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Article

Affine Type AQ Functional Equation in Various Banach Spaces

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Abstract: In this paper, we analyze the generalized Ulam-Hyers stability of affine type AQ Functional Equation in various Banach Spaces using Direct and Fixed Methods.

Keywords: Mixed functional equations, Generalized Ulam - Hyers stability, Direct Method, Fixed Method, Banach space, Intuitionistic Fuzzy Banach space.

MSC: 39B52, 32B72, 32B82

1. Introduction

A inspirational and popular talk presented by S.M Ulam [37] in 1940, refreshed the reading of stability problems for various functional equations. He gave a wide range of talk before a Mathematical Colloquium at the University of Wisconsin in which he presented a list of unsolved problems.

The first assertive answer to Ulam's question concerning the problem of stability of functional equations was given by D.H. Hyers [20] for the case of additive mappings in Banach spaces. In growth of time, the theorem delivered by Hyers was generalized by T. Aoki [3], Th.M Rassias [30], J.M. Rassias [28], P. Gavruta [19] for additive mappings and K. Ravi [32] for quadratic mappings.

The famous additive and quadratic functional equations are

$$\mathcal{F}(w_1 + w_2) = \mathcal{F}(w_1) + \mathcal{F}(w_2), \tag{1}$$

and

$$\mathcal{F}(w_1 + w_2) + \mathcal{F}(w_1 - w_2) = 2\mathcal{F}(w_1) + 2\mathcal{F}(w_2). \tag{2}$$

The general solution and generalized Ulam - Hyers stability of several types of functional equations in various normed spaces were discussed by many authors one can see [2,16,18,21,22,31] and references there in.

Also, the general solution and Hyers-Ulam-Rassias stability of the several affine functional equations are discussed by L. Lucht, C. Methfessel [23], L. Cadariu, L. Gavruta, P. Gavruta [15], Md. Nasiruzzaman [26], M. Mursaleen, KJ. Ansari[25].

Infact, the general solution and generalized Hyers-Ulam stability of the several AQ functional equations are established in [4–12,14,29].

In this paper, the we analyze the generalized Ulam-Hyers stability of affine type AQ Functional Equation of the form

$$\begin{aligned}&\mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3)\\&= 6\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) + \frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) + \mathcal{F}\left(-\sum_{\psi=1}^{3} w_\psi\right)\right\} - \sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi) - \frac{5}{2}\Big[\mathcal{F}(w_\psi) + \mathcal{F}(-w_\psi)\Big]\right\}\end{aligned} \tag{3}$$

in various Banach Spaces using Direct and Fixed Methods .

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Lemma 1.1. *[26] Let A and B be real vector spaces. Suppose $\mathcal{F} : A \to B$ be an odd mapping satisfies (3) then $\mathcal{F}$ is additive.*

Lemma 1.2. *[17] Let A and B be real vector spaces. Suppose $\mathcal{F} : A \to B$ be an even mapping satisfies (3) then $\mathcal{F}$ is quadratic.*

Now, we present the result due to Margolis, Diaz [24] and Radu [27] for fixed point theory.

Theorem 1.3. *[24,27] Suppose that for a complete generalized metric space (Ω, δ) and a strictly contractive mapping $T : \Omega \longrightarrow \Omega$ with Lipschitz constant L. Then, for each given $x \in \Omega$, either*

$$d(T^n x, T^{n+1}x) = \infty \qquad \forall \qquad n \geq 0,$$

or there exists a natural number n_0 such that
(FPC1) $d(T^n x, T^{n+1}x) < \infty$ for all $n \geq n_0$;
(FPC2) The sequence $(T^n x)$ is convergent to a fixed point y^* of T
(FPC3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;
(FPC4) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ for all $y \in \Delta$.

2. Stability In Banach Space of (3)

In this section, we explore the generalized Ulam - Hyers stability of the functional equation (3) in Banach space. To prove stability results, let us take $\mathcal{W}_1$ be a normed space and $\mathcal{W}_2$ be a Banach space. Suppose that $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ and $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ satisfying the following functional inequalities

$$\begin{aligned} \Big\| &\mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) \\ &- 6\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) - \frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) + \mathcal{F}\left(-\sum_{\psi=1}^{3} w_\psi\right)\right\} \\ &+ \sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi) - \frac{5}{2}\Big[\mathcal{F}(w_\psi) + \mathcal{F}(-w_\psi)\Big]\right\}\Big\| \leq \Psi(w_1, w_2, w_3), \end{aligned} \tag{1}$$

and

$$\begin{aligned} &\Big\|\mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 6\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) \\ &- \frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) + \mathcal{F}\left(-\sum_{\psi=1}^{3} w_\psi\right)\right\} + \sum_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi) - \frac{5}{2}\Big[\mathcal{F}(w_\psi) + \mathcal{F}(-w_\psi)\Big]\right\}\Big\| \\ &\leq \begin{cases} \delta, \\ \delta \sum_{\psi=1}^{3} |w_\psi|^{\varphi}, \\ \delta \sum_{\psi=1}^{3} |w_\psi|^{\varphi_\psi}, \\ \delta \prod_{\psi=1}^{3} |w_\psi|^{\varphi}, \\ \delta \prod_{\psi=1}^{3} |w_\psi|^{\varphi_\psi}, \\ \delta\left\{\sum_{\psi=1}^{3} |w_\psi|^{3\varphi} + \prod_{\psi=1}^{3} |w_\psi|^{\varphi}\right\}, \end{cases} \end{aligned} \tag{2}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and δ be a positive constant.

2.1. Oddness of $\mathcal{F}$: Additive Case Stability Results : Direct Method

Theorem 2.1. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (1) where $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ with the condition*

$$\lim_{\ell\to\infty} \frac{\Psi\left(5^{\ell\mu}w_1, 5^{\ell\mu}w_2, 5^{\ell\mu}w_3\right)}{5^{\ell\mu}} = 0; \quad \mu = \pm 1 \tag{3}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality

$$\|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \le \frac{1}{5}\sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{5^{\eta\mu}}\Psi_A(5^{\eta\mu}w_1) \tag{4}$$

$$= \frac{1}{5}\sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{5^{\eta\mu}}\left\{\frac{1}{3}\left\{\Psi(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, 5^{\eta\mu}w_1) + 3\Psi(5^{\eta\mu}w_1, 5^{\eta\mu}w_1, -5^{\eta\mu}w_1)\right\}\right\} \tag{5}$$

and the mapping $\mathcal{A}(w_1)$ is obtained by

$$\mathcal{A}(w_1) = \lim_{\ell\to\infty}\frac{1}{5^{\ell\mu}}\mathcal{F}\left(5^{\ell\mu}w_1\right) \tag{6}$$

for all $w_1 \in \mathcal{W}_1$.

Proof. Using oddness of $\mathcal{F}$ in (1), we get

$$\left\|\mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 6\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) + \sum_{\psi=1}^{3}\mathcal{F}(w_\psi)\right\|$$
$$\le \Psi(w_1, w_2, w_3), \forall\, w_1, w_2, w_3 \in \mathcal{W}_1. \tag{7}$$

Interchanging (w_1, w_2, w_3) by (w_1, w_1, w_1) in (7), we obtain

$$\left\|3\mathcal{F}(5w_1) - 6\mathcal{F}(3w_1) + 3\mathcal{F}(w_1)\right\| \le \Psi(w_1, w_1, w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{8}$$

Again interchanging (w_1, w_2, w_3) by $(w_1, w_1, -w_1)$ in (7), we have

$$\left\|2\mathcal{F}(3w_1) - 6\mathcal{F}(w_1)\right\| \le \Psi(w_1, w_1, -w_1)$$
$$\Rightarrow \left\|6\mathcal{F}(3w_1) - 18\mathcal{F}(w_1)\right\| \le 3\Psi(w_1, w_1, -w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{9}$$

Combining (8) and (9), we arrive

$$\left\|3\mathcal{F}(5w_1) - 15\mathcal{F}(w_1)\right\| \le \left\|3\mathcal{F}(5w_1) - 6\mathcal{F}(3w_1) + 3\mathcal{F}(w_1)\right\| + \left\|6\mathcal{F}(3w_1) - 18\mathcal{F}(w_1)\right\|$$
$$\le \Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{10}$$

One can see from (10) that

$$\left\|\mathcal{F}(5w_1) - 5\mathcal{F}(w_1)\right\| \le \frac{1}{3}\left\{\Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1)\right\} = \Psi_A(w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{11}$$

It follows from (11) that

$$\left\|\frac{1}{5}\mathcal{F}(5w_1) - \mathcal{F}(w_1)\right\| \le \frac{1}{5}\Psi_A(w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{12}$$

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Generalizing for a positive integer ℓ, we get

$$\left\|\frac{1}{5^{\ell}}\mathcal{F}(5^{\ell}w_1)-\mathcal{F}(w_1)\right\| \leq \frac{1}{5}\sum_{\eta=0}^{\ell}\frac{1}{5^{\eta}}\ \Psi_A(5^{\eta}w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{13}$$

Now, changing w_1 by $5^{\ell_1}w_1$ in (13), we obtain

$$\begin{aligned}\left\|\frac{1}{5^{\ell+\ell_1}}\mathcal{F}(5^{\ell+\ell_1}w_1)-\frac{1}{5^{\ell_1}}\mathcal{F}(5^{\ell_1}w_1)\right\| &= \frac{1}{5^{\ell_1}}\left\|\frac{1}{5^{\ell}}\mathcal{F}(5^{\ell+\ell_1}w_1)-\mathcal{F}(5^{\ell_1}w_1)\right\| \\ &\leq \frac{1}{5}\sum_{\eta=0}^{\ell}\frac{1}{5^{\eta+\ell_1}}\ \Psi_A\left(5^{\eta+\ell_1}w_1\right) \\ &\rightarrow\ 0\ \ as\ \ \ell_1\ \ \rightarrow\ \ \infty, \forall\, w_1 \in \mathcal{W}_1. \end{aligned} \tag{14}$$

Therefore, the sequence

$$\left\{\frac{1}{5^{\ell}}\mathcal{F}(5^{\ell}w_1)\right\},$$

is a Cauchy sequence and it converges to $\mathcal{A}(w_1)$ in $\mathcal{W}_2$. So, we define

$$\mathcal{A}(w_1) = \lim_{\ell\to\infty}\frac{1}{5^{\ell}}\mathcal{F}\left(5^{\ell}w_1\right), \forall\, w_1 \in \mathcal{W}_1. \tag{15}$$

Taking limit $\ell \to \infty$ in (13), we have

$$\left\|\mathcal{A}(w_1)-\mathcal{F}(w_1)\right\| \leq \frac{1}{5}\sum_{\eta=0}^{\infty}\frac{1}{5^{\eta}}\ \Psi_A(5^{\eta}w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{16}$$

Thus, (4) and (5) holds for $\mu = 1$. Interchanging

$$(w_1, w_2, w_3) = \left(5^{\ell}w_1, 5^{\ell}w_2, 5^{\ell}w_3\right),$$

we arrive

$$\begin{aligned}\frac{1}{5^{\ell}}\ &\Big\|\mathcal{F}(5^{\ell}(3w_1+w_2+w_3))+\mathcal{F}(5^{\ell}(w_1+3w_2+w_3))+\mathcal{F}(5^{\ell}(w_1+w_2+3w_3)) \\ &\quad -6\mathcal{F}\left(\sum_{\psi=1}^{3}5^{\ell}w_{\psi}\right)-\frac{1}{2}\left\{\mathcal{F}\left(\sum_{\psi=1}^{3}5^{\ell}w_{\psi}\right)+\mathcal{F}\left(-\sum_{\psi=1}^{3}5^{\ell}w_{\psi}\right)\right\} \\ &\quad +\sum_{\psi=1}^{3}\left\{\mathcal{F}(5^{\ell}w_{\psi})-\frac{5}{2}\left[\mathcal{F}(5^{\ell}w_{\psi})+\mathcal{F}(-5^{\ell}w_{\psi})\right]\right\}\Big\| \\ &\leq \frac{1}{5^{\ell}}\Psi\left(5^{\ell}w_1, 5^{\ell}w_2, 5^{\ell}w_3\right), \forall w_1, w_2, w_3 \in \mathcal{W}_1. \end{aligned} \tag{17}$$

Taking limit $\ell \to \infty$ in (17), using (15) and (3), we get

$$\begin{aligned}&\mathcal{A}(3w_1+w_2+w_3)+\mathcal{A}(w_1+3w_2+w_3)+\mathcal{A}(w_1+w_2+3w_3) \\ &= 6\mathcal{A}\left(\sum_{\psi=1}^{3}w_{\psi}\right)+\frac{1}{2}\left\{\mathcal{A}\left(\sum_{\psi=1}^{3}w_{\psi}\right)+\mathcal{A}\left(-\sum_{\psi=1}^{3}w_{\psi}\right)\right\}-\sum_{\psi=1}^{3}\left\{\mathcal{A}(w_{\psi})-\frac{5}{2}\left[\mathcal{A}(w_{\psi})+\mathcal{A}(-w_{\psi})\right]\right\}\end{aligned}$$

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for all $w_1, w_2, w_3 \in \mathcal{W}_1$. So, $\mathcal{A}(w_1)$ satisfies (3). In order to confirm that $\mathcal{A}(w_1)$ is unique, suppose $\mathcal{B}(w_1)$ be another mapping (3), (15) and (16), we obtain

$$\begin{aligned}\left\|\mathcal{A}(w_1) - \mathcal{B}(w_1)\right\| &= \left\|\frac{1}{5^\ell}\mathcal{A}\left(5^\ell w_1\right) - \frac{1}{5^\ell}\mathcal{B}\left(5^\ell w_1\right)\right\| \\ &\leq \frac{1}{5^\ell}\left\|\mathcal{A}\left(5^\ell w_1\right) - \mathcal{F}\left(5^\ell w_1\right)\right\| + \frac{1}{5^\ell}\left\|\mathcal{F}\left(5^\ell w_1\right) - \mathcal{B}\left(5^\ell w_1\right)\right\| \\ &\leq \frac{2}{5}\sum_{\eta=0}^{\infty}\frac{1}{5^{\eta+\ell}}\,\Psi_A\left(5^{\eta+\ell}w_1\right) \rightarrow 0 \;\; as \;\; \ell_1 \rightarrow \infty,\end{aligned}$$

for all $w_1 \in \mathcal{W}_1$. Therefore $\mathcal{A}(w_1)$ is unique. So, the Theorem holds for $\mu = 1$.

Changing $w_1 = \frac{w_1}{5}$ in (11), we have

$$\left\|\mathcal{F}(w_1) - 5\mathcal{F}\left(\frac{w_1}{5}\right)\right\| \leq \Psi_A\left(\frac{w_1}{5}\right), \forall\, w_1 \in \mathcal{W}_1. \tag{18}$$

Generalizing for a positive integer ℓ, we get

$$\left\|\mathcal{F}(w_1) - 5^\ell\mathcal{F}\left(\frac{w_1}{5^\eta}\right)\right\| \leq \frac{1}{5}\sum_{\eta=1}^{\ell}5^\eta\,\Psi_A\left(\frac{w_1}{5^\eta}\right), \forall\, w_1 \in \mathcal{W}_1. \tag{19}$$

The rest of the proof is similar to that of above case. So, the Theorem holds for $\mu = -1$. Hence the proof is complete □

Corollary 2.2. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ satisfying the functional inequality (2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ which satisfying (3) and the functional inequality*

$$\|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \leq \begin{cases} \frac{\delta}{|3|}, & \\ \frac{4\delta|w_1|^\varphi}{|5-5^\varphi|}; & \varphi \neq 1, \\ \frac{4\delta}{3}\sum_{\psi=1}^{3}\frac{|w_\psi|^{\varphi_\psi}}{|5-5^{\varphi_\psi}|}; & \varphi_1, \varphi_2, \varphi_3 \neq 1, \\ \frac{4\delta|w_1|^{3\varphi}}{3|5-5^{3\varphi}|}; & 3\varphi \neq 1, \\ \frac{4\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}}{3\left|5-5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|}; & \sum_{\psi=1}^{3}\varphi_\psi \neq 1, \\ \frac{16\delta|w_1|^{3\varphi}}{3|5-5^{3\varphi}|}; & 3\varphi \neq 1, \end{cases} \tag{20}$$

for all $w_1 \in \mathcal{W}_1$.

2.2. Evenness of $\mathcal{F}$: Quadratic Case Stability Results : Direct Method

Theorem 2.3. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \rightarrow \mathcal{W}_2$ satisfying the functional inequality (1) where $\Psi : \mathcal{W}_1^3 \rightarrow [0, \infty)$ with the condition*

$$\lim_{\ell\rightarrow\infty}\frac{\Psi\left(5^{\ell\mu}w_1, 5^{\ell\mu}w_2, 5^{\ell\mu}w_3\right)}{25^{\ell\mu}} = 0; \quad \mu = \pm 1 \tag{21}$$

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for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality

$$\|\mathcal{F}(w_1) - \mathcal{Q}(w_1)\| \leq \frac{1}{25} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{25^{\eta\mu}} \Psi_Q(5^{\eta\mu} w_1) \tag{22}$$

$$= \frac{1}{25} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{25^{\eta\mu}} \left\{ \frac{1}{3} \left\{ \Psi(5^{\eta\mu} w_1, 5^{\eta\mu} w_1, 5^{\eta\mu} w_1) + \frac{7}{2} \Psi(5^{\eta\mu} w_1, 5^{\eta\mu} w_1, -5^{\eta\mu} w_1) \right\} \right\} \tag{23}$$

and the mapping $\mathcal{Q}(w_1)$ is obtained by

$$\mathcal{Q}(w_1) = \lim_{\ell \to \infty} \frac{1}{25^{\ell\mu}} \mathcal{F}\left(5^{\ell\mu} w_1\right) \tag{24}$$

for all $w_1 \in \mathcal{W}_1$.

Proof. Using evenness of $\mathcal{F}$ in (1), we get

$$\left\| \mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 7\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) - 4\sum_{\psi=1}^{3} \mathcal{F}(w_\psi) \right\| \leq \Psi(w_1, w_2, w_3), \forall\, w_1, w_2, w_3 \in \mathcal{W}_1. \tag{25}$$

Interchanging (w_1, w_2, w_3) by (w_1, w_1, w_1) in (25), we obtain

$$\left\| 3\mathcal{F}(5w_1) - 7\mathcal{F}(3w_1) - 12\mathcal{F}(w_1) \right\| \leq \Psi(w_1, w_1, w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{26}$$

Again interchanging (w_1, w_2, w_3) by $(w_1, w_1, -w_1)$ in (25), we have

$$\left\| 2\mathcal{F}(3w_1) - 18\mathcal{F}(w_1) \right\| \leq \Psi(w_1, w_1, -w_1)$$
$$\Rightarrow \left\| 7\mathcal{F}(3w_1) - 63\mathcal{F}(w_1) \right\| \leq \frac{7}{2}\Psi(w_1, w_1, -w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{27}$$

Combining (26) and (27), we arrive

$$\left\| 3\mathcal{F}(5w_1) - 75\mathcal{F}(w_1) \right\| \leq \left\| 3\mathcal{F}(5w_1) - 7\mathcal{F}(3w_1) - 12\mathcal{F}(w_1) \right\| + \left\| 7\mathcal{F}(3w_1) - 63\mathcal{F}(w_1) \right\|$$
$$\leq \Psi(w_1, w_1, w_1) + \frac{7}{2}\Psi(w_1, w_1, -w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{28}$$

One can see from (28) that

$$\left\| \mathcal{F}(5w_1) - 25\mathcal{F}(w_1) \right\| \leq \frac{1}{3}\left\{ \Psi(w_1, w_1, w_1) + \frac{7}{2}\Psi(w_1, w_1, -w_1) \right\} = \Psi_Q(w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{29}$$

It follows from (29) that

$$\left\| \frac{1}{25}\mathcal{F}(5w_1) - \mathcal{F}(w_1) \right\| \leq \frac{1}{25}\Psi_Q(w_1), \forall\, w_1 \in \mathcal{W}_1. \tag{30}$$

The rest of the proof is similar to that of Theorem 2.1. Hence the proof is complete. □

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Corollary 2.4. *Suppose that an even function $\mathcal{F}:\mathcal{W}_1\to\mathcal{W}_2$ satisfying the functional inequality (2) for all $w_1,w_2,w_3\in\mathcal{W}_1$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1):\mathcal{W}_1\to\mathcal{W}_2$ which satisfying (3) and the functional inequality*

$$\|\mathcal{F}(w_1)-\mathcal{Q}(w_1)\|\leq\begin{cases}\frac{3\delta}{2|24|}, & \\ \frac{27\delta|w_1|^{\varphi}}{6|25-5^{\varphi}|}; & \varphi\neq 2,\\ \frac{9\delta}{6}\sum\limits_{\psi=1}^{3}\frac{|w_\psi|^{\varphi_\psi}}{|25-5^{\varphi_\psi}|}; & \varphi_1,\varphi_2,\varphi_3\neq 2,\\ \frac{9\delta|w_1|^{3\varphi}}{6|25-5^{3\varphi}|}; & 3\varphi\neq 2,\\ \frac{9\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}}{6\left|25-5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|}; & \sum\limits_{\psi=1}^{3}\varphi_\psi\neq 2,\\ \dfrac{36\delta|w_1|^{3\varphi}}{6|25-5^{3\varphi}|}; & 3\varphi\neq 2,\end{cases}\tag{31}$$

for all $w_1\in\mathcal{W}_1$.

2.3. Oddness and Evenness of $\mathcal{F}$: Additive Quadratic Case Stability Results : Direct Method

Theorem 2.5. *Suppose that a function $\mathcal{F}:\mathcal{W}_1\to\mathcal{W}_2$ satisfying the functional inequality (1) where $\Psi:\mathcal{W}_1^3\to[0,\infty)$ with the conditions (3) and (21) for all $w_1,w_2,w_3\in\mathcal{W}_1$. Then there exists a unique additive mapping $\mathcal{A}(w_1):\mathcal{W}_1\to\mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1):\mathcal{W}_1\to\mathcal{W}_2$ which satisfying (3) and the functional inequality*

$$\begin{aligned}&\|\mathcal{F}(w_1)-\mathcal{A}(w_1)-\mathcal{Q}(w_1)\|\\ &\leq\frac{1}{2}\left\{\frac{1}{5}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{5^{\eta\mu}}\left\{\Psi_A(5^{\eta\mu}w_1)+\Psi_A(-5^{\eta\mu}w_1)\right\}+\frac{1}{25}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{25^{\eta\mu}}\left\{\Psi_Q(5^{\eta\mu}w_1)+\Psi_Q(-5^{\eta\mu}w_1)\right\}\right\}\\ &\leq\frac{1}{2}\left\{\frac{1}{5}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{5^{\eta\mu}}\left\{\frac{1}{3}\left\{\Psi(5^{\eta\mu}w_1,5^{\eta\mu}w_1,5^{\eta\mu}w_1)+3\Psi(5^{\eta\mu}w_1,5^{\eta\mu}w_1,-5^{\eta\mu}w_1)\right\}\right.\right.\\ &\qquad\left.+\frac{1}{3}\left\{\Psi(-5^{\eta\mu}w_1,-5^{\eta\mu}w_1,-5^{\eta\mu}w_1)+3\Psi(-5^{\eta\mu}w_1,-5^{\eta\mu}w_1,5^{\eta\mu}w_1)\right\}\right\}\\ &\quad+\frac{1}{25}\sum_{\eta=\frac{1-\mu}{2}}^{\infty}\frac{1}{25^{\eta\mu}}\left\{\frac{1}{3}\left\{\Psi(5^{\eta\mu}w_1,5^{\eta\mu}w_1,5^{\eta\mu}w_1)+\frac{7}{2}\Psi(5^{\eta\mu}w_1,5^{\eta\mu}w_1,-5^{\eta\mu}w_1)\right\}\right.\\ &\qquad\left.\left.+\frac{1}{3}\left\{\Psi(-5^{\eta\mu}w_1,-5^{\eta\mu}w_1,-5^{\eta\mu}w_1)+\frac{7}{2}\Psi(-5^{\eta\mu}w_1,-5^{\eta\mu}w_1,5^{\eta\mu}w_1)\right\}\right\}\right\}\end{aligned}\tag{32}$$

and the mapping $\mathcal{A}(w_1)$ and $\mathcal{Q}(w_1)$ are given in (6) and (24) for all $w_1\in\mathcal{W}_1$.

Proof. Consider a function $\mathcal{F}_{odd}(w_1)$ by

$$\mathcal{F}_{odd}(w_1)=\frac{1}{2}\left\{\mathcal{F}(w_1)-\mathcal{F}(-w_1)\right\},\forall\, w_1\in\mathcal{W}_1,\tag{33}$$

which gives

$$\mathcal{F}_{odd}(0)=0;\qquad\mathcal{F}_{odd}(-w_1)=-\mathcal{F}_{odd}(w_1),\forall\, w_1\in\mathcal{W}_1.\tag{34}$$

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By Theorem 2.1, it follows from (33), (1), (5) and (6), we arrive

$$\begin{aligned}
&\|\mathcal{F}_{odd}(w_1) - \mathcal{A}(w_1)\| \\
&\leq \frac{1}{2} \cdot \frac{1}{5} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{5^{\eta\mu}} \left\{ \Psi_A(5^{\eta\mu} w_1) + \Psi_A(-5^{\eta\mu} w_1) \right\} \qquad (35)\\
&= \frac{1}{2} \cdot \frac{1}{5} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{5^{\eta\mu}} \left\{ \frac{1}{3}\left\{ \Psi(5^{\eta\mu} w_1, 5^{\eta\mu} w_1, 5^{\eta\mu} w_1) + 3\Psi(5^{\eta\mu} w_1, 5^{\eta\mu} w_1, -5^{\eta\mu} w_1) \right\} \right. \\
&\qquad \left. + \frac{1}{3}\left\{ \Psi(-5^{\eta\mu} w_1, -5^{\eta\mu} w_1, -5^{\eta\mu} w_1) + 3\Psi(-5^{\eta\mu} w_1, -5^{\eta\mu} w_1, 5^{\eta\mu} w_1) \right\} \right\} \qquad (36)
\end{aligned}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Consider a function $\mathcal{F}_{even}(w_1)$ by

$$\mathcal{F}_{even}(w_1) = \frac{1}{2}\left\{ \mathcal{F}(w_1) + \mathcal{F}(-w_1) \right\}, \forall\, w_1 \in \mathcal{W}_1, \qquad (37)$$

which gives

$$\mathcal{F}_{even}(0) = 0; \qquad \mathcal{F}_{even}(-w_1) = \mathcal{F}_{even}(w_1), \forall\, w_1 \in \mathcal{W}_1. \qquad (38)$$

By Theorem 2.3, it follows from (37), (1), (22) and (23), we see

$$\begin{aligned}
&\|\mathcal{F}_{even}(w_1) - \mathcal{Q}(w_1)\| \\
&\leq \frac{1}{2} \cdot \frac{1}{25} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{25^{\eta\mu}} \left\{ \Psi_Q(5^{\eta\mu} w_1) + \Psi_Q(-5^{\eta\mu} w_1) \right\} \qquad (39)\\
&= \frac{1}{2} \cdot \frac{1}{25} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{25^{\eta\mu}} \left\{ \frac{1}{3}\left\{ \Psi(5^{\eta\mu} w_1, 5^{\eta\mu} w_1, 5^{\eta\mu} w_1) + \frac{7}{2}\Psi(5^{\eta\mu} w_1, 5^{\eta\mu} w_1, -5^{\eta\mu} w_1) \right\} \right. \\
&\qquad \left. + \frac{1}{3}\left\{ \Psi(-5^{\eta\mu} w_1, -5^{\eta\mu} w_1, -5^{\eta\mu} w_1) + \frac{7}{2}\Psi(-5^{\eta\mu} w_1, -5^{\eta\mu} w_1, 5^{\eta\mu} w_1) \right\} \right\} \qquad (40)
\end{aligned}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Assume a function $\mathcal{F}(w_1)$ by

$$\mathcal{F}(w_1) = \mathcal{F}_{odd}(w_1) + \mathcal{F}_{even}(w_1), \forall\, w_1 \in \mathcal{W}_1. \qquad (41)$$

Now, it follows from (35), (36), (39), (40) and (41), we have

$$\begin{aligned}
&\|\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1)\| \\
&\leq \|\mathcal{F}_{odd}(w_1) - \mathcal{A}(w_1)\| + \|\mathcal{F}_{even}(w_1) - \mathcal{Q}(w_1)\| \\
&\leq \frac{1}{2}\left\{ \frac{1}{5} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{5^{\eta\mu}} \left\{ \Psi_A(5^{\eta\mu} w_1) + \Psi_A(-5^{\eta\mu} w_1) \right\} + \frac{1}{25} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{25^{\eta\mu}} \left\{ \Psi_Q(5^{\eta\mu} w_1) + \Psi_Q(-5^{\eta\mu} w_1) \right\} \right\} \\
&\leq \frac{1}{2}\left\{ \frac{1}{5} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{5^{\eta\mu}} \left\{ \frac{1}{3}\left\{ \Psi(5^{\eta\mu} w_1, 5^{\eta\mu} w_1, 5^{\eta\mu} w_1) + 3\Psi(5^{\eta\mu} w_1, 5^{\eta\mu} w_1, -5^{\eta\mu} w_1) \right\} \right. \right. \\
&\qquad \left. + \frac{1}{3}\left\{ \Psi(-5^{\eta\mu} w_1, -5^{\eta\mu} w_1, -5^{\eta\mu} w_1) + 3\Psi(-5^{\eta\mu} w_1, -5^{\eta\mu} w_1, 5^{\eta\mu} w_1) \right\} \right\} \\
&\quad + \frac{1}{25} \sum_{\eta=\frac{1-\mu}{2}}^{\infty} \frac{1}{25^{\eta\mu}} \left\{ \frac{1}{3}\left\{ \Psi(5^{\eta\mu} w_1, 5^{\eta\mu} w_1, 5^{\eta\mu} w_1) + \frac{7}{2}\Psi(5^{\eta\mu} w_1, 5^{\eta\mu} w_1, -5^{\eta\mu} w_1) \right\} \right. \\
&\qquad \left. \left. + \frac{1}{3}\left\{ \Psi(-5^{\eta\mu} w_1, -5^{\eta\mu} w_1, -5^{\eta\mu} w_1) + \frac{7}{2}\Psi(-5^{\eta\mu} w_1, -5^{\eta\mu} w_1, 5^{\eta\mu} w_1) \right\} \right\} \right\}
\end{aligned}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$. □

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Corollary 2.6. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality*

$$\|\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1)\| \leq \begin{cases} \frac{\delta}{|3|} + \frac{3\delta}{2|24|}, & \\ \frac{4\delta|w_1|^{\varphi}}{|5-5^{\varphi}|} + \frac{27\delta|w_1|^{\varphi}}{6|25-5^{\varphi}|}; & \varphi \neq 1,2, \\ \frac{4\delta}{3}\sum_{\psi=1}^{3}\frac{|w_\psi|^{\varphi_\psi}}{|5-5^{\varphi_\psi}|} + \frac{9\delta}{6}\sum_{\psi=1}^{3}\frac{|w_\psi|^{\varphi_\psi}}{|25-5^{\varphi_\psi}|}; & \varphi_1, \varphi_2, \varphi_3 \neq 1,2, \\ \frac{4\delta|w_1|^{3\varphi}}{3|5-5^{3\varphi}|} + \frac{9\delta|w_1|^{3\varphi}}{6|25-5^{3\varphi}|}; & 3\varphi \neq 1,2, \\ \frac{4\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}}{3\left|5-5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|} + \frac{9\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}}{6\left|25-5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|}; & \sum_{\psi=1}^{3}\varphi_\psi \neq 1,2, \\ \frac{16\delta|w_1|^{3\varphi}}{3|5-5^{3\varphi}|} + \frac{36\delta|w_1|^{3\varphi}}{6|25-5^{3\varphi}|}; & 3\varphi \neq 1,2, \end{cases} \tag{42}$$

for all $w_1 \in \mathcal{W}_1$.

2.4. Oddness of $\mathcal{F}$: Additive Case Stability Results : Fixed Method

Theorem 2.7. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (1) where $\Psi : \mathcal{W}_1^3 \to [0,\infty)$ with the condition*

$$\lim_{\ell\to\infty} \frac{\Psi\left(\tau_\nu^\ell w_1, \tau_\nu^\ell w_2, \tau_\nu^\ell w_3\right)}{\tau_\nu^\ell} = 0; \quad \tau_\nu = \begin{cases} 5; \nu = 0 \\ \frac{1}{5}; \nu = 1 \end{cases}, \forall\, w_1, w_2, w_3 \in \mathcal{W}_1. \tag{43}$$

If there exists $L = L(\nu)$ be a function have the property

$$\Psi_A(w_1) = \Psi_A\left(\frac{w_1}{5}\right) \quad \text{and} \quad \frac{1}{\tau_\nu}\Psi_A(\tau_\nu w_1) = L\,\Psi_A(w_1), \ \forall\, w_1 \in \mathcal{W}_1. \tag{44}$$

Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality

$$\|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \leq \frac{L^{1-\nu}}{1-L}\,\Psi_A(w_1) \tag{45}$$

$$= \frac{L^{1-\nu}}{1-L}\left\{\frac{1}{3}\left\{\Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1)\right\}\right\} \tag{46}$$

and the mapping $\mathcal{A}(w_1)$ is obtained by

$$\mathcal{A}(w_1) = \lim_{\ell\to\infty} \frac{1}{\tau_\nu^\ell}\mathcal{F}\left(\tau_\nu^\ell w_1\right) \tag{47}$$

for all $w_1 \in \mathcal{W}_1$.

Proof. Assume a set

$$\mathcal{G} = \{\mathcal{F}/\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2,\ \mathcal{F}(0) = 0\} \tag{48}$$

and introduce the generalized metric on the above set $\mathcal{G}$ as

$$d(\mathcal{F}, \mathcal{F}_1) = \inf\{K \in (0,\infty) : \|\mathcal{F}(w_1) - \mathcal{F}_1(w_1)\| \leq K\,\Psi(w_1, w_1, w_1), w_1 \in \mathcal{W}_1\}. \tag{49}$$

It is easy to see that $(\mathcal{G}, d)$ is complete. Define a function $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ by

$$\mathcal{H}\mathcal{F}(w_1) = \frac{1}{\tau_\nu}\mathcal{F}(\tau_\nu\, w_1), \quad \text{for all} \ \ w_1 \in \mathcal{W}_1. \tag{50}$$

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Now $\mathcal{F}, \mathcal{F}_1 \in \mathcal{G}$ and $w_1 \in \mathcal{W}_1$, we see

$$\begin{aligned} d(\mathcal{F}, \mathcal{F}_1) \leq K \Rightarrow & \parallel \mathcal{F}(w_1) - \mathcal{F}_1(w_1) \parallel \leq K\, \Psi(w_1, w_1, w_1), \\ & \Rightarrow \left\| \frac{1}{\tau_\nu} \mathcal{F}(\tau_\nu w_1) - \frac{1}{\tau_\nu} \mathcal{F}_1(\tau_\nu w_1) \right\| \leq \frac{1}{\tau_\nu} K\, \Psi(\tau_\nu w_1, \tau_\nu w_1, \tau_\nu w_1), \\ & \Rightarrow \parallel \mathcal{HF}(w_1) - \mathcal{HF}_1(\ w_1\ ) \parallel \leq L\, K\, \Psi(w_1, w_1, w_1), \\ & \Rightarrow d(\mathcal{HF}, \mathcal{HF}_1) \leq L\, K, \end{aligned}$$

i.e., $\mathcal{H}$ is a strictly contractive mapping on $\mathcal{G}$ with Lipschitz constant L (see [24]).
For the case $\nu = 0$, it follows from (12) and with the help of (44), (50), (49), we get

$$\left\| \frac{1}{5} \mathcal{F}(5w_1) - \mathcal{F}(w_1) \right\| \leq \frac{1}{5} \Psi_A(w_1), \Rightarrow d(\mathcal{HF}, \mathcal{F}) \leq L = L^{1-\nu},\ \ \forall\, w_1 \in \mathcal{W}_1. \tag{51}$$

For the case $\nu = 1$, it follows from (18) and with the help of (44), (50), (49), we obtain

$$\left\| \mathcal{F}(w_1) - 5\mathcal{F}\left(\frac{w_1}{5}\right) \right\| \leq \Psi_A\left(\frac{w_1}{5}\right), \Rightarrow d(\mathcal{F}, \mathcal{HF}) \leq 1 = L^{1-\nu},\ \ \forall\, w_1 \in \mathcal{W}_1. \tag{52}$$

Combining (51) and (52), we have

$$d(\mathcal{F}, \mathcal{HF}) \leq 1 = L^{1-\nu}. \tag{53}$$

Therefore $(FPC1)$ of Theorem 1.3 holds. The rest of the proof follows by Theorem 1.3. Hence the proof is complete. □

Corollary 2.8. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality (20) for all $w_1 \in \mathcal{W}_1$.*

Proof. If we take

$$\Psi(w_1, w_2, w_3) = \begin{cases} \delta, \\ \delta \sum_{\psi=1}^{3} |w_\psi|^{\varphi}, \\ \delta \sum_{\psi=1}^{3} |w_\psi|^{\varphi_\psi}, \\ \delta \prod_{\psi=1}^{3} |w_\psi|^{\varphi}, \\ \delta \prod_{\psi=1}^{3} |w_\psi|^{\varphi_\psi}, \\ \delta \left\{ \sum_{\psi=1}^{3} |w_\psi|^{3\varphi} + \prod_{\psi=1}^{3} |w_\psi|^{\varphi} \right\}, \end{cases} \tag{54}$$

in Theorem 2.7 and changing (w_1, w_2, w_3) by $\left(\tau_\nu^\ell w_1, \tau_\nu^\ell w_2, \tau_\nu^\ell w_3\right)$ and dividing by τ_ν^ℓ in (54), one can see

$$\frac{1}{\tau_\nu^\ell}\, \Psi\left(\tau_\nu^\ell w_1, \tau_\nu w_2^\ell, \tau_\nu^\ell w_3\right) = \begin{cases} \frac{\delta}{\tau_\nu^\ell} & \to 0\ \ as\ \ \ell\ \ to\ \ \infty, \\ \frac{\delta}{\tau_\nu^\ell} \sum_{\psi=1}^{3} \left|\tau_\nu^\ell w_\psi\right|^{\varphi}, & \to 0\ \ as\ \ \ell\ \ to\ \ \infty, \\ \frac{\delta}{\tau_\nu^\ell} \sum_{\psi=1}^{3} \left|\tau_\nu^\ell w_\psi\right|^{\varphi_\psi}, & \to 0\ \ as\ \ \ell\ \ to\ \ \infty, \\ \frac{\delta}{\tau_\nu^\ell} \prod_{\psi=1}^{3} \left|\tau_\nu^\ell w_\psi\right|^{\varphi}, & \to 0\ \ as\ \ \ell\ \ to\ \ \infty, \\ \frac{\delta}{\tau_\nu^\ell} \prod_{\psi=1}^{3} \left|\tau_\nu^\ell w_\psi\right|^{\varphi_\psi}, & \to 0\ \ as\ \ \ell\ \ to\ \ \infty, \\ \frac{\delta}{\tau_\nu^\ell} \left\{ \sum_{\psi=1}^{3} \left|\tau_\nu^\ell w_\psi\right|^{3\varphi} + \prod_{\psi=1}^{3} \left|\tau_\nu^\ell w_\psi\right|^{\varphi} \right\}, & \to 0\ \ as\ \ \ell\ \ to\ \ \infty. \end{cases}$$

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Therefore (43) holds for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Now, from (44), we have

$$\Psi_A(w_1) = \Psi_A\left(\frac{w_1}{5}\right) = \frac{1}{3}\left\{\Psi\left(\frac{w_1}{5}, \frac{w_1}{5}, \frac{w_1}{5}\right) + 3\Psi\left(\frac{w_1}{5}, \frac{w_1}{5}, -\frac{w_1}{5}\right)\right\} = \begin{cases} \frac{4\delta}{3}, \\ \frac{12|\frac{w_1}{5}|^{\varphi}}{3}, \\ \frac{4\delta}{3}\sum_{\psi=1}^{3}\left|\frac{w_1}{5}\right|^{\varphi_\psi}, \\ \frac{4\delta|\frac{w_1}{5}|^{3\varphi}}{3}, \\ \frac{4\delta|\frac{w_1}{5}|^{\sum_{\psi=1}^{3}\varphi_\psi}}{3}, \\ \frac{16\delta|\frac{w_1}{5}|^{3\varphi}}{3}, \end{cases} \tag{55}$$

$$\begin{aligned} \frac{1}{\tau_\nu}\Psi_A(\tau_\nu w_1) &= \frac{1}{\tau_\nu}\frac{1}{3}\left\{\Psi(\tau_\nu w_1, \tau_\nu w_1, \tau_\nu w_1) + 3\Psi(\tau_\nu w_1, \tau_\nu w_1, -\tau_\nu w_1)\right\} \\ &= \begin{cases} \frac{4\delta}{\tau_\nu \cdot 3}, \\ \frac{12\delta|\tau_\nu w_1|^{\varphi}}{\tau_\nu \cdot 3}, \\ \frac{4\delta}{\tau_\nu \cdot 3}\sum_{\psi=1}^{3}|\tau_\nu w_\psi|^{\varphi_\psi}, \\ \frac{4\delta|\tau_\nu w_1|^{3\varphi}}{\tau_\nu \cdot 3}, \\ \frac{4\delta|\tau_\nu w_1|^{\sum_{\psi=1}^{3}\varphi_\psi}}{\tau_\nu \cdot 3}, \\ \frac{16\delta|\tau_\nu w_1|^{3\varphi}}{\tau_\nu \cdot 3}, \end{cases} = \begin{cases} \tau_\nu^{-1}\Psi_A(w_1), \\ \tau_\nu^{\varphi-1}\Psi_A(w_1), \\ \sum_{\psi=1}^{3}\tau_\nu^{\varphi_\psi-1}\Psi_A(w_1), \\ \tau_\nu^{3\varphi-1}\Psi_A(w_1), \\ \tau_\nu^{\sum_{\psi=1}^{3}\varphi_\psi-1}\Psi_A(w_1), \\ \tau_\nu^{3\varphi-1}\Psi_A(w_1), \end{cases} = \begin{cases} L\,\Psi_A(w_1) \\ L\,\Psi_A(w_1) \\ L\,\Psi_A(w_1) \\ L\,\Psi_A(w_1) \\ L\,\Psi_A(w_1) \\ L\,\Psi_A(w_1) \end{cases} \end{aligned} \tag{56}$$

for all $w_1 \in \mathcal{W}_1$.
For the case $\nu = 0$, we have $L = \tau_0^{-1} = 5^{-1}$ and from (46), we arrive

$$\begin{aligned} \|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \le \frac{L^{1-\nu}}{1-L}\,\Psi_A(w_1) &= \frac{L^{1-\nu}}{1-L}\left\{\frac{1}{3}\left\{\Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1)\right\}\right\} \\ &= \frac{(5^{-1})^{1-0}}{1-5^{-1}}\left\{\frac{4\delta}{3}\right\} = \frac{\delta}{3}. \end{aligned}$$

For the case $\nu = 1$, we have $L = \tau_1^{-1} = (\frac{1}{5})^{-1} = 5$ and from (46), we obtain

$$\begin{aligned} \|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \le \frac{L^{1-\nu}}{1-L}\,\Psi_A(w_1) &= \frac{L^{1-\nu}}{1-L}\left\{\frac{1}{3}\left\{\Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1)\right\}\right\} \\ &= \frac{(5)^{1-1}}{1-5}\left\{\frac{4\delta}{3}\right\} = \frac{\delta}{-3}. \end{aligned}$$

For the case $\nu = 0$, we have $L = \tau_0^{\varphi-1} = 5^{\varphi-1}$ and from (46), we arrive

$$\begin{aligned} \|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \le \frac{L^{1-\nu}}{1-L}\,\Psi_A(w_1) &= \frac{L^{1-\nu}}{1-L}\left\{\frac{1}{3}\left\{\Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1)\right\}\right\} \\ &= \frac{(5^{\varphi-1})^{1-0}}{1-5^{\varphi-1}}\left\{\frac{12\delta|\frac{w_1}{5}|^{\varphi}}{3}\right\} = \frac{4\delta}{5-5^{\varphi}}. \end{aligned}$$

For the case $\nu = 1$, we have $L = \tau_1^{\varphi-1} = (\frac{1}{5})^{\varphi-1} = 5^{1-\varphi}$ and from (46), we get

$$\begin{aligned} \|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \le \frac{L^{1-\nu}}{1-L}\,\Psi_A(w_1) &= \frac{L^{1-\nu}}{1-L}\left\{\frac{1}{3}\left\{\Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1)\right\}\right\} \\ &= \frac{(5^{1-\varphi})^{1-1}}{1-5^{1-\varphi}}\left\{\frac{12\delta|\frac{w_1}{5}|^{\varphi}}{3}\right\} = \frac{4\delta}{5^{\varphi}-5}. \end{aligned}$$

For the case $\nu = 0$, we have $L = \tau_0^{3\varphi-1} = 5^{3\varphi-1}$ and from (46), we arrive

$$\begin{aligned}\|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \leq \frac{L^{1-\nu}}{1-L}\,\Psi_A(w_1) &= \frac{L^{1-\nu}}{1-L}\left\{\frac{1}{3}\Big\{\Psi(w_1,w_1,w_1) + 3\Psi(w_1,w_1,-w_1)\Big\}\right\}\\ &= \frac{(5^{3\varphi-1})^{1-0}}{1-5^{3\varphi-1}}\left\{\frac{4\delta|\frac{w_1}{5}|^{3\varphi}}{3}\right\} = \frac{4\delta|w_1|^{3\varphi}}{3(5-5^{3\varphi})}.\end{aligned}$$

For the case $\nu = 1$, we have $L = \tau_1^{3\varphi-1} = (\frac{1}{5})^{3\varphi-1} = 5^{1-3\varphi}$ and from (46), we obtain

$$\begin{aligned}\|\mathcal{F}(w_1) - \mathcal{A}(w_1)\| \leq \frac{L^{1-\nu}}{1-L}\,\Psi_A(w_1) &= \frac{L^{1-\nu}}{1-L}\left\{\frac{1}{3}\Big\{\Psi(w_1,w_1,w_1) + 3\Psi(w_1,w_1,-w_1)\Big\}\right\}\\ &= \frac{(5^{1-3\varphi})^{1-1}}{1-5^{1-3\varphi}}\left\{\frac{4\delta|\frac{w_1}{5}|^{3\varphi}}{3}\right\} = \frac{4\delta|w_1|^{3\varphi}}{3(5^{3\varphi}-5)}.\end{aligned}$$

Similarly, we can prove for rest of the cases. □

2.5. Evenness of $\mathcal{F}$: Quadratic Case Stability Results : Fixed Method

Theorem 2.9. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (1) where $\Psi : \mathcal{W}_1^3 \to [0,\infty)$ with the condition*

$$\lim_{\ell\to\infty} \frac{\Psi\left(\tau_\nu^\ell w_1, \tau_\nu^\ell w_2, \tau_\nu^\ell w_3\right)}{\tau_\nu^{2\ell}} = 0; \quad \tau_\nu = \begin{cases} 5; \nu = 0 \\ \frac{1}{5}; \nu = 1 \end{cases}, \forall\, w_1, w_2, w_3 \in \mathcal{W}_1. \tag{57}$$

If there exists $L = L(\nu)$ be a function have the property

$$\Psi_Q(w_1) = \Psi_Q\left(\frac{w_1}{5}\right) \qquad and \qquad \frac{1}{\tau_\nu^2}\Psi_Q(\tau_\nu w_1) = L\,\Psi_Q(w_1),\ \ \forall\, w_1 \in \mathcal{W}_1. \tag{58}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality

$$\|\mathcal{F}(w_1) - \mathcal{Q}(w_1)\| \leq \frac{L^{1-\nu}}{1-L}\,\Psi_Q(w_1) \tag{59}$$

$$= \frac{L^{1-\nu}}{1-L}\left\{\frac{1}{3}\Big\{\Psi(w_1,w_1,w_1) + \frac{7}{2}\Psi(w_1,w_1,-w_1)\Big\}\right\} \tag{60}$$

and the mapping $\mathcal{Q}(w_1)$ is obtained by

$$\mathcal{Q}(w_1) = \lim_{\ell\to\infty} \frac{1}{\tau_\nu^{2\ell}}\mathcal{F}\left(\tau_\nu^\ell w_1\right) \tag{61}$$

for all $w_1 \in \mathcal{W}_1$.

Proof. By Theorem 2.7, define a function $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ by

$$\mathcal{H}\mathcal{F}(w_1) = \frac{1}{\tau_\nu^2}\mathcal{F}(\tau_\nu\, w_1), \quad for\ \ all\ \ w_1 \in \mathcal{W}_1. \tag{62}$$

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Now $\mathcal{F}, \mathcal{F}_1 \in \mathcal{G}$ and $w_1 \in \mathcal{W}_1$, we see

$$\begin{aligned} d(\mathcal{F}, \mathcal{F}_1) \leq K \Rightarrow & \parallel \mathcal{F}(w_1) - \mathcal{F}_1(w_1) \parallel \leq K\, \Psi(w_1, w_1, w_1), \\ & \Rightarrow \left\| \frac{1}{\tau_v^2}\mathcal{F}(\tau_v w_1) - \frac{1}{\tau_v^2}\mathcal{F}_1(\tau_v w_1) \right\| \leq \frac{1}{\tau_v^2} K\, \Psi(\tau_v w_1, \tau_v w_1, \tau_v w_1), \\ & \Rightarrow \parallel \mathcal{H}\mathcal{F}(w_1) - \mathcal{H}\mathcal{F}_1(\ w_1\ ) \parallel \leq L\, K\, \Psi(w_1, w_1, w_1), \\ & \Rightarrow d(\mathcal{H}\mathcal{F}, \mathcal{H}\mathcal{F}_1) \leq L\, K, \end{aligned}$$

i.e., $\mathcal{H}$ is a strictly contractive mapping on $\mathcal{G}$ with Lipschitz constant L (see [24]). The rest of the proof is similar to that of Theorem 2.7. Hence the proof is complete. □

Corollary 2.10. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality (31) for all $w_1 \in \mathcal{W}_1$.*

2.6. Oddness and Evenness of $\mathcal{F}$: Additive Quadratic Case Stability Results : Fixed Method

Theorem 2.11. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (1) where $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ with the conditions (43) and (57) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. If there exists $L = L(v)$ be function have the properties (44) and (58) Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality*

$$\begin{aligned} & \|\mathcal{F}(w_1) - \mathcal{A}(w_1) - \mathcal{Q}(w_1)\| \\ & \leq \frac{1}{2} \cdot \frac{L^{1-v}}{1-L} \Big\{ \Psi_A(w_1) + \Psi_A(-w_1) + \Psi_Q(w_1) + \Psi_Q(-w_1) \Big\} \qquad (63) \\ & = \frac{1}{2} \cdot \frac{L^{1-v}}{1-L} \left\{ \frac{1}{3} \Big\{ \Psi(w_1, w_1, w_1) + 3\Psi(w_1, w_1, -w_1) + \Psi(-w_1, -w_1, -w_1) + 3\Psi(-w_1, -w_1, w_1) \right. \\ & \qquad \left. + \Psi(w_1, w_1, w_1) + \frac{7}{2}\Psi(w_1, w_1, -w_1) + \Psi(-w_1, -w_1, -w_1) + \frac{7}{2}\Psi(-w_1, -w_1, w_1) \Big\} \right\} \qquad (64) \end{aligned}$$

and the mapping $\mathcal{A}(w_1)$ and $\mathcal{Q}(w_1)$ are given in (47) and (61) for all $w_1 \in \mathcal{W}_1$.

Proof. The proof is similar ideas to that of Theorem 2.5. □

Corollary 2.12. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality (42) for all $w_1 \in \mathcal{W}_1$.*

3. Stability In Intuitionistic Fuzzy Banach Space of (3)

In this section, we explore the generalized Ulam - Hyers stability of the functional equations (3) in Intuitionistic Fuzzy Banach Space.

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In order to prove stability results, assume $(\mathcal{W}_1, \mu, \nu)$ and $(\mathcal{W}_2, \mu', \nu')$ are Intuitionistic Fuzzy normed space and Intuitionistic Fuzzy Banach space respectively. Suppose that $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ and $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ satisfying the following functional inequalities

$$\left.\begin{array}{l}
\mu\Bigg(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum\limits_{\psi=1}^{3} w_\psi\right) \\
\qquad -\frac{1}{2}\left\{\mathcal{F}\left(\sum\limits_{\psi=1}^{3} w_\psi\right)+\mathcal{F}\left(-\sum\limits_{\psi=1}^{3} w_\psi\right)\right\}+\sum\limits_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\Bigg) \\
\qquad\qquad \geq \mu'(\Psi(w_1,w_2,w_3),\Lambda) \\
\nu\Bigg(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum\limits_{\psi=1}^{3} w_\psi\right) \\
\qquad -\frac{1}{2}\left\{\mathcal{F}\left(\sum\limits_{\psi=1}^{3} w_\psi\right)+\mathcal{F}\left(-\sum\limits_{\psi=1}^{3} w_\psi\right)\right\}+\sum\limits_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\Bigg) \\
\qquad\qquad \leq \nu'(\Psi(w_1,w_2,w_3),\Lambda)
\end{array}\right\} \quad (1)$$

and

$$\left.\begin{array}{l}
\mu\Bigg(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum\limits_{\psi=1}^{3} w_\psi\right) \\
\qquad -\frac{1}{2}\left\{\mathcal{F}\left(\sum\limits_{\psi=1}^{3} w_\psi\right)+\mathcal{F}\left(-\sum\limits_{\psi=1}^{3} w_\psi\right)\right\}+\sum\limits_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\Bigg) \\
\qquad\qquad \geq \begin{cases}
\mu'(\delta,\Lambda), \\
\mu'\left(\delta\sum\limits_{\psi=1}^{3}|w_\psi|^{\varphi},\Lambda\right), \\
\mu'\left(\delta\sum\limits_{\psi=1}^{3}|w_\psi|^{\varphi_\psi},\Lambda\right), \\
\mu'\left(\delta\prod\limits_{\psi=1}^{3}|w_\psi|^{\varphi},\Lambda\right), \\
\mu'\left(\delta\prod\limits_{\psi=1}^{3}|w_\psi|^{\varphi_\psi},\Lambda\right), \\
\mu'\left(\delta\left\{\sum\limits_{\psi=1}^{3}|w_\psi|^{3\varphi}+\prod\limits_{\psi=1}^{3}|w_\psi|^{\varphi}\right\},\Lambda\right),
\end{cases} \\
\nu\Bigg(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-6\mathcal{F}\left(\sum\limits_{\psi=1}^{3} w_\psi\right) \\
\qquad -\frac{1}{2}\left\{\mathcal{F}\left(\sum\limits_{\psi=1}^{3} w_\psi\right)+\mathcal{F}\left(-\sum\limits_{\psi=1}^{3} w_\psi\right)\right\}+\sum\limits_{\psi=1}^{3}\left\{\mathcal{F}(w_\psi)-\frac{5}{2}\left[\mathcal{F}(w_\psi)+\mathcal{F}(-w_\psi)\right]\right\},\Lambda\Bigg) \\
\qquad\qquad \leq \begin{cases}
\nu'(\delta,\Lambda), \\
\nu'\left(\delta\sum\limits_{\psi=1}^{3}|w_\psi|^{\varphi},\Lambda\right), \\
\nu'\left(\delta\sum\limits_{\psi=1}^{3}|w_\psi|^{\varphi_\psi},\Lambda\right), \\
\nu'\left(\delta\prod\limits_{\psi=1}^{3}|w_\psi|^{\varphi},\Lambda\right), \\
\nu'\left(\delta\prod\limits_{\psi=1}^{3}|w_\psi|^{\varphi_\psi},\Lambda\right), \\
\nu'\left(\delta\left\{\sum\limits_{\psi=1}^{3}|w_\psi|^{3\varphi}+\prod\limits_{\psi=1}^{3}|w_\psi|^{\varphi}\right\},\Lambda\right),
\end{cases}
\end{array}\right\} \quad (2)$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with δ be a positive constant.

3.1. Definitions and Notations of Intuitionistic Fuzzy Banach Space

Now, we recall the basic definitions and notations in the setting of intuitionistic fuzzy normed space given in [33].

Definition 3.1. *[33] A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be continuous t-norm if $*$ satisfies the following conditions:*

($*$1) $*$ *is commutative and associative;*
($*$2) $*$ *is continuous;*
($*$3) $a * 1 = a$ *for all* $a \in [0,1]$;
($*$4) $a * b \le c * d$ *whenever* $a \le c$ *and* $b \le d$ *for all* $a, b, c, d \in [0,1]$.

Definition 3.2. *[33] A binary operation $\diamond : [0,1] \times [0,1] \longrightarrow [0,1]$ is said to be continuous t-conorm if $\diamond$ satisfies the following conditions:*

($\diamond$1) $\diamond$ *is commutative and associative;*
($\diamond$2) $\diamond$ *is continuous;*
($\diamond$3) $a \diamond 0 = a$ *for all* $a \in [0,1]$;
($\diamond$4) $a \diamond b \le c \diamond d$ *whenever* $a \le c$ *and* $b \le d$ *for all* $a, b, c, d \in [0,1]$.

Definition 3.3. *[33] The five-tuple $(X, \mu, \nu, *, \diamond)$ is said to be an intuitionistic fuzzy normed space (for short, IFNS) if X is a vector space, $*$ is a continuous t-norm, $\diamond$ is a continuous $t-$ conorm, and μ, ν are fuzzy sets on $X \times (0, \infty)$ satisfying the following conditions. For every $x, y \in X$ and $s, t > 0$*

(IFN1) $\mu(x,t) + \nu(x,t) \le 1$;
(IFN2) $\mu(x,t) > 0$;
(IFN3) $\mu(x,t) = 1$, *if and only if* $x = 0$;
(IFN4) $\mu(dx,t) = \mu(x, \frac{t}{d})$ *for each* $d \neq 0$;
(IFN5) $\mu(x,t) * \mu(y,s) \le \mu(x+y, t+s)$;
(IFN6) $\mu(x,\cdot) : (0,\infty) \to [0,1]$ *is continuous;*
(IFN7) $\lim_{t\to\infty} \mu(x,t) = 1$ *and* $\lim_{t\to 0} \mu(x,t) = 0$;
(IFN8) $\nu(x,t) < 1$;
(IFN9) $\nu(x,t) = 0$, *if and only if* $x = 0$;
(IFN10) $\nu(dx,t) = \nu(x, \frac{t}{d})$ *for each* $d \neq 0$;
(IFN11) $\nu(x,t) \diamond \nu(y,s) \ge \nu(x+y, t+s)$;
(IFN12) $\nu(x,\cdot) : (0,\infty) \to [0,1]$ *is continuous;*
(IFN13) $\lim_{t\to\infty} \nu(x,t) = 0$ *and* $\lim_{t\to 0} \nu(x,t) = 1$.

In this case, (μ, ν) is called an intuitionistic fuzzy norm.

Example 3.4. *[33] Let $(X, \|\cdot\|)$ be a normed space. Let $a * b = ab$ and $a \diamond d = \min\{a+b, 1\}$ for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, consider*

$$\mu(x,t) = \begin{cases} \frac{t}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \le 0; \end{cases} \qquad and \qquad \nu(x,t) = \begin{cases} \frac{\|x\|}{t+\|x\|} & if \quad t > 0; \\ 0 & if \quad t \le 0. \end{cases}$$

Then $(X, \mu, \nu, *, \diamond)$ is an IFN-space.

Definition 3.5. *[33] Let $(X, \mu, \nu, *, \diamond)$ be an IFNS. Then, a sequence $x = \{x_k\}$ is said to be intuitionistic fuzzy convergent to a point $L \in X$ if*

$$\lim \quad \mu(x_k - L, t) = 1 \quad and \quad \lim \quad \nu(x_k - L, t) = 0$$

for all $\rho > 0$. In this case, we write

$$x_k \xrightarrow{IF} L \qquad as \qquad k \to \infty$$

Definition 3.6. *[33] Let $(X, \mu, \nu, *, \diamond)$ be an IFN-space. Then, $x = \{x_k\}$ is said to be intuitionistic fuzzy Cauchy sequence if*

$$\mu\Big(x_{k+p} - x_k, t\Big) = 1 \qquad and \qquad \nu\Big(x_{k+p} - x_k, t\Big) = 0$$

for all $\rho > 0$, and $p = 1, 2 \cdots$.

Definition 3.7. *[33] Let $(X, \mu, \nu, *, \diamond)$ be an IFN-space. Then $(X, \mu, \nu, *, \diamond)$ is said to be complete if every intuitionistic fuzzy Cauchy sequence in $(X, \mu, \nu, *, \diamond)$ is intuitionistic fuzzy convergent $(X, \mu, \nu, *, \diamond)$.*

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3.2. Oddness of $\mathcal{F}$: Additive Case Stability Results : Direct Method

Theorem 3.8. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (1) where $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ with the conditions*

$$\left.\begin{array}{l} \mu'\left(\Psi\left(5^{\ell\mu}w_1, 5^{\ell\mu}w_2, 5^{\ell\mu}w_3\right), \Lambda\right) \geq \mu'\left(I^{\ell\mu}\Psi(w_1, w_2, w_3), \Lambda\right) \\ \nu'\left(\Psi\left(5^{\ell\mu}w_1, 5^{\ell\mu}w_2, 5^{\ell\mu}w_3\right), \Lambda\right) \leq \nu'\left(I^{\ell\mu}\Psi(w_1, w_2, w_3), \Lambda\right) \end{array}\right\} \tag{3}$$

and

$$\left.\begin{array}{l} \lim_{\ell\to\infty} \mu'\left(\Psi\left(5^{\ell\mu}w_1, 5^{\ell\mu}w_2, 5^{\ell\mu}w_3\right), 5^{\ell\mu}\Lambda\right) = 1 \\ \lim_{\ell\to\infty} \nu'\left(\Psi\left(5^{\ell\mu}w_1, 5^{\ell\mu}w_2, 5^{\ell\mu}w_3\right), 5^{\ell\mu}\Lambda\right) = 0 \end{array}\right\} \tag{4}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with $\mu = \pm 1$ and $0 < \left(\frac{I}{5}\right)^{\mu} < 1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality

$$\left.\begin{array}{rl} \mu(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda) \geq & \mu'\left(\Psi_A(w_1), \frac{3\Lambda}{4}\,|5 - I|\right) \\ = & \mu'\left(\Psi(w_1, w_1, w_1), \frac{3\Lambda}{4}\,|5 - I|\right) * \mu'\left(\Psi(w_1, w_1, -w_1), \frac{3\Lambda}{4}\,|5 - I|\right) \\ \nu(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda) \leq & \nu'\left(\Psi_A(w_1), \frac{3\Lambda}{4}\,(5 - I)\right) \\ = & \nu'\left(\Psi(w_1, w_1, w_1), \frac{3\Lambda}{4}\,(5 - I)\right) \diamond \nu'\left(\Psi(w_1, w_1, -w_1), \frac{3\Lambda}{4}\,(5 - I)\right) \end{array}\right\} \tag{5}$$

and the mapping $\mathcal{A}(w_1)$ is obtained by

$$\left.\begin{array}{l} \lim_{\ell\to\infty} \mu\left(\frac{1}{5^{\ell\mu}}\mathcal{F}\left(5^{\ell\mu}w_1\right) - \mathcal{A}(w_1), \Lambda\right) = 1 \\ \lim_{\ell\to\infty} \nu\left(\frac{1}{5^{\ell\mu}}\mathcal{F}\left(5^{\ell\mu}w_1\right) - \mathcal{A}(w_1), \Lambda\right) = 0 \end{array}\right\} \tag{6}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Proof. Using oddness of $\mathcal{F}$ in (1), we get

$$\left.\begin{array}{c} \mu\left(\mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 6\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) + \sum_{\psi=1}^{3}\mathcal{F}(w_\psi), \Lambda\right) \\ \geq \mu'(\Psi(w_1, w_2, w_3), \Lambda) \\ \nu\left(\mathcal{F}(3w_1 + w_2 + w_3) + \mathcal{F}(w_1 + 3w_2 + w_3) + \mathcal{F}(w_1 + w_2 + 3w_3) - 6\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right) + \sum_{\psi=1}^{3}\mathcal{F}(w_\psi), \Lambda\right) \\ \leq \nu'(\Psi(w_1, w_2, w_3), \Lambda) \end{array}\right\} \tag{7}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. Interchanging (w_1, w_2, w_3) by (w_1, w_1, w_1) in (7), we obtain

$$\left.\begin{array}{l} \mu(3\mathcal{F}(5w_1) - 6\mathcal{F}(3w_1) + 3\mathcal{F}(w_1), \Lambda) \geq \mu'(\Psi(w_1, w_1, w_1), \Lambda) \\ \nu(3\mathcal{F}(5w_1) - 6\mathcal{F}(3w_1) + 3\mathcal{F}(w_1), \Lambda) \leq \nu'(\Psi(w_1, w_1, w_1), \Lambda) \end{array}\right\} \tag{8}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Again interchanging (w_1, w_2, w_3) by $(w_1, w_1, -w_1)$ in (7) and using (IFN4), (IFN10), we have

$$\left.\begin{array}{l}\mu(2\mathcal{F}(3w_1) - 6\mathcal{F}(w_1), \Lambda) \geq \mu'(\Psi(w_1, w_1, w_1), \Lambda) \\ \nu(2\mathcal{F}(3w_1) - 6\mathcal{F}(w_1), \Lambda) \leq \nu'(\Psi(w_1, w_1, -w_1), \Lambda)\end{array}\right\}$$
$$\Rightarrow \left.\begin{array}{l}\mu(6\mathcal{F}(3w_1) - 18\mathcal{F}(w_1), 3\Lambda) \geq \mu'(\Psi(w_1, w_1, w_1), \Lambda) \\ \nu(6\mathcal{F}(3w_1) - 18\mathcal{F}(w_1), 3\Lambda) \leq \nu'(\Psi(w_1, w_1, -w_1), \Lambda)\end{array}\right\} \tag{9}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Combining (8) and (9) using (IFN5), (IFN11), we arrive

$$\left.\begin{array}{r}\mu(3\mathcal{F}(5w_1) - 15\mathcal{F}(w_1), 4\Lambda) \geq \mu(3\mathcal{F}(5w_1) - 6\mathcal{F}(3w_1) + 3\mathcal{F}(w_1), \Lambda) * \mu(6\mathcal{F}(3w_1) - 18\mathcal{F}(w_1), 3\Lambda) \\ \geq \mu'(\Psi(w_1, w_1, w_1), \Lambda) * \mu'(\Psi(w_1, w_1, -w_1), \Lambda) = \mu'(\Psi_A(w_1), \Lambda) \\ \nu(3\mathcal{F}(5w_1) - 15\mathcal{F}(w_1), 4\Lambda) \leq \nu(3\mathcal{F}(5w_1) - 6\mathcal{F}(3w_1) + 3\mathcal{F}(w_1), \Lambda) \diamond \nu(6\mathcal{F}(3w_1) - 18\mathcal{F}(w_1), 3\Lambda) \\ \leq \nu'(\Psi(w_1, w_1, w_1), \Lambda) \diamond \nu'(\Psi(w_1, w_1, -w_1), \Lambda) = \nu'(\Psi_A(w_1), \Lambda)\end{array}\right\} \tag{10}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Using (IFN4), (IFN10), one can see from (10) that

$$\left.\begin{array}{l}\mu\left(\frac{1}{5}\mathcal{F}(5w_1) - \mathcal{F}(w_1), \frac{4}{3} \cdot \frac{1}{5}\Lambda\right) \geq \mu'(\Psi_A(w_1), \Lambda) \\ \nu\left(\frac{1}{5}\mathcal{F}(5w_1) - \mathcal{F}(w_1), \frac{4}{3} \cdot \frac{1}{5}\Lambda\right) \leq \nu'(\Psi_A(w_1), \Lambda)\end{array}\right\} \tag{11}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Changing w_1 by $5^\ell w_1$ in (11), and using (IFN4), (IFN10), (3), we get

$$\left.\begin{array}{r}\mu\left(\frac{1}{5^{\ell+1}}\mathcal{F}(5^{\ell+1}w_1) - \frac{1}{5^\ell}\mathcal{F}(5^\ell w_1), \frac{4}{3\cdot 5} \cdot \frac{1}{5^\ell}\Lambda\right) \geq \mu'\left(\Psi_A\left(5^\ell w_1\right), \Lambda\right) \geq \mu'\left(I^\ell \Psi_A(w_1), \Lambda\right) \\ = \mu'\left(\Psi_A(w_1), \frac{1}{I^\ell}\Lambda\right) \\ \nu\left(\frac{1}{5^{\ell+1}}\mathcal{F}(5^{\ell+1}w_1) - \frac{1}{5^\ell}\mathcal{F}(5^\ell w_1), \frac{4}{3\cdot 5} \cdot \frac{1}{5^\ell}\Lambda\right) \leq \nu'\left(\Psi_A\left(5^\ell w_1\right), \Lambda\right) \leq \nu'\left(I^\ell \Psi_A(w_1), \Lambda\right) \\ = \nu'\left(\Psi_A(w_1), \frac{1}{I^\ell}\Lambda\right)\end{array}\right\} \tag{12}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$ also $\ell > 0$. Changing Λ by $I^\ell \Lambda$ in (12), we see

$$\left.\begin{array}{l}\mu\left(\frac{1}{5^{\ell+1}}\mathcal{F}(5^{\ell+1}w_1) - \frac{1}{5^\ell}\mathcal{F}(5^\ell w_1), \frac{4}{3\cdot 5} \cdot \left(\frac{I}{5}\right)^\ell \Lambda\right) \geq \mu'(\Psi_A(w_1), \Lambda) \\ \nu\left(\frac{1}{5^{\ell+1}}\mathcal{F}(5^{\ell+1}w_1) - \frac{1}{5^\ell}\mathcal{F}(5^\ell w_1), \frac{4}{3\cdot 5} \cdot \left(\frac{I}{5}\right)^\ell \Lambda\right) \leq \nu'(\Psi_A(w_1), \Lambda)\end{array}\right\} \tag{13}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. It is easy to check that

$$\frac{1}{5^\ell}\mathcal{F}(5^\ell w_1) - \mathcal{F}(w_1) = \sum_{\eta=0}^{\ell-1} \frac{1}{5^{\eta+1}}\mathcal{F}(5^{\eta+1}w_1) - \frac{1}{5^\eta}\mathcal{F}(5^\eta w_1) \tag{14}$$

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for all $w_1 \in \mathcal{W}_1$. Using (IFN5), (IFN11), it follows from (13) and (14), we obtain

$$\left.\begin{aligned}
&\mu\left(\frac{1}{5^\ell}\mathcal{F}(5^\ell w_1)-\mathcal{F}(w_1),\sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\eta\Lambda\right)\\
&\qquad=\mu\left(\sum_{\eta=0}^{\ell-1}\frac{1}{5^{\eta+1}}\mathcal{F}(5^{\eta+1}w_1)-\frac{1}{5^\eta}\mathcal{F}(5^\eta w_1),\sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\eta\Lambda\right)\\
&\qquad\geq\prod_{\eta=0}^{\ell-1}\mu\left(\frac{1}{5^{\eta+1}}\mathcal{F}(5^{\eta+1}w_1)-\frac{1}{5^\eta}\mathcal{F}(5^\eta w_1),\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\eta\Lambda\right)\\
&\qquad\geq\textstyle\prod_{\eta=0}^{\ell-1}\mu'(\Psi_A(w_1),\Lambda)=\mu'(\Psi_A(w_1),\Lambda)\\
&\nu\left(\frac{1}{5^\ell}\mathcal{F}(5^\ell w_1)-\mathcal{F}(w_1),\sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\eta\Lambda\right)\\
&\qquad=\nu\left(\sum_{\eta=0}^{\ell-1}\frac{1}{5^{\eta+1}}\mathcal{F}(5^{\eta+1}w_1)-\frac{1}{5^\eta}\mathcal{F}(5^\eta w_1),\sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\eta\Lambda\right)\\
&\qquad\leq\coprod_{\eta=0}^{\ell-1}\nu\left(\frac{1}{5^{\eta+1}}\mathcal{F}(5^{\eta+1}w_1)-\frac{1}{5^\eta}\mathcal{F}(5^\eta w_1),\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^\eta\Lambda\right)\\
&\qquad\leq\textstyle\coprod_{\eta=0}^{\ell-1}\nu'(\Psi_A(w_1),\Lambda)=\nu'(\Psi_A(w_1),\Lambda)
\end{aligned}\right\}\tag{15}$$

where

$$\prod_{\eta=0}^{\ell-1}\mu=\mu*\mu*\mu*\ldots\quad\textit{and}\quad\coprod_{\eta=0}^{\ell-1}\nu=\nu\diamond\nu\diamond\nu\diamond\ldots$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Again changing w_1 by $5^{\ell_1}w_1$ in (15), and using (IFN4), (IFN10), (3) in that changing Λ by $I^{\ell_1}\Lambda$, we have

$$\left.\begin{aligned}
&\mu\left(\frac{1}{5^{\ell+\ell_1}}\mathcal{F}(5^{\ell+\ell_1}w_1)-\frac{1}{5^{\ell_1}}\mathcal{F}(\ell_1 w_1),\sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^{\eta+\ell_1}\Lambda\right)\geq\mu'(\Psi_A(w_1),\Lambda)\\
&\nu\left(\frac{1}{5^{\ell+\ell_1}}\mathcal{F}(5^{\ell+\ell_1}w_1)-\frac{1}{5^{\ell_1}}\mathcal{F}(\ell_1 w_1),\sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^{\eta+\ell_1}\Lambda\right)\leq\nu'(\Psi_A(w_1),\Lambda)
\end{aligned}\right\}\tag{16}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$ also $\ell, \ell_1 > 0$. It follows from (16) that

$$\left.\begin{aligned}
&\mu\left(\frac{1}{5^{\ell+\ell_1}}\mathcal{F}(5^{\ell+\ell_1}w_1)-\frac{1}{5^{\ell_1}}\mathcal{F}(\ell_1 w_1),\Lambda\right)\geq\mu'\left(\Psi_A(w_1),\frac{\Lambda}{\sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^{\eta+\ell_1}}\right)\\
&\nu\left(\frac{1}{5^{\ell+\ell_1}}\mathcal{F}(5^{\ell+\ell_1}w_1)-\frac{1}{5^{\ell_1}}\mathcal{F}(\ell_1 w_1),\Lambda\right)\leq\nu'\left(\Psi_A(w_1),\frac{\Lambda}{\sum_{\eta=0}^{\ell-1}\frac{4}{3\cdot 5}\cdot\left(\frac{I}{5}\right)^{\eta+\ell_1}}\right)
\end{aligned}\right\}\tag{17}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. By data, the Cauchy criterion for convergence in Intuitionistic Fuzzy normed space gives that the sequence $\left\{\frac{1}{5^\ell}\mathcal{F}(5^\ell w_1)\right\}$, is Cauchy in $(\mathcal{W}_2, \mu', \nu')$ and it is a complete Intuitionistic Fuzzy normed space, this sequence converges to some point $\mathcal{A}(w_1)$ in $(\mathcal{W}_2, \mu', \nu')$ for all $w_1 \in \mathcal{W}_1$. So, by notation, we write

$$\left.\begin{aligned}
&\lim_{\ell\to\infty}\mu\left(\frac{1}{5^\ell}\mathcal{F}\left(5^\ell w_1\right)-\mathcal{A}(w_1),\Lambda\right)=1\\
&\lim_{\ell\to\infty}\nu\left(\frac{1}{5^\ell}\mathcal{F}\left(5^\ell w_1\right)-\mathcal{A}(w_1),\Lambda\right)=0
\end{aligned}\right\}\tag{18}$$

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for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Letting $\ell_1 = 0$ and $\ell \to \infty$ in (17) and using (18), we arrive

$$\left.\begin{aligned} \mu(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda) &\geq \mu'\left(\Psi_A(w_1), \frac{3\Lambda}{4}\,(5-I)\right) \\ &= \mu'\left(\Psi(w_1, w_1, w_1), \frac{3\Lambda}{4}\,(5-I)\right) * \mu'\left(\Psi(w_1, w_1, -w_1), \frac{3\Lambda}{4}\,(5-I)\right) \\ \nu(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda) &\leq \nu'\left(\Psi_A(w_1), \frac{3\Lambda}{4}\,(5-I)\right) \\ &= \nu'\left(\Psi(w_1, w_1, w_1), \frac{3\Lambda}{4}\,(5-I)\right) \diamond \nu'\left(\Psi(w_1, w_1, -w_1), \frac{3\Lambda}{4}\,(5-I)\right) \end{aligned}\right\} \quad (19)$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Thus, (5) and (6) holds for $\mu = 1$. Interchanging

$$(w_1, w_2, w_3) = \left(5^\ell w_1, 5^\ell w_2, 5^\ell w_3\right),$$

in (1) and using (IFN4), (IFN10), we have

$$\left.\begin{aligned} &\mu\Bigg(\tfrac{1}{5^\ell}\Big\{\mathcal{F}(5^\ell(3w_1 + w_2 + w_3)) + \mathcal{F}(5^\ell(w_1 + 3w_2 + w_3)) + \mathcal{F}(5^\ell(w_1 + w_2 + 3w_3)) - 6\mathcal{F}\Big(\sum_{\psi=1}^{3} 5^\ell w_\psi\Big) \\ &\quad -\tfrac{1}{2}\Big\{\mathcal{F}\Big(\sum_{\psi=1}^{3} 5^\ell w_\psi\Big) + \mathcal{F}\Big(-\sum_{\psi=1}^{3} 5^\ell w_\psi\Big)\Big\} + \sum_{\psi=1}^{3}\Big\{\mathcal{F}(5^\ell w_\psi) - \tfrac{5}{2}\Big[\mathcal{F}(5^\ell w_\psi) + \mathcal{F}(-5^\ell w_\psi)\Big]\Big\}\Big\}, \Lambda\Bigg) \\ &\qquad\qquad \geq \mu'\left(\Psi\left(5^\ell w_1, 5^\ell w_2, 5^\ell w_3\right), 5^\ell\,\Lambda\right) \\ &\nu\Bigg(\tfrac{1}{5^\ell}\Big\{\mathcal{F}(5^\ell(3w_1 + w_2 + w_3)) + \mathcal{F}(5^\ell(w_1 + 3w_2 + w_3)) + \mathcal{F}(5^\ell(w_1 + w_2 + 3w_3)) - 6\mathcal{F}\Big(\sum_{\psi=1}^{3} 5^\ell w_\psi\Big) \\ &\quad -\tfrac{1}{2}\Big\{\mathcal{F}\Big(\sum_{\psi=1}^{3} 5^\ell w_\psi\Big) + \mathcal{F}\Big(-\sum_{\psi=1}^{3} 5^\ell w_\psi\Big)\Big\} + \sum_{\psi=1}^{3}\Big\{\mathcal{F}(5^\ell w_\psi) - \tfrac{5}{2}\Big[\mathcal{F}(5^\ell w_\psi) + \mathcal{F}(-5^\ell w_\psi)\Big]\Big\}\Big\}, \Lambda\Bigg) \\ &\qquad\qquad \leq \nu'\left(\Psi\left(5^\ell w_1, 5^\ell w_2, 5^\ell w_3\right), 5^\ell\,\Lambda\right) \end{aligned}\right\} \quad (20)$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. Now,

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$$\left.\begin{array}{l}
\mu\Big(\mathcal{A}(3w_1+w_2+w_3)+\mathcal{A}(w_1+3w_2+w_3)+\mathcal{A}(w_1+w_2+3w_3)-6\mathcal{A}\Big(\sum_{\psi=1}^{3} w_\psi\Big) \\
-\frac{1}{2}\Big\{\mathcal{A}\Big(\sum_{\psi=1}^{3} w_\psi\Big)-\mathcal{A}\Big(-\sum_{\psi=1}^{3} w_\psi\Big)\Big\}+\sum_{\psi=1}^{3}\Big\{\mathcal{A}(w_\psi)-\frac{5}{2}\Big[\mathcal{A}(w_\psi)+\mathcal{A}(-w_\psi)\Big]\Big\},\Lambda\Big) \\
\geq \mu\Big(\mathcal{A}(3w_1+w_2+w_3)-\frac{1}{5^\ell}\mathcal{F}(5^\ell(3w_1+w_2+w_3)),\frac{\Lambda}{7}\Big)* \\
\quad \mu\Big(\mathcal{A}(w_1+3w_2+w_3)-\frac{1}{5^\ell}\mathcal{F}(5^\ell(w_1+3w_2+w_3)),\frac{\Lambda}{7}\Big)* \\
\quad \mu\Big(\mathcal{A}(w_1+w_2+3w_3)-\frac{1}{5^\ell}\mathcal{F}(5^\ell(w_1+w_2+3w_3)),\frac{\Lambda}{7}\Big)* \\
\quad \mu\Big(-6\mathcal{A}\Big(\sum_{\psi=1}^{3} w_\psi\Big)+\frac{1}{5^\ell}6\mathcal{F}\Big(\sum_{\psi=1}^{3} 5^\ell w_\psi,\frac{\Lambda}{7}\Big)\Big)* \\
\quad \mu\Big(\frac{1}{2}\Big\{\mathcal{A}\Big(\sum_{\psi=1}^{3} w_\psi\Big)+\mathcal{A}\Big(-\sum_{\psi=1}^{3} w_\psi\Big)\Big\} \\
\qquad -\frac{1}{5^\ell}\frac{1}{2}\Big\{\mathcal{F}\Big(\sum_{\psi=1}^{3} 5^\ell w_\psi\Big)+\mathcal{F}\Big(-\sum_{\psi=1}^{3} 5^\ell w_\psi\Big)\Big\},\frac{\Lambda}{7}\Big)* \\
\quad \mu\Big(\sum_{\psi=1}^{3}\Big\{\mathcal{A}(w_\psi)-\frac{5}{2}\Big[\mathcal{A}(w_\psi)+\mathcal{A}(-w_\psi)\Big]\Big\} \\
\qquad -\frac{1}{5^\ell}\sum_{\psi=1}^{3}\Big\{\mathcal{F}(5^\ell w_\psi)-\frac{5}{2}\Big[\mathcal{F}(5^\ell w_\psi)+\mathcal{F}(-5^\ell w_\psi)\Big]\Big\},\frac{\Lambda}{7}\Big)* \\
\quad \mu\Big(\frac{1}{5^\ell}\Big\{\mathcal{F}(5^\ell(3w_1+w_2+w_3))+\mathcal{F}(5^\ell(w_1+3w_2+w_3))+\mathcal{F}(5^\ell(w_1+w_2+3w_3)) \\
\qquad -6\mathcal{F}\Big(\sum_{\psi=1}^{3} 5^\ell w_\psi\Big)-\frac{1}{2}\Big\{\mathcal{F}\Big(\sum_{\psi=1}^{3} 5^\ell w_\psi\Big)+\mathcal{F}\Big(-\sum_{\psi=1}^{3} 5^\ell w_\psi\Big)\Big\} \\
\qquad +\sum_{\psi=1}^{3}\Big\{\mathcal{F}(5^\ell w_\psi)-\frac{5}{2}\Big[\mathcal{F}(5^\ell w_\psi)+\mathcal{F}(-5^\ell w_\psi)\Big]\Big\}\Big\},\frac{\Lambda}{7}\Big) \\
\nu\Big(\mathcal{A}(3w_1+w_2+w_3)+\mathcal{A}(w_1+3w_2+w_3)+\mathcal{A}(w_1+w_2+3w_3)-6\mathcal{A}\Big(\sum_{\psi=1}^{3} w_\psi\Big) \\
-\frac{1}{2}\Big\{\mathcal{A}\Big(\sum_{\psi=1}^{3} w_\psi\Big)-\mathcal{A}\Big(-\sum_{\psi=1}^{3} w_\psi\Big)\Big\}+\sum_{\psi=1}^{3}\Big\{\mathcal{A}(w_\psi)-\frac{5}{2}\Big[\mathcal{A}(w_\psi)+\mathcal{A}(-w_\psi)\Big]\Big\},\Lambda\Big) \\
\leq \nu\Big(\mathcal{A}(3w_1+w_2+w_3)-\frac{1}{5^\ell}\mathcal{F}(5^\ell(3w_1+w_2+w_3)),\frac{\Lambda}{7}\Big)\diamond \\
\quad \nu\Big(\mathcal{A}(w_1+3w_2+w_3)-\frac{1}{5^\ell}\mathcal{F}(5^\ell(w_1+3w_2+w_3)),\frac{\Lambda}{7}\Big)\diamond \\
\quad \nu\Big(\mathcal{A}(w_1+w_2+3w_3)-\frac{1}{5^\ell}\mathcal{F}(5^\ell(w_1+w_2+3w_3)),\frac{\Lambda}{7}\Big)\diamond \\
\quad \nu\Big(-6\mathcal{A}\Big(\sum_{\psi=1}^{3} w_\psi\Big)+\frac{1}{5^\ell}6\mathcal{F}\Big(\sum_{\psi=1}^{3} 5^\ell w_\psi,\frac{\Lambda}{7}\Big)\Big)\diamond \\
\quad \nu\Big(\frac{1}{2}\Big\{\mathcal{A}\Big(\sum_{\psi=1}^{3} w_\psi\Big)+\mathcal{A}\Big(-\sum_{\psi=1}^{3} w_\psi\Big)\Big\} \\
\qquad -\frac{1}{5^\ell}\frac{1}{2}\Big\{\mathcal{F}\Big(\sum_{\psi=1}^{3} 5^\ell w_\psi\Big)+\mathcal{F}\Big(-\sum_{\psi=1}^{3} 5^\ell w_\psi\Big)\Big\},\frac{\Lambda}{7}\Big)\diamond \\
\quad \nu\Big(\sum_{\psi=1}^{3}\Big\{\mathcal{A}(w_\psi)-\frac{5}{2}\Big[\mathcal{A}(w_\psi)+\mathcal{A}(-w_\psi)\Big]\Big\} \\
\qquad -\frac{1}{5^\ell}\sum_{\psi=1}^{3}\Big\{\mathcal{F}(5^\ell w_\psi)-\frac{5}{2}\Big[\mathcal{F}(5^\ell w_\psi)+\mathcal{F}(-5^\ell w_\psi)\Big]\Big\},\frac{\Lambda}{7}\Big)\diamond \\
\quad \nu\Big(\frac{1}{5^\ell}\Big\{\mathcal{F}(5^\ell(3w_1+w_2+w_3))+\mathcal{F}(5^\ell(w_1+3w_2+w_3))+\mathcal{F}(5^\ell(w_1+w_2+3w_3)) \\
\qquad -6\mathcal{F}\Big(\sum_{\psi=1}^{3} 5^\ell w_\psi\Big)-\frac{1}{2}\Big\{\mathcal{F}\Big(\sum_{\psi=1}^{3} 5^\ell w_\psi\Big)+\mathcal{F}\Big(-\sum_{\psi=1}^{3} 5^\ell w_\psi\Big)\Big\} \\
\qquad +\sum_{\psi=1}^{3}\Big\{\mathcal{F}(5^\ell w_\psi)-\frac{5}{2}\Big[\mathcal{F}(5^\ell w_\psi)+\mathcal{F}(-5^\ell w_\psi)\Big]\Big\}\Big\},\frac{\Lambda}{7}\Big)
\end{array}\right\} \tag{21}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. Taking limit $\ell \to \infty$ in (21), using (18) and (20), we get

$$\left.\begin{array}{l}
\mu\Big(\mathcal{A}(3w_1+w_2+w_3)+\mathcal{A}(w_1+3w_2+w_3)+\mathcal{A}(w_1+w_2+3w_3)-6\mathcal{A}\Big(\sum_{\psi=1}^{3} w_\psi\Big) \\
-\frac{1}{2}\Big\{\mathcal{A}\Big(\sum_{\psi=1}^{3} w_\psi\Big)+\mathcal{A}\Big(-\sum_{\psi=1}^{3} w_\psi\Big)\Big\}+\sum_{\psi=1}^{3}\Big\{\mathcal{A}(w_\psi)-\frac{5}{2}\Big[\mathcal{A}(w_\psi)+\mathcal{A}(-w_\psi)\Big]\Big\},\Lambda\Big)=1 \\
\nu\Big(\mathcal{A}(3w_1+w_2+w_3)+\mathcal{A}(w_1+3w_2+w_3)+\mathcal{A}(w_1+w_2+3w_3)-6\mathcal{A}\Big(\sum_{\psi=1}^{3} w_\psi\Big) \\
-\frac{1}{2}\Big\{\mathcal{A}\Big(\sum_{\psi=1}^{3} w_\psi\Big)+\mathcal{A}\Big(-\sum_{\psi=1}^{3} w_\psi\Big)\Big\}+\sum_{\psi=1}^{3}\Big\{\mathcal{A}(w_\psi)-\frac{5}{2}\Big[\mathcal{A}(w_\psi)+\mathcal{A}(-w_\psi)\Big]\Big\},\Lambda\Big)=0
\end{array}\right\} \tag{22}$$

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for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. Using (IFN3), (IFN9) in (22), we see, $\mathcal{A}(w_1)$ satisfies (3). In order to confirm that $\mathcal{A}(w_1)$ is unique, suppose $\mathcal{B}(w_1)$ be another mapping (3), (18) and (19), we obtain

$$\left.\begin{aligned}
\mu(\mathcal{A}(w_1)-\mathcal{B}(w_1),2\Lambda) &= \mu\left(\mathcal{A}\left(5^\ell w_1\right)-\mathcal{B}\left(5^\ell w_1\right),5^\ell\, 2\Lambda\right)\\
&\geq \mu\left(\mathcal{A}\left(5^\ell w_1\right)-\mathcal{F}\left(5^\ell w_1\right),5^\ell\Lambda\right)*\mu\left(\mathcal{F}\left(5^\ell w_1\right)-\mathcal{B}\left(5^\ell w_1\right),5^\ell\Lambda\right)\\
&\geq \mu'\left(\Psi_A\left(5^\ell w_1\right),\tfrac{3\Lambda}{4}\,5^\ell(5-I)\right)*\mu'\left(\Psi_A\left(5^\ell w_1\right),\tfrac{3\Lambda}{4}\,5^\ell(5-I)\right)\\
&\geq \mu'\left(\Psi_A(w_1),\tfrac{3\Lambda}{4}\tfrac{5^\ell}{I^\ell}\,(5-I)\right)\\
\nu(\mathcal{A}(w_1)-\mathcal{B}(w_1),2\Lambda) &= \nu\left(\mathcal{A}\left(5^\ell w_1\right)-\mathcal{B}\left(5^\ell w_1\right),5^\ell\, 2\Lambda\right)\\
&\leq \nu\left(\mathcal{A}\left(5^\ell w_1\right)-\mathcal{F}\left(5^\ell w_1\right),5^\ell\Lambda\right)\diamond\nu\left(\mathcal{F}\left(5^\ell w_1\right)-\mathcal{B}\left(5^\ell w_1\right),5^\ell\Lambda\right)\\
&\leq \nu'\left(\Psi_A\left(5^\ell w_1\right),\tfrac{3\Lambda}{4}\,5^\ell(5-I)\right)\diamond\nu'\left(\Psi_A\left(5^\ell w_1\right),\tfrac{3\Lambda}{4}\,5^\ell(5-I)\right)\\
&\leq \nu'\left(\Psi_A(w_1),\tfrac{3\Lambda}{4}\tfrac{5^\ell}{I^\ell}\,(5-I)\right)
\end{aligned}\right\} \quad (23)$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Taking limit $\ell \to \infty$ in (23), and using (IFN7), (IFN13), we arrive

$$\left.\begin{aligned}
\mu(\mathcal{A}(w_1)-\mathcal{B}(w_1),2\Lambda) &= 1\\
\nu(\mathcal{A}(w_1)-\mathcal{B}(w_1),2\Lambda) &= 0
\end{aligned}\right\} \quad (24)$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. By (IFN4) and (IFN10), we get $\mathcal{A}(w_1)$ is unique. So, the Theorem holds for $\mu = 1$.

Changing $w_1 = \dfrac{w_1}{5}$ in (10) and using (IFN4), (IFN10), (3), in that changing Λ by $\frac{\Lambda}{I}$, we have

$$\left.\begin{aligned}
\mu\left(\mathcal{F}(w_1)-5\mathcal{F}\left(\frac{w_1}{5}\right),\frac{4}{3\cdot I}\Lambda\right) &\geq \mu'(\Psi_A(w_1),\Lambda)\\
\nu\left(\mathcal{F}(w_1)-5\mathcal{F}\left(\frac{w_1}{5}\right),\frac{4}{3\cdot I}\Lambda\right) &\leq \nu'(\Psi_A(w_1),\Lambda)
\end{aligned}\right\} \quad (25)$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Changing w_1 by $\dfrac{w_1}{5^\ell}$ in (25), and using (IFN4), (IFN10), (3) in that changing Λ by $\dfrac{\Lambda}{I^\ell}$, we get

$$\left.\begin{aligned}
\mu\left(5^\ell\mathcal{F}\left(\frac{w_1}{5^\ell}\right)-5^{\ell+1}\mathcal{F}\left(\frac{w_1}{5^{\ell+1}}\right),\frac{4}{3\cdot I}\left(\frac{5}{I}\right)^\ell\Lambda\right) &\leq \mu'(\Psi_A(w_1),\Lambda)\\
\nu\left(5^\ell\mathcal{F}\left(\frac{w_1}{5^\ell}\right)-5^{\ell+1}\mathcal{F}\left(\frac{w_1}{5^{\ell+1}}\right),\frac{4}{3\cdot I}\left(\frac{5}{I}\right)^\ell\Lambda\right) &\leq \nu'(\Psi_A(w_1),\Lambda)
\end{aligned}\right\} \quad (26)$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$ also $\ell > 0$. It is easy to check that

$$\mathcal{F}(w_1)-5^\ell\mathcal{F}\left(\frac{w_1}{5^\ell}\right)=\sum_{\eta=0}^{\ell-1}5^\eta\mathcal{F}\left(\frac{w_1}{5^\eta}\right)-5^{\eta+1}\mathcal{F}\left(\frac{w_1}{5^{\eta+1}}\right) \quad (27)$$

for all $w_1 \in \mathcal{W}_1$. The rest of the proof is similar to that of above case. So, the Theorem holds for $\mu = -1$. Hence the proof is complete $\square$

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Corollary 3.9. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality*

$$\left.\begin{aligned} \mu(\mathcal{A}(w_1)-\mathcal{F}(w_1),\Lambda) &\geq \begin{cases} \mu'(\delta, |3|\ \Lambda), & \\ \mu'\left(\delta|w_1|^{\varphi}, \frac{\Lambda}{4}\ |5-5^{\varphi}|\right), & \varphi \neq 1, \\ \mu'\left(\delta \sum_{\psi=1}^{3} |w_1|^{\varphi_\psi}, \frac{3\Lambda}{4}\ \sum_{\psi=1}^{3} |5-5^{\varphi_\psi}|\right), & \varphi_1,\varphi_2,\varphi_3 \neq 1, \\ \mu'\left(\delta|w_1|^{3\varphi}, \frac{3\Lambda}{4}\ |5-5^{3\varphi}|\right), & 3\varphi \neq 1, \\ \mu'\left(\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}, \frac{3\Lambda}{4}\ \left|5-5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|\right), & \sum_{\psi=1}^{3}\varphi_\psi \neq 1, \\ \mu'\left(2\delta|w_1|^{3\varphi}, \frac{3\Lambda}{4}\ |5-5^{3\varphi}|\right), & 3\varphi \neq 1, \end{cases} \\ \nu(\mathcal{A}(w_1)-\mathcal{F}(w_1),\Lambda) &\leq \begin{cases} \nu'(\delta, |3|\ \Lambda), & \\ \nu'\left(\delta|w_1|^{\varphi}, \frac{\Lambda}{4}\ |5-5^{\varphi}|\right), & \varphi \neq 1, \\ \nu'\left(\delta \sum_{\psi=1}^{3} |w_1|^{\varphi_\psi}, \frac{3\Lambda}{4}\ \sum_{\psi=1}^{3} |5-5^{\varphi_\psi}|\right), & \varphi_1,\varphi_2,\varphi_3 \neq 1, \\ \nu'\left(\delta|w_1|^{3\varphi}, \frac{3\Lambda}{4}\ |5-5^{3\varphi}|\right), & 3\varphi \neq 1, \\ \nu'\left(\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}, \frac{3\Lambda}{4}\ \left|5-5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|\right), & \sum_{\psi=1}^{3}\varphi_\psi \neq 1, \\ \nu'\left(2\delta|w_1|^{3\varphi}, \frac{3\Lambda}{4}\ |5-5^{3\varphi}|\right), & 3\varphi \neq 1, \end{cases} \end{aligned}\right\} \tag{28}$$

for all $w_1 \in \mathcal{W}_1$.

3.3. Evenness of $\mathcal{F}$: Quadratic Case Stability Results : Direct Method

Theorem 3.10. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (1) where $\Psi : \mathcal{W}_1^3 \to [0,\infty)$ with the conditions (3) and*

$$\left.\begin{aligned} \lim_{\ell\to\infty} \mu'\left(\Psi\left(5^{\ell\mu}w_1, 5^{\ell\mu}w_2, 5^{\ell\mu}w_3\right), 25^{\ell\mu}\Lambda\right) = 1 \\ \lim_{\ell\to\infty} \nu'\left(\Psi\left(5^{\ell\mu}w_1, 5^{\ell\mu}w_2, 5^{\ell\mu}w_3\right), 25^{\ell\mu}\Lambda\right) = 0 \end{aligned}\right\} \tag{29}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with $\mu = \pm 1$ and $0 < \left(\frac{I}{25}\right)^{\mu} < 1$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality

$$\left.\begin{aligned} \mu(\mathcal{Q}(w_1)-\mathcal{F}(w_1),\Lambda) &\geq \mu'\left(\Psi_Q(w_1), \frac{7\Lambda}{3}\ |25-I|\right) \\ &= \mu'\left(\Psi(w_1,w_1,w_1), \frac{7\Lambda}{3}\ |25-I|\right) * \mu'\left(\Psi(w_1,w_1,-w_1), \frac{7\Lambda}{3}\ |25-I|\right) \\ \nu(\mathcal{Q}(w_1)-\mathcal{F}(w_1),\Lambda) &\leq \nu'\left(\Psi_Q(w_1), \frac{7\Lambda}{3}\ (25-I)\right) \\ &= \nu'\left(\Psi(w_1,w_1,w_1), \frac{7\Lambda}{3}\ (25-I)\right) \diamond \nu'\left(\Psi(w_1,w_1,-w_1), \frac{7\Lambda}{3}\ (25-I)\right) \end{aligned}\right\} \tag{30}$$

and the mapping $\mathcal{Q}(w_1)$ is obtained by

$$\left.\begin{aligned} \lim_{\ell\to\infty} \mu\left(\frac{1}{25^{\ell\mu}}\mathcal{F}\left(5^{\ell\mu}w_1\right) - \mathcal{Q}(w_1), \Lambda\right) = 1 \\ \lim_{\ell\to\infty} \nu\left(\frac{1}{25^{\ell\mu}}\mathcal{F}\left(5^{\ell\mu}w_1\right) - \mathcal{Q}(w_1), \Lambda\right) = 0 \end{aligned}\right\} \tag{31}$$

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for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Proof. Using evenness of $\mathcal{F}$ in (1), we get

$$\left.\begin{array}{c} \mu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-7\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right)-4\sum_{\psi=1}^{3}\mathcal{F}(w_\psi),\Lambda\right) \\ \geq \mu'(\Psi(w_1,w_2,w_3),\Lambda) \\ \nu\left(\mathcal{F}(3w_1+w_2+w_3)+\mathcal{F}(w_1+3w_2+w_3)+\mathcal{F}(w_1+w_2+3w_3)-7\mathcal{F}\left(\sum_{\psi=1}^{3} w_\psi\right)-4\sum_{\psi=1}^{3}\mathcal{F}(w_\psi),\Lambda\right) \\ \leq \nu'(\Psi(w_1,w_2,w_3),\Lambda) \end{array}\right\} \tag{32}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. Interchanging (w_1, w_2, w_3) by (w_1, w_1, w_1) in (32), we obtain

$$\left.\begin{array}{l} \mu(3\mathcal{F}(5w_1)-7\mathcal{F}(3w_1)-12\mathcal{F}(w_1),\Lambda) \geq \mu'(\Psi(w_1,w_1,w_1),\Lambda) \\ \nu(3\mathcal{F}(5w_1)-7\mathcal{F}(3w_1)-12\mathcal{F}(w_1),\Lambda) \leq \nu'(\Psi(w_1,w_1,w_1),\Lambda) \end{array}\right\} \tag{33}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Again interchanging (w_1, w_2, w_3) by $(w_1, w_1, -w_1)$ in (32) and using (IFN4), (IFN10), we have

$$\begin{array}{l} \left.\begin{array}{l} \mu(2\mathcal{F}(3w_1)-18\mathcal{F}(w_1),\Lambda) \geq \mu'(\Psi(w_1,w_1,w_1),\Lambda) \\ \nu(2\mathcal{F}(3w_1)-18\mathcal{F}(w_1),\Lambda) \leq \nu'(\Psi(w_1,w_1,-w_1),\Lambda) \end{array}\right\} \\ \qquad \Rightarrow \left.\begin{array}{l} \mu\left(7\mathcal{F}(3w_1)-63\mathcal{F}(w_1),\frac{2}{7}\Lambda\right) \geq \mu'(\Psi(w_1,w_1,w_1),\Lambda) \\ \nu\left(7\mathcal{F}(3w_1)-63\mathcal{F}(w_1),\frac{2}{7}\Lambda\right) \leq \nu'(\Psi(w_1,w_1,-w_1),\Lambda) \end{array}\right\} \end{array} \tag{34}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Combining (33) and (34) using (IFN5), (IFN11), we arrive

$$\left.\begin{array}{l} \mu\left(3\mathcal{F}(5w_1)-75\mathcal{F}(w_1),\frac{9}{7}\Lambda\right) \geq \mu(3\mathcal{F}(5w_1)-7\mathcal{F}(3w_1)-12\mathcal{F}(w_1),\Lambda) * \mu\left(7\mathcal{F}(3w_1)-63\mathcal{F}(w_1),\frac{2}{7}\Lambda\right) \\ \qquad \geq \mu'(\Psi(w_1,w_1,w_1),\Lambda) * \mu'(\Psi(w_1,w_1,-w_1),\Lambda) = \mu'\left(\Psi_Q(w_1),\Lambda\right) \\ \nu\left(3\mathcal{F}(5w_1)-15\mathcal{F}(w_1),\frac{9}{7}\Lambda\right) \leq \nu(3\mathcal{F}(5w_1)-7\mathcal{F}(3w_1)-12\mathcal{F}(w_1),\Lambda) \diamond \nu\left(7\mathcal{F}(3w_1)-63\mathcal{F}(w_1),\frac{2}{7}\Lambda\right) \\ \qquad \leq \nu'(\Psi(w_1,w_1,w_1),\Lambda) \diamond \nu'(\Psi(w_1,w_1,-w_1),\Lambda) = \nu'\left(\Psi_Q(w_1),\Lambda\right) \end{array}\right\} \tag{35}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Using (IFN4), (IFN10), one can see from (35) that

$$\left.\begin{array}{l} \mu\left(\frac{1}{25}\mathcal{F}(5w_1)-\mathcal{F}(w_1),\frac{9}{7\cdot 3}\cdot\frac{1}{25}\Lambda\right) \geq \mu'\left(\Psi_Q(w_1),\Lambda\right) \\ \nu\left(\frac{1}{25}\mathcal{F}(5w_1)-\mathcal{F}(w_1),\frac{9}{7\cdot 3}\cdot\frac{1}{25}\Lambda\right) \leq \nu'\left(\Psi_Q(w_1),\Lambda\right) \end{array}\right\} \tag{36}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. The rest of the proof is similar to that of Theorem 3.8. Hence the proof is complete. □

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Corollary 3.11. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality*

$$\left.\begin{array}{l}
\mu(\mathcal{Q}(w_1)-\mathcal{F}(w_1),\Lambda) \geq \begin{cases} \mu'(\delta, |8|\, 7\Lambda), & \\ \mu'\left(\delta|w_1|^{\varphi}, \frac{7\Lambda}{9}\, |25-5^{\varphi}|\right), & \varphi \neq 2, \\ \mu'\left(\delta\sum_{\psi=1}^{3}|w_1|^{\varphi_\psi}, \frac{7\Lambda}{3}\, \sum_{\psi=1}^{3}|25-5^{\varphi_\psi}|\right), & \varphi_1,\varphi_2,\varphi_3 \neq 2, \\ \mu'\left(\delta|w_1|^{3\varphi}, \frac{7\Lambda}{3}\, |25-5^{3\varphi}|\right), & 3\varphi \neq 2, \\ \mu'\left(\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}, \dfrac{7\Lambda}{3}\, \left|25-5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|\right), & \sum_{\psi=1}^{3}\varphi_\psi \neq 2, \\ \mu'\left(2\delta|w_1|^{3\varphi}, \frac{7\Lambda}{3}\, |25-5^{3\varphi}|\right), & 3\varphi \neq 2, \end{cases} \\
\nu(\mathcal{Q}(w_1)-\mathcal{F}(w_1),\Lambda) \leq \begin{cases} \nu'(\delta, |8|\, 7\Lambda), & \\ \nu'\left(\delta|w_1|^{\varphi}, \frac{7\Lambda}{9}\, |25-5^{\varphi}|\right), & \varphi \neq 2, \\ \nu'\left(\delta\sum_{\psi=1}^{3}|w_1|^{\varphi_\psi}, \frac{7\Lambda}{3}\, \sum_{\psi=1}^{3}|25-5^{\varphi_\psi}|\right), & \varphi_1,\varphi_2,\varphi_3 \neq 2, \\ \nu'\left(\delta|w_1|^{3\varphi}, \frac{7\Lambda}{3}\, |25-5^{3\varphi}|\right), & 3\varphi \neq 2, \\ \nu'\left(\delta|w_\psi|^{\sum_{\psi=1}^{3}\varphi_\psi}, \frac{7\Lambda}{3}\, \left|25-5^{\sum_{\psi=1}^{3}\varphi_\psi}\right|\right), & \sum_{\psi=1}^{3}\varphi_\psi \neq 2, \\ \nu'\left(2\delta|w_1|^{3\varphi}, \frac{7\Lambda}{3}\, |25-5^{3\varphi}|\right), & 3\varphi \neq 2, \end{cases}
\end{array}\right\} \tag{37}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

3.4. Oddness and Evenness of $\mathcal{F}$: Additive Quadratic Case Stability Results : Direct Method

Theorem 3.12. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (1) where $\Psi : \mathcal{W}_1^3 \to [0,\infty)$ with the conditions (3), (4), and (29) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ with $\mu = \pm 1$ and $0 < \left(\frac{I}{5}\right)^{\mu} < 1, 0 < \left(\frac{I}{25}\right)^{\mu} < 1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality*

$$\left.\begin{array}{l}
\mu(\mathcal{F}(w_1)-\mathcal{A}(w_1)-\mathcal{Q}(w_1),4\Lambda) \\
\quad \geq \mu'\left(\Psi_A(w_1), \frac{3\Lambda}{4}\, |5-I|\right) * \mu'\left(\Psi_A(-w_1), \frac{3\Lambda}{4}\, |5-I|\right) * \\
\qquad\qquad \mu'\left(\Psi_Q(w_1), \frac{7\Lambda}{3}\, |25-I|\right) * \mu'\left(\Psi_Q(-w_1), \frac{7\Lambda}{3}\, |25-I|\right) \\
\quad = \mu'\left(\Psi(w_1,w_1,w_1), \frac{3\Lambda}{4}\, |5-I|\right) * \mu'\left(\Psi(w_1,w_1,-w_1), \frac{3\Lambda}{4}\, |5-I|\right) * \\
\qquad \mu'\left(\Psi(-w_1,-w_1,-w_1), \frac{3\Lambda}{4}\, |5-I|\right) * \mu'\left(\Psi(-w_1,-w_1,w_1), \frac{3\Lambda}{4}\, |5-I|\right) * \\
\qquad \mu'\left(\Psi(w_1,w_1,w_1), \frac{7\Lambda}{3}\, |5-I|\right) * \mu'\left(\Psi(w_1,w_1,-w_1), \frac{7\Lambda}{3}\, |5-I|\right) * \\
\qquad \mu'\left(\Psi(-w_1,-w_1,-w_1), \frac{7\Lambda}{3}\, |5-I|\right) * \mu'\left(\Psi(-w_1,-w_1,w_1), \frac{7\Lambda}{3}\, |5-I|\right) \\
\nu(\mathcal{F}(w_1)-\mathcal{A}(w_1)-\mathcal{Q}(w_1),4\Lambda) \\
\quad \leq \nu'\left(\Psi_A(w_1), \frac{3\Lambda}{4}\, |5-I|\right) \diamond \nu'\left(\Psi_A(-w_1), \frac{3\Lambda}{4}\, |5-I|\right) \diamond \\
\qquad\qquad \nu'\left(\Psi_Q(w_1), \frac{7\Lambda}{3}\, |25-I|\right) \diamond \nu'\left(\Psi_Q(-w_1), \frac{7\Lambda}{3}\, |25-I|\right) \\
\quad = \nu'\left(\Psi(w_1,w_1,w_1), \frac{3\Lambda}{4}\, |5-I|\right) \diamond \nu'\left(\Psi(w_1,w_1,-w_1), \frac{3\Lambda}{4}\, |5-I|\right) \diamond \\
\qquad \nu'\left(\Psi(-w_1,-w_1,-w_1), \frac{3\Lambda}{4}\, |5-I|\right) \diamond \nu'\left(\Psi(-w_1,-w_1,w_1), \frac{3\Lambda}{4}\, |5-I|\right) \diamond \\
\qquad \nu'\left(\Psi(w_1,w_1,w_1), \frac{7\Lambda}{3}\, |5-I|\right) \diamond \nu'\left(\Psi(w_1,w_1,-w_1), \frac{7\Lambda}{3}\, |5-I|\right) \diamond \\
\qquad \nu'\left(\Psi(-w_1,-w_1,-w_1), \frac{7\Lambda}{3}\, |5-I|\right) \diamond \nu'\left(\Psi(-w_1,-w_1,w_1), \frac{7\Lambda}{3}\, |5-I|\right)
\end{array}\right\} \tag{38}$$

and the mapping $\mathcal{A}(w_1)$ and $\mathcal{Q}(w_1)$ are given in (6) and (31) for all $w_1 \in \mathcal{W}_1$.

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Proof. By Theorem 3.8, it follows from (33), (1) and (5), we arrive

$$\left.\begin{aligned} \mu(\mathcal{A}(w_1)-\mathcal{F}_{odd}(w_1),2\Lambda) &\geq \mu'\left(\Psi_A(w_1),\frac{3\Lambda}{4}\,|5-I|\right)*\mu'\left(\Psi_A(-w_1),\frac{3\Lambda}{4}\,|5-I|\right)\\ \nu(\mathcal{A}(w_1)-\mathcal{F}_{odd}(w_1),2\Lambda) &\leq \nu'\left(\Psi_A(w_1),\frac{3\Lambda}{4}\,|5-I|\right)\diamond\nu'\left(\Psi_A(-w_1),\frac{3\Lambda}{4}\,|5-I|\right)\end{aligned}\right\}\tag{39}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. By Theorem 3.10, it follows from (37), (1), and (30), we see

$$\left.\begin{aligned} \mu(\mathcal{Q}(w_1)-\mathcal{F}_{even}(w_1),2\Lambda) &\geq \mu'\left(\Psi_Q(w_1),\frac{7\Lambda}{3}\,|25-I|\right)*\mu'\left(\Psi_Q(-w_1),\frac{7\Lambda}{3}\,|25-I|\right)\\ \nu(\mathcal{Q}(w_1)-\mathcal{F}_{even}(w_1),2\Lambda) &\leq \nu'\left(\Psi_Q(w_1),\frac{7\Lambda}{3}\,|25-I|\right)\diamond\nu'\left(\Psi_Q(-w_1),\frac{7\Lambda}{3}\,|25-I|\right)\end{aligned}\right\}\tag{40}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Now, it follows from (39), (40) and (40), we have

$$\left.\begin{aligned} &\mu(\mathcal{F}(w_1)-\mathcal{A}(w_1)-\mathcal{Q}(w_1),4\Lambda)\\ &\geq \mu(\mathcal{A}(w_1)-\mathcal{F}_{odd}(w_1),2\Lambda)*\mu(\mathcal{Q}(w_1)-\mathcal{F}_{even}(w_1),2\Lambda)\\ &\geq \mu'\left(\Psi_A(w_1),\frac{3\Lambda}{4}\,|5-I|\right)*\mu'\left(\Psi_A(-w_1),\frac{3\Lambda}{4}\,|5-I|\right)*\\ &\qquad\qquad \mu'\left(\Psi_Q(w_1),\frac{7\Lambda}{3}\,|25-I|\right)*\mu'\left(\Psi_Q(-w_1),\frac{7\Lambda}{3}\,|25-I|\right)\\ &\nu(\mathcal{F}(w_1)-\mathcal{A}(w_1)-\mathcal{Q}(w_1),4\Lambda)\\ &\leq \nu(\mathcal{A}(w_1)-\mathcal{F}_{odd}(w_1),2\Lambda)\diamond\nu(\mathcal{Q}(w_1)-\mathcal{F}_{even}(w_1),2\Lambda)\\ &\leq \nu'\left(\Psi_A(w_1),\frac{3\Lambda}{4}\,|5-I|\right)\diamond\nu'\left(\Psi_A(-w_1),\frac{3\Lambda}{4}\,|5-I|\right)\diamond\\ &\qquad\qquad \nu'\left(\Psi_Q(w_1),\frac{7\Lambda}{3}\,|25-I|\right)\diamond\nu'\left(\Psi_Q(-w_1),\frac{7\Lambda}{3}\,|25-I|\right)\end{aligned}\right\}\tag{41}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. □

Corollary 3.13. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality*

$$\left.\begin{aligned} &\mu(\mathcal{F}(w_1)-\mathcal{A}(w_1)-\mathcal{Q}(w_1),4\Lambda)\\ &\geq \begin{cases} \mu'(2\delta,(|3|+7\,|8|)\Lambda) & \\ \mu'\left(2\delta|w_1|^{\varphi},\Lambda\left\{\frac{1}{4}\,|5-5^{\varphi}|+\frac{7}{9}\,|25-5^{\varphi}|\right\}\right), & \varphi\neq 1,2,\\ \mu'\left(2\delta\sum_{\psi=1}^{3}|w_\psi|^{\varphi_\psi},\left\{\frac{3}{4}\sum_{\psi=1}^{3}|5-5^{\varphi_\psi}|+\frac{7}{3}\sum_{\psi=1}^{3}|25-5^{\varphi_\psi}|\right\}\right), & \varphi_1,\varphi_2,\varphi_3\neq 1,2,\\ \mu'\left(2\delta|w_1|^{3\varphi},\Lambda\left\{\frac{3}{4}|5-5^{3\varphi}|+\frac{7}{3}|25-5^{3\varphi}|\right\}\right), & 3\varphi\neq 1,2,\\ \mu'\left(4\delta|w_\psi|^{\sum\limits_{\psi=1}^{3}\varphi_\psi},2\Lambda\left\{\frac{3}{4}\left|5-5^{\sum\limits_{\psi=1}^{3}\varphi_\psi}\right|+\frac{7}{3}\left|25-5^{\sum\limits_{\psi=1}^{3}\varphi_\psi}\right|\right\}\right) & \sum\limits_{\psi=1}^{3}\varphi_\psi\neq 1,2,\\ \mu'\left(4\delta|w_1|^{3\varphi},\Lambda\left\{\frac{3}{4}|5-5^{3\varphi}|+\frac{7}{3}|25-5^{3\varphi}|\right\}\right) & 3\varphi\neq 1,2, \end{cases}\\ &\nu(\mathcal{F}(w_1)-\mathcal{A}(w_1)-\mathcal{Q}(w_1),4\Lambda)\\ &\leq \begin{cases} \nu'(2\delta,(|3|+7\,|8|)\Lambda) & \\ \nu'\left(2\delta|w_1|^{\varphi},\Lambda\left\{\frac{1}{4}\,|5-5^{\varphi}|+\frac{7}{9}\,|25-5^{\varphi}|\right\}\right), & \varphi\neq 1,2,\\ \nu'\left(2\delta\sum_{\psi=1}^{3}|w_\psi|^{\varphi_\psi},\left\{\frac{3}{4}\sum_{\psi=1}^{3}|5-5^{\varphi_\psi}|+\frac{7}{3}\sum_{\psi=1}^{3}|25-5^{\varphi_\psi}|\right\}\right), & \varphi_1,\varphi_2,\varphi_3\neq 1,2,\\ \nu'\left(2\delta|w_1|^{3\varphi},\Lambda\left\{\frac{3}{4}|5-5^{3\varphi}|+\frac{7}{3}|25-5^{3\varphi}|\right\}\right), & 3\varphi\neq 1,2,\\ \nu'\left(4\delta|w_\psi|^{\sum\limits_{\psi=1}^{3}\varphi_\psi},2\Lambda\left\{\frac{3}{4}\left|5-5^{\sum\limits_{\psi=1}^{3}\varphi_\psi}\right|+\frac{7}{3}\left|25-5^{\sum\limits_{\psi=1}^{3}\varphi_\psi}\right|\right\}\right) & \sum\limits_{\psi=1}^{3}\varphi_\psi\neq 1,2,\\ \nu'\left(4\delta|w_1|^{3\varphi},\Lambda\left\{\frac{3}{4}|5-5^{3\varphi}|+\frac{7}{3}|25-5^{3\varphi}|\right\}\right) & 3\varphi\neq 1,2, \end{cases}\end{aligned}\right\}\tag{42}$$

for all $w_1 \in \mathcal{W}_1$.

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3.5. Oddness of $\mathcal{F}$: Additive Case Stability Results : Fixed Method

Theorem 3.14. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (1) where $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ with the condition*

$$\left.\begin{array}{l} \lim\limits_{\ell\to\infty} \mu'\left(\Psi\left(\tau_\nu^\ell w_1, \tau_\nu^\ell w_2, \tau_\nu^\ell w_3\right), \tau_\nu^\ell \Lambda\right) = 1 \\ \lim\limits_{\ell\to\infty} \nu'\left(\Psi\left(\tau_\nu^\ell w_1, \tau_\nu^\ell w_2, \tau_\nu^\ell w_3\right), \tau_\nu^\ell \Lambda\right) = 0 \end{array}\right\}; \quad \tau_\nu = \begin{cases} 5; \nu = 0 \\ \frac{1}{5}; \nu = 1 \end{cases} \tag{43}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$ If there exists $L = L(\nu)$ be a function have the property

$$\left.\begin{array}{l} \mu(\Psi_A(w_1), \Lambda) = \mu(\Psi_A(\frac{w_1}{5}), \Lambda) \\ \nu(\Psi_A(w_1), \Lambda) = \nu(\Psi_A(\frac{w_1}{5}), \Lambda) \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} \mu\left(\frac{1}{\tau_\nu}\Psi_A(\tau_\nu w_1), \Lambda\right) = \mu(L\,\Psi_A(w_1), \Lambda) \\ \nu\left(\frac{1}{\tau_\nu}\Psi_A(\tau_\nu w_1), \Lambda\right) = \nu(L\,\Psi_A(w_1), \Lambda) \end{array}\right\}, \tag{44}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality

$$\left.\begin{array}{l} \mu(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda) \geq \mu'\left(\dfrac{L^{1-\nu}}{1-L}\Psi_A(w_1), \dfrac{3\Lambda}{4}\right) \\ \qquad = \mu'\left(\dfrac{L^{1-\nu}}{1-L}\Psi(w_1, w_1, w_1), \dfrac{3\Lambda}{4}\right) * \mu'\left(\dfrac{L^{1-\nu}}{1-L}\Psi(w_1, w_1, -w_1), \dfrac{3\Lambda}{4}\right) \\ \nu(\mathcal{A}(w_1) - \mathcal{F}(w_1), \Lambda) \leq \nu'\left(\dfrac{L^{1-\nu}}{1-L}\Psi_A(w_1), \dfrac{3\Lambda}{4}\right) \\ \qquad = \nu'\left(\dfrac{L^{1-\nu}}{1-L}\Psi(w_1, w_1, w_1), \dfrac{3\Lambda}{4}\right) \diamond \nu'\left(\dfrac{L^{1-\nu}}{1-L}\Psi(w_1, w_1, -w_1), \dfrac{3\Lambda}{4}\right) \end{array}\right\} \tag{45}$$

and the mapping $\mathcal{A}(w_1)$ is obtained by

$$\left.\begin{array}{l} \lim\limits_{\ell\to\infty} \mu\left(\dfrac{1}{\tau_\nu^\ell}\mathcal{F}\left(\tau_\nu^\ell w_1\right) - \mathcal{A}(w_1), \Lambda\right) = 1 \\ \lim\limits_{\ell\to\infty} \nu\left(\dfrac{1}{\tau_\nu^\ell}\mathcal{F}\left(\tau_\nu^\ell w_1\right) - \mathcal{A}(w_1), \Lambda\right) = 0 \end{array}\right\} \tag{46}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Proof. Assume a set $\mathcal{G}$ as in Theorem 2.7 of (48) and introduce the generalized metric on the above set $\mathcal{G}$ as

$$d(\mathcal{F}, \mathcal{F}_1) = \inf\left\{K \in (0, \infty) : \left\{\begin{array}{l} \mu(\mathcal{F}(w_1) - \mathcal{F}_1(w_1), \Lambda) \geq \mu(K\,\Psi(w_1, w_1, w_1), \Lambda) \\ \nu(\mathcal{F}(w_1) - \mathcal{F}_1(w_1), \Lambda) \leq \nu(K\,\Psi(w_1, w_1, w_1), \Lambda) \end{array}\right\}\right\}. \tag{47}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. It is easy to see that $(\mathcal{G}, d)$ is complete. Define a function $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ as by Theorem 2.7 of (50) and for $\mathcal{F}, \mathcal{F}_1 \in \mathcal{G}$ and $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$, we see

$$\begin{aligned} d(\mathcal{F}, \mathcal{F}_1) \leq K \Rightarrow & \left\{\begin{array}{l} \mu(\mathcal{F}(w_1) - \mathcal{F}_1(w_1), \Lambda) \geq \mu(K\,\Psi(w_1, w_1, w_1), \Lambda) \\ \nu(\mathcal{F}(w_1) - \mathcal{F}_1(w_1), \Lambda) \leq \nu(K\,\Psi(w_1, w_1, w_1), \Lambda) \end{array}\right\} \\ \Rightarrow & \left\{\begin{array}{l} \mu\left(\left\|\frac{1}{\tau_\nu}\mathcal{F}(\tau_\nu w_1) - \frac{1}{\tau_\nu}\mathcal{F}_1(\tau_\nu w_1)\right\|, \Lambda\right) \geq \mu\left(\tau_\nu K\,\Psi(\frac{1}{\tau_\nu}w_1, \tau_\nu w_1, \tau_\nu w_1), \Lambda\right) \\ \nu\left(\left\|\frac{1}{\tau_\nu}\mathcal{F}(\tau_\nu w_1) - \frac{1}{\tau_\nu}\mathcal{F}_1(\tau_\nu w_1)\right\|, \Lambda\right) \leq \nu\left(\tau_\nu K\,\Psi(\frac{1}{\tau_\nu}w_1, \tau_\nu w_1, \tau_\nu w_1), \Lambda\right) \end{array}\right\} \\ \Rightarrow & \left\{\begin{array}{l} \mu(\mathcal{H}\mathcal{F}(w_1) - \mathcal{H}\mathcal{F}_1(w_1), \Lambda) \geq \mu(L\,K\,\Psi(w_1, w_1, w_1), \Lambda) \\ \nu(\mathcal{H}\mathcal{F}(w_1) - \mathcal{H}\mathcal{F}_1(w_1), \Lambda) \leq \nu(L\,K\,\Psi(w_1, w_1, w_1), \Lambda) \end{array}\right\} \\ \Rightarrow & d(\mathcal{H}\mathcal{F}, \mathcal{H}\mathcal{F}_1) \leq L\,K, \end{aligned}$$

i.e., $\mathcal{H}$ is a strictly contractive mapping on $\mathcal{G}$ with Lipschitz constant L (see [24]).
For the case $\nu = 0$, it follows from (11) and with the help of (44), (50), (47), we get

$$\left.\begin{array}{l} \mu\left(\frac{1}{5}\mathcal{F}(5w_1) - \mathcal{F}(w_1), \frac{4}{3}\Lambda\right) \geq \mu'\left(\frac{1}{5}\Psi_A(w_1), \Lambda\right) \\ \nu\left(\frac{1}{5}\mathcal{F}(5w_1) - \mathcal{F}(w_1), \frac{4}{3}\Lambda\right) \leq \nu'\left(\frac{1}{5}\Psi_A(w_1), \Lambda\right) \end{array}\right\} \Rightarrow d(\mathcal{HF}, \mathcal{F}) \leq L = L^{1-\nu}, \tag{48}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.
For the case $\nu = 1$, it follows from (17) and with the help of (44), (50), (47), we obtain

$$\left.\begin{array}{l} \mu\left(\mathcal{F}(w_1) - 5\mathcal{F}\left(\frac{w_1}{5}\right), \frac{4}{3\cdot I}\Lambda\right) \geq \mu'\left(\Psi_A\left(\frac{w_1}{5}\right), \Lambda\right) \\ \nu\left(\mathcal{F}(w_1) - 5\mathcal{F}\left(\frac{w_1}{5}\right), \frac{4}{3\cdot I}\Lambda\right) \leq \nu'\left(\Psi_A\left(\frac{w_1}{5}\right), \Lambda\right) \end{array}\right\} \Rightarrow d(\mathcal{F}, \mathcal{HF}) \leq 1 = L^{1-\nu}, \tag{49}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Combining (48) and (49), we have

$$d(\mathcal{F}, \mathcal{HF}) \leq 1 = L^{1-\nu}. \tag{50}$$

Therefore $(FPC1)$ of Theorem 1.3 holds. The rest of the proof follows by Theorem 1.3. Hence the proof is complete. □

Corollary 3.15. *Suppose that an odd function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality (20) for all $w_1 \in \mathcal{W}_1$.*

3.6. Evenness of $\mathcal{F}$: Quadratic Case Stability Results : Fixed Method

Theorem 3.16. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (1) where $\Psi : \mathcal{W}_1^3 \to [0, \infty)$ with the condition*

$$\left.\begin{array}{l} \lim\limits_{\ell\to\infty} \mu'\left(\Psi\left(\tau_\nu^\ell w_1, \tau_\nu^\ell w_2, \tau_\nu^\ell w_3\right), \tau_\nu^{2\ell}\Lambda\right) = 1 \\ \lim\limits_{\ell\to\infty} \nu'\left(\Psi\left(\tau_\nu^\ell w_1, \tau_\nu^\ell w_2, \tau_\nu^\ell w_3\right), \tau_\nu^{2\ell}\Lambda\right) = 0 \end{array}\right\}; \quad \tau_\nu = \left\{\begin{array}{l} 5; \nu = 0 \\ \frac{1}{5}; \nu = 1 \end{array}\right. \tag{51}$$

for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. If there exists $L = L(\nu)$ be function have the property

$$\left.\begin{array}{l} \mu(\Psi_Q(w_1), \Lambda) = \mu(\Psi_Q(\frac{w_1}{5}), \Lambda) \\ \nu(\Psi_Q(w_1), \Lambda) = \nu(\Psi_Q(\frac{w_1}{5}), \Lambda) \end{array}\right\} \quad \textit{and} \quad \left.\begin{array}{l} \mu\left(\frac{1}{\tau_\nu^2}\Psi_Q(\tau_\nu w_1), \Lambda\right) = \mu(L\,\Psi_Q(w_1), \Lambda) \\ \nu\left(\frac{1}{\tau_\nu^2}\Psi_Q(\tau_\nu w_1), \Lambda\right) = \nu(L\,\Psi_Q(w_1), \Lambda) \end{array}\right\}, \tag{52}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality

$$\left.\begin{array}{l} \mu(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda) \geq \mu'\left(\frac{L^{1-\nu}}{1-L}\Psi_Q(w_1), \frac{7\Lambda}{3}\right) \\ \qquad = \mu'\left(\frac{L^{1-\nu}}{1-L}\Psi(w_1, w_1, w_1), \frac{7\Lambda}{3}\right) * \mu'\left(\frac{L^{1-\nu}}{1-L}\Psi(w_1, w_1, -w_1), \frac{7\Lambda}{3}\right) \\ \nu(\mathcal{Q}(w_1) - \mathcal{F}(w_1), \Lambda) \leq \nu'\left(\frac{L^{1-\nu}}{1-L}\Psi_Q(w_1), \frac{7\Lambda}{3}\right) \\ \qquad = \nu'\left(\frac{L^{1-\nu}}{1-L}\Psi(w_1, w_1, w_1), \frac{7\Lambda}{3}\right) \diamond \nu'\left(\frac{L^{1-\nu}}{1-L}\Psi(w_1, w_1, -w_1), \frac{7\Lambda}{3}\right) \end{array}\right\} \tag{53}$$

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and the mapping $\mathcal{Q}(w_1)$ is obtained by

$$\left.\begin{array}{l}\lim_{\ell\to\infty}\mu\left(\frac{1}{\tau_\nu^{2\ell}}\mathcal{F}\left(\tau_\nu^\ell w_1\right)-\mathcal{Q}(w_1),\Lambda\right)=1\\ \lim_{\ell\to\infty}\nu\left(\frac{1}{\tau_\nu^{2\ell}}\mathcal{F}\left(\tau_\nu^\ell w_1\right)-\mathcal{Q}(w_1),\Lambda\right)=0\end{array}\right\} \tag{54}$$

for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Proof. Define a function $\mathcal{H} : \mathcal{G} \to \mathcal{G}$ as by Theorem 2.9 of (62) and for $\mathcal{F}, \mathcal{F}_1 \in \mathcal{G}$ and $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$, we see

$$\begin{aligned}d(\mathcal{F},\mathcal{F}_1)\le K\Rightarrow&\left\{\begin{array}{l}\mu(\mathcal{F}(w_1)-\mathcal{F}_1(w_1),\Lambda)\ge\mu(K\,\Psi(w_1,w_1,w_1),\Lambda)\\ \nu(\mathcal{F}(w_1)-\mathcal{F}_1(w_1),\Lambda)\le\nu(K\,\Psi(w_1,w_1,w_1),\Lambda)\end{array}\right\}\\ \Rightarrow&\left\{\begin{array}{l}\mu\left(\left\|\frac{1}{\tau_\nu^2}\mathcal{F}(\tau_\nu w_1)-\frac{1}{\tau_\nu^2}\mathcal{F}_1(\tau_\nu w_1)\right\|,\Lambda\right)\ge\mu(\tau_\nu K\,\Psi(\tau_\nu w_1,\tau_\nu w_1,\tau_\nu w_1),\Lambda)\\ \nu\left(\left\|\frac{1}{\tau_\nu^2}\mathcal{F}(\tau_\nu w_1)-\frac{1}{\tau_\nu^2}\mathcal{F}_1(\tau_\nu w_1)\right\|,\Lambda\right)\le\nu(\tau_\nu^2 K\,\Psi(\tau_\nu w_1,\tau_\nu w_1,\tau_\nu w_1),\Lambda)\end{array}\right\}\\ \Rightarrow&\left\{\begin{array}{l}\mu(\mathcal{H}\mathcal{F}(w_1)-\mathcal{H}\mathcal{F}_1(w_1),\Lambda)\ge\mu(L\,K\,\Psi(w_1,w_1,w_1),\Lambda)\\ \nu(\mathcal{H}\mathcal{F}(w_1)-\mathcal{H}\mathcal{F}_1(w_1),\Lambda)\le\nu(L\,K\,\Psi(w_1,w_1,w_1),\Lambda)\end{array}\right\}\\ \Rightarrow& d(\mathcal{H}\mathcal{F},\mathcal{H}\mathcal{F}_1)\le L\,K,\end{aligned}$$

i.e., $\mathcal{H}$ is a strictly contractive mapping on $\mathcal{G}$ with Lipschitz constant L (see [24]). The rest of the proof is similar to that of Theorem 3.14. Hence the proof is complete. □

Corollary 3.17. *Suppose that an even function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality (37) for all $w_1 \in \mathcal{W}_1$.*

3.7. Oddness and Evenness of $\mathcal{F}$: Additive Quadratic Case Stability Results : Fixed Method

Theorem 3.18. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (1) where $\Psi : \mathcal{W}_1^3 \to [0,\infty)$ with the conditions (43) and (51) for all $w_1, w_2, w_3 \in \mathcal{W}_1$ and all $\Lambda > 0$. If there exists $L = L(\nu)$ be function have the properties (44) and (52) for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality*

$$\left.\begin{array}{l}\mu(\mathcal{F}(w_1)-\mathcal{A}(w_1)-\mathcal{Q}(w_1),4\Lambda)\\ \ge\mu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi_A(w_1),\frac{3\Lambda}{4}\right)*\mu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi_A(-w_1),\frac{3\Lambda}{4}\right)*\\ \qquad\mu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi_Q(w_1),\frac{7\Lambda}{3}\right)*\mu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi_Q(-w_1),\frac{7\Lambda}{3}\right)\\ =\mu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(w_1,w_1,w_1),\frac{3\Lambda}{4}\right)*\mu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(w_1,w_1,-w_1),\frac{3\Lambda}{4}\right)*\\ \quad\mu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(-w_1,-w_1,-w_1),\frac{3\Lambda}{4}\right)*\mu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(-w_1,-w_1,w_1),\frac{3\Lambda}{4}\right)*\\ \quad\mu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(w_1,w_1,w_1),\frac{7\Lambda}{3}\right)*\mu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(w_1,w_1,-w_1),\frac{7\Lambda}{3}\right)*\\ \quad\mu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(-w_1,-w_1,-w_1),\frac{7\Lambda}{3}\right)*\mu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(-w_1,-w_1,w_1),\frac{7\Lambda}{3}\right)\\ \nu(\mathcal{F}(w_1)-\mathcal{A}(w_1)-\mathcal{Q}(w_1),4\Lambda)\\ \le\nu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi_A(w_1),\frac{3\Lambda}{4}\right)\diamond\nu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi_A(-w_1),\frac{3\Lambda}{4}\right)\diamond\\ \qquad\nu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi_Q(w_1),\frac{7\Lambda}{3}\right)\diamond\nu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi_Q(-w_1),\frac{7\Lambda}{3}\right)\\ =\nu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(w_1,w_1,w_1),\frac{3\Lambda}{4}\right)\diamond\nu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(w_1,w_1,-w_1),\frac{3\Lambda}{4}\right)\diamond\\ \quad\nu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(-w_1,-w_1,-w_1),\frac{3\Lambda}{4}\right)\diamond\nu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(-w_1,-w_1,w_1),\frac{3\Lambda}{4}\right)\diamond\\ \quad\nu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(w_1,w_1,w_1),\frac{7\Lambda}{3}\right)\diamond\nu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(w_1,w_1,-w_1),\frac{7\Lambda}{3}\right)\diamond\\ \quad\nu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(-w_1,-w_1,-w_1),\frac{7\Lambda}{3}\right)\diamond\nu'\left(\frac{L^{1-\nu}}{1-L}\,\Psi(-w_1,-w_1,w_1),\frac{7\Lambda}{3}\right)\end{array}\right\} \tag{55}$$

and the mapping $\mathcal{A}(w_1)$ and $\mathcal{Q}(w_1)$ are given in (45) and (54) for all $w_1 \in \mathcal{W}_1$ and all $\Lambda > 0$.

Proof. The proof is similar ideas to that of Theorem 3.12. □

Corollary 3.19. *Suppose that a function $\mathcal{F} : \mathcal{W}_1 \to \mathcal{W}_2$ satisfying the functional inequality (2) for all $w_1, w_2, w_3 \in \mathcal{W}_1$. Then there exists a unique additive mapping $\mathcal{A}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ and a unique quadratic mapping $\mathcal{Q}(w_1) : \mathcal{W}_1 \to \mathcal{W}_2$ which satisfying (3) and the functional inequality (42) for all $w_1 \in \mathcal{W}_1$.*

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