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Article

On Proving an Inequality of Ramanujan Using Explicit Order Estimates for the Mertens Function

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Abstract: This research article provides an unconditional proof of an inequality proposed by *Srinivasa Ramanujan* involving the Prime Counting Function $\pi(x)$, $(\pi(x))^2 < \frac{ex}{\log x} \pi\left(\frac{x}{e}\right)$ for every real $x \geq \exp(547)$, using specific order estimates for the *Mertens Function*, $M(x)$. The proof primarily hinges upon investigating the underlying relation between $M(x)$ and the *Second Chebyshev Function*, $\psi(x)$, in addition to applying the meromorphic properties of the *Riemann Zeta Function*, $\zeta(s)$ with an intention of deriving an improved approximation for $\pi(x)$.

Keywords: Riemann Zeta Function; Mertens Function; Chebyshev Function; Arithmetic Function; Error Estimates; Perron's Formula; Möbius Inversion Formula; Dirichlet Partial Summation Formula

MSC: [2020] Primary 11A41; 11A25; 11N05; 11N37; 11N56; Secondary 11M06; 11M26

1. Introduction and Motivation

The motivation for investigating the distribution of prime numbers over the real line \mathbb{R} first reflected in the writings of famous mathematician *Ramanujan*, as evident from his letters [20] [pp. xxiii-xxx, 349-353] to one of the most prominent mathematicians of 20th century, *G. H. Hardy* during the months of Jan/Feb of 1913, which are testaments to several strong assertions about *prime numbers*, especially the *Prime Counting Function*, $\pi(x)$ [cf. Definition (3)].

In the following years, Hardy himself analyzed some of those results [21,22] [, pp. 234-238] and even wholeheartedly acknowledged them in many of his publications, one such notable result is the *Prime Number Theorem* [cf. Theorem 2].

Ramanujan provided several inequalities regarding the behaviour and the asymptotic nature of $\pi(x)$. One of such relation can be found in the notebooks written by Ramanujan himself has the following claim.

Theorem 1. (*Ramanujan's Inequality* [1]) For x sufficiently large, we shall have,

$$(\pi(x))^2 < \frac{ex}{\log x} \pi\left(\frac{x}{e}\right) \quad (1)$$

Worth mentioning that, Ramanujan indeed provided a simple, yet unique solution in support of his claim. Furthermore, it has been well established that, the result is not true for every positive real x . Thus, the most intriguing question that the statement of Theorem 1 poses is, *is there any x_0 such that, Ramanujan's Inequality will be unconditionally true for every $x \geq x_0$?*

A brilliant effort put up by *F. S. Wheeler*, *J. Keiper*, and *W. Galway* in search for such x_0 using tools such as MATHEMATICA went in vain, although independently *Galway* successfully computed the largest prime counterexample below 10^{11} at $x = 38\,358\,837\,677$. However, *Hassani* [19] [Theorem 1.2] proposed a more inspiring answer to the question in a way that, \exists such $x_0 = 138\,766\,146\,692\,471\,228$ with (1) being satisfied for every $x \geq x_0$, but *one has to necessarily assume the Riemann Hypothesis*. In a recent paper by *A. W. Dudek* and *D. J. Platt* [2] [Theorem 1.2], it has been established that, *Ramanujan's Inequality* holds true unconditionally for every $x \geq \exp(9658)$. Although this can be considered as an exceptional achievement in this area, efforts of further improvements to this bound are already underway. For instance, *Mossinghoff* and *Trudgian* [24] made significant progress in this endeavour, when they established a better estimate as, $x \geq \exp(9394)$. One recent even better result

by Axler [23] suggests that, the lower bound for x , namely $\exp(9658)$ can in fact be further improved upto $\exp(3158.442)$ using similar techniques as described in [2], although modifying the error term accordingly.

This article shall provide in detail, a new proof of Ramanujan's Inequality, using a completely different technique by introducing the notion of *Mertens Function* [11,12]. We shall utilize one of the most significant order properties of $M(x)$ [13], namely, $M(x) = O(\sqrt{x})$ in order to find an improved estimate for $\pi(x)$ [cf. Section 3.3]. Thus in turn, we shall verify the inequality in the final part of the article [cf. Section 4]. As an application to this method, we shall be able to refine the lower bound for x even further in order for Theorem 1 to hold true without any further assumptions.

2. Arithmetic Functions

As for definition, *Arithmetic Functions* are in fact complex-valued functions on the set of Natural Numbers \mathbb{N} .

For the convenience of the readers, let us first introduce some notations, under standard assumption that, $x \in \mathbb{R}$.

Definition 1. We say $f(x)$ is asymptotic to $g(x)$, and denote it by, $f(x) \sim g(x)$ if, $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 1$.

Definition 2. (Big O Notation) Given $g(x) > 0 \quad \forall x \geq a$, the notation, $f(x) = O(g(x))$ implies that, the quotient, $\frac{f(x)}{g(x)}$ is bounded for all $x \geq a$; i.e., \exists a constant $M > 0$ such that,

$$|f(x)| \leq M.g(x) \quad , \forall x \geq a .$$

In this section, we shall discuss about a few specific important such type of arithmetic functions pertaining to the context of the paper and the proof of the original result.

2.1. Prime Counting Function

Definition 3. For each $x \geq 0$, we define,

$$\pi(x) := \text{The number of primes } \leq x .$$

The most important contribution of $\pi(x)$ is undoubtedly to the *Prime Number Theorem* [8,16], which can be stated as follows.

Theorem 2. For every real $x \geq 0$, the following estimate is valid.

$$\pi(x) \sim \frac{x}{\log x} \tag{2}$$

Equivalently,

$$\lim_{x \rightarrow \infty} \frac{\pi(x) \log x}{x} = 1 \tag{3}$$

For an elementary proof of above, readers can refer to [17].

2.2. Chebyshev Function

Chebyshev ψ Function [7] has the following definition.

Definition 4. For each $x \geq 0$, we define,

$$\psi(x) := \sum_{n \leq x} \Lambda(n) ,$$

Where ,

$$\Lambda(n) := \begin{cases} \log p, & \text{if } n = p^m, p^m \leq x, m \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases} \quad (4)$$

$\Lambda(n)$ is said to be the "Mangoldt Function" .

An important observation is,

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{m=1}^{\infty} \sum_{p, p^m \leq x} \Lambda(p^m) = \sum_{m=1}^{\infty} \sum_{p \leq x^{\frac{1}{m}}} \log p \quad (5)$$

In fact, one can use the ψ function in order to simplify the statement of the *Prime Number Theorem* (2). In other words, one can deduce that, proving the theorem is equivalent to proving the following statement [6],

$$\psi(x) \sim x \quad \text{as } x \rightarrow \infty. \quad (6)$$

2.3. Möbius Function

We start with the formal definition.

Definition 5. (Möbius Function) $\mu : \mathbb{N} \rightarrow \{0, \pm 1\}$ is defined as follows:

$$\mu(n) := \begin{cases} (-1)^k & \text{if } n = \prod_{i=1}^k p_i^{a_i} \text{ such that, } \gcd(p_i, p_j) = 1 \quad \forall i \neq j \\ 1 & \text{if, } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

One can in fact use definition (5) to deduce the following property regarding the Möbius Function.

Proposition 1. [15] [Theorem 2.1 , pp. 25]

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if, } n = 1 \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

2.4. Mertens Function

Definition 6. The Mertens Function [13] $M : \mathbb{N} \rightarrow \mathbb{Z}$ has the representation,

$$M(n) := \sum_{d=1}^n \mu(d) \quad (8)$$

Remark 1. In general, there's a notion of the Extended Mertens Function,

$$M(x) := \sum_{1 \leq n \leq x} \mu(n), \quad \forall x \in \mathbb{R}$$

In his paper [4], Mertens conjectured that, for all $M(n)$ with $1 \leq n \leq 10^4$, we shall have,

$$|M(n)| < \sqrt{n} \quad (9)$$

This is also known as the *Mertens Hypothesis*. [Interested readers can refer to [3, Theorem 14.28, pp. 374]]

Extending *Mertens'* results further upto $n = 5 \times 10^6$, *Sterneck* [5] conjectured that,

$$|M(n)| < \frac{1}{2}\sqrt{n}, \quad \forall n > 200.$$

The primary objective for *Mertens* behind introducing the function $M(x)$ (As defined in (1)) was its underlying relation to the location of the zeros of the *Riemann Zeta Function* $\zeta(s)$, the reason being largely due to it's consequences for the distribution of the primes, also hailed as one of the most important unsolved problems in Analytic Number Theory. We shall be working with a particular order estimate of $M(x)$ in later sections of the text, although readers are encouraged to consult [9,10,13,14] for further details.

2.5. Some Necessary Derivations

Proposition 2. *The Dirichlet Series Representation* [3, Theorem 3.13, pp. 62] for $\mu(n)$ is given by:

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \quad \Re(s) > 1. \quad (10)$$

$\zeta(s)$ denoting the Riemann Zeta Function.

Proposition 3. *The following order estimates hold true:*

1. $\sum_{d \leq x} \frac{\mu(d)}{d} = O\left(\frac{1}{\log x}\right)$
2. $\sum_{d \leq x} \frac{\mu(d) \log(d)}{d} = O(1)$
3. $\sum_{d \leq x} \mu(d) = M(x) = O(\sqrt{x})$

Proof. 1. Near $s = 1$, we have the expansion for $\zeta(s)$:

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$$

Where, γ denotes the *Euler Constant*. Thus,

$$\frac{1}{\zeta(s)} = (s-1) - \gamma(s-1)^2 + O((s-1)^3)$$

Using *Perron's Formula*, we have:

$$\sum_{n \leq x} \frac{\mu(n)}{n} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{\zeta(s)} \frac{x^s}{s} ds + O\left(\frac{x \log^2 x}{T}\right) \quad (11)$$

where $c > 1$ and T is a parameter to be chosen later.

We evaluate the integral using the following steps.

Step 1: Integral around $s = 1$

Consider a small semicircle Γ (say) of radius ϵ around $s = 1$ having the following parametrization, $s = 1 + \epsilon e^{i\theta}$, $-\pi/2 \leq \theta \leq \pi/2$.

In this region,

$$\frac{1}{\zeta(s)} = \epsilon e^{i\theta} - \gamma \epsilon^2 e^{2i\theta} + O(\epsilon^3)$$

and,

$$\begin{aligned}\frac{x^s}{s} &= \frac{x^{1+\epsilon e^{i\theta}}}{1+\epsilon e^{i\theta}} = x \cdot \frac{x^{\epsilon e^{i\theta}}}{1+\epsilon e^{i\theta}} = x \cdot e^{\epsilon e^{i\theta} \log x} (1 - \epsilon e^{i\theta} + O(\epsilon^2)) \\ &= x(1 + \epsilon e^{i\theta} \log x + O(\epsilon^2))(1 - \epsilon e^{i\theta} + O(\epsilon^2)) \\ &= x(1 + \epsilon e^{i\theta}(\log x - 1) + O(\epsilon^2))\end{aligned}$$

Thus, the integrand becomes:

$$\begin{aligned}\frac{1}{\zeta(s)} \frac{x^s}{s} &= x(\epsilon e^{i\theta} - \gamma \epsilon^2 e^{2i\theta} + O(\epsilon^3))(1 + \epsilon e^{i\theta}(\log x - 1) + O(\epsilon^2)) \\ &= x(\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}(\log x - 1 - \gamma) + O(\epsilon^3))\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\Gamma} \frac{1}{\zeta(s)} \frac{x^s}{s} &= \int_{-\pi/2}^{\pi/2} x(\epsilon e^{i\theta} + \epsilon^2 e^{2i\theta}(\log x - 1 - \gamma) + O(\epsilon^3)) i e^{i\theta} d\theta \\ &= \int_{-\pi/2}^{\pi/2} x(\epsilon^2 e^{2i\theta}(\log x - 1 - \gamma) + O(\epsilon^3)) i e^{i\theta} d\theta \quad \left[\because \int_{-\pi/2}^{\pi/2} e^{2i\theta} d\theta = 0 \right] \\ &= \epsilon^2 x(\log x - 1 - \gamma) \int_{-\pi/2}^{\pi/2} i e^{3i\theta} d\theta + O(\epsilon^3)\end{aligned}$$

Again, the integral of $e^{3i\theta}$ over a symmetric interval around zero is zero. Therefore, the integral around the small semicircle contributes a negligible amount of $O(\epsilon^3 x)$.

Step 2: Integral along the vertical line $s = c + it$

For the part of the integral along the vertical line, say, $L: s = c + it, -T \leq t \leq T$, where $c > 1$, i.e.,

$$\int_{c-iT}^{c+iT} \frac{1}{\zeta(s)} \frac{x^s}{s} ds$$

It can indeed be verified that, $\frac{1}{\zeta(s)}$ is bounded on L as, $\zeta(s)$ does not have any pole for $\text{Re}(s) > 1$. Specifically, for $s = c + it$ with $c > 1$, $\zeta(s)$ is bounded away from zero, so $\frac{1}{\zeta(s)}$ is bounded.

An appropriate choice of $T = \sqrt{x}$ gives us a bound on the integral:

$$\begin{aligned}\left| \int_{c-iT}^{c+iT} \frac{1}{\zeta(s)} \frac{x^s}{c+it} ds \right| &\leq \int_{-T}^T \left| \frac{1}{\zeta(c+it)} \frac{x^c x^{it}}{c+it} \right| dt \leq K x^c \int_{-T}^T \frac{1}{\sqrt{c^2 + t^2}} dt \\ &\leq K x^c \cdot \frac{2T}{c}\end{aligned}$$

Since $\frac{1}{\zeta(c+it)}$ is bounded by some constant K and $|x^{it}| = 1$. Subsequently,

$$\left| \int_L \frac{1}{\zeta(s)} \frac{x^s}{s} ds \right| = \left| \int_{c-iT}^{c+iT} \frac{1}{\zeta(s)} \frac{x^s}{s} ds \right| \leq K x^c \cdot \frac{2\sqrt{x}}{c} = O(x^{c-\frac{1}{2}})$$

Since $c > 1, c - \frac{1}{2} > \frac{1}{2}$, and for large x , this term is small.

Step 3: Error term from the integral

The total error term combining both parts is:

$$O\left(\frac{x \log^2 x}{T}\right) = O\left(\frac{x \log^2 x}{\sqrt{x}}\right) = O(x^{1/2} \log^2 x)$$

It can be observed that, the residue term around $s = 1$ contributes to x , and the error terms contribute to $= O(\epsilon^3 x) + O(x^{1/2} \log^2 x)$.

Choosing ϵ small enough (such as $\epsilon = x^{-1/6}$),

$$O(\epsilon^3 x) = O(x^{1-1/2}) = O(x^{1/2})$$

Thus, combining all terms and dividing by x to normalize, we obtain.

$$\sum_{d \leq x} \frac{\mu(d)}{d} = x + O(x^{1/2} \log^2 x) = O\left(\frac{1}{\log x}\right)$$

2. As for the proof, we can use **Perron's formula** for any arithmetic function $a(n)$,

$$\sum_{n \leq x} a(n) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(\sum_{n=1}^{\infty} \frac{a(n)}{n^s} \right) \frac{x^s}{s} ds + O\left(\sum_{n=1}^{\infty} \frac{|a(n)|}{n^c} \min\left(1, \frac{x}{T|s|}\right) \right) \quad (12)$$

where $c > 1$ and T be a suitably chosen parameter.

Consider the Dirichlet series involving $\mu(d) \log d$:

$$-\frac{\zeta'(s)}{\zeta^2(s)} = \sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n^s}$$

Applying this to (12):

$$\sum_{d \leq x} \frac{\mu(d) \log d}{d} = -\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta'(s)}{\zeta^2(s)} \frac{x^s}{s} ds + O\left(\sum_{d=1}^{\infty} \frac{|\mu(d) \log d|}{d^c} \min\left(1, \frac{x}{T|s|}\right) \right)$$

We thus study the following integral,

$$-\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\zeta'(s)}{\zeta^2(s)} \frac{x^s}{s} ds$$

using following steps.

Step 1: Integral around $s = 1$

Consider a small semicircle Γ (say) of radius ϵ around $s = 1$ having the following parametrization, $s = 1 + \epsilon e^{i\theta}$, $-\pi/2 \leq \theta \leq \pi/2$, and a priori from the fact that, near $s = 1$,

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s-1)$$

and,

$$\zeta'(s) = -\frac{1}{(s-1)^2} + O(1)$$

Thus, on Γ , we have,

$$\frac{\zeta'(s)}{\zeta^2(s)} = \frac{-\frac{1}{(\epsilon e^{i\theta})^2} + O(1)}{\left(\frac{1}{\epsilon e^{i\theta}} + \gamma + O(\epsilon)\right)^2} = -\epsilon^{-1} e^{-i\theta} + O(\epsilon)$$

Hence, the integrand becomes,

$$\begin{aligned} & -\frac{1}{2\pi i} \int_{-\pi/2}^{\pi/2} \left(-\epsilon^{-1} e^{-i\theta} + O(\epsilon)\right) \frac{x^{1+\epsilon e^{i\theta}}}{(1+\epsilon e^{i\theta})} i\epsilon e^{i\theta} d\theta \\ &= -\frac{1}{2\pi i} \int_{-\pi/2}^{\pi/2} \left(-\epsilon^{-1} e^{-i\theta} + O(\epsilon)\right) \cdot x \left(1 + \epsilon e^{i\theta} \log x + O(\epsilon^2)\right) \left(1 - \epsilon e^{i\theta} + O(\epsilon^2)\right) i\epsilon e^{i\theta} d\theta \\ &= -\frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \left(-\epsilon^{-1} e^{-i\theta} + O(\epsilon)\right) \left(x\epsilon e^{i\theta} (\log x - 1) + O(\epsilon^2)\right) i\epsilon e^{i\theta} d\theta \\ &= O(\epsilon^2 x (\log x - 1)) \end{aligned}$$

(Expanding and simplifying, the leading term integrates to zero due to symmetry of the integrand around zero).

Step 2: Integral along the vertical line $s = c + it$

For the part of the integral along the vertical line, say, L : $s = c + it$, $-T \leq t \leq T$, where $c > 1$, i.e.,

$$\int_{c-iT}^{c+iT} \frac{\zeta'(s)}{\zeta^2(s)} \frac{x^s}{s} ds$$

It can be checked that, $\frac{\zeta'(s)}{\zeta^2(s)}$ is bounded on L . Specifically, for $s = c + it$ with $c > 1$, both $\zeta(s)$ and $\zeta'(s)$ are bounded, so $\frac{\zeta'(s)}{\zeta^2(s)}$ is bounded by some constant K . Hence,

$$\left| \int_{c-iT}^{c+iT} \frac{\zeta'(s)}{\zeta^2(s)} \frac{x^{c+it}}{c+it} ds \right| \leq K \int_{-T}^T \frac{x^c}{\sqrt{c^2 + t^2}} dt \leq K \cdot \frac{2T}{c}$$

Choosing $T = \sqrt{x}$,

$$\left| \int_{c-iT}^{c+iT} \frac{\zeta'(s)}{\zeta^2(s)} \frac{x^s}{s} ds \right| \leq Kx^c \cdot \frac{2\sqrt{x}}{c} = O(x^{c-\frac{1}{2}})$$

Since $c > 1$, $c - \frac{1}{2} > \frac{1}{2}$, and for large x , this term is small.

Step 3: Error term from the integral

The total error term combining both parts is:

$$O\left(\frac{x \log^2 x}{T}\right) = O\left(\frac{x \log^2 x}{\sqrt{x}}\right) = O(x^{1/2} \log^2 x)$$

Important to note that the residue term around $s = 1$ contributes negligibly as $= O(\epsilon^2 x (\log x - 1))$, and the error terms contribute upto $O(x^{1/2} \log^2 x) = O(x^{1/3} (\log x - 1))$. (This can be achieved by considering ϵ small enough (such as $\epsilon = x^{-1/6}$). Therefore, we conclude that,

$$\sum_{d \leq x} \frac{\mu(d) \log d}{d} = O(1)$$

3. A priori from the definition of $M(x)$ and applying *Perron's Formula* for $a(n) = \mu(n)$ yields,

$$M(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{\zeta(s)} \frac{x^s}{s} ds + O\left(\sum_{n=1}^{\infty} \frac{1}{n^c} \min\left(1, \frac{x}{T|s|}\right)\right) \quad (13)$$

As for computing the integral in (13) over the vertical line $L: s = c + it, -T \leq t \leq T$, where $c > 1$, our aim is to try shifting the contour of integration to a vertical line closer to the critical strip. For our convenience, we choose $c = 1 + \epsilon$ where, $\epsilon > 0$. Using the fact that $\zeta(s)$ has no zeros for $\Re(s) > 1$, we intend on obtaining a suitable bound for $\frac{1}{\zeta(s)}$ in this region.

Observe that for $\Re(s) = 1 + \epsilon$, $\zeta(s)$ is bounded away from zero, implying $\frac{1}{\zeta(s)}$ is also bounded. Specifically, for $s = 1 + \epsilon + it$,

$$\left| \frac{1}{\zeta(s)} \right| \leq A$$

for some constant A . On the other hand,

$$\begin{aligned} \left| \int_{1+\epsilon-iT}^{1+\epsilon+iT} \frac{1}{\zeta(s)} \frac{x^s}{s} ds \right| &= \left| \int_{-T}^T \frac{x^{1+\epsilon+it}}{1+\epsilon+it} \cdot \frac{1}{\zeta(1+\epsilon+it)} dt \right| \\ &\leq A \cdot \int_{-T}^T \frac{x^{1+\epsilon}}{\sqrt{(1+\epsilon)^2 + t^2}} dt \\ &\leq A \cdot x^{1+\epsilon} \int_{-\infty}^{\infty} \frac{1}{\sqrt{(1+\epsilon)^2 + t^2}} dt = A \cdot x^{1+\epsilon} \frac{\pi}{1+\epsilon} = O(x^{1+\epsilon}) \end{aligned}$$

Subsequently, the error term from the vertical line integral can be estimated as, $O(x^{1+\epsilon})$, for any small $\epsilon > 0$.

Hence, we need to choose T appropriately to control the error term in Perron's formula. Using the **Cauchy Residue Theorem** [18] [Chapt. 5.1, pp. 120] and estimating the integral, we set $T = \sqrt{x}$ and consider the main term and error terms:

$$\int_{1+\epsilon-i\sqrt{x}}^{1+\epsilon+i\sqrt{x}} \frac{1}{\zeta(s)} \frac{x^s}{s} ds = O(x^{1/2+\epsilon})$$

This ensures that the main contribution comes from the vertical integral and the error terms are bounded appropriately. Accordingly,

$$M(x) = O(x^{1/2+\epsilon})$$

By choosing $\epsilon > 0$ sufficiently small, we can make the bound as close to $O(\sqrt{x})$ as desired.

□

3. Order Estimates involving $M(x)$

In this section, we shall rely upon the definitions of *Chebyshev ψ -Function*, $\psi(x)$ in order to come up with a suitable estimate for $\pi(x)$ in terms of the *Mertens Function*, $M(x)$.

3.1. Relation between $\psi(x)$ and $M(x)$

Theorem 3. *The following holds true for the Chebyshev ψ function, $\psi(x)$:*

$$\psi(x) = x - \sum_{n \leq x} M\left(\frac{x}{n}\right) + O(\sqrt{x}) \quad (14)$$

Proof. A priori from the definition of $\psi(x)$ involving the Von Mangoldt Function, $\Lambda(n)$, we apply the *Möbius Inversion Formula* [15] [Section 14.1, pp. 30] on $\Lambda(n)$ to obtain,

$$\Lambda(n) = \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right) = \sum_{n \leq x} \sum_{d|n} \mu(d) \log\left(\frac{n}{d}\right) \quad (15)$$

$$= \sum_{d \leq x} \mu(d) \sum_{k \leq \frac{x}{d}} \log(k) \quad [\text{Interchanging order of summation}] \quad (16)$$

(N.B. Here, we set $n = dk$, so that the inner sum is over k with $k \leq \frac{x}{d}$.)

We approximate the sum of $\log(k)$ by integrating the logarithm function from 1 to $\frac{x}{d}$, and then applying Integration by Parts.

$$\begin{aligned} \sum_{k \leq \frac{x}{d}} \log(k) &= \int_1^{\frac{x}{d}} \log(t) dt + O\left(\log\left(\frac{x}{d}\right)\right) = [t \log(t) - t]_1^{\frac{x}{d}} + O\left(\log\left(\frac{x}{d}\right)\right) \\ &= \frac{x}{d} \log\left(\frac{x}{d}\right) - \frac{x}{d} + 1 + O\left(\log\left(\frac{x}{d}\right)\right) = \frac{x}{d} \log\left(\frac{x}{d}\right) - \frac{x}{d} + 1 + O(\log x) \\ &\quad [\text{Since } \log\left(\frac{x}{d}\right) \text{ is of the same order as } \log x \text{ for large } x] \end{aligned}$$

Subsequently, from (16) we get,

$$\begin{aligned} \psi(x) &= \sum_{d \leq x} \mu(d) \left(\frac{x}{d} \log\left(\frac{x}{d}\right) - \frac{x}{d} + 1 + O(\log(x)) \right) \\ &= x \sum_{d \leq x} \frac{\mu(d)}{d} \log\left(\frac{x}{d}\right) - x \sum_{d \leq x} \frac{\mu(d)}{d} + \sum_{d \leq x} \mu(d) + O(x \log x) \sum_{d \leq x} \mu(d) \end{aligned}$$

A priori using the results obtained in proposition (3),

$$x \sum_{d \leq x} \frac{\mu(d)}{d} \log\left(\frac{x}{d}\right) = x \log(x) \sum_{d \leq x} \frac{\mu(d)}{d} - x \sum_{d \leq x} \frac{\mu(d) \log(d)}{d} \quad (17)$$

Furthermore,

$$O(x \log x) \sum_{d \leq x} \mu(d) = O(x \log x) \cdot O(\sqrt{x}) = O(x^{3/2} \log x) \quad (18)$$

Combining (17) and (18) yields,

$$\psi(x) = x - \sum_{n \leq x} M\left(\frac{x}{n}\right) + O(x) + O(x^{3/2} \log x)$$

Since $x \ll x^{3/2} \log x$ asymptotically, hence we conclude that,

$$\psi(x) = x - \sum_{n \leq x} M\left(\frac{x}{n}\right) + O(\sqrt{x})$$

And the proof is thus complete. \square

3.2. An important approximation for $\psi(x)$

A tricky application of the *Prime Number Theorem* (2) yields the following estimate,

$$\psi(x) = x + O\left(xe^{-c\sqrt{\log x}}\right) \quad (19)$$

or some constant $c > 0$. However, it is indeed possible to obtain a simpler, and more effective bound for the *Chebyshev ψ -Function*.

Lemma 1. *We have,*

$$\psi(x) = x + O\left(x \log^2 x\right) \quad (20)$$

Proof. This proof thoroughly utilizes results from Analytic Number Theory, specifically the properties of the Chebyshev function $\psi(x)$ and the distribution of primes. We shall also leverage results from the analytic properties of the Riemann zeta function $\zeta(s)$.

Important to note that, the proof relies on properties of the Riemann zeta function $\zeta(s)$ and its non-trivial zeros. However, we **do not** assume the Riemann Hypothesis (RH) here explicitly.

The explicit formula for $\psi(x)$ involves the zeros of $\zeta(s)$:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \log(2\pi),$$

where the sum is over the non-trivial zeros ρ of $\zeta(s)$, and the term $\log(2\pi)$ arises due to the existence of pole at $s = 1$.

Suppose, $\rho = \beta + i\gamma$ be a non-trivial zero of $\zeta(s)$. The zeros are symmetric about the real axis, so we consider only the upper half-plane. For each zero ρ of $\zeta(s)$, the term $\frac{x^{\rho}}{\rho}$ contributes to $\psi(x)$. To estimate the error term, consider the sum over the non-trivial zeros ρ ,

$$\sum_{\rho} \frac{x^{\rho}}{\rho}$$

In addition to above, we use the fact that the non-trivial zeros ρ have, $\beta < 1$. The contribution of each such ρ can be bounded by,

$$\left| \frac{x^{\rho}}{\rho} \right| \leq x^{\beta} |\rho|^{-1}$$

Now, the number of zeros with $|\gamma| \leq T$ is $O(T \log T)$. We choose $T = x$ to cover the relevant range of zeros.

$$\sum_{|\gamma| \leq x} |\rho|^{-1} = \sum_{|\gamma| \leq x} (\beta^2 + \gamma^2)^{-1/2} \leq \sum_{|\gamma| \leq x} (\gamma^2)^{-1/2} \leq \sum_{|\gamma| \leq x} \frac{1}{|\gamma|} = O(\log x)$$

Combining these estimates, we obtain,

$$\sum_{\rho} \frac{x^{\rho}}{\rho} = O(x \log x) \quad (21)$$

Including the logarithmic term from the pole at $s = 1$, the error term in $\psi(x)$ becomes,

$$\psi(x) = x + O(x \log x)$$

Since we know that the error term actually involves $\log^2 x$ due to the density of the zeros of $\zeta(s)$ and more refined approximations, thus, we can further improve our estimate to,

$$\psi(x) = x + O(x \log^2 x)$$

as desired. \square

As the title of this section suggests, we shall now proceed towards understanding how we can approximate $\pi(x)$ using properties of $\psi(x)$.

Theorem 4. *The following holds for the Prime counting function, $\pi(x)$:*

$$\pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right) \quad (22)$$

Proof. Before we delve into the proof, notice that $\Lambda(n)$ is zero except when n is a power of a prime, specifically $n = p^k$. We can hence rewrite $\psi(x)$ as,

$$\psi(x) = \sum_{p \leq x} \log p \cdot \left(1 + \left\lfloor \frac{\log x}{\log p} \right\rfloor\right) = \sum_{p \leq x} \log p + \sum_{k=2}^{\infty} \sum_{p \leq x^{1/k}} \log p \quad (23)$$

The error term in (23) comes from the higher powers of primes,

$$\sum_{k=2}^{\infty} \sum_{p \leq x^{1/k}} \log p$$

Which is then dominated by the term corresponding to $k = 2$. In other words,

$$\sum_{k=2}^{\infty} \sum_{p \leq x^{1/k}} \log p = O(\sqrt{x} \log x).$$

Combining the main term and the error terms in (23),

$$\psi(x) = \sum_{p \leq x} \log p + O(\sqrt{x} \log x) \quad (24)$$

On the contrary, an application of the *Prime Number Theorem* allows us to have the following approximation,

$$\sum_{p \leq x} \log p = \pi(x) \log x + \epsilon(x) \quad (25)$$

Given our earlier discussion, the *error term*, $\epsilon(x)$ can be bounded by,

$$\epsilon(x) = O(\sqrt{x} \log x).$$

To isolate $\pi(x)$ from (25), we divide both sides by $\log x$,

$$\pi(x) = \frac{\psi(x)}{\log x} - \frac{\epsilon(x)}{\log x}$$

Substituting the error term bound:

$$\pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{\sqrt{x} \log x}{\log x}\right) = \frac{\psi(x)}{\log x} + O(\sqrt{x})$$

A priori using the fact that, $\sqrt{x} \ll \frac{x}{\log^2 x}$ asymptotically, and combining the error terms, and applying Theorem 2 again enables us to assert that,

$$\pi(x) = \frac{\psi(x)}{\log x} + O\left(\frac{x}{\log^2 x}\right).$$

□

3.3. An Improved Estimate for $\pi(x)$

First, let us recall that, (14) in Theorem 3 gives us an order estimate for $\psi(x)$ in terms of $M(x)$, whereas, we have derived a unique representation of $\pi(x)$ applying properties of $\psi(x)$, and analytic properties of $\zeta(s)$, as mentioned in (22) in Theorem 4.

We substitute (14) into (22) to obtain,

$$\pi(x) = \frac{x}{\log x} - \frac{1}{\log x} \sum_{n \leq x} M\left(\frac{x}{n}\right) + \frac{O(\sqrt{x})}{\log x} + O\left(\frac{x}{\log^2 x}\right)$$

Combining the error terms,

$$\pi(x) = \frac{x}{\log x} - \frac{1}{\log x} \sum_{n \leq x} M\left(\frac{x}{n}\right) + O\left(\frac{\sqrt{x}}{\log x}\right) + O\left(\frac{x}{\log^2 x}\right). \quad (26)$$

Note that, $\frac{\sqrt{x}}{\log x} \ll \frac{x}{\log^2 x}$ asymptotically. Thus, the dominant error term is $O\left(\frac{x}{\log^2 x}\right)$. As a consequence, we have the following improved estimate for $\pi(x)$ as follows.

Theorem 5.

$$\pi(x) = \frac{x}{\log x} - \frac{1}{\log x} \sum_{n \leq x} M\left(\frac{x}{n}\right) + O\left(\frac{x}{\log^2 x}\right) \quad (27)$$

4. Proving Ramanujan's Inequality

In order to prove *Ramanujan's Inequality*, our primary intention will be to investigate the sign of the function,

$$\mathcal{G}(x) := (\pi(x))^2 - \frac{ex}{\log(x)} \pi\left(\frac{x}{e}\right) \quad (28)$$

for large values of x using the relationship between $\pi(x)$ and the Mertens function $M(x)$. Given the complexity of the order relations and the extent of robust computations involving these functions as discussed in the previous section, we'll have to analyze the expressions and error terms cautiously.

4.1. An order expression for $\mathcal{G}(x)$

A priori from (27) of Theorem 5, we can deduce a simplified expression for $\pi(x/e)$ as,

$$\begin{aligned}\pi(x/e) &= \frac{x/e}{\log(x/e)} - \frac{1}{\log(x/e)} \sum_{n \leq x/e} M\left(\frac{x/e}{n}\right) + O\left(\frac{x/e}{(\log(x/e))^2}\right) \\ &= \frac{x/e}{\log x - 1} - \frac{1}{\log x - 1} \sum_{n \leq x/e} M\left(\frac{x/e}{n}\right) + O\left(\frac{x/e}{(\log x - 1)^2}\right)\end{aligned}$$

Subsequently,

$$\begin{aligned}\frac{ex}{\log x} \pi(x/e) &= \frac{x^2}{\log x(\log x - 1)} - \frac{ex}{\log x(\log x - 1)} \sum_{n \leq x/e} M\left(\frac{x/e}{n}\right) \\ &\quad + O\left(\frac{x^2}{\log x(\log x - 1)^2}\right)\end{aligned}\quad (29)$$

Furthermore, squaring $\pi(x)$ yields,

$$\begin{aligned}(\pi(x))^2 &= \left(\frac{x}{\log x} - \frac{1}{\log x} \sum_{n \leq x} M\left(\frac{x}{n}\right) + O\left(\frac{x}{\log^2 x}\right)\right)^2 \\ &= \frac{x^2}{\log^2 x} - \frac{2x}{(\log x)^2} \sum_{n \leq x} M\left(\frac{x}{n}\right) + \frac{1}{(\log x)^2} \left(\sum_{n \leq x} M\left(\frac{x}{n}\right)\right)^2 + O\left(\frac{x^2}{\log^3 x}\right)\end{aligned}\quad (30)$$

Therefore, using (29) and (30), we deduce that,

$$\begin{aligned}\mathcal{G}(x) &= \left(\frac{x^2}{\log^2 x} - \frac{2x}{(\log x)^2} \sum_{n \leq x} M\left(\frac{x}{n}\right) + \frac{1}{(\log x)^2} \left(\sum_{n \leq x} M\left(\frac{x}{n}\right)\right)^2 + O\left(\frac{x^2}{\log^3 x}\right)\right) \\ &\quad - \left(\frac{x^2}{\log x(\log x - 1)} - \frac{ex}{\log x(\log x - 1)} \sum_{n \leq x/e} M\left(\frac{x/e}{n}\right) + O\left(\frac{x^2}{\log x(\log x - 1)^2}\right)\right).\end{aligned}$$

Simplifying each term:

$$\begin{aligned}&= \frac{x^2}{(\log x)^2(\log x - 1)} - \frac{2x}{(\log x)^2} \sum_{n \leq x} M\left(\frac{x}{n}\right) + \frac{1}{(\log x)^2} \left(\sum_{n \leq x} M\left(\frac{x}{n}\right)\right)^2 \\ &\quad + \frac{ex}{\log x(\log x - 1)} \sum_{n \leq x/e} M\left(\frac{x/e}{n}\right) + O\left(\frac{x^2}{\log^3 x}\right)\end{aligned}\quad (31)$$

4.2. Asymptotic Behavior of individual Terms

A priori using (iii) (cf. Prop. (3)) involving the Mertens function $M(x)$,

$$\sum_{n \leq x} M\left(\frac{x}{n}\right) = O\left(\sqrt{x} \sum_{n \leq x} \frac{1}{\sqrt{n}}\right) = O(x) \quad (32)$$

So, we can further approximate each term of (31) as follows,

$$\frac{1}{(\log x)^2} \left(\sum_{n \leq x} M\left(\frac{x}{n}\right) \right)^2 = O\left(\frac{x^2}{\log^2 x}\right),$$

$$\frac{ex}{\log x(\log x - 1)} \sum_{n \leq x/e} M\left(\frac{x/e}{n}\right) = O\left(\frac{x^2}{\log x(\log x - 1)}\right).$$

As a consequence,

$$\begin{aligned} \mathcal{G}(x) &= -\frac{2x}{(\log x)^2} \sum_{n \leq x} M\left(\frac{x}{n}\right) + O\left(\frac{x^2}{\log^2 x}\right) + O\left(\frac{x^2}{\log x(\log x - 1)}\right) + O\left(\frac{x^2}{\log^3 x}\right). \\ &= -\frac{2x}{(\log x)^2} \sum_{n \leq x} M\left(\frac{x}{n}\right) + O\left(\frac{x^2}{\log^2 x}\right) \end{aligned} \quad (33)$$

For sufficiently large values of x . Therefore, we conclude that, $\mathcal{G}(x) < 0$, for large values of x , and this concludes our proof of the inequality.

Later, we shall try to establish a better range of the values of x for which *Ramanujan's Inequality* does hold true.

5. A modified Bound for $\pi(x)$

The derivations which we've made in the previous sections yielded several order estimates involving $M(x)$, $\psi(x)$ and especially $\pi(x)$. In this section, we shall discuss how this method enables us to find more optimal bounds for $\pi(x)$.

5.1. Upper Bound for $\pi(x)$

A priori from Lemma (1), we have that for some positive constant $\alpha > 0$,

$$\psi(x) \leq x + \alpha x(\log x)^2 \quad (34)$$

Substituting this into the the estimate (22) of Theorem 4 involving $\pi(x)$,

$$\pi(x) \leq \frac{x + \alpha x(\log x)^2}{\log x} + O\left(\frac{x}{(\log x)^2}\right) = \frac{x}{\log x} + \alpha x \log x + O\left(\frac{x}{(\log x)^2}\right). \quad (35)$$

5.2. Lower Bound for $\pi(x)$

A priori using (32) we can derive,

$$\frac{1}{\log x} \sum_{n \leq x} M\left(\frac{x}{n}\right) = O\left(\frac{x}{\log x}\right).$$

Therefore, applying the above derivation to (27) in Theorem 5, we assert that,

$$\pi(x) \geq \frac{x}{\log x} - \frac{\beta x}{\log x} + O\left(\frac{x}{(\log x)^2}\right) = (1 - \beta) \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right). \quad (36)$$

for some positive constant $\beta > 0$.

In summary, (35) and (36) allows us to state the following.

Theorem 6. *The following bounds on $\pi(x)$ is valid for sufficiently large values of x :*

$$(1 - \beta) \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right) \leq \pi(x) \leq \frac{x}{\log x} + \alpha x \log x + O\left(\frac{x}{(\log x)^2}\right) \quad (37)$$

for positive constants $\alpha, \beta > 0$.

Remark 2. *The result (37) in Theorem 6 provides an alternative justification to the validity of the famous **Prime Number Theorem**. This can be observed from the fact that, $\pi(x)$ is bounded on both sides by a constant multiple of $\frac{x}{\log x}$ plus an error term, which can be minimized for large values of x .*

6. An improved condition for Ramanujan's Inequality

Adhering to *Sterneck's* deduction [5], as mentioned in section 2.4 of this text, we shall investigate the function \mathcal{G} for its sign for large values of x , with every intention of improving the claim made by *Dudek and Platt* [2, cf. Th. 2].

6.1. Monotonicity of the function \mathcal{G}

Our aim in this section is to establish the following claim.

Proposition 4. *The function $\mathcal{G}(x)$ as defined in (28) is monotone decreasing for $x \geq 201$.*

Proof. We intend on verifying that, $\mathcal{G}(x + \epsilon) - \mathcal{G}(x) < 0$ for every large x and for every $\epsilon > 0$ arbitrarily chosen.

Observe from definition that the difference,

$$\mathcal{G}(x + \epsilon) - \mathcal{G}(x) = \left[(\pi(x + \epsilon))^2 - (\pi(x))^2 \right] - \left[\frac{e(x + \epsilon)}{\log(x + \epsilon)} \pi\left(\frac{x + \epsilon}{e}\right) - \frac{ex}{\log x} \pi(x/e) \right] \quad (38)$$

Now we evaluate,

$$(\pi(x + \epsilon))^2 - (\pi(x))^2 = (\pi(x + \epsilon) - \pi(x))(\pi(x + \epsilon) + \pi(x)) \quad (39)$$

Using the explicit forms (27):

$$\pi(x + \epsilon) = \frac{x}{\log x} + \frac{\epsilon}{\log x} - \frac{x\epsilon}{(\log x)^2} - \frac{1}{\log x} \sum_{n \leq x + \epsilon} M\left(\frac{x + \epsilon}{n}\right) + O\left(\frac{x + \epsilon}{(\log(x + \epsilon))^2}\right)$$

Therefore,

$$\begin{aligned} \pi(x + \epsilon) - \pi(x) &= \frac{\epsilon}{\log x} - \frac{x\epsilon}{(\log x)^2} - \frac{1}{\log x} \sum_{n \leq x + \epsilon} M\left(\frac{x + \epsilon}{n}\right) + \frac{1}{\log x} \sum_{n \leq x} M\left(\frac{x}{n}\right) \\ &\quad + O\left(\frac{x + \epsilon}{(\log(x + \epsilon))^2} - \frac{x}{(\log x)^2}\right) \end{aligned} \quad (40)$$

For small ϵ , we use the linear approximation:

$$\frac{x + \epsilon}{\log(x + \epsilon)} \approx \frac{x}{\log x} + \frac{\epsilon}{\log x} - \frac{x\epsilon}{(\log x)^2}$$

And,

$$\frac{1}{\log(x + \epsilon)} \approx \frac{1}{\log x} - \frac{\epsilon}{x(\log x)^2}$$

Thus, we can compute further in (40) as follows,

$$\begin{aligned} \pi(x + \epsilon) - \pi(x) &= \frac{\epsilon}{\log x} - \frac{x\epsilon}{(\log x)^2} - \frac{1}{\log x} \sum_{n \leq x + \epsilon} M\left(\frac{x + \epsilon}{n}\right) \\ &\quad + \frac{1}{\log x} \sum_{n \leq x} M\left(\frac{x}{n}\right) + O\left(\frac{x\epsilon}{(\log x)^2}\right) \end{aligned} \quad (41)$$

Similarly,

$$\begin{aligned} \pi(x + \epsilon) + \pi(x) &= 2\frac{x}{\log x} + \frac{\epsilon}{\log x} - \frac{x\epsilon}{(\log x)^2} - \frac{1}{\log x} \sum_{n \leq x + \epsilon} M\left(\frac{x + \epsilon}{n}\right) \\ &\quad + \frac{1}{\log x} \sum_{n \leq x} M\left(\frac{x}{n}\right) + O\left(\frac{x\epsilon}{(\log x)^2}\right) \end{aligned} \quad (42)$$

Combining (41) and (42) yields,

$$\begin{aligned} (\pi(x + \epsilon))^2 - (\pi(x))^2 &= \left(\frac{\epsilon}{\log x} - \frac{x\epsilon}{(\log x)^2} - \frac{1}{\log x} \sum_{n \leq x + \epsilon} M\left(\frac{x + \epsilon}{n}\right) + \frac{1}{\log x} \sum_{n \leq x} M\left(\frac{x}{n}\right) \right) \\ &\quad \left(2\frac{x}{\log x} + \frac{\epsilon}{\log x} - \frac{x\epsilon}{(\log x)^2} \right) + O\left(\frac{x\epsilon}{(\log x)^2}\right) \\ &= 2\frac{x\epsilon}{(\log x)^2} - 2\frac{x^2\epsilon}{(\log x)^3} - 2\frac{x}{(\log x)^2} \sum_{n \leq x + \epsilon} M\left(\frac{x + \epsilon}{n}\right) + 2\frac{x}{(\log x)^2} \sum_{n \leq x} M\left(\frac{x}{n}\right) \\ &\quad + O\left(\frac{x\epsilon}{(\log x)^2}\right) \end{aligned} \quad (43)$$

As for the second term in (38),

$$\begin{aligned} \frac{e(x + \epsilon)}{\log(x + \epsilon)} \pi\left(\frac{x + \epsilon}{e}\right) - \frac{ex}{\log x} \pi(x/e) &\approx \frac{e(x + \epsilon)}{\log(x + \epsilon)} \left(\frac{x + \epsilon}{e \log(x + \epsilon/e)} \right) \\ &\quad - \frac{ex}{\log x} \left(\frac{x}{e \log(x/e)} \right) \end{aligned}$$

(Using the earlier approximations for $\pi\left(\frac{x+\epsilon}{e}\right)$ and $\pi\left(\frac{x}{e}\right)$)

$$\begin{aligned} &= \frac{(x+\epsilon)^2}{(\log(x+\epsilon))^2} - \frac{x^2}{(\log x)^2} + O\left(\frac{x\epsilon}{(\log x)^3}\right) = \frac{(x+\epsilon)^2 - x^2}{(\log x)^2} + O\left(\frac{x\epsilon}{(\log x)^3}\right) \\ &= \frac{2x\epsilon + \epsilon^2}{(\log x)^2} + O\left(\frac{x\epsilon}{(\log x)^3}\right) \end{aligned} \quad (44)$$

Hence, substituting (43) and (44) in (38) and simplifying,

$$\begin{aligned} &\mathcal{G}(x+\epsilon) - \mathcal{G}(x) \\ &= \left(2\frac{x\epsilon}{(\log x)^2} - 2\frac{x^2\epsilon}{(\log x)^3} - 2\frac{x}{(\log x)^2} \sum_{n \leq x+\epsilon} M\left(\frac{x+\epsilon}{n}\right) + 2\frac{x}{(\log x)^2} \sum_{n \leq x} M\left(\frac{x}{n}\right)\right) \\ &\quad - \left(\frac{2x\epsilon + \epsilon^2}{(\log x)^2}\right) + O\left(\frac{x\epsilon}{(\log x)^3}\right) \\ &= -2\frac{x}{(\log x)^2} \left(\sum_{n \leq x+\epsilon} M\left(\frac{x+\epsilon}{n}\right) - \sum_{n \leq x} M\left(\frac{x}{n}\right)\right) - \frac{\epsilon^2}{(\log x)^2} + O\left(\frac{x\epsilon}{(\log x)^3}\right) \end{aligned} \quad (45)$$

We utilize *Sterneck's conjecture*,

$$|M(x)| < \frac{1}{2}\sqrt{x}, \quad \text{for } x \geq 201.$$

to bound the difference as follows,

$$-2\frac{x}{(\log x)^2} \left(\sum_{n \leq x+\epsilon} M\left(\frac{x+\epsilon}{n}\right) - \sum_{n \leq x} M\left(\frac{x}{n}\right)\right) \leq -2\frac{x}{(\log x)^2} \left(\frac{1}{2}\sqrt{x+\epsilon} - \frac{1}{2}\sqrt{x}\right) \quad (46)$$

Subsequently, $\epsilon > 0$ enables us to conclude from (45) and (46),

$$\begin{aligned} \mathcal{G}(x+\epsilon) - \mathcal{G}(x) &\leq -2\frac{x}{(\log x)^2} \left(\frac{1}{2}(\sqrt{x+\epsilon} - \sqrt{x})\right) - \frac{\epsilon^2}{(\log x)^2} + O\left(\frac{x\epsilon}{(\log x)^3}\right) \\ &= -\frac{x}{(\log x)^2} (\sqrt{x+\epsilon} - \sqrt{x}) - \frac{\epsilon^2}{(\log x)^2} + O\left(\frac{x\epsilon}{(\log x)^3}\right) \end{aligned} \quad (47)$$

Since $\sqrt{x+\epsilon} - \sqrt{x} > 0$ for any $\epsilon > 0$ and the error term $O\left(\frac{x\epsilon}{(\log x)^3}\right)$ is smaller, it implies,

$$\mathcal{G}(x+\epsilon) - \mathcal{G}(x) < 0 \text{ for every } x+\epsilon > x, \epsilon > 0$$

Thus, $\mathcal{G}(x)$ is monotone decreasing for large x such that, $x \geq 201$.

□

Remark 3. Note that, Sterneck's Conjecture only establishes the fact that,

$$|M(n)| < \frac{1}{2}\sqrt{n}, \quad \text{for } n > 200.$$

where, $n \in \mathbb{N}$. But in this case, since, we're dealing with $x \in \mathbb{R}$, hence, we've modified the lower bound for x accordingly.

6.2. A better range for the values of x

Just to recall, Proposition 4 comments on the monotonicity of \mathcal{G} for x within a certain interval. Thus for every sufficiently large $x \geq 201$, we must have,

$$\mathcal{G}(x) < \mathcal{G}\left(\frac{\log x}{e}\right) < 0$$

provided, $\frac{\log x}{e} \geq 201$, i.e., $\log x \geq 546.3746475$. Therefore, we have our following improved bound on x in order to satisfy (1).

Theorem 7. The Ramanujan's Inequality (1) is unconditionally true for every $x \geq \exp(547)$.

6.3. Numerical Estimates for $\mathcal{G}(x)$

A priori from Proposition 4, we have formally established that, the function \mathcal{G} as defined in (28) is in fact *monotone decreasing* for large values of x . Applying MATHEMATICA¹, we can indeed provide ample numerical evidence in support of our claim proposed in Theorem 7.

In fact, using Table 1 and Figure 1 we can observe and study the decrement in the values of \mathcal{G} for increasing values of x in $[\exp(547), \exp(3247)]$ scaled appropriately. Another important comment to make is that, (1) in Theorem 1 has been well established [23] for every $x \geq \exp(3158.442)$. Hence, it only suffices to check the sign changes for \mathcal{G} in the above chosen interval. Furthermore, we can assert from the data given in the Table (1) that,

$$\mathcal{G}(\exp(547)) = -5.0283287 \times 10^{458} < 0$$

which along with Proposition 4, establishes our result in Theorem 7.

¹ Codes are available at: <https://github.com/subhamde1/Paper-11.git>

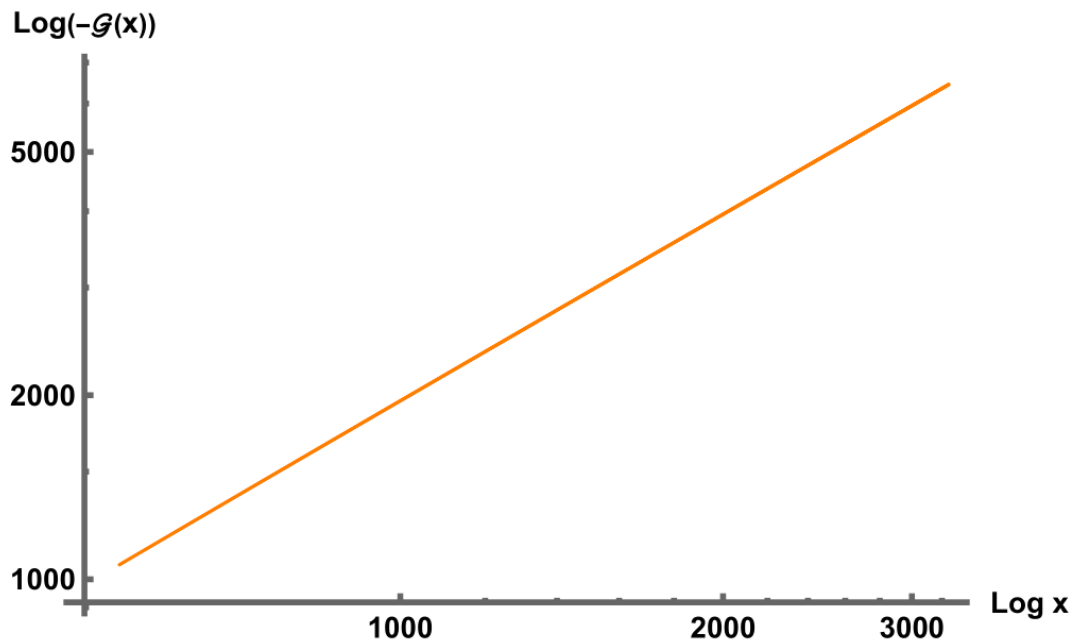


Figure 1. Plot of $\log(-\mathcal{G}(x))$ with respect to $\log x$

Table 1. Values of $\mathcal{G}(x)$

x	$\mathcal{G}(x)$	x	$\mathcal{G}(x)$
e^{547}	$-5.0283287 \times 10^{458}$	e^{1947}	$-2.5682958 \times 10^{1671}$
e^{647}	$-1.3215407 \times 10^{545}$	e^{2047}	$-1.3736529 \times 10^{1758}$
e^{747}	$-4.0198388 \times 10^{631}$	e^{2147}	$-7.4533134 \times 10^{1844}$
e^{847}	$-1.3638260 \times 10^{718}$	e^{2247}	$-4.0972567 \times 10^{1931}$
e^{947}	$-5.0360748 \times 10^{804}$	e^{2347}	$-2.2793647 \times 10^{2018}$
e^{1047}	$-1.9897153 \times 10^{891}$	e^{2447}	$-1.2819727 \times 10^{2105}$
e^{1147}	$-8.3076362 \times 10^{977}$	e^{2547}	$-7.2829524 \times 10^{2191}$
e^{1247}	$-3.6318520 \times 10^{1064}$	e^{2647}	$-4.1760285 \times 10^{2278}$
e^{1347}	$-1.6506455 \times 10^{1151}$	e^{2747}	$-2.4151700 \times 10^{2365}$
e^{1447}	$-7.7558806 \times 10^{1237}$	e^{2847}	$-1.4079697 \times 10^{2452}$
e^{1547}	$-3.7508004 \times 10^{1324}$	e^{2947}	$-8.2691531 \times 10^{2538}$
e^{1647}	$-1.8602058 \times 10^{1411}$	e^{3047}	$-4.8903016 \times 10^{2625}$
e^{1747}	$-9.4329900 \times 10^{1497}$	e^{3147}	$-2.9108696 \times 10^{2712}$
e^{1847}	$-4.8787962 \times 10^{1584}$	e^{3247}	$-1.7431960 \times 10^{2799}$

7. Future Research Prospects

In summary, we've utilized specific order estimates for the *Prime Counting Function* $\pi(x)$, the *Second Chebyshev Function* $\psi(x)$ and the *Mertens Function* $M(x)$ in order to conjure up an improved bound for the famous *Ramanujan's Inequality*. Although, one can derive other approximations for $M(x)$ in order to improve this result even further. It'll surely be interesting to observe whether it's at all feasible to apply any other techniques for this purpose.

On the other hand, one can surely work on some modifications of *Ramanujan's Inequality* For instance, *Hassani* studied (1) extensively for different cases [19], and eventually claimed that, the inequality does in fact reverses if one can replace e by some α satisfying, $0 < \alpha < e$, although it retains the same sign for every $\alpha \geq e$.

In addition to above, it is very much possible to come up with certain generalizations of Theorem 1. In this context, we can study *Hassani's* stellar effort in this area where, he apparently increased

the power of $\pi(x)$ from 2 upto 2^n and provided us with this wonderful inequality stating that for sufficiently large values of x [25],

$$(\pi(x))^{2^n} < \frac{e^n}{\prod_{k=1}^n \left(1 - \frac{k-1}{\log x}\right)^{2^{n-k}}} \left(\frac{x}{\log x}\right)^{2^n-1} \pi\left(\frac{x}{e^n}\right)$$

Finally, and most importantly, we can choose to broaden our horizon, and proceed towards studying the *prime counting function* in much more detail in order to establish other results analogous to Theorem 1, or even study some specific polynomial functions in $\pi(x)$ and also their powers if possible. One such example which can be found in [26] eventually proves that, for sufficiently large values of x ,

$$\frac{3ex}{\log x} \left(\pi\left(\frac{x}{e}\right)\right)^{3^n-1} < (\pi(x))^{3^n} + \frac{3e^2x}{(\log x)^2} \left(\pi\left(\frac{x}{e^2}\right)\right)^{3^n-2}, \quad n > 1$$

Whereas, significantly the inequality reverses for the specific case when, $n = 1$ (*Cubic Polynomial Inequality*) (cf. Theorem (3.1) [26]).

Hopefully, further research in this context might lead the future researchers to resolve some of the unsolved mysteries involving *prime numbers*, or even solve some of the unsolved problems surrounding the iconic field of Number Theory.

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