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Keywords: Heteroclinic solutions; L1-Carathéodory functions and sequences; equiconvergence at infinity and at the impulsive points

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Existence of Heteroclinic Solutions in Nonlinear Differential Equations of the Second-Order Incorporating Generalized Impulse Effects with the Possibility of Application to Bird Population Growth

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Abstract: This work considers the existence of solutions of the heteroclinic type in nonlinear second order differential equations with ϕ-Laplacians, incorporating generalized impulsive conditions on the real line. For the construction of the results, it was only imposed that ϕ be a homeomorphism, using the fixed point theorem of Schauder, coupled with concepts of L^1-Carathéodory sequences and functions, along with impulsive points equiconvergence and equiconvergence at infinity. Finally, a practical part illustrates the main theorem, and a possible application to bird population growth.

Keywords: heteroclinic solutions; impulsive points equiconvergence and equiconvergence at infinity; L^1-Carathéodory sequences and functions

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1. Introduction

In the theory of differential equations, there are several aspects and interesting features to study or explore further. Among these is the study of impulsive differential equations or systems and the qualitative analysis of solutions in the real line. In this context, this article aims to study the existence of solutions of the heteroclinic type in nonlinear second order differential equations with generalized, infinite impulse effects, which to the best of our knowledge is a rare occurrence.

Impulses incorporated in differential equations are intended to describe and represent the effects of small and sudden changes in a given system over certain periods of time. In the literature, there are several areas of study associated with impulses such as biotechnology, medicine, population dynamics, logging, etc (see, [12,29]) and references therein. Various theoretical approaches, as well as numerous applications of second-order nonlinear differential equations featuring impulses can be found in ([2,11,19,21–24,27]).

On the other hand, in the analysis of the qualitative aspects of differential equations, the investigation into the existence of heteroclinic or homoclinic solutions is useful and necessary. When a system of ordinary differential equations has equilibria (that is, constant solutions), studying the connections between them through the trajectories of the system's solutions, known as homoclinic or heteroclinic solutions, becomes an essential task. It is common for homoclinic and heteroclinic solutions to emerge in mathematical models dealing with dynamical systems, bifurcations, mechanics, chemistry and biology [4,13,14]. The existence of heteroclinic orbits is also crucial for analyzing the spatio-temporal chaotic patterns of nonlinear evolution equations [16].

Over the years, some studies have been carried out on the topic of heteroclinic solutions. In [26], the author investigates heteroclinic solutions pertaining to a second-order equation that is asymptoti-

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cally autonomous $\ddot{x} = a(t)V'(x(t))$. In [9], Coti Zelati along with Rabinowitz investigated heteroclinic orbits for a non-autonomous differential equation that connects stationary points with distinct energy levels. For a fourth-degree ordinary differential equation, heteroclinic solutions linking nonconsecutive equilibria of a triple-well potential were found [5]. Cabada et al., in [6] examine the existence of heteroclinic type solutions in semi-linear second-order difference equations pertaining to the Fisher-Kolmogorov's equation. Monotonicity and continuity arguments form the basis of the proof for these results. Hale and Rybakowski also demonstrated the existence of heteroclinic solutions for retarded functional differential equations. [15]. Furthermore, works involving heteroclinics and impulses can be found in [3,7,8,17,20,25,28] and references therein.

Using findings concerning the existence of non-principal solutions, in [1], the authors study Leighton and Wong theorems of oscillation regarding a class of second-order impulsive equations having the form

$$\begin{cases} (p(t)x')' + q(t)x = 0, & t \neq \theta_i \\ \Delta x + a_i x = 0, \;\; \Delta p(t)x' + b_i x + c_i x' = 0, & t = \theta_i, \end{cases}$$

and

$$\begin{cases} (p(t)x\prime)' + q(t)x = f(t), & t \neq \theta_i \\ \Delta x + a_i x = f_i, \;\; \Delta p(t)x' + b_i x + c_i x' = g_i, & t = \theta_i, \end{cases}$$

in which $p > 0$, q, f are left continuous piece-wise functions in $[0, \infty)$, and $\{a_i\}$, $\{b_i\}$ and $\{c_i\}$ are real number sequences with $i \geq 1$. The set $\{\theta_i\}$, of impulse points constitutes a strictly increasing, unbounded sequence of positive real numbers.

In [10], Cupini, Marcelli and Papalini present the strongly nonlinear boundary-value problem

$$\begin{cases} (a(x(t))\Phi(x'(t)))' = f(t, x(t), x'(t)) \;\text{ a.e. } t \in \mathbb{R} \\ x(-\infty) = v^-, \;\; x(+\infty) = v^+ \end{cases}$$

In this work, the authors consider nonlinear mixed differential operators depending both on x and x'. Where $v^- < v^+$, and with $\Phi : \mathbb{R} \to \mathbb{R}$ being a general increasing homeomorphism, $\Phi(0) = 0$, a being a positive continuous function and f a nonlinear Carateódory function.

In a recent paper [11], Sousa and Minhós consider the following coupled system

$$\begin{cases} (a(t)\phi(u'(t)))' = f(t, u(t), v(t), u'(t), v'(t)), \\ (b(t)\psi(v'(t)))' = h(t, u(t), v(t), u'(t), v'(t)), t \in \mathbb{R}. \end{cases}$$

where ϕ and ψ are increasing homeomorphisms satisfying adequate relations on their inverses, with $a, b : \mathbb{R} \to (0, +\infty[$ being continuous functions, and $f, h : \mathbb{R}^5 \to \mathbb{R}$, L^1-Carathéodory functions, along with the following asymptotic conditions

$$u(-\infty) = A, \;\; u'(+\infty) = 0, v(-\infty) = B, v'(+\infty) = 0$$

for $A, B \in \mathbb{R}$.

Motivated by these works, in our paper we consider a similar problem but with the inclusion of infinite impulsive conditions, more precisely, we study the following real nonlinear second-order differential equation

$$\left(a(t)\phi(u'(t))\right)' = f(t, u(t), u'(t)), \; t \in \mathbb{R} \backslash \{t_k\}, \;\; k \in \mathbb{Z} \tag{1}$$

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with ϕ being an increasing homeomorphism satisfying adequate relations on its inverse, $a : \mathbb{R} \to (0, +\infty[$ a continuous function, and $f : \mathbb{R}^3 \to \mathbb{R}$ an L^1-Carathéodory function, considering the following asymptotic conditions

$$u(-\infty) = C, \; u(+\infty) = L \tag{2}$$

for $C, L \in \mathbb{R}$, together with the generalized and infinite impulse conditions

$$\begin{cases} \Delta u(t_k) = I_k(t_k, u(t_k), u'(t_k)), \\ \Delta\phi(u'(t_k)) = J_k(t_k, u(t_k), u'(t_k)), \end{cases} \tag{3}$$

where, for $k \in \mathbb{Z}$ ($\mathbb{Z}$ is the set of all integers), $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, $\Delta\phi(u'(t_k)) = \phi(u'(t_k^+)) - \phi(u'(t_k^-))$, and $u(t_k^+)$, $u(t_k^-)$ are the right and left limits for $u(t_k)$, respectively, $\Delta\phi(u'(t_k^+))$ and $\Delta\phi(u'(t_k^-))$ has a similar meaning for $\phi(u'(t_k))$. $I_k, J_k \in C(\mathbb{R}^3, \mathbb{R})$, are Carathéodory sequences and t_k are moments such that $\cdots < t_k < t_{k+1} < t_{k+2} < \cdots$, and

$$\lim_{k\to-\infty} t_k = -\infty, \quad \lim_{k\to+\infty} t_k = +\infty.$$

Our results are based on [11,20], extending the results for a ϕ-Laplacian operator with infinite impulses of more or less intensity. This can be very interesting for modeling phenomena with minor changes and with different intensities, which occur very quickly and for long periods of time, opening new fields for investigation on the subject.

The outline of the present paper is given as follows: Section 2 comprises of the functional backgrounds. In Section 3, we present a result of existence. Lastly, an example application illustrates the main result.

2. Auxiliary Results and Definitions

Let us define

$$u(t_k^\pm) := \lim_{t\to t_k^\pm} u(t),$$

and consider the set

$$PC(\mathbb{R}) = \left\{ \begin{array}{c} u : u \in C^n(\mathbb{R}) \text{ is continuous for } t \neq t_k, u^{(n)}(t_k) = u^{(n)}(t_k^-), \\ u^{(n)}(t_k^+) \text{ exists for } k \in \mathbb{Z} \text{ and } n = 0, 1 \end{array} \right\}.$$

Considering the space

$$X := \left\{ x : x \in PC(\mathbb{R}) : \lim_{t\to\pm\infty} x^{(i)}(t) \in \mathbb{R}, \; i = 0, 1 \right\}, \tag{4}$$

with the norm

$$\|u\|_X = \max\{\|u\|_\infty, \|u'\|_\infty\},$$

being

$$\|Y\|_\infty := \sup_{t\in\mathbb{R}} |Y(t)|.$$

Lemma 1. *$(X, \|\cdot\|_X)$ given in (4) is a real Banach space.*

Proof. Let $x, y \in X$, and $\lambda \in \mathbb{R}$. In order to show that the space $(X, \|\cdot\|_X)$ is Banach, the following points need to be proven:

1. *X is a vector space.*

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(a) *Vector addition*

$$\lim_{t\to\pm\infty}(x+y)^{(i)}(t) = \lim_{t\to\pm\infty}\left(x^{(i)}(t)+y^{(i)}(t)\right)$$
$$= \left(\lim_{t\to\pm\infty}x^{(i)}(t)+\lim_{t\to\pm\infty}y^{(i)}(t)\right)\in\mathbb{R}$$

Hence X is closed under addition, and one can see that this addition is commutative, associative and there exists a *zero vector* 0, and $-x$ for each x such that:

$$\lim_{t\to\pm\infty}(x+(-x)+0)^{(i)}(t) = \lim_{t\to\pm\infty}\left(x^{(i)}(t)+(-x)^{(i)}(t)\right)$$
$$= \left(\lim_{t\to\pm\infty}x^{(i)}(t)-\lim_{t\to\pm\infty}x^{(i)}(t)\right) = 0\in\mathbb{R}$$

(b) *Multiplication by scalars*

$$\lim_{t\to\pm\infty}(\lambda x)^{(i)}(t) = \lim_{t\to\pm\infty}\lambda(x)^{(i)}(t)$$
$$= \lambda\left(\lim_{t\to\pm\infty}(x)^{(i)}(t)\right)\in\mathbb{R}$$

Thus X is also closed under multiplication by scalars, satisfying the commutative and distributive laws as well. And therefore X is a vector space.

2. $(X, \|\cdot\|_X)$ is a normed space.
We can see that $\|x\|_X \geq 0$, $\forall t\in\mathbb{R}$. When $\|x\|_X = 0$ then by definition we must have $\|x\|_\infty = \|x'\|_\infty = 0$, which in turn means that $x(t) = 0$ for all $t\in\mathbb{R}$. Conversely, when $x = 0$ for all $t\in\mathbb{R}$ we get that $\|x\|_\infty = \|x'\|_\infty = 0$, and therefore $\|x\|_X = 0$.

Given some $\lambda\in\mathbb{R}$ we have that,

$$\|\lambda x\|_X = \max\{\sup\|\lambda x(t)\|, \sup\|(\lambda x)'(t)\|\}$$
$$= \max\{|\lambda|\sup\|x(t)\|, |\lambda|\sup\|x'(t)\|\}$$
$$= |\lambda|\|x\|_X.$$

Now we show the triangle inequality. We know that:

$$\sup|x| + \sup|y| \geq |x| + |y| \geq |x+y|,$$

and

$$\sup|x'| + \sup|y'| \geq |x'| + |y'| \geq |x'+y'|.$$

But since the supremum is the least upper bound, we get

$$|x+y| \leq \sup|x+y|,$$

$$|x'+y'| \leq \sup|x'+y'|.$$

And then finally

$$\sup|x| + \sup|y| \geq \sup|x+y|$$

and

$$\sup|x'| + \sup|y'| \geq \sup|x'+y'|.$$

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So

$$\begin{aligned}\sup|x+y| &\leq \sup|x| + \sup|y| \\ &\leq \max\{\sup|x|, \sup|x'|\} + \max\{\sup|y|, \sup|y'|\}\end{aligned}$$

and

$$\begin{aligned}\sup|(x+y)'| &\leq \sup|x'| + \sup|y'| \\ &\leq \max\{\sup|x|, \sup|x'|\} + \max\{\sup|y|, \sup|y'|\}\end{aligned}$$

So the maximum between them is bounded by:

$$\max\{\sup|x+y|, \sup|(x+y)'|\} \leq \max\{\sup|x|, \sup|x'|\} + \max\{\sup|y|, \sup|y'|\},$$

giving us,

$$\|x+y\|_X \leq \|x\|_X + \|y\|_X$$

thus showing that the norm is well defined.

3. $(X, \|\cdot\|_X)$ is complete in the metric induced by the norm:

$$d(x,y) = \|x-y\|_X = \max\{\sup|x-y|, \sup|(x-y)'|\}.$$

Let (x_m) be an arbitrary Cauchy sequence in X. Then, for any $\varepsilon > 0$, there exists an $N \in \mathbb{R}$ such that for all $m, n > N$,

$$d(x_m, x_n) = \max\{\sup|x_m - x_n|, \sup|(x_m - x_n)'|\} < \varepsilon.$$

So for every fixed $t_0 \in \mathbb{R}$ we have:

$$\max\{|x_m(t_0) - x_n(t_0)|, |(x_m(t_0) - x_n(t_0))'|\} < \varepsilon$$

that is,

$$|x_m(t_0) - x_n(t_0)| < \varepsilon$$

and

$$|(x_m(t_0) - x_n(t_0))'| < \varepsilon$$

And so the sequence of numbers $(x_1(t_0), x_2(t_0), x_3(t_0), ...)$ and $(x_1'(t_0), x_2'(t_0), x_3'(t_0), ...)$ are Cauchy, and each of them converge, (see [18], Theorem $1.4-4$), say

$$x_m(t_0) \to x(t_0) \in \mathbb{R}$$

and

$$x_m'(t_0) \to x'(t_0) \in \mathbb{R},$$

as $m \to \infty$. Therefore,

$$\lim_{t \to \pm\infty} x(t) \in \mathbb{R}$$

and

$$\lim_{t \to \pm\infty} x'(t) \in \mathbb{R};$$

And so $x(t) \in X$, the space X is complete, and because it satisfies all the conditions, it is also a Banach space.

□

For the reader's convenience we consider the definition of L^1- Carathéodory functions:

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Definition 1. *A function* $g : \mathbb{R}^3 \to \mathbb{R}$ *is* L^1- *Carathéodory if*

i) for each $(x, y) \in \mathbb{R}^2$, $t \mapsto g(t, x, y)$ *is measurable on* $\mathbb{R}$;
ii) for a.e. $t \in \mathbb{R}$, $(x, y) \mapsto g(t, x, y)$ *is continuous on* $\mathbb{R}^2$;
iii) for each $\rho > 0$, *there exists a positive function* $\omega_\rho \in L^1(\mathbb{R})$ *such that, whenever* $x, y \in [-\rho, \rho]$, *then*

$$|g(t, x, y)| \le \omega_\rho(t), \text{ a.e. } t \in \mathbb{R}. \tag{5}$$

Definition 2. *A sequence* $(\alpha_k)_{k\in\mathbb{Z}} : \mathbb{R}^3 \to \mathbb{R}$ *is a Carathéodory sequence if it verifies*

i) for each $(a, b) \in \mathbb{R}^2$, $(a, b) \longmapsto \alpha_k(t_k, a, b)$ *is continuous for all* $k \in \mathbb{Z}$;
ii) for each $\rho > 0$, *there are nonnegative constants* $\chi_{k,\rho} \ge 0$ *with* $\sum_{-\infty<k<+\infty} \chi_{k,\rho} < +\infty$ *such that for* $|a| < \rho$ *and* $|b| < \rho$ *we have* $|\alpha_k(t_k, a, b)| \le \chi_{k,\rho}$, *for every* $k \in \mathbb{Z}$.

Lemma 2. *Assume that* $f : \mathbb{R}^3 \to \mathbb{R}$ *is an* L^1- *Carathéodory function and* $I_k, J_k : \mathbb{R}^3 \to \mathbb{R}$ *are Carathéodory sequences, for* $k \in \mathbb{Z}$. *Then the equation (1) with conditions (2), (3), has a solution* $u \in X$ *expressed by*

$$\begin{aligned} u(t) = &\int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r, u(r), u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k, u(s_k), u'(s_k))}{a(s)}\right)ds \\ &+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k, u(t_k), u'(t_k)), \end{aligned}$$

with $C, M \in \mathbb{R}$ *satisfying condition (2). Namely,* M *is such that the following expression is verified:*

$$\begin{aligned} u(+\infty) = &\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r, u(r), u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k, u(s_k), u'(s_k))}{a(s)}\right)ds \\ &+C + \sum_{-\infty<t_k<+\infty} I_k(t_k, u(t_k), u'(t_k)) = L. \end{aligned} \tag{6}$$

Proof. Assuming the appropriate convergence conditions are met, the first boundary condition is satisfied as:

$$\begin{aligned} u(-\infty) = &\int_{-\infty}^{-\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r, u(r), u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k, u(s_k), u'(s_k))}{a(s)}\right)ds \\ &+C + \sum_{-\infty<t_k<-\infty} I_k(t_k, u(t_k), u'(t_k)) = C. \end{aligned}$$

Also, $M \in \mathbb{R}$ is such that:

$$\begin{aligned} u(+\infty) = &\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r, u(r), u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k, u(s_k), u'(s_k))}{a(s)}\right)ds \\ &+C + \sum_{-\infty<t_k<+\infty} I_k(t_k, u(t_k), u'(t_k)) = L. \end{aligned}$$

Working the expression with $u(t)$ we get:

$$\begin{aligned} &u'(t) = \phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r, u(r), u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k, u(t_k), u'(t_k))}{a(t)}\right) \\ &\Leftrightarrow a(t)\phi(u'(t)) = \int_{-\infty}^{t} f(r, u(r), u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k, u(t_k), u'(t_k)) \\ &\Leftrightarrow (a(t)\phi(u'(t)))' = f(t, u(t), u'(t)). \end{aligned}$$

□

The following theorem presents a useful criterion for the operator's compactness:

Theorem 1. *([11], Theorem 3) A set* $M \subset X$ *is relatively compact if the following conditions hold:*

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i) *both* $\{t \to x(t) : x \in M\}$ *and* $\{t \to x'(t) : x \in M\}$ *are uniformly bounded;*
ii) *both* $\{t \to x(t) : x \in M\}$ *and* $\{t \to x'(t) : x \in M\}$ *are equicontinuous on any compact interval of* $\mathbb{R}$*;*
iii) *both* $\{t \to x(t) : x \in M\}$ *and* $\{t \to x'(t) : x \in M\}$ *are equiconvergent at* $\pm\infty$*, that is, for any given* $\epsilon > 0$*, there exists* $t_\epsilon > 0$ *such that*

$$|f(t) - f(\pm\infty)| < \epsilon, \ \left|f'(t) - f'(\pm\infty)\right| < \epsilon, \forall |t| > t_\epsilon, f \in M.$$

Schauder's fixed point theorem will provide the means to establish existence:

Theorem 2. *([30]) Let* Y *be a nonempty, closed, bounded and convex subset of a Banach space* X*, and suppose that* $P : Y \to Y$ *is a compact operator. Then* P *has at least one fixed point in* Y*.*

Along this paper we assume that

(H1) $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing homeomorphism such that

a) $\phi(\mathbb{R}) = \mathbb{R}, \quad \phi(0) = 0.$

(H2) $a : \mathbb{R} \to (0, +\infty[$ is a positive continuous function such that

$$\lim_{t\to\pm\infty} \frac{1}{a(t)} \in \mathbb{R}.$$

3. Main Theorem

Here we present the main result of this work, that is, the Theorem that guarantees the existence of a solution to the problem (1)-(3), for $C, L \in \mathbb{R}$.

Theorem 3. *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be an increasing homeomorphism and* $a : \mathbb{R} \to (0, +\infty[$ *a continuous function satisfying (H1) and (H2). Assume that* $f : \mathbb{R}^3 \to \mathbb{R}$ *is an* L^1-*Carathéodory function,* $I_k, J_k : \mathbb{R}^3 \to \mathbb{R}$ *are Carathéodory sequences and there are* $\rho > 0$*,* $\omega_\rho \in L^1(\mathbb{R})$ *and non-negative constants* $\chi_{k,\rho}, \Psi_{k,\rho} \geq 0$ *such that*

$$\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_\rho(r)dr + M + \chi_{k,\rho}}{a(s)}\right) ds + C + \Psi_{k,\rho} < +\infty, \tag{7}$$

with

$$\sup_{t\in\mathbb{R}} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_\rho(r)dr + M + \chi_{k,\rho}}{a(t)}\right) < +\infty, \tag{8}$$

$$\begin{aligned} |f(t,x,y)| &\leq \omega_\rho(t), \\ |J_k(t_k, x(t_k), y(t_k))| &\leq \chi_{k,\rho}, \\ |I_k(t_k, x(t_k), y(t_k))| &\leq \Psi_{k,\rho}. \end{aligned}$$

when $x, y \in [-\rho, \rho]$*.*

Then, for $C, L \in \mathbb{R}$*, satisfying condition (2) and* $M \in \mathbb{R}$ *such that (6) is satisfied, the problem (1)-(3) has, at least heteroclinic solutions* $u \in X$*.*

Proof. Let us define the operator

$$\begin{aligned} T : \quad X &\to \quad X \\ u &\to \quad T(u) \end{aligned}$$

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with

$$(T(u))(t) = \int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds$$
$$+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)),$$

with $C \in \mathbb{R}$, satisfying condition (2) and $M \in \mathbb{R}$ such that (6) is satisfied.

To apply the Theorem 2, we will prove that T is compact and that it has a fixed point, that is, the proof follows five steps.

Step 1. *T is well defined and it is continuous in X.*

Allow $u \in X$ and let us take $\rho > 0$ such that $\|u\|_X < \rho$. Being f an L^1-Carathéodory function and I_k, J_k are Carathéodory sequence, there exists a positive function $\omega_\rho \in L^1(\mathbb{R})$, and non-negative constants $\chi_{k,\rho}, \Psi_{k,\rho} \geq 0$ such that

$$|J_k(t_k,u(t_k),u'(t_k))| \leq \chi_{k,\rho}, \quad |I_k(t_k,u(t_k),u'(t_k))| \leq \Psi_{k,\rho}.$$

Thus, $T \in C^1(\mathbb{R})$, as

$$\int_{-\infty}^{t} |f(r,u(r),u'(r))|dr + |M| + \sum_{-\infty<t_k<t<+\infty} |J_k(t_k,u(t_k),u'(t_k))|$$
$$\leq \int_{-\infty}^{+\infty} |f(r,u(r),u'(r))|dr + |M| + \sum_{-\infty<t_k<t<+\infty} |J_k(t_k,u(t_k),u'(t_k))|$$
$$\leq \int_{-\infty}^{+\infty} \omega_\rho(t)\,dt + |M| + \sum_{-\infty<t_k<+\infty} \chi_{k,\rho} < +\infty,$$

$$\sum_{-\infty<t_k<t<+\infty} |I_k(t_k,u(t_k),u'(t_k))| \leq \sum_{-\infty<t_k<+\infty} \Psi_{k,\rho} < +\infty$$

and

$$(T(u))'(t) = \phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)}\right)$$
$$\leq \sup_{t\in\mathbb{R}} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_\rho(t)\,dt + M + \sum_{-\infty<t_k<+\infty} \chi_{k,\rho}}{a(t)}\right) < +\infty.$$

Furthermore, by (2), (7), (8) and (H2),

$$\lim_{t\to-\infty} T(u)(t)$$
$$= \lim_{t\to-\infty}\left(\int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds\right.$$
$$\left.+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k))\right) = C \in \mathbb{R},$$

$$\lim_{t\to+\infty} T(u)(t)$$
$$= \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds$$
$$+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)) = L \in \mathbb{R},$$

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and

$$\begin{aligned}
&\lim_{t\to\pm\infty}(T(u))'(t) \\
&= \lim_{t\to\pm\infty}\phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)}\right) \\
&\le \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty}\omega_\rho(r)\,dr + M + \sum_{-\infty<t_k<+\infty}\chi_{k,\rho}}{a(+\infty)}\right) < +\infty.
\end{aligned}$$

Therefore, $T(u) \in X$.

Step 2. *TK is uniformly bounded on $K \subseteq X$, for some bounded K.*

Take K to be a bounded set of X, with the definition

$$K := \{u \in X : \max\{\|u\|_\infty, \|u'\|_\infty\} \le \rho_1\} \tag{9}$$

for some $\rho_1 > 0$.
By (7), (8), (H1) and (H2), we have

$$\begin{aligned}
&\|T(u)(t)\|_\infty \\
&= \sup_{t\in\mathbb{R}}\left(\left|\int_{-\infty}^{t}\phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds\right.\right. \\
&\left.\left.+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k))\right|\right) \\
&\le \sup_{t\in\mathbb{R}}\int_{-\infty}^{t}\left|\phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)\right|ds \\
&+|C| + \sup_{t\in\mathbb{R}}\left(\sum_{-\infty<t_k<t<+\infty}|I_k(t_k,u(t_k),u'(t_k))|\right) \\
&\le \sup_{t\in\mathbb{R}}\int_{-\infty}^{t}\phi^{-1}\left(\frac{\int_{-\infty}^{s}|f(r,u(r),u'(r))|dr + |M| + \sum_{-\infty<s_k<s<+\infty}|J_k(s_k,u(s_k),u'(s_k))|}{a(s)}\right)ds \\
&+|C| + \sup_{t\in\mathbb{R}}\left(\sum_{-\infty<t_k<t<+\infty}|I_k(t_k,u(t_k),u'(t_k))|\right) \\
&\le \int_{-\infty}^{+\infty}\phi^{-1}\left(\frac{\int_{-\infty}^{s}\omega_{\rho_1}(r)\,dr + |M| + \sum_{-\infty<s_k<s<+\infty}\chi_{k,\rho_1}}{a(s)}\right)ds \\
&+|C| + \sum_{-\infty<t_k<+\infty}\Psi_{k,\rho_1} < +\infty
\end{aligned}$$

and

$$\begin{aligned}
&\|T(u)'(t)\|_\infty \\
&= \sup_{t\in\mathbb{R}}\left|\phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)}\right)\right| \\
&\le \sup_{t\in\mathbb{R}}\phi^{-1}\left(\frac{\int_{-\infty}^{t}|f(r,u(r),u'(r))|dr + |M| + \sum_{-\infty<t_k<t<+\infty}|J_k(t_k,u(t_k),u'(t_k))|}{a(t)}\right) \\
&\le \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty}\omega_{\rho_1}(r)\,dr + |M| + \sum_{-\infty<t_k<+\infty}\chi_{k,\rho_1}}{a(+\infty)}\right) < +\infty
\end{aligned}$$

So, $\|T(u)(t)\|_X < +\infty$, that is, TK is uniformly bounded on X.

Step 3. *TK is equicontinuous, on each* $]t_k, t_{k+1}]$ interval, *for $k \in \mathbb{Z}$.*

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Consider $t_1, t_2 \in I \subseteq]t_k, t_{k+1}]$ and let us suppose, without losing generality, that $t_1 \leq t_2$. So, for $u \in K$ and by (7), (8), and (H1), follow

$$\begin{aligned}
&|T(u)(t_1) - T(u)(t_2)| \\
&= \left| \int_{-\infty}^{t_1} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)} \right) ds \right. \\
&+C + \sum_{-\infty<t_k<t_1<+\infty} I_k(t_k,u(t_k),u'(t_k)) - \\
&\left(\int_{-\infty}^{t_2} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)} \right) ds \right. \\
&\left. \left. +C + \sum_{-\infty<t_k<t_2<+\infty} I_k(t_k,u(t_k),u'(t_k)) \right) \right| \\
&\leq \int_{t_1}^{t_2} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)} \right) ds \\
&+ \sum_{t_1 \leq t_k < t_2 < +\infty} I_k(t_k,u(t_k),u'(t_k)) \\
&\leq \int_{t_1}^{t_2} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_{\rho_1}(t)\,dt + |M| + \sum_{-\infty<s_k<s<+\infty} \chi_{k,\rho_1}}{a(s)} \right) ds + \sum_{t_1 \leq t_k < t_2 < +\infty} \Psi_{k,\rho_1} \to 0
\end{aligned}$$

uniformly for $u \in K$, as $t_1 \to t_2$,

$$\begin{aligned}
&|T(u)'(t_1) - T(u)'(t_2)| \\
&= \left| \phi^{-1}\left(\frac{\int_{-\infty}^{t_1} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t_1<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t_1)} \right) \right. \\
&\left. -\phi^{-1}\left(\frac{\int_{-\infty}^{t_2} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t_2<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t_2)} \right) \right| \to 0
\end{aligned}$$

uniformly for $u \in K$, as $t_1 \to t_2$. Then, *TK is equicontinuous on each interval $]t_k, t_{k+1}]$, for $k \in \mathbb{Z}$.*

Step 4. *TK is equiconvergent at each impulse point, and at $t = \pm\infty$, that is TK, is equiconvergent at $t = t_i^+$, ($i \in \mathbb{Z}$) and at infinity.*

First, let us prove that TK is equiconvergent at $t = t_i^+$, for $i \in \mathbb{Z}$. Let $u \in K$. So, by (7), (8), and (H1), it follows

$$\begin{aligned}
&|T(u)(t) - \lim_{t \to t_i^+} T(u)(t)| \\
&= \left| \int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)} \right) ds \right. \\
&+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)) - \\
&\left(\int_{-\infty}^{t_i^+} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s_i^+<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)} \right) ds \right. \\
&\left. \left. +C + \sum_{-\infty<t_k<t_i^+<+\infty} I_k(t_k,u(t_k),u'(t_k)) \right) \right| \to 0
\end{aligned}$$

uniformly in $u \in K$, as $t \to t_i^+$, for $i \in \mathbb{Z}$ and

$$\begin{aligned}
&|T(u)'(t) - \lim_{t \to t_i^+} T(u)'(t)| \\
&= \left| \phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)} \right) \right. \\
&\left. -\phi^{-1}\left(\frac{\int_{-\infty}^{t_i^+} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t_i^+<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t_i^+)} \right) \right| \to 0
\end{aligned}$$

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uniformly in $u \in K$, as $t \to t_i^+$, for $i \in \mathbb{Z}$. Therefore, TK is equiconvergent at each point $t = t_i^+$, for $i \in \mathbb{Z}$.

Identically, we will prove that *TK is equiconvergent at $t = \pm\infty$*. In this way we have,

$$\begin{aligned}
&|T(u)(t) - \lim_{t\to-\infty} T(u)(t)| \\
&= \Bigg| \int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds \\
&+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)) - \lim_{t\to-\infty} \\
&\left(\int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds \right. \\
&\left. +C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k))\right)\Bigg| \\
&= \Bigg| \int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<s_k<s<+\infty} J_k(s_k,u(s_k),u'(s_k))}{a(s)}\right)ds \\
&+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)) - C\Bigg| \to 0
\end{aligned}$$

uniformly in $u \in K$, as $t \to -\infty$.

In turn,

$$\begin{aligned}
&|T(u)(t) - \lim_{t\to+\infty} T(u)(t)| \\
&= \Bigg| \int_{-\infty}^{t} \phi^{-1}\left(\frac{\int_{-\infty}^{s} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<s<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(s)}\right)ds \\
&+C + \sum_{-\infty<t_k<t<+\infty} I_k(t_k,u(t_k),u'(t_k)) - L\Bigg| \to 0
\end{aligned}$$

uniformly in $u \in K$, as $t \to +\infty$.
It follows for the derivative that,

$$\begin{aligned}
&|T(u)'(t) - \lim_{t\to+\infty} T(u)'(t)| \\
&= \Bigg|\phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)}\right) \\
&-\phi^{-1}\left(\lim_{t\to+\infty} \frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)}\right)\Bigg| \to 0
\end{aligned}$$

uniformly in $u \in K$, as $t \to +\infty$, and

$$\begin{aligned}
&|T(u)'(t) - \lim_{t\to-\infty} T(u)'(t)| \\
&= \Bigg|\phi^{-1}\left(\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)}\right) \\
&-\phi^{-1}\left(\lim_{t\to-\infty} \frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)}\right)\Bigg| \\
&\le \phi^{-1}\left(\left|\frac{\int_{-\infty}^{t} f(r,u(r),u'(r))dr + M + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,u(t_k),u'(t_k))}{a(t)}\right|\right) \\
&\le \phi^{-1}\left(\frac{\int_{-\infty}^{t} \omega_{\rho_1}(r)\,dr + |M| + \sum_{-\infty<t_k<t<+\infty} \chi_{n_1,\rho_1}}{a(t)}\right) \to 0
\end{aligned}$$

uniformly in $u \in K$, as $t \to -\infty$.
Therefore, TK is equiconvergent at $\pm\infty$ and by Theorem 1, TK is relatively compact.

Step 5. *$T : X \to X$ has a fixed point.*

To be able to apply Schauder's fixed point Theorem for the operator $T(u)$, we have to prove that

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$TD \subset D$, for some, bounded, closed and convex $D \subset X$.
Let us consider

$$D := \{u \in X : \|u\|_X \le \rho_2\},$$

with $\rho_2 > 0$ such that

$$\rho_2 \ge \max \begin{cases} \rho_1, \\ \int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_{\rho_1}(r)dr + |M| + \sum_{-\infty<t_k<+\infty} \chi_{k,\rho}}{a(s)}\right) ds + |C| + \sum_{-\infty<t_k<+\infty} \Psi_{k,\rho}, \\ \sup_{t\in\mathbb{R}} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty} \omega_{\rho_1}(r)dr + |M| + \sum_{-\infty<t_k<+\infty} \chi_{k,\rho}}{a(t)}\right) \end{cases} \quad \text{with } \rho_1 \text{ given by (9).}$$

Following arguments similar to step 2, we have that for $u \in D$

$$\|T(u)\|_X = \max\{\|T(u)\|_\infty, \|(T(u))'\|_\infty\} \le \rho_2,$$

and $TD \subset D$. Then, the operator $T(u)$, by Theorem 2, has a fixed point $u \in X$. Using standard arguments, we can demonstrate that this fixed point determines a pair of heteroclinic or homoclinic solutions for the problem (1)-(3). □

4. Example of Application of the Main Result and a Concrete Case of Application: Model for Studying the Dynamics of Bird Population Growth in the Natural Reserve

4.1. Example of Application of the Main Result

Let us consider the following second-order nonlinear system

$$\left((1+t^6)(u'(t))^5\right)' = \frac{t^2}{(1+t^4)^2}\left[(u(t))^2 + (u'(t))^3 + u(t)u'(t)\right] \tag{10}$$

together with the boundary conditions

$$u(-\infty) = C, \;\; u(+\infty) = L. \tag{11}$$

for $C, L \in \mathbb{R}$, and the generalized, impulse conditions

$$\begin{cases} \Delta u(t_k) = \frac{1}{k^4}(\alpha_1 \sqrt[5]{u(t_k)} + \alpha_2 \sqrt[5]{u'(t_k)}), \\ \\ \Delta\phi(u')(t_k) = \frac{1}{k^2}(\alpha_3 u(t_k) + \alpha_4 u'(t_k)), \end{cases} \tag{12}$$

with $\alpha_i \in \mathbb{R}$, $i = 1, 2, 3, 4.$ and for $k \in \mathbb{N}$, $\cdots < t_1 < \cdots < t_k < \cdots, .$

The above system (10)-(12) happens to be a particular case of problem (1)-(3). For example, for $\rho > 0$ so that

$$\rho := \max\{|x|, |y|\}, \tag{13}$$

taking $x = u(t), y = u'(t)$ and $f : \mathbb{R}^3 \to \mathbb{R}$ is an L^1-Carathéodory function being

$$\begin{aligned} f(t,x,y) &= \frac{t^2}{(1+t^4)^2}(x^2 + y^3 + xy), \\ &\le \frac{t^2}{(1+t^4)^2}(2\rho^2 + \rho^3) := \omega_\rho(t), \end{aligned}$$

where $\omega_\rho(t) \in L^1(\mathbb{R})$,

$$\phi(y) = y^5, \;\; a(t) = 1 + t^6,$$

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and I_k, J_k, are Carathéodory sequences that satisfy Definition 2, as for each k we set

$$I_k(t_k,x,y) \le \frac{\sqrt[5]{\rho}(|\alpha_1|+|\alpha_2|)}{k^4}, \quad J_k(t_k,x,y) \le \frac{\rho(|\alpha_3|+|\alpha_4|)}{k^2}.$$

with $t_k = k,\ k \in \mathbb{N}, \alpha_i \in \mathbb{R}, i = 1,2,3,4.$

So evaluating the following series we get

$$\sum_{k=1}^{+\infty} I_k(t_k,x,y) \le \sum_{k=1}^{+\infty} \frac{\sqrt[5]{\rho}(|\alpha_1|+|\alpha_2|)}{k^4} = \frac{\pi^4\sqrt[5]{\rho}(|\alpha_1|+|\alpha_2|)}{90}$$

and

$$\sum_{k=1}^{+\infty} J_k(t_k,x,y) \le \sum_{k=1}^{+\infty} \frac{\rho(|\alpha_3|+|\alpha_4|)}{k^2} = \frac{\pi^2\rho(|\alpha_3|+|\alpha_4|)}{6}$$

Besides, conditions (H1), (H2) hold, being that,

- $\phi(\mathbb{R}) = \mathbb{R},\ \phi(0) = 0,\ |\phi^{-1}(y)| = |\sqrt[5]{y}| = \phi^{-1}(|y|) = \sqrt[5]{|y|};$
- $\lim\limits_{t\to\pm\infty} \frac{1}{a(t)} = \lim\limits_{t\to\pm\infty} \frac{1}{1+t^6} = 0.$

Finally, note that

$$\int_{-\infty}^{+\infty} \phi^{-1}\left(\frac{\int_{-\infty}^{+\infty}\omega_\rho(r)dr + |M| + \sum_{-\infty<t_k<t<+\infty} J_k(t_k,x,y)}{a(s)}\right)ds + |C| +$$
$$\sum_{-\infty<t_k<t<+\infty} I_k(t_k,x,y)$$
$$\le \int_{-\infty}^{+\infty}\left(\sqrt[5]{\frac{\int_{-\infty}^{+\infty}\frac{r^2}{(1+r^4)^2}(2\rho^2+\rho^3)dr + |M| + \frac{\pi^2\rho(|\alpha_3|+|\alpha_4|)}{6}}{1+s^6}}\right)ds + |C| +$$
$$\frac{\pi^4\sqrt[5]{\rho}(|\alpha_1|+|\alpha_2|)}{90} < \rho$$

is finite. For example, taking $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, $M = C = 0$ and using the Maple software, we find an approximated fixed point for $\rho = 382.8775379$. For slightly smaller values of ρ, the above inequality is false. So, by Theorem 3, there is at least $u \in X$, that is a solution to the problem (10)-(12).

4.2. A Possible and Specific Case of Application: Model for Studying the Dynamics of Bird Population Growth in the Nature Reserve

Let us consider a specific example of an ordinary differential equations system that describes the population growth of a bird species in a specific area.

Suppose we are studying the population of a bird species in a nature reserve over time. We can model the population growth of this species with the following system of ordinary differential equations (ODEs):

$$\left((1+t^2)k\sqrt{u'(t)}\right)' = \frac{t}{1+t^4}\left[u(t)^2 + (u'(t))^3 + u(t)u'(t)\right], \tag{14}$$

where:

- $t \in \mathbb{R} \setminus \{t_k\}$: Time t, excluding specific moments t_k,
- $u(t)$: Denotes the population size of the bird species at time t,
- $u'(t) = \frac{du}{dt}$: Represents the derivative of $u(t)$ with respect to time,
- $\phi(u'(t)) = k\sqrt{u'(t)}$: Represents the migration rate of birds, influencing population dynamics, chosen as an increasing homeomorphism, where k is a constant that represents the migration rate,

- $a(t) = 1 + \alpha \cdot \sin(\omega t)$: Represents the environmental influence on population growth, modeled by a continuous seasonal sinusoidal function, with α as amplitude and ω as the variation frequency,
- $f(t, u, u') = \frac{t}{(1+t^4)^2}\left[u^2 + u'^3 + uu'\right]$: Represents an L^1-Carathéodory function.

The initial condition indicates that the population starts with 1000 individuals:

$$u(-\infty) = 1000.$$

After a long period, the population stabilizes at 50000 individuals:

$$u(+\infty) = 50000.$$

The model also incorporates thrust conditions to account for specific events that affect the bird population:

$$\begin{cases} \Delta u(t_k) = \frac{1}{k^2}\left(\alpha_1\sqrt{u(t_k)} + \alpha_2\sqrt{u'(t_k)}\right), \\ \Delta\phi(u'(t_k)) = \frac{1}{k}(\beta_1 u(t_k) + \beta_2 u'(t_k)), \end{cases}$$

where:

- $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$: Represents the change in population size at discrete time points t_k,
- $\Delta\phi(u'(t_k)) = \phi(u'(t_k^+)) - \phi(u'(t_k^-))$: Represents the change in migration rate in t_k.

5. Discussion

This work presents the existence of heteroclinic solutions of strongly nonlinear second order equations. While the existence of the solutions is guaranteed by Schauder's fixed point theorem, which is very well known, our work extends the results in the literature, using a ϕ-Laplacian operator in the equations with conditions of infinite impulses of greater or lesser intensity.

This can be very interesting for modeling phenomena with small changes and different intensities, which occur very quickly and for long periods of time, opening new fields of investigation on the topic. As illustrated in the example and application section, an example of this is that this type of models can incorporate at the same time, the dynamics of continuous growth governed by a differential equation, as well as the discrete changes represented by infinite impulse conditions. Therefore, they provide a broad vision and field to investigate the complex dynamics of bird population growth in a natural reserve, integrating biological, environmental and behavioral factors.

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