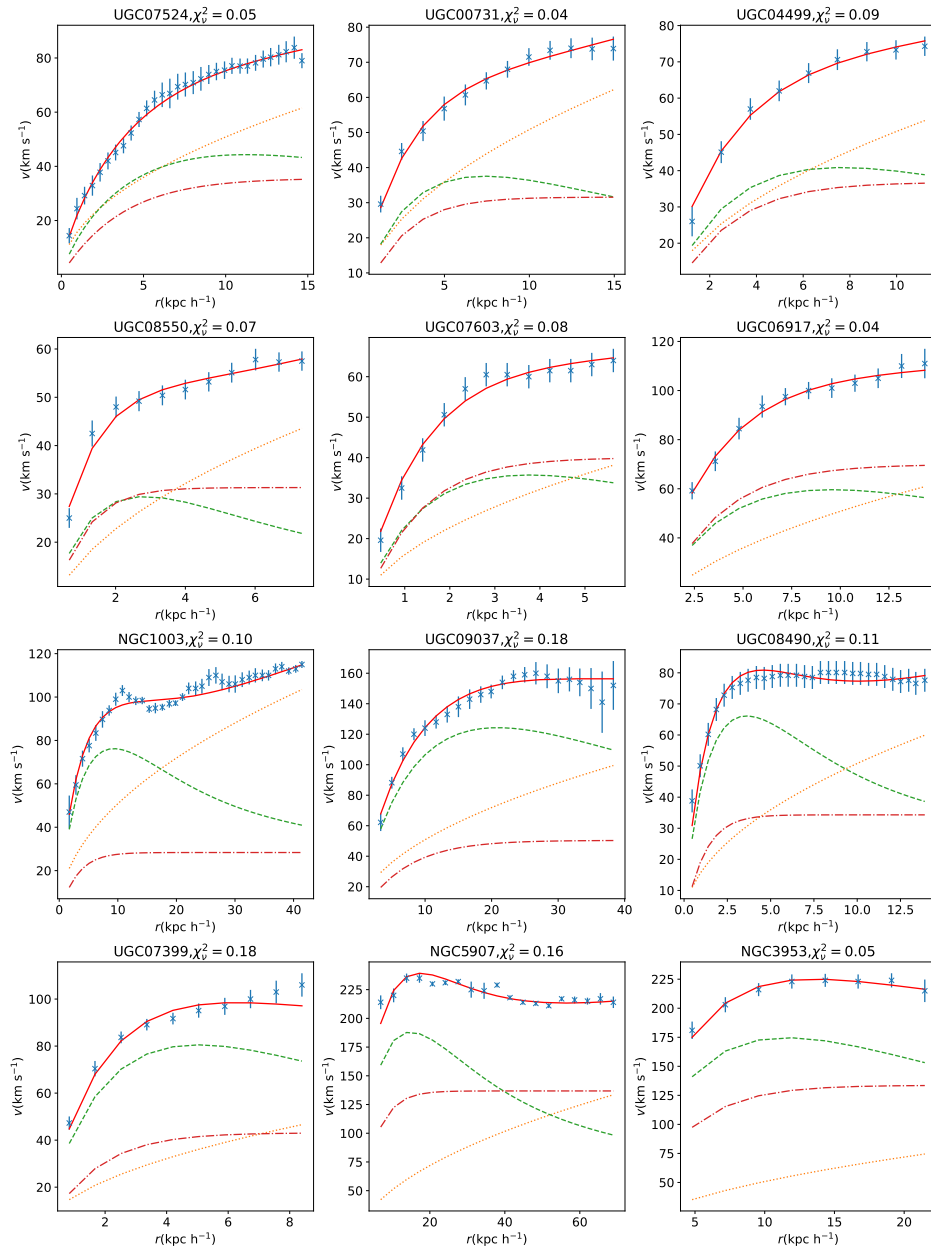


Figure 2. The best-fitting rotation curves of LSB (upper 6 panels) and HSB (lower 6 panels) galaxies with the model presented in this paper. The panels are listed by the increasing effective surface brightness of the galaxies. In each panel, the dotted curve shows the contribution from cosmological quantum effect, given by (77); the dashed curve indicates the contribution from luminous Newtonian potential given by (82); the dash-dotted curve shows the contribution from quantum effect of the galaxies themselves given by (83); and finally, the solid curve is the total circular velocity given by (84).

The fittings reveal that $A_0 = 0.02$ for all galaxies, with $B = 3.0$ for LSB galaxies and $B = 0.3$ for HSB galaxies. The cosmological effect on local galaxies thus suggests an acceleration of $A_0 c H_0 \sim 10^{-9} \text{ cm/sec}^2$, significantly smaller than the universal value of $c H_0 = 5.74 \times 10^{-8} \text{ cm/sec}^2$. Our fittings further suggest that A_0 can be substituted with

$$A(R) = \frac{A_0 + 2(R/\text{Mpc})^2}{1 + (R/\text{Mpc})^2}. \quad (85)$$

This slowly varying function provides a good fit to the rotation curve data and also fulfil the requirement that $A(R)$ converges to 2 as R approaches cosmological distances. On the other hand, the value of B for LSB galaxies exceeds that for HSB galaxies, suggesting that LSB galaxies require relatively more QPE than HSB galaxies, and $B(\lambda)$ is also scale dependent.

All these results indicate that, in terms of mass-energy, scale-dependent quantum phenomena do not exhibit a linear superposition, but rather a hierarchical process.

Despite the very simple model used to describe the matter distribution in LSB and HSB galaxies, the fittings can still capture the main phenomenological results observed in other theories of gravity. For instance, the quantum effect contributions ($v_L^2 + V_Q^2$) dominate the most regions of LSB galaxies, whereas this is only true for the outer regions of HSB galaxies.

Our proposed quantum effect on large scales has proven to be successful in explaining the mass discrepancy problem in galaxies.

4. Conclusions and Discussions

We have proposed the quantum nature of spacetime on cosmic scales. To investigate this postulation, we examine the spherically symmetric and static gravitational systems composed of free-falling particles with identical mass m . We assume that the matter distribution can be smoothed using the number density $\rho(r)$ and mass density $\rho_m(r)$. We further assume that the Dirac spinor $\psi(r)$ can fully capture all the physical aspects of the particles and satisfies the covariant Dirac equation. Therefore, we recognise $\rho(r) = \psi(r)\tilde{\psi}(r)$ and $\rho_m(r) = m\psi(r)\tilde{\psi}(r) = m\rho(r)$. For a given mass density $\rho_m(r)$, the interplay between the number density $\rho(r)$ and the particle mass m offers us a inherent mechanism to establish a connection between the spin density $S(r) = \frac{\hbar}{2}\psi(r)i\gamma_3\tilde{\psi}(r)$ and the scale λ of the gravitational systems.

We utilize the gauge theory gravity (GTG) [13,29] to achieve the subsequent derivations. After solving Einstein equations, the metric of the spacetime for the radially-symmetric and static gravitational systems has been derived, as shown in (46). The fundamental results obtained thus far require additional physical interpretations and practical applications. We find that the quantum potential energy (QPE) defined by (49)

$$\rho_Q(r, \lambda) = -\frac{3}{4}\kappa S^2(r) = \rho_m^2(r)\alpha(\lambda)/\rho_c$$

is contained in the metric via the mass (energy) $M(r)$, where $\alpha(\lambda)$ is a dimensionless function of the scale of the system involved. The mass within radius r can then be expressed in (50).

It is important to note that at each radius r , we consider $M(r)$ as the mass of a gravitational system, therefore we identify r as the scale λ of the "system". As such, for any given mass density $\rho_m(r)$ of a gravitational system, we can derive the metric which encompasses not only the classical mass but also the quantum potential energy (QPE). When the scale of the gravitational systems are macroscopic, for instance, the solar system, the QPE can be neglected, and the Einstein-Newtonian theory of gravity is recovered.

The dimensionless function $\alpha(\lambda)$ is universal, and it must be determined by observations. However, we have discovered that the general expression presented in (53) successfully achieves our objective.

While our fundamental results are initially derived for radially-symmetric and static systems, they can be applied to any static gravitational system. By substituting the mass density ρ_m with $\rho_m + \rho_Q + \rho_c + \rho_c\alpha_c(\lambda)$, where ρ_c is the constant mass density of the universe, we establish a fundamental boundary condition that spacetime metric encompassing any local gravitational system must adhere to. Remarkably, this boundary condition inherently gives rise to the cosmological impact on local galaxies, as evidenced by observations of galactic rotation curves [15,30,34,36,38,39].

When applied to cosmology, our model yields a static universe. The predicted luminosity distance-redshift relation fits remarkably well with the SNe Ia data, providing evidence that the

cosmological redshift may be due to QPE-induced time dilation rather than the expansion of the universe. It is not surprising that when we extend Dirac's theory for free electrons to macroscopic particles (including stars within galaxies and galaxies within the entire universe), we are effectively moving away from the notion of a preferred absolute spacetime as suggested by general relativity and commonly accepted by many physicists, as illustrated in the standard Λ CDM cosmology. Without the existence of absolute spacetime, physical processes should be described from the perspective of any observer in the universe independently and equivalently, without relying on the absolute spacetime reference frame of the universe. Therefore, the state of a celestial body should be characterized in the frame associated with a specific observer rather than in the frame of the absolute spacetime of the universe. Consequently, we assume that, for any observer in the universe, a distant celestial body behaves akin to a micro-particle and can be effectively described using Dirac's theory. On the contrary, in a scenario where absolute spacetime exists, all matter particles, including observers, must be described in relation to it. As a result, the motion of distant celestial bodies is typically explained by a combination of the local velocity and the overall expansion velocity. In this context, we should not anticipate quantum effects on large scales for any observers.

When considering galaxies, it is intriguing to note that quantum effects can serve as a substitute for dark matter. To validate our theory, the most effective approach is to utilize the extensive observational data on the rotation curves of dwarf and spiral galaxies. Remarkably, our proposed quantum effect on large scales has successfully addressed the mass discrepancy issue in galaxies. Specifically, our theory presents a fitting formula as outlined in (85), which elegantly explains the correlation between the universal acceleration cH_0 suggested by cosmology, a_0 proposed by MOND, and $0.02cH_0$ resulting from the cosmological effect on local galaxies [34].

In conclusion, we expand the quantum characteristics of microparticles to encompass macroscopic matter on cosmic scales. As illustrated, this expansion is remarkably coherent. The subsequent outcomes are derived within the framework of the firmly established theory of gravity (GTG) and mathematical language (STA). While there is room for improvement in the technical details of our derivations, the fundamental discoveries are credible and, notably, have been verified through astronomical observations of SNe Ia and galactic rotation curve data. In particular, our proposal provide an alternative view point to understand the quantum nature of spacetime.

The manifestation of physical laws should be independent of the mathematical language used, and thus we believe that our proposal can also be accomplished, for instance, through the tensor analysis approach. Of course, using an unfamiliar language may make it challenging for readers to comprehend our theory. However, it has been shown that STA and GC can significantly streamline the calculations in the area of our concern. Additionally, STA can illustrate the cohesive link between classical mechanics and quantum mechanics, thereby enhancing our comprehension of the enigmatic aspects of quantum theory and spacetime.

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Appendix A. Spacetime Algebra and Geometric Calculus

The spacetime algebra (STA) is generated by a 4-dimensional Minkowski vector space. The inner and outer products of the four orthonormal basis vectors $\{\gamma_\mu, \mu = 0 \dots 3\}$ are defined to be

$$\begin{aligned}\gamma_\mu \cdot \gamma_\nu &\equiv \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) \equiv \eta_{\mu\nu} = \text{diag}(+ - - -) \\ \gamma_\mu \wedge \gamma_\nu &\equiv \frac{1}{2}(\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu).\end{aligned}\tag{A1}$$

A full basis for the STA is

$$1, \{\gamma_\mu\}, \{\sigma_k, i\sigma_k\}, \{i\gamma_\mu\}, i\tag{A2}$$

where $\sigma_k \equiv \gamma_k \gamma_0, k = 1 \dots 3$ constitute the orthonormal basis of a 3-dimensional vector space relative to γ_0 , and $i = \gamma_0 \gamma_1 \gamma_2 \gamma_3 = \sigma_1 \sigma_2 \sigma_3$ is called pseudoscalar. The STA is a linear space of dimension 16. We refer to the general elements of STA as multivectors, and each multivector decomposes into a sum of elements of different grades. If a multivector contains only grade- r components, we call it homogeneous, and is denoted by A_r . We call grade-0 multivectors scalars, grade-1 vectors, grade-2 bivectors and grade-3 trivectors. A grade- r multivector is called simple or an r -blade if and only if it can be written as

$$A_r = a_1 \wedge a_2 \wedge \dots \wedge a_r = \frac{1}{r!} \sum_{\pi} (\text{sgn } \pi) a_{\pi(1)} a_{\pi(2)} \dots a_{\pi(r)}, \quad (\text{A3})$$

where π is a permutation of 1 through r , 'sgn π ' is the sign of the permutation, 1 for even and -1 for odd, and the sum is over all $r!$ possible permutations. For $r = 2$ this reduces the outer product of two vectors

$$a \wedge b = \frac{1}{2}(ab - ba), \quad (\text{A4})$$

which is in agreement with the definition in (A1), and clearly $a \wedge a = 0$.

The geometric product of a grade- r multivector A_r with a grade- s multivector B_s is defined simply by $A_r B_s$, which decomposes into

$$A_r B_s = \langle A_r B_s \rangle_{|r-s|} + \langle A_r B_s \rangle_{|r-s|+2} + \dots + \langle A_r B_s \rangle_{r+s}, \quad (\text{A5})$$

where $\langle X \rangle_r$ denotes the projection onto the grade- r part of X . The grade-0 (scalar) part of X is written as $\langle X \rangle$. We employ "." and " $\wedge$ " symbols to denote the lowest-grade and highest-grade terms in (A5), so that

$$A_r \cdot B_s = \langle A_r B_s \rangle_{|r-s|}, \quad (\text{A6})$$

$$A_r \wedge B_s = \langle A_r B_s \rangle_{r+s}, \quad (\text{A7})$$

which are called inner and outer products respectively. They represent a generalization of the geometric product between two vectors a and b , which can be expressed as follows

$$ab = a \cdot b + a \wedge b. \quad (\text{A8})$$

We define the reverse of a geometric product AB by $(AB)^{\sim} = \tilde{B}\tilde{A}$, so that for vectors $a_1, a_2, \dots, a_r$, we have

$$(a_1 a_2 \dots a_r)^{\sim} = a_r a_{r-1} \dots a_1. \quad (\text{A9})$$

This reverse operator obeys the rule

$$(A + B)^{\sim} = \tilde{A} + \tilde{B}, \quad (\text{A10})$$

$$\langle \tilde{A} \rangle = \langle A \rangle, \quad (\text{A11})$$

$$\tilde{a} = a, \quad (\text{A12})$$

where a is a vector. It is easy to show that

$$\tilde{A}_r = (-1)^{r(r-1)/2} A_r. \quad (\text{A13})$$

Thus suppose $r \leq s$, the inner and outer products satisfy the symmetry properties

$$A_r \cdot B_s = (-1)^{r(s-1)} B_s \cdot A_r, \quad (\text{A14})$$

$$A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r. \quad (\text{A15})$$

The scalar product is defined by

$$A * B = \langle AB \rangle. \quad (\text{A16})$$

From (A6), $A_r * B_s$ is nonzero only if $r = s$, thus the scalar product (A16) is commutative

$$\langle AB \rangle = \langle BA \rangle, \quad (\text{A17})$$

which proves to be very useful in our calculations. The inner product and outer product obey the distributive rule

$$A \cdot (B + C) = A \cdot B + A \cdot C \quad (\text{A18})$$

$$A \wedge (B + C) = A \wedge B + A \wedge C. \quad (\text{A19})$$

The outer product is associative

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C. \quad (\text{A20})$$

The inner product is not associative, but homogeneous multivectors obey

$$A_r \cdot (B_s \cdot C_t) = (A_r \wedge B_s) \cdot C_t, \text{ for } r + s \leq t, \quad (\text{A21})$$

$$A_r \cdot (B_s \cdot C_t) = (A_r \cdot B_s) \cdot C_t, \text{ for } r + t \leq s. \quad (\text{A22})$$

For any vector a it can be proved that

$$a \cdot A_r = \langle a A_r \rangle_{r-1} = \frac{1}{2} (a A_r - (-1)^r A_r a), \quad (\text{A23})$$

$$a \wedge A_r = \langle a A_r \rangle_{r+1} = \frac{1}{2} (a A_r + (-1)^r A_r a). \quad (\text{A24})$$

From which we immediately have

$$a A_r = a \cdot A_r + a \wedge A_r. \quad (\text{A25})$$

We also have the following useful identities

$$a \cdot (A_r B) = (a \cdot A_r) B + (-1)^r A_r (a \cdot B) \quad (\text{A26a})$$

$$= (a \wedge A_r) B - (-1)^r A_r (a \wedge B) \quad (\text{A26b})$$

$$a \wedge (A_r B) = (a \wedge A_r) B - (-1)^r A_r (a \cdot B) \quad (\text{A26c})$$

$$= (a \cdot A_r) B + (-1)^r A_r (a \wedge B). \quad (\text{A26d})$$

These identities imply the following particularly useful relations,

$$a \cdot (A_r i) = (a \wedge A_r) i, \quad (\text{A27})$$

$$a \wedge (A_r i) = (a \cdot A_r) i, \quad (\text{A28})$$

where i is a pseudoscalar, and we have used the fact that $a \wedge i = 0$.

Any vector a can be decomposed in terms of $\{\gamma_\mu\}$ into

$$a = a \cdot \gamma_\mu \gamma^\mu = a \cdot \gamma^\mu \gamma_\mu, \quad (\text{A29})$$

where the summation convention is implied. Similarly, an arbitrary multivector A can be decomposed into

$$A = \sum_{\mu < \dots < \nu} A_{\mu \dots \nu} \gamma^\mu \wedge \dots \wedge \gamma^\nu, \quad (\text{A30})$$

where

$$A_{\mu\dots\nu} = A \cdot (\gamma_\nu \wedge \dots \wedge \gamma_\mu). \quad (\text{A31})$$

We further define the commutator product

$$A \times B = \frac{1}{2}(AB - BA), \quad (\text{A32})$$

which satisfies the Jacobi identity

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0. \quad (\text{A33})$$

In addition, we have the Leibnitz rule for the commutator product

$$A \times (BC) = (A \times B)C + B(A \times C). \quad (\text{A34})$$

This rule is particular useful with bivectors. It can be proved that

$$A_2 \times A_r = \frac{1}{2}(A_2 A_r - A_r A_2) = \langle A_2 A_r \rangle_r, \quad (\text{A35})$$

which means that if one of the factors is a bivector then the commutator product preserves grade. Thus for $\langle A \rangle_1 = 0$, we have

$$A_2 A = A_2 \cdot A + A_2 \times A + A_2 \wedge A. \quad (\text{A36})$$

Since the commutator product with a bivector is grade preserving, the identity in (A34) still holds if $A = A_2$ and we replace all geometric products with either inner or outer products:

$$\begin{aligned} A_2 \times (B \cdot C) &= (A_2 \times B) \cdot C + B \cdot (A_2 \times C), \\ A_2 \times (B \wedge C) &= (A_2 \times B) \wedge C + B \wedge (A_2 \times C). \end{aligned} \quad (\text{A37})$$

Finally, for any vector a and trivector T , it can be proved that

$$(a \cdot T) \times T = 0. \quad (\text{A38})$$

Geometric calculus (GC) is the extension of a geometric algebra (like STA) to include differentiation and integration. Let multivector F be an arbitrary function of some multivector argument X , then the derivative of $F(X)$ with respect to X in the A direction is defined by

$$A * \partial_X F(X) \equiv \lim_{\tau \rightarrow 0} \frac{F(X + \tau A) - F(X)}{\tau}, \quad (\text{A39})$$

where the multivector partial derivative ∂_X inherits the multivector properties of its argument X . Suppose that the set $\{e_k\}$ form a vector frame (which is not necessarily orthonormal), the reciprocal frame is determined by [18]

$$e^i = (-1)^{i-1} e_1 \wedge e_2 \wedge \dots \wedge \check{e}_i \wedge \dots \wedge e_n e^{-1} \quad (\text{A40})$$

$$e \equiv e_1 \wedge e_2 \wedge \dots \wedge e_n \quad (\text{A41})$$

and the check on $\check{e}$ denotes that this term is missing from the expression. As usual, the two frames are related by

$$e_j \cdot e^k = \delta_j^k. \quad (\text{A42})$$

From this frame the multivector derivative ∂_X in equation (A39) is defined by

$$\partial_X \equiv \sum_{i < \dots < j} e^i \wedge \dots \wedge e^j (e^i \wedge \dots \wedge e^j) * \partial_X. \quad (\text{A43})$$

The Leibnitz rule can be written in the form

$$\partial_X(AB) = \dot{\partial}_X \dot{A}B + \dot{\partial}_X A \dot{B}, \quad (\text{A44})$$

where the overdot indicates the scope of the multivector derivative.

We have from (A39) and (A43)

$$\partial_X \langle XA \rangle = P_X(A), \quad \partial_X \langle \tilde{X}A \rangle = P_X(\tilde{A}), \quad (\text{A45})$$

where $P_X(A)$ is the projection of A onto the grades contained in X . These results are combined using Leibnitz's rule to give

$$\partial_X \langle X\tilde{X} \rangle = \dot{\partial}_X \langle \dot{X}\tilde{X} \rangle + \dot{\partial}_X \langle X\dot{\tilde{X}} \rangle = 2\tilde{X}. \quad (\text{A46})$$

For a vector argument x and a constant vector a , (A39) and (A45) yield

$$a \cdot \partial_x x = a = \partial_x(x \cdot a). \quad (\text{A47})$$

For a vector variable $a = a^\mu \gamma_\mu = a \cdot \gamma_\mu \gamma^\mu = a \cdot \gamma^\mu \gamma_\mu$, where γ^μ constitute the reciprocal basis and satisfy $\gamma_\mu \cdot \gamma^\nu = \delta_\mu^\nu$, the vector derivative can be defined as

$$\partial_a \equiv \gamma^\mu \frac{\partial}{\partial a^\mu}, \quad (\text{A48})$$

For the derivative with respect to a spacetime position vector x we use the symbol $\nabla \equiv \partial_x = \gamma^\mu \frac{\partial}{\partial x^\mu}$, if $x = x^\mu \gamma_\mu$. From (A47) and (A48) we can obtain useful results

$$\partial_a = \partial_b b \cdot \partial_a = \gamma^\mu \gamma_\mu \cdot \partial_a. \quad (\text{A49})$$

Some results for the derivative with respect to position vector x in an n -dimensional space are [18]

$$\partial_x(x \cdot A_r) = r A_r, \quad (\text{A50a})$$

$$\partial_x(x \wedge A_r) = (n - r) A_r, \quad (\text{A50b})$$

$$\dot{\partial}_x A_r \dot{x} = (-1)^r (n - 2r) A_r. \quad (\text{A50c})$$

From (A21) and (A50a), an r -blade A_r can be expressed as

$$A_r = \frac{1}{r!} \partial_{a_1} \wedge \dots \wedge \partial_{a_r} (a_r \wedge \dots \wedge a_1) * A_r. \quad (\text{A51})$$

When considering a vector argument x and a constant vector a , (A39) becomes the definition of the directional derivative $a \cdot \nabla$, thus

$$a \cdot \nabla F = a \cdot \partial_x F(x) = \left. \frac{d}{d\tau} F(x + a\tau) \right|_{\tau=0} = \lim_{\tau \rightarrow 0} \frac{F(x + a\tau) - F(x)}{\tau}. \quad (\text{A52})$$

Then the general vector derivative can be obtained from the directional derivative by using (A49) as

$$\nabla F = \partial_x F(x) = \partial_a a \cdot \partial_x F(x). \quad (\text{A53})$$

The directional derivative (A52) produces from F a tensor field termed differential of F , denoted variously by

$$\underline{F}(a) = F_a \equiv a \cdot \nabla F. \quad (\text{A54})$$

The underbar notation serves to indicate that $\underline{F}(a)$ is a linear function of a . This induced linear function is very important for us to describe the apparatus of GC for handling transformations of spacetime and the induced transformations of multivector fields on spacetime.

Suppose there is a diffeomorphism which transforms each point x in some region of spacetime into another point x' as

$$x' = f(x). \quad (\text{A55})$$

This induces a linear transformation of tangent vectors at x to tangent vectors at x' given by the differential

$$a'(x') = \underline{f}(a) = a \cdot \nabla f. \quad (\text{A56})$$

If we regard x as a map representing the ordering of points in spacetime, then x' can be interpreted as a different map, or a remapping of the same spacetime. The transformation f also induces an adjoint transformation $\bar{f}$, which takes a tangent vector b' at x' back to a tangent vector b at x , as defined by

$$b(x) = \bar{f}(b') \equiv \partial_x f(x) \cdot b'(x'). \quad (\text{A57})$$

The differential and its adjoint are related by

$$b' \cdot \underline{f}(a) = a \cdot \bar{f}(b'). \quad (\text{A58})$$

By using this relation we can find one transformation from another by

$$\bar{f}(a) = \partial_b b \cdot \bar{f}(a) = \partial_b (\underline{f}(b) \cdot a), \quad (\text{A59})$$

$$\underline{f}(a) = \partial_b b \cdot \underline{f}(a) = \partial_b (\bar{f}(b) \cdot a). \quad (\text{A60})$$

In addition to the induced linear transformations $\underline{f}(a)$ and $\bar{f}(a)$ of tangent vectors, by the rule of direct substitution, (A55) can also induce a transformation of a multivector field $F(x)$ defined by

$$F'(x') \equiv F'(f(x)) = F(x), \quad (\text{A61})$$

in which directional derivatives of the two functions are related by the chain rule

$$a \cdot \nabla F = a \cdot \partial_x F'(f(x)) \quad (\text{A62})$$

$$= (a \cdot \nabla_x f(x)) \cdot \partial_{x'} F'(x') \quad (\text{A63})$$

$$= \underline{f}(a) \cdot \nabla' F' = a \cdot \bar{f}(\nabla') F' \quad (\text{A64})$$

$$= a' \cdot \nabla' F'. \quad (\text{A65})$$

The operator identity is

$$a \cdot \nabla = \underline{f}(a) \cdot \nabla' = a \cdot \bar{f}(\nabla') = a' \cdot \nabla'. \quad (\text{A66})$$

Differentiation with respect to the vector a yields

$$\nabla = \bar{f}(\nabla') \text{ or } \nabla' = \bar{f}^{-1}(\nabla). \quad (\text{A67})$$

Now is an opportune moment to discuss linear algebra. In fact, GC enables us to carry out coordinate-free calculations in linear algebra, eliminating the need for matrices. Every linear transformation $\underline{f}$ on spacetime has a unique extension to a linear function on the whole STA, called

outermorphism. For arbitrary multivectors A, B , and any scalar α , the outermorphism is defined by the property

$$\underline{f}(A \wedge B) = \underline{f}(A) \wedge \underline{f}(B), \quad \underline{f}(\alpha) = \alpha. \quad (\text{A68})$$

It follows that for an r -blade $A_r = a_1 \wedge \dots \wedge a_r$,

$$\underline{f}(A_r) = \underline{f}(a_1) \wedge \dots \wedge \underline{f}(a_r). \quad (\text{A69})$$

Since the outermorphism preserves the outer product, it also preserves grade:

$$\underline{f}(\langle A \rangle_r) = \langle \underline{f}(A) \rangle_r \quad (\text{A70})$$

for any multivector A . This implies that $\underline{f}$ alters the pseudoscalar i only by a scalar multiple:

$$\underline{f}(i) = (\det \underline{f})i, \quad (\text{A71})$$

which defines the determinant of $\underline{f}$. The product of two linear transformations $\underline{h} = \underline{g}\underline{f}$ also applies to their outermorphisms. It follows from (A71) that

$$\underline{h}(i) = \underline{g}(\underline{f}(i)) = (\det \underline{f})\underline{g}(i) = (\det \underline{f})(\det \underline{g})i \quad (\text{A72})$$

$$\det(\underline{g}\underline{f}) = \det \underline{g} \det \underline{f} \quad (\text{A73})$$

$$\det \underline{f}^{-1} = (\det \underline{f})^{-1}. \quad (\text{A74})$$

Every $\underline{f}$ has an adjoint $\bar{f}$ which can be extended to an outermorphism denoted by the same symbol

$$\langle A \bar{f}(B) \rangle = \langle B \underline{f}(A) \rangle \quad (\text{A75})$$

for any multivectors A and B . Unlike the outer product, the inner product is not generally preserved by outermorphisms. However, it obeys the law

$$A_r \cdot \bar{f}(B_s) = \bar{f}(\underline{f}(A_r) \cdot B_s) \quad \text{for } r \leq s, \quad (\text{A76})$$

$$\underline{f}(A_r) \cdot B_s = \underline{f}(A_r \cdot \bar{f}(B_s)) \quad \text{for } r \geq s. \quad (\text{A77})$$

From these identities we can construct a formula for the inverse of a linear transformation. Consider a multivector B , lying entirely in the algebra defined by the pseudoscalar i , we have

$$\underline{f}(i)B = (\det \underline{f})iB = \underline{f}(i\bar{f}(B)). \quad (\text{A78})$$

Replacing iB by A we find

$$(\det \underline{f})A = \underline{f}(i\bar{f}(i^{-1}A)) \quad (\text{A79})$$

with a similar result holding for the adjoint. It follows immediately that

$$\underline{f}^{-1}(A) = i\bar{f}(i^{-1}A)(\det \underline{f})^{-1}, \quad (\text{A80})$$

$$\bar{f}^{-1}(A) = i\underline{f}(i^{-1}A)(\det \underline{f})^{-1}. \quad (\text{A81})$$

Once again, one great advantage of GC is that it eliminates unnecessary conceptual barriers between classical, quantum and relativistic physics.

Appendix B. stress-Energy Tensor Derived from Dirac Theory

In this Appendix, we show that the stress-energy tensor derived from the Dirac equation for free spin- $\frac{1}{2}$ particles can be decomposed into the sum of a symmetric part and an antisymmetric part. The

symmetric part represents a classical pressureless ideal fluid, while the antisymmetric part represents the contribution of quantum potential energy. As a result, when spin is zero, the stress-energy tensor becomes that of a pressureless ideal fluid. We follow Hestenes's work [22], except that we set $\beta = 0$ for free spin- $\frac{1}{2}$ particles. As mentioned in our previous paper, the spinor field at spacetime point x takes the form [10]

$$\psi(x) = \rho(x)^{1/2} R(x), \quad (\text{B1})$$

where $\rho(x)$ is a scalar, represents the proper probability density and $R(x)$ is a rotor (Lorentz rotation) satisfying $R\tilde{R} = 1$. The rotor R can be used to transform a fixed frame $\{\gamma_\mu\}$ into a new frame $\{e_\mu\}$

$$e_\mu = R\gamma_\mu\tilde{R}. \quad (\text{B2})$$

We identify $v = e_0$ as the proper velocity associated with the expected history $x(\tau)$ of a particle, $v = \frac{dx}{d\tau}$. Our present objective is to derive an stress-energy tensor for a pressureless ideal fluid from the Dirac theory. The desired form of the tensor is as follows:

$$T_{\mu\nu} = \rho_m v_\mu v_\nu, \quad (\text{B3})$$

where $v = v_\mu \gamma^\mu$, and $\rho_m = m\rho$ as explained in the previous paper.

For later use, we define

$$\text{spin bivector : } S = \frac{1}{2} R\gamma_2\gamma_1\tilde{R} = \frac{1}{2} e_2 e_1 \quad (\text{B4})$$

$$\text{spin density trivector : } S_3 = \frac{1}{2} \psi i \gamma_3 \tilde{\psi} = \frac{1}{2} \rho e_2 e_1 v = \rho S v. \quad (\text{B5})$$

The Dirac equation is

$$\nabla \psi i \gamma_3 = m \psi. \quad (\text{B6})$$

An stress-energy tensor $T(a)$ is a linear vector function of a vector variable a , which denotes a flux of energy-momentum through a hypersurface with normal a at spacetime point x . The tensor $T(a)$ for the free Dirac field is given by [12,22]

$$T(a) = \gamma^\nu \langle a \partial_\nu \psi i \gamma_3 \tilde{\psi} \rangle = \gamma^\nu a \cdot \langle \partial_\nu \psi i \gamma_3 \tilde{\psi} \rangle_1. \quad (\text{B7})$$

This tensor is not symmetric, and its adjoint (transposed tensor) is

$$\bar{T}(a) = \partial_b \langle T(b) a \rangle = \langle a \cdot \nabla \psi i \gamma_3 \tilde{\psi} \rangle_1. \quad (\text{B8})$$

We can decompose $T(a)$ into a sum of a symmetric part and an anti-symmetric part as [18]

$$T(a) = T_S(a) + T_A(a), \quad (\text{B9})$$

where

$$T_S(a) = \frac{1}{2} (T(a) + \bar{T}(a)) = \frac{1}{2} \partial_a (a \cdot T(a)), \quad (\text{B10})$$

$$T_A(a) = \frac{1}{2} (T(a) - \bar{T}(a)) = \frac{1}{2} a \cdot (\partial_b \wedge T(b)) = -\frac{1}{2} a \cdot (\partial_b \wedge \bar{T}(b)). \quad (\text{B11})$$

From (B8) and (B5) we write

$$\begin{aligned}
 \partial_a \wedge \bar{T}(a) &= \dot{\nabla} \wedge \langle \psi i \gamma_3 \tilde{\psi} \rangle_1 \\
 &= \frac{1}{2} \langle \nabla \psi i \gamma_3 \tilde{\psi} - \gamma^\mu (\psi i \gamma_3 (\partial_\mu \psi)^\sim) \rangle_2 \\
 &= -\frac{1}{2} \langle \nabla \psi i \gamma_3 \tilde{\psi} + \gamma^\mu (\psi i \gamma_3 (\partial_\mu \psi)^\sim) \rangle_2 \\
 &= -\frac{1}{2} \langle \nabla (\psi i \gamma_3 \tilde{\psi}) \rangle_2 \\
 &= -\nabla \cdot S_3,
 \end{aligned} \tag{B12}$$

where we have used the fact that Dirac equation (B6) implies $\langle \nabla \psi i \gamma_3 \tilde{\psi} \rangle_2 = 0$. Thus according to (B11) and (B12), the anti-symmetric part of $T(a)$ can be written as

$$T_A(a) = \frac{1}{2} a \cdot (\nabla \cdot S_3) = \frac{1}{2} (a \wedge \nabla) \cdot S_3. \tag{B13}$$

We are now ready to define the local energy-momentum p , one of the most fundamental quantities of the Dirac theory, as [22]

$$\rho p = T(v) = v^\mu T_\mu, \tag{B14}$$

where $v = v^\mu \gamma_\mu$ and $T_\mu = T(\gamma_\mu)$ are understood. Now T_μ can be decomposed in the form [22]

$$T_\mu = \rho v_\mu p + N_\mu, \tag{B15}$$

where $N_\mu = N(\gamma_\mu)$ describe the flow of energy momentum normal to the velocity. This can be verified from (B14) and (B15), which reads $N(v) = N(v^\mu \gamma_\mu) = v^\mu N_\mu = 0$, since $v^\mu v_\mu = v^2 = 1$.

According to the conservation of energy and momentum, the divergence of an stress-energy tensor must be equal to the force density exerted on the particle. This relationship holds true even for free particles, where no external forces are acting on them. We thus write

$$\dot{T}(\dot{\nabla}) = \dot{T}(\dot{\nabla}) = \dot{T}(\gamma^\mu \dot{\partial}_\mu) = \partial_\mu \bar{T}^\mu = 0, \tag{B16}$$

where we have used the fact that the divergence of the anti-symmetric part of the stress-energy tensor vanishes. This is because, according to (B13), $\dot{T}_A(\dot{\nabla}) = 0$.

One advantage of STA is that the formulations of physical laws in Dirac theory, such as conservation laws and dynamics, are expressed in the same form as in classical mechanics. From (B2) it follows that

$$a \cdot \nabla e_\mu = \Omega(a) \times e_\mu, \tag{B17}$$

where

$$\Omega(a) = 2a \cdot \nabla R \tilde{R} \tag{B18}$$

is a bivector valued function of a vector variable, which can be explained as the angular velocity of the frame $\{e_\mu\}$ rotates in the a direction, and $A \times B = \frac{1}{2}(AB - BA)$. From (B7) and (B15) we write

$$T_{\mu\nu} = T_\mu \cdot \gamma_\nu = \gamma_\mu \cdot \langle \partial_\nu \psi i \gamma_3 \tilde{\psi} \rangle_1 = \rho v_\mu p_\nu + N_{\mu\nu}, \tag{B19}$$

where $p_\mu = p \cdot \gamma_\mu$ and $N_{\mu\nu} = N_\mu \cdot \gamma_\nu$. To express $T_{\mu\nu}$ with observables, we need to calculate $\partial_\nu \psi i \gamma_3 \tilde{\psi}$ from (2). We have

$$\begin{aligned}\partial_\nu \psi i \gamma_3 \tilde{\psi} &= \partial_\nu (\rho^{1/2} R) i \gamma_3 \rho^{1/2} \tilde{R} \\ &= \frac{1}{2} \partial_\nu \rho R i \gamma_3 \tilde{R} + \rho \partial_\nu R i \gamma_3 \tilde{R} \\ &= (\partial_\nu \ln \rho) S_3 + \Omega_\nu S_3 \\ &= (\partial_\nu \ln \rho) \rho S v + \rho \Omega_\nu S v \\ &= \rho (W_\nu - \partial_\nu S + \Omega_\nu S) v,\end{aligned}\tag{B20}$$

here, $\Omega_\nu = \Omega(\gamma_\nu) = 2\partial_\nu R \tilde{R}$; W_ν represents a bivector defined as

$$W_\nu \equiv \frac{1}{\rho} \partial_\nu (\rho S),\tag{B21}$$

which is called quantum potential, and it holds a pivotal position in quantum mechanics in the approach of causal interpretation [22]. The vector part of (B20) gives

$$\begin{aligned}\langle \partial_\nu \psi i \gamma_3 \tilde{\psi} \rangle_1 &= \Omega_\nu \cdot S_3 \\ &= \rho (W_\nu \cdot v - \partial_\nu S \cdot v + \Omega_\nu \cdot S v + (\Omega_\nu \times S) \cdot v) \\ &= \rho (W_\nu \cdot v + \Omega_\nu \cdot S v),\end{aligned}\tag{B22}$$

where we have used $\partial_\nu S = \Omega_\nu \times S$, which can be proved as follows

$$\begin{aligned}\partial_\mu S &= \frac{1}{2} \partial_\mu (R \gamma_2 \gamma_1 \tilde{R}) \\ &= \frac{1}{2} ((\partial_\mu R) \gamma_2 \gamma_1 \tilde{R} + R \gamma_2 \gamma_1 \partial_\mu \tilde{R}) \\ &= \frac{1}{2} (\Omega_\mu S - S \Omega_\mu) \\ &= \Omega_\mu \times S.\end{aligned}\tag{B23}$$

We thus get from (B7), (B14) and (B22),

$$T(v) = v \cdot \langle \partial_\nu \psi i \gamma_3 \tilde{\psi} \rangle_1 \gamma^\nu = (v \wedge \Omega_\nu) \cdot S_3 \gamma^\nu = \rho \Omega_\nu \cdot S \gamma^\nu = \rho p,\tag{B24}$$

which means

$$p_\nu = \Omega_\nu \cdot S.\tag{B25}$$

At last, from (B19), (B22) and (B25) we obtain [22]

$$T_{\mu\nu} = \rho v_\mu p_\nu + \rho (v \wedge \gamma_\mu) \cdot W_\nu.\tag{B26}$$

To achieve the desired result (B3), we need to identify the constraints that would align the momentum and velocity (i.e., $v \wedge p = 0$), and ensure the symmetry of the stress-energy tensor (i.e., $T(a) = \bar{T}(a)$ or $T_{\mu\nu} = T_{\nu\mu}$).

Let us first multiply the Dirac equation (B6) on the right by $\tilde{\psi}$ to get

$$\nabla \psi \gamma_2 \gamma_1 \tilde{\psi} = m \rho v.\tag{B27}$$

Next, we evaluate $\nabla\psi\gamma_2\gamma_1\tilde{\psi}$ by substituting ψ from (2), resulting in

$$\begin{aligned}\gamma^\mu(\partial_\mu\psi\gamma_2\gamma_1\tilde{\psi}) &= \gamma^\mu((\partial_\mu\sqrt{\rho})R\gamma_2\gamma_1\tilde{\psi} + \sqrt{\rho}(\partial_\mu R)\gamma_2\gamma_1\tilde{\psi}) \\ &= \gamma^\mu((\partial_\mu\rho)S + \rho\Omega_\mu S) \\ &= (\nabla\rho)S + \rho\gamma^\mu\Omega_\mu S.\end{aligned}\tag{B28}$$

The vector part of this equation gives

$$\begin{aligned}\langle\nabla\psi\gamma_2\gamma_1\tilde{\psi}\rangle_1 &= \langle(\nabla\rho)S + \rho\gamma^\mu\Omega_\mu S\rangle_1 \\ &= (\nabla\rho) \cdot S + \rho\gamma^\mu\Omega_\mu \cdot S + \rho\gamma^\mu \cdot (\Omega_\mu \times S) \\ &= (\nabla\rho) \cdot S + \rho p + \rho\nabla \cdot S \\ &= \nabla \cdot (\rho S) + \rho p.\end{aligned}\tag{B29}$$

From (B27) and (B29) we get

$$p = mv - \frac{1}{\rho}\nabla \cdot (\rho S).\tag{B30}$$

Decomposing the stress-energy tensor into its symmetric and anti-symmetric parts in terms of observables proves to be highly advantageous. In the main text, we specifically utilize the adjoint form outlined in this Appendix, hence we express $\tilde{T}_{\mu\nu}$ as derived from (B26) as follows:

$$\tilde{T}_\mu = \rho p_\mu v + [\partial_\mu(S_3 v)] \cdot v.\tag{B31}$$

By substituting p given by (B30), we have

$$\begin{aligned}\tilde{T}(a) &= \rho a \cdot p v + [a \cdot \nabla(S_3 v)] \cdot v \\ &= \rho \left[ma \cdot v - \frac{1}{\rho}(a \wedge \nabla) \cdot (S_3 v) \right] v + [a \cdot \nabla(S_3 v)] \cdot v \\ &= \rho_m a \cdot v v + [a \cdot \nabla(S_3 v)] \cdot v - (a \wedge \nabla) \cdot (S_3 v) v,\end{aligned}\tag{B32}$$

where $\rho_m = \rho m$ represents the mass density. We see that if $S_3 = 0$, we promptly obtain the desired expression for an stress-energy tensor that is appropriate to describe a pressureless ideal fluid

$$T(a) = \tilde{T}(a) = \rho_m a \cdot v v.\tag{B33}$$

Appendix C. GTG and Matter

By virtue of GC, GTG is constructed such that gravitational effects are described by a pair of gauge fields, $\bar{h}(a) = \bar{h}(a, x)$ and $\omega(a) = \omega(a, x)$, defined over a flat Minkowski background spacetime [29], where x is the STA position vector and is usually suppressed for short.

The first of them, $\bar{h}(a)$, is a position-dependent linear function mapping the vector argument a to vectors. The introduction of $\bar{h}(a)$ ensures covariance of the equations under arbitrary local displacements (or an arbitrary remapping $x' = f(x)$) of the matter fields in the background spacetime. In order to understand the physical meaning of the $\bar{h}$ field, we first define the covariant displacement transformation as

$$M(x) \xrightarrow{x'=f(x)} M'(x) = M(x'),\tag{C34}$$

so that the equations $A(x) = B(x)$ and $A(x') = B(x')$ have exactly the same physical content. Suppose we have a vector field $b(x) = \nabla\phi(x)$, where $\phi(x)$ is a scalar field that is already covariant under

displacement, i.e., $\phi'(x) = \phi(x')$. Now can we write $b'(x) = b(x')$ or $\nabla\phi'(x) = \nabla_{x'}\phi(x')$? By the chain rule we find

$$\begin{aligned} a \cdot \nabla\phi'(x) &= a \cdot \nabla\phi(x') \\ &= (a \cdot \nabla f(x)) \cdot \nabla_{x'}\phi(x') \\ &= \underline{f}(a) \cdot \nabla_{x'}\phi(x') \\ &= a \cdot \bar{f}(\nabla_{x'}\phi(x')), \end{aligned} \quad (C35)$$

where $\underline{f}(a) = a \cdot \nabla f(x)$ is a linear function of a and an arbitrary function of x , and $\bar{f}(\nabla_{x'}) = \nabla f(x) \cdot \nabla_{x'}$, and we call $\bar{f}$ the adjoint of $\underline{f}$, satisfying $a \cdot \underline{f}(b) = \bar{f}(a) \cdot b$, or $\bar{f}(a) = \partial_b \langle \underline{f}(b) a \rangle$. It follows that $\nabla\phi'(x) = \bar{f}(\nabla_{x'}\phi(x'))$, or

$$\nabla_x = \bar{f}(\nabla_{x'}) \quad \text{and} \quad \bar{f}^{-1}(\nabla_x) = \nabla_{x'}, \quad (C36)$$

which shows us that $b(x)$ is not covariant under displacement. In order to make the objects such as $b(x)$ covariant, we must introduce a position-gauge field $\bar{h}(a, x)$, which is a linear function of a and arbitrary function of x , so that

$$\bar{h}(a, x) \xrightarrow{x'=f(x)} \bar{h}'(a, x) = \bar{h}(\bar{f}^{-1}(a), x'). \quad (C37)$$

Now if we redefine $b(x) = \bar{h}(\nabla\phi(x))$, then

$$\begin{aligned} b(x) = \bar{h}(\nabla\phi(x)) \xrightarrow{x'=f(x)} b'(x) &= \bar{h}'(\nabla\phi'(x)) \\ &= \bar{h}(\bar{f}^{-1}(\nabla\phi'(x))) \\ &= \bar{h}(\nabla_{x'}\phi(x')) = b(x'), \end{aligned} \quad (C38)$$

which becomes covariant. The $\bar{h}(a)$ field plays the same role of vierbein in the tensor calculus approach of gauge theory of gravity [17,29]. For later use, we give the relationship between $\bar{h}(a)$ and the metric tensor $g_{\mu\nu}$ in GR. We define a position gauge invariant directional derivative as [25,29]

$$L_a = a \cdot \bar{h}(\nabla) = \underline{h}(a) \cdot \nabla, \quad (C39)$$

where $\underline{h}$ is the adjoint of $\bar{h}$ defined by $\underline{h}(b) \equiv \partial_c(\bar{h}(c) \cdot b)$, and a is an invariant vector (i.e., for $x' = f(x)$, $a(x)$ is transformed to $a'(x) = a(x')$). So $\underline{h}$ maps tangent vectors to tangent vectors and $\bar{h}$ maps cotangent vectors to cotangent vectors. For a given set of coordinates $\{x^\mu, \mu = 0, 1, 2, 3\}$, we introduce the basis vectors

$$e_\mu \equiv \frac{\partial x}{\partial x^\mu}, \quad e^\mu \equiv \nabla x^\mu, \quad (C40)$$

which satisfy $e_\mu \cdot e^\nu = \delta_\mu^\nu$. From these vectors we further define vectors

$$g_\mu \equiv \underline{h}^{-1}(e_\mu), \quad g^\mu \equiv \bar{h}(e^\mu). \quad (C41)$$

These vectors satisfy the relation

$$g_\mu \cdot g^\nu = \underline{h}^{-1}(e_\mu) \cdot \bar{h}(e^\nu) = e_\mu \cdot \bar{h}^{-1}(\bar{h}(e^\nu)) = e_\mu \cdot e^\nu = \delta_\mu^\nu. \quad (C42)$$

The metric tensor then is given by

$$g_{\mu\nu} \equiv g_\mu \cdot g_\nu. \quad (C43)$$

Let $x(\tau)$ be a time like curve (where τ is the proper time), a mapping $f: x \rightarrow x' = f(x)$ induces the transformation

$$\dot{x} = \frac{dx}{d\tau} \rightarrow \dot{x}' = \frac{dx'}{d\tau} = \frac{dx}{d\tau} \cdot \nabla f(x). \quad (C44)$$

Comparing Equation (C44) with Equation (C39), we introduce an invariant velocity $v = v(x(\tau))$ as [25]

$$\dot{x} = \underline{h}(v), \quad v = \underline{h}^{-1}(\dot{x}). \quad (\text{C45})$$

From the known formula $dx = dx^\mu e_\mu$, the invariant normalization $v^2 = 1$ induces the invariant line element on a timelike curve in GR

$$d\tau^2 = [\underline{h}^{-1}(dx)]^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (\text{C46})$$

Another gauge field, $\omega(a)$, is a position-dependent linear function mapping the vector argument a to bivectors. Its introduction ensures covariance of the equations of physics under local Lorentz rotations described by the rotor R . Under local Lorentz rotations, the multivector M transforms as $M' = RM\tilde{R}$ and the spinor ψ transforms as $\psi' = R\psi$. To ensure covariance of the quantities like $\bar{h}(\nabla)M$ and $\bar{h}(\nabla)\psi$ under local Lorentz rotations, $\bar{h}(\nabla) = \bar{h}(\partial_a)a \cdot \nabla$ should be replaced by a covariant derivative D [29]. To achieve this we focus attention on $a \cdot \nabla\psi' = a \cdot \nabla(R\psi)$ and write

$$a \cdot \nabla(R\psi) = Ra \cdot \nabla\psi + (a \cdot \nabla R)\psi. \quad (\text{C47})$$

Clearly, the presence of the term $(a \cdot \nabla R)\psi$ renders the operator $a \cdot \nabla$ non-covariant. Since the rotor R satisfies $R\tilde{R} = 1$ we find that

$$a \cdot \nabla R\tilde{R} + Ra \cdot \nabla\tilde{R} = 0, \quad (\text{C48})$$

which implies

$$a \cdot \nabla R\tilde{R} = -Ra \cdot \nabla\tilde{R} = -(a \cdot \nabla R\tilde{R})^\sim. \quad (\text{C49})$$

Hence $a \cdot \nabla R\tilde{R}$ is equal to minus its reverse and thus must be a bivector in the STA. We can therefore rewrite (C47) as

$$a \cdot \nabla(R\psi) = Ra \cdot \nabla\psi + \frac{1}{2}(2a \cdot \nabla R\tilde{R})(R\psi). \quad (\text{C50})$$

This suggests that, to achieve a covariant derivative we must add a connection term to $a \cdot \nabla$ to construct an operator

$$D_a\psi = a \cdot \nabla\psi + \frac{1}{2}\Omega(a)\psi \quad (\text{C51})$$

which must be covariant under local Lorentz rotations. The connection $\Omega(a) = \Omega(a, x)$ is a bivector valued linear function of a with an arbitrary x dependence. Under local rotations we expect that the operator D_a be unchanged in form, namely,

$$D'_a = a \cdot \nabla + \frac{1}{2}\Omega'(a), \quad (\text{C52})$$

here we have used fact that $a \cdot \nabla$ cannot change under local rotations. Nevertheless, the property that the covariant derivative must satisfy is

$$\begin{aligned} D'_a\psi' &= (a \cdot \nabla + \frac{1}{2}\Omega'(a))(R\psi) \\ &= RD_a\psi = R(a \cdot \nabla\psi + \frac{1}{2}\Omega(a)\psi). \end{aligned} \quad (\text{C53})$$

From these identities it follows that $\Omega(a)$ transforms as

$$\Omega'(a) = R\Omega(a)\tilde{R} - 2a \cdot \nabla R\tilde{R}. \quad (\text{C54})$$

Now we reassemble the covariant derivative (C51) with the term $\bar{h}(\partial_a)$ to form

$$D \equiv \bar{h}(\partial_a)D_a, \quad (\text{C55})$$

and write

$$D\psi = \hbar(\partial_a)(a \cdot \nabla\psi + \frac{1}{2}\Omega(a)\psi). \quad (\text{C56})$$

Note that the vector derivative D is fully covariant, but D_a is not since it contains the $\Omega(a)$ field, which must transform in the same way as $a \cdot \nabla R\tilde{R}$ under displacement and thus picks up a term in $\underline{f}$ (A66). Recall the definition of the position gauge invariant directional derivative (C39), we thus define

$$a \cdot D\psi = a \cdot \hbar(\nabla)\psi + \frac{1}{2}\omega(a)\psi, \quad (\text{C57})$$

where $\omega(a)$ is defined by

$$\omega(a) = \Omega(h(a)). \quad (\text{C58})$$

It should be pointed out that the vector a is declared to be position gauge invariant, as stated below equation (C39). Therefore, D_a is related to $a \cdot D$ by

$$\begin{aligned} D_a\psi &= \underline{h}^{-1}(a) \cdot D\psi \\ &= \underline{h}^{-1}(a) \cdot \hbar(\nabla)\psi + \frac{1}{2}\omega(\underline{h}^{-1}(a))\psi \\ &= a \cdot \nabla\psi + \frac{1}{2}\Omega(a)\psi, \end{aligned} \quad (\text{C59})$$

or simply

$$D_a = \underline{h}^{-1}(a) \cdot D. \quad (\text{C60})$$

We thus also have

$$D\psi \equiv \partial_a \left(a \cdot \hbar(\nabla)\psi + \frac{1}{2}\omega(a)\psi \right). \quad (\text{C61})$$

We are now ready to establish the form of the covariant derivatives of the observables formed from a spinor field. In general, such observables have the form

$$M = \psi\Gamma\tilde{\psi}, \quad (\text{C62})$$

where Γ is a constant multivector formed from combinations of γ_0 , γ_3 and $i\sigma_3$. Under displacements $x' = f(x)$, the spinor fields are covariant: $\psi'(x) = \psi(x')$. Thus the observable M inherits its transformation properties from the spinor ψ ,

$$M(x) \xrightarrow{x'=f(x)} M'(x) = M(x'), \quad (\text{C63})$$

exactly the same form in (C34). Under rotations, the spinor transforms as $\psi' = R\psi$. So it follows from (C62) that, under rotations M transforms as

$$M(x) \xrightarrow{R} M' = \psi'\Gamma\tilde{\psi}' = R\psi\Gamma\tilde{\psi}R = RM\tilde{R}. \quad (\text{C64})$$

To achieve the fully covariant derivative for M , we first write

$$L_a M = L_a\psi\Gamma\tilde{\psi} + \psi\Gamma(L_a\psi)^\sim. \quad (\text{C65})$$

This equation is displacement covariant. We simply replace L_a with $a \cdot D$ to obtain

$$\begin{aligned} a \cdot DM &= a \cdot D\psi\Gamma\tilde{\psi} + \psi\Gamma(a \cdot D\psi)^\sim \\ &= L_a(\psi\Gamma\tilde{\psi}) + \omega(a) \times (\psi\Gamma\tilde{\psi}) \\ &= L_a M + \omega(a) \times M. \end{aligned} \quad (\text{C66})$$

Note the difference between the forms of $a \cdot D$ acting on M and ψ .

The field strength corresponding to the $\omega(a)$ gauge field is defined by

$$[a \cdot D, b \cdot D]\psi = \frac{1}{2}\mathcal{R}(a \wedge b)\psi, \quad (\text{C67})$$

where

$$\mathcal{R}(a \wedge b) \equiv L_a\omega(b) - L_b\omega(a) + \omega(a) \times \omega(b), \quad (\text{C68})$$

a and b are constant vectors. Ricci tensor $\mathcal{R}(a)$, Ricci scalar $\mathcal{R}$ and Einstein tensor $\mathcal{G}(a)$ are defined, respectively, as:

$$\mathcal{R}(a) = \partial_b \cdot \mathcal{R}(b \wedge a), \quad (\text{C69})$$

$$\mathcal{R} = \partial_a \cdot \mathcal{R}(a), \quad (\text{C70})$$

$$\mathcal{G}(a) = \mathcal{R}(a) - \frac{1}{2}a\mathcal{R}. \quad (\text{C71})$$

The overall action integral is of the form

$$I = \int |d^4x| \det(h)^{-1} \left(\frac{1}{2}\mathcal{R} - \kappa\mathcal{L}_m \right), \quad (\text{C72})$$

where $\mathcal{L}_m$ describes the matter content and $\kappa = 8\pi$. In this paper, we adopt the covariant $\mathcal{L}_m = Di\gamma_3\tilde{\psi} - m\psi\tilde{\psi}$ from Dirac theory to describe the macroscopic matter, which has been well-studied [9,13,29] for electrons. From (C72), we obtain the following equations that describe the field coupled self-consistently to gravity [13]:

$$\text{torsion: } D \wedge \bar{h}(a) = \kappa \bar{h}(a) \cdot S, \quad (\text{C73})$$

$$\text{Einstein: } \mathcal{G}(a) = \kappa \mathcal{T}(a) \quad (\text{C74})$$

$$\text{Dirac: } D\psi i\sigma_3 = m\psi\gamma_0, \quad (\text{C75})$$

where $D \wedge \bar{h}(a)$ is the gravitational torsion which is determined by the trivector spin density $S \equiv \frac{1}{2}\psi i\gamma\tilde{\psi}$, $\kappa = 8\pi$, and

$$\mathcal{T}(a) = \langle a \cdot D\psi i\gamma_3\tilde{\psi} \rangle_1 \quad (\text{C76})$$

is the matter stress-energy tensor. We can solve Equation (C73) for $\omega(a)$ to obtain [13]

$$\omega(a) = \omega'(a) + \frac{1}{2}\kappa a \cdot S = -H(a) + \frac{1}{2}a \cdot [\partial_b \wedge H(b)] + \frac{1}{2}\kappa a \cdot S, \quad (\text{C77})$$

this defines $\omega'(a)$ as the ω -function in the absence of torsion, and

$$H(a) \equiv \bar{h}(\dot{\nabla} \wedge \dot{h}^{-1}(a)) = -\bar{h}(\dot{\nabla}) \wedge \dot{h}(\bar{h}^{-1}(a)), \quad (\text{C78})$$

where ‘overdot’ notation is employed to denote the scope of a differential operator. It proves convenient if we employ the primed symbols to denote the torsion free part of the curvature tensors, we obtain [13]

$$\begin{aligned} \mathcal{R}(a \wedge b) &= \mathcal{R}'(a \wedge b) + \frac{1}{4}\kappa^2[(a \wedge b) \cdot S] \cdot S \\ &\quad - \frac{1}{2}\kappa[(a \wedge b) \cdot D] \cdot S, \end{aligned} \quad (\text{C79})$$

$$\begin{aligned} \mathcal{R}(a) &= \mathcal{R}'(a) + \frac{1}{2}\kappa^2(a \cdot S) \cdot S \\ &\quad - \frac{1}{2}\kappa a \cdot (D \cdot S), \end{aligned} \quad (\text{C80})$$

$$\mathcal{R} = \mathcal{R}' + \frac{3}{2}\kappa^2 S^2. \quad (\text{C81})$$

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