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Article

$2^m - 1$ as the Integer Formulation to Govern the Dynamics of Collatz-Type Sequences

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Abstract: It is discovered that writing an odd integer as $2^m - 1$ for $m \geq 1$ helps in understanding the dynamics of the Collatz-type sequences. Starting with the original Collatz sequence $3n + 1$, it is found that when the odd step is applied to an odd integer $2^m - 1$, an even integer $2^{m+1} + 2^m - 2$ is obtained, which is exactly once divisible by 2. This implies that the sequence alternates between odd and even steps m times. This governs the dynamics of the Collatz-type sequences because: (i) for the sequence to diverge to infinity, m must be infinite, and (ii) the value of m determines the number of times the integer can be divided by 2 in each even step. This formulation allows construction of odd integers like $\sum 2^m - 1$ to follow specific patterns of odd and even steps. Applying this understanding to the modified Collatz sequence $5n + 1$ reveals that for $m = 3$, the integer $2^{m+1} + 2^m - 1$ is obtained, indicating that any integer formulation ending with $2^3 - 1$ in $5n + 1$ will diverge to infinity.

Keywords: Collatz; $3n+1$

1. Introduction

The Collatz problem [1–4], defines the following set of rules: If n is odd, multiply it by 3, and add 1. If n is even, it is divided by 2.

The associated Collatz conjecture states that every integer ultimately reduces to unity. To prove this conjecture, it must be shown not only that the sequence eventually cycles through 1, 4, 2, 1, but also that no integer diverges to infinitely larger integers [5,6].

While a complete proof may be impossible, this article attempts to understand the working of Collatz-type sequences. For this, the odd integers are expressed as $2^m - 1$ for $m \geq 1$. By examining the integers that result from this seed form, insights are gained into the patterns of the odd-even steps. The conditions that govern the progression of Collatz-type sequences are immediately made clear through the use of the modified binary form $2^m - 1$. For this reason, $2^m - 1$ is stated as the governing integer formulation for Collatz-type sequences.

2. Behavior of Collatz Sequence $3n + 1$ with $2^m - 1$

In the following text, $\mathcal{O}1, \mathcal{O}2, \dots$ denote the resulting integer at the end of the odd step while $\mathcal{E}1, \mathcal{E}2, \dots$ denote the resulting integer at the end of the even step. The modified binary expression of integers at the end of each step is:

$$\begin{aligned}
 n &= 2^m - 1 \\
 \mathcal{O}1 &= (2 + 1)(2^m - 1) + 1 \\
 &= 2^{m+1} + 2^m - 2 \\
 \mathcal{E}1 &= 2^m + 2^{m-1} - 1 \\
 \mathcal{O}2 &= 2^{m+2} + 2^{m-1} - 2 \\
 \mathcal{E}2 &= 2^{m+1} + 2^{m-2} - 1 \\
 \mathcal{O}3 &= 2^{m+2} + 2^{m+1} + 2^{m-1} + 2^{m-2} - 2 \\
 \mathcal{E}3 &= 2^{m+1} + 2^m + 2^{m-2} + 2^{m-3} - 1 \\
 \mathcal{O}4 &= 2^{m+3} + 2^{m+1} + 2^{m-3} - 2 \\
 \mathcal{E}4 &= 2^{m+2} + 2^m + 2^{m-4} - 1
 \end{aligned}$$

The following observations are made:

- The number of divisions in each even step depends on m . If the lowest index of 2 vanishes, then the negative 1 is cancelled by 2^0 . The resulting integer is even, leading to an additional $\mathcal{E}$ step.
- The reduction in index at an $\mathcal{E}$ step is compensated by multiplying by $(2 + 1)$ at the $\mathcal{O}$ step. However, the reduction in index by an additional $\mathcal{E}$ step cannot be compensated and is carried forward for the remainder of the cycle. However the reduction in index by an additional $\mathcal{E}$ step cannot be compensated and is carried as is for the remainder of cycle.
- Once the first 2^0 is reached, additional $\mathcal{E}$ steps occur after fewer $\mathcal{OE}$ cycles since the lowest index is now less than m .
- After m additional $\mathcal{E}$ steps, the term 2^m is reduced to 2^0 , and all lower index terms vanish. The value m is also deducted from higher indices.
- The seed integer $2^m - 1$ cannot occur as the last term in the modified binary expression of any odd integer. The reasons are: (1) If the end term results in 2^0 , it cancels the negative 1.. (2) The lower indices cannot coalesce together to produce 2^m as there is always a missing index. (3) If lower indices somehow coalesce together and result in 2^m , then the resulting 2^m will merge with the already existing 2^m to produce 2^{m+1} .

3. Controlling Collatz Sequence Using $2^m - 1$

Since the behavior of the Collatz sequence is understood, it is now possible to estimate the starting integer based on a given pattern of $\mathcal{OE}$ cycles, or to estimate the cycle pattern based on the integer. An example of estimating the integer based on a given $\mathcal{OE}$ cycle is presented.

3.1. Estimating Integer Based on a Cycle Pattern

Suppose the following cycle pattern is desired:

$$\underbrace{\mathcal{OEE}}_{\alpha} \underbrace{\mathcal{OEE}}_{\beta} \underbrace{\mathcal{OEEOEE}}_{\gamma}$$

The cycle is segmented into α , β , and γ blocks depending on termination by the extra $\mathcal{E}$ step. Let the integer be of the form $2^\gamma + 2^\beta + 2^\alpha - 1$. The value of α is visually determined as 1.

The residue of $2^\alpha - 1$ at the end of α block is one. The lowest index term at the end of α block determine the cycle pattern for the β block. The lowest index terms resulting from 2^β and 2^γ at the end of the α block are estimated using binary formulation in previous section.

$$\mathcal{OEE}(n) = \dots + 2^{\alpha-1} + 2^{\alpha-2} + 2^{\beta-1} + 2^{\beta-2} + 1$$

The higher index terms are ignored. The integer formulation should end in $2^1 - 1$ since there is one $\mathcal{OE}$ cycle. Additionally, let $\gamma > \beta$, therefore, the lowest index term is $2^{\beta-2} + 1$ and this should be equivalent to $2^1 - 1$.

$$2^{\beta-2} + 1 \equiv 2^1 - 1$$

$$\beta \geq 4$$

In a similar manner, the lowest index at the end of β block are

$$\begin{aligned} \mathcal{OEEOE}(n) &= \dots + 2^{\alpha-2} + 2^{\alpha-3} + 2^{\alpha-3} + 2^{\alpha-4} + 2^{\beta-2} + 2^{\beta-3} + 2^{\beta-3} + 2^{\beta-4} + 1 \\ \mathcal{OEEOE}(n) &= \dots + 2^{\alpha-1} + 2^{\alpha-4} + 2^{\beta-1} + 2^{\beta-4} + 1 \\ \mathcal{OEEOE}(n) &= \dots + 2^{\alpha-4} + 2^{\beta-4} + 1 \end{aligned}$$

The higher index terms are obtained from coalescing lower index terms are removed. The lowest index terms are $2^{\alpha-4} + 2^{\beta-4} + 1$ and this should be equivalent to $2^2 - 1$ since the $\mathcal{OE}$ continues for two cycle.

$$2^{\alpha-4} + 2^{\beta-4} + 1 \equiv 2^2 - 1$$

If $\beta = 5$, the term $2^{\beta} + 1$ becomes equal to $2^2 - 1$. Therefore, the value of α is greater than 4. Some of the integers obtained for different values of α along with their cycle are given.

3.1.1. $\alpha = 5$

The integer is $2^5 + 2^5 + 2^1 - 1 = 65$. The Collatz cycle is $\mathcal{OE}\mathcal{E}\mathcal{O}\mathcal{E}\mathcal{E}\mathcal{O}\mathcal{E}\mathcal{E}\mathcal{E}$. The cycle is correct till the β block but then differs for the γ block. It happens because the value of α and β is taken equal that results in index coalescing.

3.1.2. $\alpha = 6$

The integer is $2^6 + 2^5 + 2^1 - 1 = 97$. The Collatz cycle is $\mathcal{OE}\mathcal{E}\mathcal{O}\mathcal{E}\mathcal{E}\mathcal{O}\mathcal{E}\mathcal{O}\mathcal{E}\mathcal{E}$. The cycle is improvement over the previous one but still not correct. This occurs because $2^{\alpha-4} + 2^{\beta-4} + 1 = 2^2 + 2^2 - 1$. As one can see, the indices are same and coalesce to become $2^3 - 1$ for which the $\mathcal{OE}$ repeat three times.

3.1.3. $\alpha = 7$

The integer is $2^7 + 2^5 + 2^1 - 1 = 161$ and the Collatz cycle is $\mathcal{OE}\mathcal{E}\mathcal{O}\mathcal{E}\mathcal{E}\mathcal{O}\mathcal{E}\mathcal{O}\mathcal{E}\mathcal{E}$ which is the desired cycle.

3.1.4. $\alpha = 8$

The integer is $2^8 + 2^5 + 2^1 - 1 = 289$ and the Collatz cycle is $\mathcal{OE}\mathcal{E}\mathcal{O}\mathcal{E}\mathcal{E}\mathcal{O}\mathcal{E}\mathcal{O}\mathcal{E}\mathcal{E}$ which is also the desired cycle. Similarly, any integer $2^{\alpha} + 2^5 + 2^1 - 1$ will follow the desired cycle for any $\alpha \geq 7$.

4. Application to Collatz-Type $5n + 1$ Sequence

Consider the binary expression of integers obtained when $5n + 1$ is applied to $2^m - 1$:

$$\begin{aligned} n &= 2^m - 1 \\ \mathcal{O}1 &= (2^2 + 1)(2^m - 1) + 1 \\ &= 2^{m+2} + 2^m - 4 \\ \mathcal{E}1 &= 2^{m+1} + 2^{m-1} - 2 \\ \mathcal{E}1 &= 2^m + 2^{m-2} - 1 \\ \mathcal{O}2 &= 2^{m+2} + 2^{m+1} + 2^{m-2} - 2^2 \\ \mathcal{E}2 &= 2^{m+1} + 2^m + 2^{m-3} - 2 \end{aligned}$$

In contrast to $3n + 1$, the sequence $5n + 1$ is different because:

- At least two $\mathcal{E}$ steps are required to obtain the negative 1.
- If the lowest index of 2 is zero after the first $\mathcal{E}$ step, the resulting integer is not even since $2^0 - 2 = -1$.

Consider the integer obtained after the first $\mathcal{E}_2$ step. If the lowest index of 2 is let zero, the resulting integer will be $2^{m+1} + 2^m - 1$. It is seen that the seed integer $2^m - 1$ reoccurs with a higher index term. If the cycle $\mathcal{OE}\mathcal{E}\mathcal{O}\mathcal{E}$ is applied again, the resulting integer is $\dots + 2^{m+2} + 2^m - 1$. This pattern suggests

that any integer formulation ending in $2^3 - 1$ will diverge to infinity for the Collatz-type sequence $5n + 1$.

The lowest integer that diverges for the sequence $5n + 1$ is $2^3 - 1 = 7$.

5. Conclusion

It has been recently discovered that expressing odd integers in the alternate binary form $2^m - 1$ aids in understanding the workings of Collatz-type sequences. For the original Collatz sequence $3n + 1$, the odd step $(2 + 1)(2^m - 1) + 1$ yields an even integer that ends in -2 , thus exactly followed by one even step. A second even step occurs if the lowest index in the binary representation is reduced to zero, in which case 2^0 cancels the negative 1.

The benefit of the alternate binary form is that it helps in predicting the patterns of odd and even steps. This knowledge can be applied in reverse to predict the seed integer that generates the desired step patterns.

Finally, the same method is applied to the Collatz-type sequence $5n + 1$, showing that any integer formulation ending in $2^3 - 1$ will yield a never-ending $\mathcal{O}\mathcal{E}\mathcal{E}\mathcal{O}\mathcal{E}$ pattern in which the integer values diverge.

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