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Article

Generalized Weighted Group Inverse in Banach Algebras with Proper Involution

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Abstract: In this paper, we introduce the notion of the generalized w -group inverse in a Banach algebra with proper involution. This is a natural generalization of generalized (weak) group inverse in Banach $*$ -algebras. Many elementary properties of such new weighted generalized inverse are presented. We further establish the equivalent relations between generalized weighted group inverse and weighted generalized Drazin inverse. Related properties of generalized (weak) group inverse are extended to the wider cases.

Keywords: weighted generalized Drazin inverse; weighted group inverse; weak group inverse; complex matrix; Banach algebra

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1. Introduction

An involution of a Banach algebra \mathcal{A} is an anti-automorphism whose square is the identity map 1. A Banach algebra \mathcal{A} with involution $*$ is called a Banach $*$ -algebra. Let $a, w \in \mathcal{A}$. Recall that $a \in \mathcal{A}$ has generalized w -Drazin inverse x if there exists unique $x \in \mathcal{A}$ such that

$$awx = xwa, xwawx = x \text{ and } a - awxwa \in \mathcal{A}^{qnil}.$$

We denote x by $a^{d,w}$ (see [10]). Here, $\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid 1 + \lambda x \in \mathcal{A}^{-1}\}$. As is well known, $x \in \mathcal{A}^{qnil} \Leftrightarrow \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}} = 0$. If we replace \mathcal{A}^{qnil} by the set \mathcal{A}^{nil} of all nilpotents in \mathcal{A} , $a^{d,w}$ is called the w -Drazin inverse of a , and denote it by $a^{D,w}$. If the weight $w = 1$, we call $a^{d,w}$ ($a^{D,w}$) the g -Drazin (Drazin) inverse of a , and denote it by a^d (a^D). Evidently, $a^{d,w} = [(aw)^d]^2 a = a[(wa)^d]^2 = (aw)^d a (wa)^d$.

Definition 1.1. (see [4]) An element $a \in \mathcal{A}$ has w -group inverse if there exist $x \in \mathcal{A}$ such that

$$awxwa = a, xwawx = x, awx = xwa.$$

The preceding x is unique if it exists, and we denote it by $a_w^\#$. The set of all generalized w -group invertible elements in \mathcal{A} is denoted by $\mathcal{A}_w^\#$.

An element a in a Banach algebra \mathcal{A} has group inverse provided that it has w -group inverse for the weight $w = 1$, i.e., there exists $x \in \mathcal{A}$ such that

$$xa^2 = a, ax^2 = x, ax = xa.$$

Such x is unique if exists, denoted by $a^\#$, and called the group inverse of a . As is well known, a square complex matrix A has group inverse if and only if $\text{rank}(A) = \text{rank}(A^2)$.

A square complex matrix X is the core-EP inverse of A if

$$XAX = X, \text{Im}(X) = \text{Im}X^* = \text{Im}(A^m),$$

where $m = \text{ind}(A)$ (see [8]). Recently, Wang and Chen (see [15]) introduced and studied a weak group inverse for square complex matrices. A square complex matrix A has weak group inverse X if it satisfies the equations

$$AX^2 = X, AX = A^\oplus A.$$

The involution $*$ is proper, that is, $x^*x = 0 \implies x = 0$, e.g., in a Rickart $*$ -ring, the involution is always proper. Let $\mathbb{C}^{n \times n}$ be the Banach algebra of all $n \times n$ complex matrices, with conjugate transpose $*$ as the involution. Then the involution $*$ is proper. Zou et al. extended the notion of weak group inverse to a proper $*$ -Banach algebra element. We refer the reader for weak group inverse in [3,13,18,19]. In [2], the generalized group inverse in a Banach algebra with proper involution was introduced. An element a in a Banach $*$ -algebra has generalized group inverse if there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (a^*a^2x)^* = a^*a^2x, \lim_{n \rightarrow \infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Such x is unique if it exists and is called the generalized group inverse of a . We denote it by a^\oplus . Many properties of generalized group inverse were presented in [2]. Recently, Mosić and Zhang introduced and studied weighted weak group inverse for a Hilbert space operator A in $\mathcal{B}(X)^d$ (see [12]).

The motivation of this paper is to introduce and study a new kind of generalized weighted inverse as a natural generalization of generalized inverses mentioned above.

In Section 2, we introduce generalized weighted group inverse by using a new kind of weighted group decomposition. Let $\mathcal{A}_w^{qnil} = \{x \in \mathcal{A} \mid xw \in \mathcal{A}^{qnil}\}$.

Definition 1.2. An element $a \in \mathcal{A}$ has generalized w -group decomposition if there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}_w^\#, y \in \mathcal{A}_w^{qnil}.$$

We prove that $a \in \mathcal{A}$ has generalized w -group decomposition if and only if $aw \in \mathcal{A}^d$ and there exists unique $x \in \mathcal{A}$ such that

$$x = a(wx)^2, ((aw)^d)^*(aw)^2x = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

The preceding x is called generalized w -group inverse of a .

In Section 3, we establish elementary properties of generalized w -group inverse. Many properties of the weak group inverse generalized group inverse are thereby generalized to wider cases.

In Section 3, we investigate equivalences between generalized w -group inverse and weighted g -Drazin inverse for a Banach algebra element. We prove that $a \in \mathcal{A}_w^\oplus$ if and only if $a \in \mathcal{A}^{d,w}$ and $(a^{d,w})^*a^{d,w}x = (a^{d,w})^*a$ for some $x \in \mathcal{A}$.

Throughout the paper, all Banach algebras are complex with a proper involution $*$. We use $\mathcal{A}^{-1}, \mathcal{A}^\#, \mathcal{A}^d$ and \mathcal{A}^D to denote the sets of all invertible, group invertible, g -Drazin invertible and Drazin invertible elements in \mathcal{A} , respectively. We use $\ell(a)$ to stand for the left annihilator of a in \mathcal{A} .

2. Generalized w -Group Inverse

The purpose of this section is to introduce a new generalized inverse which is a natural generalization of (weak) group inverse in a $*$ -Banach algebra.

Lemma 2.1. Let $a \in \mathcal{A}_w^\#$. Then $aw, wa \in \mathcal{A}^\#$. In this case,

$$a_w^\# = (aw)^\#a(wa)^\#, (aw)^2a_w^\# = a \text{ and } a[wa_w^\#]^2 = a_w^\#.$$

Proof. Let $x = a_w^\#$. Then

$$awxwa = a, xwawx = x, awx = xwa.$$

Hence, $(aw)^2x = aw(awx) = (aw)x(wa) = a$, and so $(aw)^2xw = aw$. Likewise, $xw(aw)^2 = aw$. This implies that $aw \in \mathcal{A}^\#$. Moreover, we have

$$a[wa_w^\#]^2 = a(wx)^2 = (awx)wx = (xwa)wx = x,$$

as desired. \square

Theorem 2.2. Let $a \in \mathcal{A}$. Then the following are equivalent:

- (1) $a \in \mathcal{A}$ has generalized w -group decomposition.
- (2) $aw \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, ((aw)^d)^*(aw)^2x = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

Proof. (1) \Rightarrow (2) By hypothesis, a has the generalized w -group decomposition $a = a_1 + a_2$. Let $x = (a_1)_w^\#$. By virtue of Lemma 2.1, we have

$$awx = (a_1w + a_2w)(a_1)_w^\# = a_1w(a_1)_w^\#, \\ a(wx)^2 = a_1[w(a_1)_w^\#]^2 = (a_1)_w^\# = x,$$

Since $a_2wa_1 = 0$, we have

$$aw - xw(aw)^2 \\ = (a_1w + a_2w) - [(a_1)_w^\#wa_1w + (a_1)_w^\#wa_2w](a_1w + a_2w) \\ = [1 - (a_1)_w^\#wa_1w - (a_1)_w^\#wa_2w]a_2w,$$

and then

$$\|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} \\ = \|(aw - xw(aw)^2)(aw)^{n-1}\|^{\frac{1}{n}} \\ = \|[1 - (a_1)_w^\#wa_1w - (a_1)_w^\#wa_2w]a_2w(aw)^{n-1}\|^{\frac{1}{n}} \\ = \|[1 - (a_1)_w^\#wa_1w - (a_1)_w^\#wa_2w](a_2w)^n\|^{\frac{1}{n}} \\ \leq \|1 - (a_1)_w^\#wa_1w - (a_1)_w^\#wa_2w\|^{\frac{1}{n}} \|(a_2w)^n\|^{\frac{1}{n}}.$$

Since $a_2 \in \mathcal{A}_w^{qnil}$, we have $\lim_{n \rightarrow \infty} \|(a_2w)^n\|^{\frac{1}{n}} = 0$. Accordingly, we have

$$\lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

Since $a_2wa_1 = 0$, $(a_1w)(a_1w)^\pi = 0$ and $(a_2w)^d = 0$, Then $aw \in \mathcal{A}^d$ and

$$(aw)^d = (a_1w)^\# + \sum_{n=1}^{\infty} ((a_1w)^\#)^{n+1}(a_2w)^n.$$

Then

$$((aw)^d)^*a_2 \\ = ((a_1w)^\#)^*a_2 + \sum_{n=1}^{\infty} [((a_1w)^\#)^{n+1}(a_2w)^n]^*a_2 \\ = [((a_1w)^\#)^2]^*((a_1w)^\#)^*a_2 + \sum_{n=1}^{\infty} [((a_1w)^\#)^{n+2}(a_2w)^n]^*((a_1w)^\#)^*a_2 \\ = 0;$$

hence, $((aw)^d)^*a_1 = ((aw)^d)^*a$. Accordingly,

$$((aw)^d)^*(aw)^2x \\ = ((aw)^d)^*(a_1w + a_2w)(a_1w + a_2w)(a_1)_w^\# \\ = ((aw)^d)^*(a_1w)^2(a_1)_w^\# = ((aw)^d)^*a_1 \\ = ((aw)^d)^*a.$$

(2) \Rightarrow (1) By hypothesis, there exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, ((aw)^d)^*(aw)^2x = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^n\|^{\frac{1}{n}} = 0.$$

Then

$$x = (aw)x(wx) = (aw)[(aw)x(wx)](wx) = (aw)^2x(wx)^2 \\ = \dots = (aw)^{n-1}x(wx)^{n-1}.$$

Let $a_1 = (aw)^2x$ and $a_2 = a - (aw)^2x$. Then we check that

$$\|a_2wa_1\| = \|[a - (aw)^2x]w(aw)^2x\| \\ = \|[(aw)^2 - (aw)^2xw(aw)](aw)x\| \\ = \|(aw)^2[1 - xw(aw)](aw)x\| \\ = \|(aw)^2[1 - xw(aw)](aw)^n x(wx)^{n-1}\| \\ \leq \|(aw)^2\| \|(aw)^n - xw(aw)^{n+1}\| \|x(wx)^{n-1}\|.$$

Hence

$$\|a_2wa_1\|^{\frac{1}{n}} \leq \|(aw)^2\|^{\frac{1}{n}} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} \|x(wx)^{n-1}\|^{\frac{1}{n}}.$$

Since $\lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} = 0$, we deduce that

$$\lim_{n \rightarrow \infty} \|a_2wa_1\|^{\frac{1}{n}} = 0.$$

Thus, $a_2wa_1 = 0$.

Moreover, we have

$$(aw)^2x \\ = (aw)^2[(aw)^{n-1}x(wx)^{n-1}] = (aw)^{n+1}x(wx)^{n-1} \\ = [(aw)^{n+1} - (aw)^d(aw)^{n+2}]x(wx)^{n-1} + (aw)^d(aw)^{n+2}x(wx)^{n-1} \\ = [(aw)^{n+1} - (aw)^d(aw)^{n+2}]x(wx)^{n-1} + (aw)^d(aw)^3[(aw)^{n-1}x(wx)^{n-1}] \\ = [(aw)^{n+1} - (aw)^d(aw)^{n+2}]x(wx)^{n-1} + (aw)^d(aw)^3x.$$

Then we verify that

$$\|a_1^*a_2\| \\ = \|((aw)^2x)^*(a - (aw)^2x)\| \\ = \|([(aw)^{n+1} - (aw)^d(aw)^{n+2}]x(wx)^{n-1} + (aw)^d(aw)^3x)^*(a - (aw)^2x)\| \\ \leq \|((aw)^{n+1} - (aw)^d(aw)^{n+2})^*\| \|(x(wx)^{n-1})^*\| \|a - (aw)^2x\| \\ + \|((aw)^d(aw)^3x)^*(a - (aw)^2x)\| \\ \leq \|((aw)^{n+1} - (aw)^d(aw)^{n+2})^*\| \|(x(wx)^{n-1})^*\| \|a - (aw)^2x\| \\ + \|(aw)^3x\| \|((aw)^d)^*(a - (aw)^2x)\| \\ = \|((aw)^{n+1} - (aw)^d(aw)^{n+2})^*\| \|(x(wx)^{n-1})^*\| \|a - (aw)^2x\|.$$

Then

$$\|a_1^*a_2\|^{\frac{1}{n+1}} \leq \|((aw)^{n+1} - (aw)^d(aw)^{n+2})^*\|^{\frac{1}{n+1}} \|(x(wx)^{n-1})^*\|^{\frac{1}{n+1}} \\ \|a - (aw)^2x\|^{\frac{1}{n+1}}.$$

Therefore $\lim_{n \rightarrow \infty} \|a_1^*a_2\|^{\frac{1}{n+1}} = 0$, and so $a_1^*a_2 = 0$. Clearly, we have

$$\|awx - xw(aw)^2x\| = \|(aw - xw(aw)^2)x\| \\ = \|[(aw)^n - xw(aw)^{n+1}]x(wx)^{n-1}\| \\ \leq \|(aw)^n - xw(aw)^{n+1}\| \|x(wx)^{n-1}\|.$$

Hence, $\lim_{n \rightarrow \infty} \|awx - xw(aw)^2x\|^{\frac{1}{n}} = 0$, and then $awx = xw(aw)^2x$. Moreover, we check that

$$\begin{aligned} a_1wx &= (aw)^2xwx = awawxwx = awa(wx)^2 \\ &= awx = xw(aw)^2x = xwa_1, \\ a_1wxwa_1 &= (aw)^2xwxw(aw)^2x = awa(wx)^2wawawx \\ &= awxw(aw)^2x = (aw)^2x = a_1, \\ xwa_1wx &= xw(aw)^2xwx = xwa_1wa_1(wx)^2 = xwawx = x. \end{aligned}$$

Thus, $a_1 \in \mathcal{A}_w^\#$. By hypothesis, we have $aw \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that

$$\begin{aligned} xw &= aw(xw)^2, ((aw)^d)^*(aw)^2xw = ((aw)^d)^*aw, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} &= 0. \end{aligned}$$

As in the proof of ([2] Theorem 2.2), we have $aw - (aw)^2xw \in \mathcal{A}^{qnil}$, and then $a_2w = (a - (aw)^2x)w \in \mathcal{A}^{qnil}$. This implies that $a_2 \in \mathcal{A}_w^{qnil}$. Accordingly, we have the generalized w -group inverse $a = a_1 + a_2$, as required. \square

Corollary 2.3. *Let $a \in \mathcal{A}$ and $w \in \mathcal{A}^{-1}$. Then the following are equivalent:*

- (1) $a \in \mathcal{A}$ has generalized w -group decomposition.
- (2) There exists $x \in \mathcal{A}$ such that

$$\begin{aligned} x &= a(wx)^2, [(aw)^*(aw)^2xw]^* = (aw)^*(aw)^2xw, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} &= 0. \end{aligned}$$

Proof. (1) \Rightarrow (2) By hypothesis, a has generalized w -group decomposition $a = a_1 + a_2$. Let $x = (a_1)_w^\#$. By virtue of Theorem 2.2, we have $aw \in \mathcal{A}^d$ and

$$\begin{aligned} x &= a(wx)^2, ((aw)^d)^*(aw)^2x = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} &= 0. \end{aligned}$$

Moreover, we check that

$$\begin{aligned} (aw)^2(xw) &= (a_1w + a_2w)^2(a_1)_w^\#w = (a_1w)^2(a_1)_w^\#w = a_1w, \\ (aw)^*(aw)^2(xw) &= (a_1w + a_2w)^*a_1w = (a_1w)^*a_1w. \end{aligned}$$

Therefore

$$[(aw)^*(aw)^2(xw)]^* = [(a_1w)^*a_1w]^* = (a_1w)^*a_1w = (aw)^*(aw)^2(xw),$$

as required.

- (2) \Rightarrow (1) By hypothesis, there exists $x \in \mathcal{A}$ such that

$$\begin{aligned} x &= a(wx)^2, [(aw)^*(aw)^2xw]^* = (aw)^*(aw)^2xw, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} &= 0. \end{aligned}$$

Let $a_1 = (aw)^2x$ and $a_2 = a - (aw)^2x$. As is the proof of "(3) \Rightarrow (2)", we verify that $a_2wa_1 = 0$ and $(a_1)_w^\# = x$. As in the proof of Theorem 2.2, $x = (aw)^{n-1}x(wx)^{n-1}$. Obviously, we verify that

$$\begin{aligned} \|x - (xw)(awx)\| &= \|[(aw)^{n-1} - xwa_1w(aw)^{n-1}]x(wx)^{n-1}\| \\ &= \|(aw)^{n-1} - xw(aw)^n\| \|x(wx)^{n-1}\|. \end{aligned}$$

By hypothesis, we have

$$\lim_{n \rightarrow \infty} \|(aw)^{n-1} - xw(aw)^n\|^{\frac{1}{n-1}} = 0.$$

Hence,

$$\lim_{n \rightarrow \infty} \|x - (xw)(awx)\|^{\frac{1}{n-1}} = 0.$$

This implies that $x = (xw)(awx)$. Moreover, we have

$$\begin{aligned} w^*a_1^*a_2w &= w^*[(aw)^2x]^*[a - (aw)^2x]w \\ &= [(aw)^2xw]^*aw[1 - (aw)xw] \\ &= [(aw)^*(aw)^2xw]^*[1 - (aw)xw] \\ &= [(aw)^*(aw)^2xw][1 - (aw)xw] \\ &= (aw)^*(aw)^2xw - (aw)^*(aw)^2xw(aw)xw \\ &= (aw)^*(aw)^2xw - (aw)^*(aw)^2(xwawx)w \\ &= (aw)^*(aw)^2xw - (aw)^*(aw)^2xw \\ &= 0. \end{aligned}$$

Since $w \in \mathcal{A}^{-1}$, we have $w^* \in \mathcal{A}^{-1}$. Hence, $a_1^*a_2 = 0$.

By hypothesis, there exists $x \in \mathcal{A}$ such that

$$\begin{aligned} xw &= (aw)(xw)^2, [(aw)^*(aw)^2xw]^* = (aw)^*(aw)^2xw, \\ \lim_{n \rightarrow \infty} \|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} &= 0. \end{aligned}$$

Then $xw = (aw)^{\otimes}$. As in the proof of ([2] Theorem 2.2), we have $aw - (aw)^2(xw) \in \mathcal{A}^{qnil}$. This implies that $a_2 = a - (aw)^2x \in \mathcal{A}_w^{qnil}$. Therefore a has generalized w -group decomposition $a = a_1 + a_2$, as asserted. \square

Theorem 2.4. Let $a, w \in \mathcal{A}$. Then the following are equivalent:

- (1) $a \in \mathcal{A}$ has generalized w -group decomposition.
- (2) $aw \in \mathcal{A}^d$ and there exists unique $x \in \mathcal{A}$ such that

$$\begin{aligned} x &= a(wx)^2, ((aw)^d)^*(aw)^2x = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} &= 0. \end{aligned}$$

Proof. (1) \Rightarrow (2) In view of Theorem 2.2, $aw \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that

$$\begin{aligned} x &= a(wx)^2, ((aw)^d)^*(aw)^2x = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} &= 0. \end{aligned}$$

Assume that there exists $y \in \mathcal{A}$ such that

$$\begin{aligned} y &= a(wy)^2, ((aw)^d)^*(aw)^2y = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - yw(aw)^{n+1}\|^{\frac{1}{n}} &= 0. \end{aligned}$$

Claim 1. $xw = yw$.

By hypothesis, we have

$$\begin{aligned} xw &= (aw)(xw)^2, ((aw)^d)^*(aw)^2(xw) = ((aw)^d)^*(aw), \\ \lim_{n \rightarrow \infty} \|(aw)^n - (xw)(aw)^{n+1}\|^{\frac{1}{n}} &= 0. \end{aligned}$$

Then xw is generalized group inverse of aw . Likewise, yw is generalized group inverse of aw . By virtue of ([2] Corollary 2.3), we have $xw = yw$, as desired.

Claim 2. $(aw)^2x = (aw)^2y$.

As in the proof of Theorem 2.2, $x = (aw)^{n-1}x(wx)^{n-1}$, and so $awx = (aw)^{n+1}x(wx)^{n-1}$.

$$\begin{aligned} & \|((aw)^2x)^*(aw)^2x - ((aw)^2x)^*a\| \\ \leq & \|((x(wx)^{n-1})^*)^*\| \|((aw)^{n+1})^*(aw)^2x - ((aw)^{n+1})^*a\| \\ \leq & \|((x(wx)^{n-1})^*)^*\| \|((aw)^{n+1} - (aw)^d(aw)^{n+2})^*(aw)^2x \\ & - ((aw)^{n+1} - (aw)^d(aw)^{n+2})^*a\| \\ + & \|((x(wx)^{n-1})^*)^*\| \|((aw)^{n+2})^*\| \|((aw)^d)^*(aw)^2x - ((aw)^d)^*a\| \\ = & \|((x(wx)^{n-1})^*)^*\| \|((aw)^{n+1} - (aw)^d(aw)^{n+2})^*\| \|((aw)^2x - a)\|. \end{aligned}$$

Therefore

$$\lim_{n \rightarrow \infty} \|((aw)^2x)^*(aw)^2x - ((aw)^2x)^*a\|^{\frac{1}{n}} = 0.$$

Hence, $[((aw)^2x)^*(aw)^2x]^* = ((aw)^2x)^*a$. Likewise, $[((aw)^2x)^*(aw)^2y]^* = ((aw)^2x)^*a$, $[((aw)^2y)^*(aw)^2x]^* = ((aw)^2x)^*a$ and $[((aw)^2y)^*(aw)^2y]^* = ((aw)^2x)^*a$. Let $z = (aw)^2x - (aw)^2y$. Then we check that

$$\begin{aligned} z^*z &= ((aw)^2x - (aw)^2y)^*((aw)^2x - (aw)^2y) \\ &= ((aw)^2x)^*(aw)^2x - ((aw)^2x)^*(aw)^2y \\ &\quad - ((aw)^2y)^*(aw)^2x + ((aw)^2y)^*(aw)^2y \\ &= ((aw)^2x)^*a - ((aw)^2x)^*a - ((aw)^2y)^*a^2 + ((aw)^2y)^*a^2 \\ &= 0. \end{aligned}$$

Since \mathcal{A} is a proper *-Banach algebra, we have $z = 0$; hence, $(aw)^2x = (aw)^2y$.

Claim 3. $x = y$.

As in the proof of Corollary 2.3, we have $x = (xw)(awx)$. Also we check that

$$\begin{aligned} & \|(xw)(awx) - (xw)^2(aw)^2x\| \\ = & \|(xw)[aw - (xw)(aw)^2]x\| \\ = & \|(xw)[aw - (xw)(aw)^2](aw)^{n-1}x(wx)^{n-1}\| \\ \leq & \|xw\| \|((aw)^n - (xw)(aw)^{n+1})\| \|x(wx)^{n-1}\|. \end{aligned}$$

This implies that

$$\lim_{n \rightarrow \infty} \|(xw)(awx) - (xw)^2(aw)^2x\|^{\frac{1}{n}} = 0.$$

Hence, $(xw)(awx) = (xw)^2(aw)^2x$. Thus, $x = (xw)^2(aw)^2x$. Likewise, $y = (yw)^2(aw)^2y$. By the preceding discussion, we have

$$xw = yw \text{ and } (aw)^2x = (aw)^2y.$$

Therefore $x = y$, as required.

(2) \Rightarrow (1) This is obvious by Theorem 2.1. \square

We denote x in Corollary 2.3 by a_w^{\circledast} , and call it the generalized w -group inverse of a . $\mathcal{A}_w^{\circledast}$ denotes the sets of all generalized w -group invertible elements in \mathcal{A} .

Corollary 2.5. Let $a \in \mathcal{A}_w^{\circledast}$. Then the following hold.

- (1) $a_w^{\circledast} = a_w^{\circledast}waww_a^{\circledast}$.
- (2) $a_w^{\circledast}w = (aw)^{\circledast}$.

Proof. (1) These are obvious by the proof of Theorem 2.4. \square

Let $\mathbb{C}^{n \times n}$ be the Banach algebra of all $n \times n$ complex matrices, with conjugate transpose $*$ as the involution. As is well known, $(\mathbb{C}^{n \times n})^{qnil} = (\mathbb{C}^{n \times n})^{nil}$. For a complex $A \in \mathbb{C}^{n \times n}$, the weak W -group inverse and generalized W -group inverse coincide with each other, i.e., $A_W^{\textcircled{w}} = A_W^{\textcircled{g}}$. We come now to provide a new characterization for the W -weak group inverse of a complex matrix.

Corollary 2.6. *Let $A, W \in \mathbb{C}^{n \times n}$. Then the following are equivalent:*

- (1) A has weak W -group inverse.
- (2) There exist $k \in \mathbb{N}$ and $X \in \mathbb{C}^{n \times n}$ such that $X = A(WX)^2$, $((AW)^D)^*(AW)^2X = ((AW)^D)^*A$, $(AW)^k = XW(AW)^{k+1}$.

Proof. (1) \Rightarrow (2) Since A has weak W -group inverse, there exist $k \in \mathbb{N}$ and $X \in \mathbb{C}^{n \times n}$ such that $X = A(WX)^2$, $((AW)^D)^*(AW)^2X = ((AW)^D)^*A$, $(AW)^k = XW(AW)^{k+1}$. Hence,

$$\begin{aligned} ((AW)^D)^*(AW)^2X &= [(AW)^k((AW)^D)^{k+1}]^*(AW)^2X \\ &= [((AW)^D)^{k+1}]^*((AW)^k)^*(AW)^2X \\ &= [((AW)^D)^{k+1}]^*((AW)^k)^*A \\ &= [(AW)^k((AW)^D)^{k+1}]^*A \\ &= ((AW)^D)^*A. \end{aligned}$$

(2) \Rightarrow (1) By hypothesis, there exist $k \in \mathbb{N}$ and $X \in \mathbb{C}^{n \times n}$ such that

$$\begin{aligned} (AW)^k &= XW(AW)^{k+1}, X = A(WX)^2, \\ ((AW)^D)^*(AW)^2X &= ((AW)^D)^*A. \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \|(AW)^n - XW(AW)^{n+1}\|^{\frac{1}{n}} = 0.$$

In view of Theorem 2.4, $A \in \mathbb{C}^{n \times n}$ has generalized W -group inverse. Therefore A has weak W -group inverse. \square

3. Elementary Properties

In this section we establish some elementary properties of generalized group inverses. We now derive

Theorem 3.1. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}_w^{\textcircled{g}}$ if and only if*

- (1) $a \in \mathcal{A}^{d,w}$;
- (2) There exists $x \in \mathcal{A}$ such that

$$\begin{aligned} x &= a(wx)^2, ((aw)^d)^*(aw)^2x = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - (aw)xw(aw)^n\|^{\frac{1}{n}} &= 0. \end{aligned}$$

In this case, $a_w^{\textcircled{g}} = wxw$.

Proof. \Rightarrow In view of Theorem 2.4, $a \in \mathcal{A}^{d,w}$ and there exists $x \in \mathcal{A}$ such that

$$\begin{aligned} x &= a(wx)^2, ((aw)^d)^*(aw)^2x = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|(aw)^{n-1} - xw(aw)^n\|^{\frac{1}{n}} &= 0. \end{aligned}$$

Obviously, we have

$$\begin{aligned} \|(aw)^n - (aw)xw(aw)^n\|^{\frac{1}{n}} &= \|aw[(aw)^{n-1} - xw(aw)^n]\|^{\frac{1}{n}} \\ &\leq \|aw\|^{\frac{1}{n}} \|(aw)^{n-1} - xw(aw)^n\|^{\frac{1}{n}}. \end{aligned}$$

Then $\lim_{n \rightarrow \infty} \|(aw)^n - (aw)xw(aw)^n\|^{\frac{1}{n}} = 0$, as desired.
 \Leftarrow By hypothesis, there exists $x \in \mathcal{A}$ such that

$$x = a(wx)^2, ((aw)^d)^*(aw)^2x = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - awxw(aw)^n\|^{\frac{1}{n}} = 0.$$

Let $a_1 = (aw)^2x$ and $a_2 = a - (aw)^2x$. Then we check that

$$\begin{aligned} \|a_2wa_1\| &= \|[a - (aw)^2x]w(aw)^2x\| \\ &= \|[aw - (aw)^2xw](aw)^2x\| \\ &= \|(aw)[aw - (aw)xw(aw)](aw)x\| \\ &= \|(aw)[aw - (aw)xw(aw)](aw)^{k-1}x^{k-1}\| \\ &= \|(aw)[(aw)^k - (aw)xw(aw)^k]x^{k-1}\|. \end{aligned}$$

Hence

$$\|a_2wa_1\|^{\frac{1}{k}} \leq \|aw\|^{\frac{1}{k}} \|(aw)^k - (aw)xw(aw)^k\|^{\frac{1}{k}} \|x\|^{1-\frac{1}{k}}.$$

Since $\lim_{k \rightarrow \infty} \|(aw)^k - (aw)xw(aw)^k\|^{\frac{1}{k}} = 0$, we deduce that

$$\lim_{k \rightarrow \infty} \|a_2wa_1\|^{\frac{1}{k}} = 0.$$

Thus, $a_2wa_1 = 0$.

As in the proof of Theorem 2.2, we check that $a_1^*a_2 = 0$.

Let $y = wxw$. Then we verify that

$$\begin{aligned} a_1wy &= (aw)^2xwxwx = awa(wx)^2wx = a(wx)^2 \\ &= x = xwxw(aw)^2x = xwxw(aw)^2x = ywa_1, \\ a_1wywa_1 &= (aw)^2xwxw(aw)^2x = awa(wx)^2wawawx \\ &= (awxw)(aw)^2x = (aw)^2x = a_1, \\ ywa_1wy &= a_1(wx)^2 = (aw)^2x(wx)^2 = awa(wx)^2wx \\ &= a(wx)^2 = y, \end{aligned}$$

Hence $(a_1)_w^\# = y$. Similarly to ([2] Theorem 2.2), $a_2 = a - (aw)^2x \in \mathcal{A}_w^{qnil}$. Therefore $x = a_w^\circledast$, as required. \square

Corollary 3.2. Let $A, W \in \mathbb{C}^{n \times n}$. Then the following are equivalent:

- (1) A has weak W -group inverse.
- (2) There exist $k \in \mathbb{N}$ and $X \in \mathbb{C}^{n \times n}$ such that $X = A(WX)^2$, $(AW)^k = AWXWA^k$, $((AW)^D)^*(AW)^2X = (AW)^D)^*A$.

Proof. This is clear by Theorem 3.3. \square

We are ready to prove:

Theorem 3.3. Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}_w^\circledast$ if and only if

- (1) $a \in \mathcal{A}^{d,w}$;
- (2) There exists $x \in \mathcal{A}$ such that

$$x\mathcal{A} = a^{d,w}\mathcal{A}, ((aw)^d)^*(aw)^2x = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

In this case, $a_w^\circledast = x$.

Proof. \implies In view of Theorem 2.4, $aw \in \mathcal{A}^d$ and there exists unique $x \in \mathcal{A}$ such that

$$x = a(wx)^2, ((aw)^d)^*(aw)^2x = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} = 0.$$

Hence, $a \in \mathcal{A}^{d,w}$ by 3.1. Obviously, $xw = (aw)^{\otimes}$. In light of ([2] Theorem 3.4),

$$x = a(wx)^2 = aw(aw)^{\otimes}x = (aw)^2(aw)^d(aw)^{\otimes}x \\ = [(aw)^d]^2(aw)^3(aw)^{\otimes}x = a^{d,w}[w(aw)^2(aw)^{\otimes}x].$$

One easily checks that

$$\|(aw)^d - xw(aw)(aw)^d\|^{\frac{1}{n}} \\ = \|[(aw)^n - xw(aw)^{n+1}][(aw)^d]^{n+1}\|^{\frac{1}{n}} \\ \leq \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} \|(aw)^d\|^{\frac{1}{n}} \|(aw)^d\|^{\frac{1}{n}} \\ = \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} \|(aw)^d\|^{\frac{1}{n}}.$$

We infer that

$$\lim_{n \rightarrow \infty} \|(aw)^d - xw(aw)(aw)^d\|^{\frac{1}{n}} = 0.$$

Hence, $(aw)^d = xw(aw)(aw)^d$. Therefore $x\mathcal{A} = a^{d,w}\mathcal{A}$, as required.

\Leftarrow We directly check that

$$\|awxw(aw)^d - (aw)^d\|^{\frac{1}{n}} \\ = \|awxw(aw)^{n+1}((aw)^d)^{n+2} - aw(aw)^n((aw)^d)^{n+2}\|^{\frac{1}{n}} \\ \leq \|aw\|^{\frac{1}{n}} \|(aw)^n - xw(aw)^{n+1}\|^{\frac{1}{n}} \|(aw)^d\|^{\frac{1}{n}} \|(aw)^d\|^{\frac{1}{n}}.$$

Accordingly,

$$\lim_{n \rightarrow \infty} \|awxw(aw)^d - (aw)^d\|^{\frac{1}{n}} = 0.$$

Therefore we have $awxw(aw)^d = (aw)^d$, and so $1 - awxw \in \ell(aw)^d$. Since $x\mathcal{A} = (aw)^d\mathcal{A}$, we see that $1 - awxw \in \ell(x)$. Therefore $x = awxwx = a(wx)^2$. In light of Theorem 2.2, $a \in \mathcal{A}_w^{\otimes}$. In this case, $a_w^{\otimes} = x$, as asserted. \square

Corollary 3.4. Let $A, W \in \mathbb{C}^{n \times n}$. Then the following are equivalent:

- (1) A has weak W -group inverse.
- (2) There exist $k \in \mathbb{N}$ and $X \in \mathbb{C}^{n \times n}$ such that

$$(AW)^k = XWA^{k+1}, \text{Im}(X) = \text{Im}(A^{D,W}), \\ ((AW)^D)^*(AW)^2X = (AW)^D)^*A.$$

Proof. This is obvious by Theorem 3.3. \square

We next investigate the polar-like property of generalized weighted group inverse.

Lemma 3.5. Let $a \in \mathcal{A}_w^{\otimes}$. Then

- (1) $a \in \mathcal{A}^{d,w}$;
- (2) There exists an idempotent $p \in \mathcal{A}$ such that

$$wa + p \in \mathcal{A}^{-1}, ((aw)^d)^*ap = 0 \text{ and } pwa = paw \in \mathcal{A}^{qnil}.$$

Proof. In view of Theorem 3.1, $a \in \mathcal{A}^{d,w}$. Since $a \in R_w^\otimes$, there exist $z, y \in \mathcal{A}$ such that

$$a = z + y, z^*y = ywz = 0, z \in \mathcal{A}_w^\#, y \in \mathcal{A}_w^{qnil}.$$

Set $x = z_w^\#$. Then we check that

$$\begin{aligned} awx &= (z + y)wz_w^\# = zwz_w^\#, \\ (wawx)^2 &= wz_w^\#zwz_w^\# = wz_w^\#z_w^\# = wawx, \end{aligned}$$

Let $p = 1 - wz_w^\#z_w^\#$. Then $p = 1 - wawx = p^2 \in \mathcal{A}$. Furthermore, $apw = a(1 - wz_w^\#z_w^\#)w = (z + y)(1 - wz_w^\#z_w^\#)w = yw + z(1 - wz_w^\#z_w^\#)w \in \mathcal{A}^{qnil}$, and so $pwa \in \mathcal{A}^{qnil}$ by Cline's formula (see ([7] Theorem 2.1)). Since $pwa(1 - p) = (1 - wz_w^\#z_w^\#)w(z + y)wz_w^\# = (1 - wz_w^\#z_w^\#)wz_w^\#z_w^\# = wz_w^\#z_w^\#z_w^\# - wz_w^\#z_w^\#z_w^\#z_w^\# = wz_w^\#z_w^\#z_w^\# - wz_w^\#z_w^\#z_w^\#z_w^\# = 0$, we get $pwa = pwap$.

By Theorem 2.4, $((aw)^d)^*(aw)^2a_w^\otimes = ((aw)^d)^*a$, and so $((aw)^d)^*awawz_w^\# = ((aw)^d)^*a$. This implies that $((aw)^d)^*a[1 - wz_w^\#z_w^\#] = ((aw)^d)^*a[1 - w(z + y)wz_w^\#] = ((aw)^d)^*a[1 - wawz_w^\#] = 0$, i.e., $((aw)^d)^*ap = 0$. Obviously, $wz + 1 - wz_w^\#z_w^\# = (z_w^\# + 1 - wz_w^\#z_w^\#)^{-1} \in \mathcal{A}^{-1}$. Since $y(wz_w^\# + 1 - wz_w^\#z_w^\#) = y$, we have $y(wz_w^\# + 1 - wz_w^\#z_w^\#)^{-1} = y \in \mathcal{A}_w^{qnil}$. It follows by ([7] Theorem 2.1) that $(wz_w^\# + 1 - wz_w^\#z_w^\#)^{-1}wy \in \mathcal{A}^{qnil}$. Hence $1 + (wz_w^\# + 1 - wz_w^\#z_w^\#)^{-1}wy \in \mathcal{A}^{-1}$. This implies that

$$\begin{aligned} wa + p &= wz + wy + 1 - wz_w^\#z_w^\# \\ &= (wz + 1 - wz_w^\#z_w^\#)[1 + (wz_w^\# + 1 - wz_w^\#z_w^\#)^{-1}wy] \\ &\in \mathcal{A}^{-1}. \end{aligned}$$

This completes the proof. \square

Lemma 3.6. Let $a \in \mathcal{A}$ and $w \in \mathcal{A}^{-1}$. Then $a \in \mathcal{A}_w^\#$ if and only if $aw, wa \in \mathcal{A}^\#$. In this case,

$$a_w^\# = (aw)^\#a(wa)^\#.$$

Proof. \implies This is clear by Lemma 2.1.

\impliedby Let $x = (aw)^\#a(wa)^\#$. Then we directly check that $x = a_w^\#$, as desired. \square

Theorem 3.7. Let $a \in \mathcal{A}$ and $w \in \mathcal{A}^{-1}$. Then $a \in \mathcal{A}_w^\otimes$ if and only if

- (1) $a \in \mathcal{A}^{d,w}$;
- (2) There exists an idempotent $p \in \mathcal{A}$ such that

$$a + w^{-1}p \in \mathcal{A}^{-1}, ((aw)^d)^*ap = 0 \text{ and } pwa = pwap \in \mathcal{A}^{qnil}.$$

Proof. \implies This is obvious by Lemma 3.5.

\impliedby By hypothesis, there exists an idempotent $p \in \mathcal{A}$ such that

$$wa + p \in \mathcal{A}^{-1}, ((aw)^d)^*ap = 0 \text{ and } pwa = pwap \in \mathcal{A}^{qnil}.$$

Set $x = a(1 - p)$ and $y = ap$. Then $a = x + y$. Since $pwa = pwap \in \mathcal{A}^{qnil}$, we have $yw = apw \in \mathcal{A}^{qnil}$ by ([7] Theorem 2.1). Hence, $y \in \mathcal{A}_w^{qnil}$. We also see that $pwa(1 - p) = 0$, and then $ywx = apwa(1 - p) = 0$. Clearly, $wx = wa(1 - p) = u(1 - p)$. Let $z = (1 - p)u^{-1}$. Then we check that $zwx = 1 - p$; hence, $z(wx)^2 = (1 - p)wx = (1 - p)wa(1 - p) = wa(1 - p) = wx$. Since $a \in \mathcal{A}^d, ap \in \mathcal{A}^d$ and $pwa(1 - p) = 0$, it follows by ([17] Lemma 2.2) that $wx = wa(1 - p) \in \mathcal{A}^d$. Therefore $wx = z(wx)^2 = z^n(wx)^{n+1} = z^n((wx)^n - (wx)^d(wx)^{n+1})wx + z^n(wx)^d(wx)^{n+2}$ and $(wx)^2(wx)^d = (z(wx)^2)wx(wx)^d = z^n(wx)^{n+2}(wx)^d$. Then

$$\begin{aligned} \|wx - (wx)^2(wx)^d\|_n^{\frac{1}{n}} &= \|z^n((wx)^n - (wx)^d(wx)^{n+1})x\|_n^{\frac{1}{n}} \\ &\leq \|z^n\|_n^{\frac{1}{n}} \|(wx)^n - (wx)^d(wx)^{n+1}\|_n^{\frac{1}{n}} \|wx\|_n^{\frac{1}{n}}. \end{aligned}$$

Accordingly,

$$\lim_{n \rightarrow \infty} \|wx - (wx)^2(wx)^d\|^{\frac{1}{n}} = 0.$$

Therefore we have $wx = (wx)^2(wx)^d$. By using Cline's formula (see ([7] Theorem 2.1)), we have

$$\begin{aligned} xw &= w^{-1}(wx)w = w^{-1}[(wx)^2(wx)^d]w \\ &= w^{-1}wxwxw[(xw)^d]^2xw \\ &= xwxw[(xw)^d]^2xw \\ &= (xw)^2(xw)^d. \end{aligned}$$

Hence, $x \in \mathcal{A}_w^\#$. Therefore

$$w^*x^*yw = [xw]^*apw = ((xw)^2)^*((xw)^d)^*apw = 0.$$

Since $w \in \mathcal{A}^{-1}$, we have $w^* \in \mathcal{A}^{-1}$. Thus, $x^*y = 0$. Then $a = x + y$ is a generalized w -group decomposition of a . In light of Theorem 2.2, $a \in \mathcal{A}_w^\oplus$. \square

Corollary 3.8. Let $A \in \mathbb{C}^{n \times n}$ and $W \in \mathbb{C}^{n \times n}$ be invertible, with conjugate transpose $*$ as the involution. Then the following are equivalent:

- (1) A has weak W -group inverse.
- (2) There exists an idempotent $E \in \mathbb{C}^{n \times n}$ such that $A + W^{-1}E$ is invertible, $((AW)^D)^*AE = 0$ and $EWA = EWAE$ is nilpotent.

Proof. As is well known, every complex has Drazin inverse. This completes the proof by Theorem 3.7. \square

Example 3.9. Let $\mathbb{C}^{2 \times 2}$ be the Banach algebra of all 2×2 complex matrices, with transpose $*$ as the involution.

Let $A = \begin{pmatrix} 1 & 0 \\ -i & 0 \end{pmatrix}$, $W = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \in \mathbb{C}^{2 \times 2}$. Then the equations

$$\begin{aligned} A(WX)^2 &= X, ((AW)^*(AW)^2X)^* = (AW)^*(AW)^2X, \\ \lim_{n \rightarrow \infty} \|((AW)^n - XW(AW)^{n+1})\|^{\frac{1}{n}} &= 0 \end{aligned}$$

has two solutions

$$X_1 = \begin{pmatrix} 1 & 0 \\ -i & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & -1 \\ 0 & i \end{pmatrix}.$$

Then the solution of the preceding equations is not unique. Choose an idempotent $E = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Then $A + W^{-1}E$ is invertible, $((AW)^D)^*AE = 0$, $EWA = EWAE$ is nilpotent. In this case, the involution $*$ is not proper.

4. Relations with Weighted g-Drazin Inverses

In this section we discuss the relations between generalized weighted group and weighted g-Drazin inverses. We now derive

Theorem 4.1. Let $a \in \mathcal{A}$, $w \in \mathcal{A}^{-1}$. Then $a \in \mathcal{A}_w^\oplus$ if and only if

- (1) $a \in \mathcal{A}^{d,w}$;
- (2) There exists an idempotent $q \in \mathcal{A}$ such that

$$a^{d,w}\mathcal{A} = q\mathcal{A} \text{ and } (aw)^*(aw)q = q^*(aw)^*aw.$$

In this case, $a_w^{\otimes} = a^{d,w}wqw^{-1}$.

Proof. \implies By virtue of Theorem 3.1, $a \in \mathcal{A}^{d,w}$. Let $q = awa_w^{\otimes}w$. Since $a_w^{\otimes} = a_w^{\otimes}wawaw_w^{\otimes}$. Then $q^2 = q$, i.e., $q \in \mathcal{A}$ is an idempotent. We check that

$$(aw)^*(aw)q = (aw)^*(aw)^2a_w^{\otimes}w.$$

In view of Corollary 2.3, $(aw)^*(aw)q = [(aw)^*(aw)q]^* = q^*(aw)^*(aw)$. As in the proof of Theorem 3.3, we have

$$a_w^{\otimes} = a^{d,w}[w(aw)^2(aw)^{\otimes}a_w^{\otimes}].$$

Then

$$q = awa_w^{\otimes}w = awa^{d,w}[w(aw)^2(aw)^{\otimes}a_w^{\otimes}]w.$$

Hence, $q\mathcal{A} \subseteq a^{d,w}\mathcal{A}$. In light of Theorem 3.3, $a_w^{\otimes}\mathcal{A} = a^{d,w}\mathcal{A}$. Write $a^{d,w} = a_w^{\otimes}z$ for some $z \in \mathcal{A}$. Then $a^{d,w} = a(wa_w^{\otimes})^2z = awa_w^{\otimes}waw(a_w^{\otimes})^2z = qaw(a_w^{\otimes})^2z$. Hence $a^{d,w}\mathcal{A} \subseteq q\mathcal{A}$. Therefore $q\mathcal{A} = a^{d,w}\mathcal{A}$, as required.

\longleftarrow By hypothesis, there exists an idempotent $q \in \mathcal{A}$ such that $a^{d,w}\mathcal{A} = q\mathcal{A}$ and $(aw)^*(aw)q = q^*(aw)^*aw$. Hence, $q\mathcal{A} = awa^{d,w}w\mathcal{A}$. Set $x = a^{d,w}wqw^{-1}$. Then $awxw = awa^{d,w}wq = q$. We verify that

$$\begin{aligned} ((aw)^*(aw)^2xw)^* &= ((aw)^*(aw)awxw)^* = ((aw)^*(aw)q)^* \\ &= q^*(aw)^*(aw) = (aw)^*(aw)q \\ &= ((aw)^*(aw))(awxw) = (aw)^*(aw)^2xw. \end{aligned}$$

Moreover, we have

$$a(wx)^2 = (awxw)x = qx = qa^{d,w}wqw^{-1} = a^{d,w}wqw^{-1} = x.$$

We verify that

$$\begin{aligned} &|| (aw)^n - xw(aw)^{n+1} || \\ &= || [(aw)^n - a^{d,w}wqa^{d,w}w(aw)^{n+2}] \\ &\quad - [xw((aw)^{n+1} - xwa^{d,w}w(aw)^{n+2})] || \\ &\leq || (aw)^n - (a^{d,w}w)^2(aw)^{n+2} || + || xw || || (aw)^{n+1} - a^{d,w}w(aw)^{n+2} || \\ &= || (aw)^n - (aw)^d(aw)^{n+1} || + || xw || || (aw)^{n+1} - a^{d,w}w(aw)^{n+2} || \\ &= (1 + || xw || || aw ||) || (aw)^n - (aw)^d(aw)^{n+1} ||. \end{aligned}$$

Since $aw - a^{d,w}w(aw)^2 \in \mathcal{A}^{qnil}$, we have

$$\lim_{n \rightarrow \infty} || (aw - a^{d,w}w(aw)^2)^n ||^{\frac{1}{n}} = 0.$$

Therefore

$$\lim_{n \rightarrow \infty} || (aw)^n - (aw)^d(aw)^{n+1} ||^{\frac{1}{n}} = 0.$$

In view of Corollary 2.3, $a \in \mathcal{A}_w^{\otimes}$. In this case, $a_w^{\otimes} = a^{d,w}wqw^{-1}$. \square

Corollary 4.2. Let $a \in \mathcal{A}$, $w \in \mathcal{A}^{-1}$. Then $a \in \mathcal{A}_w^{\otimes}$ if and only if

- (1) $a \in \mathcal{A}^{d,w}$;
- (2) There exists a unique idempotent $q \in \mathcal{A}$ such that

$$a^{d,w}\mathcal{A} = q\mathcal{A} \text{ and } (aw)^*(aw)q = q^*(aw)^*aw.$$

Proof. \implies By hypothesis, there exists an idempotent $q \in \mathcal{A}$ such that

$$a^{d,w}\mathcal{A} = q\mathcal{A} \text{ and } (aw)^*(aw)q = q^*(aw)^*aw.$$

In this case, $q = awa_w^{\otimes}w$. In view of Theorem 2.4, $q \in \mathcal{A}$ is unique.

\Leftarrow This is obvious by Theorem 4.1. \square

Corollary 4.3. *Let $a \in \mathcal{A}, w \in \mathcal{A}^{-1}$. Then $a \in \mathcal{A}_w^{\otimes}$ if and only if*

- (1) $a \in \mathcal{A}^{d,w}$;
- (2) There exists an idempotent $q \in \mathcal{A}$ such that

$$\ell(a^{d,w}) = \ell(q) \text{ and } (aw)^*(aw)q = q^*(aw)^*aw.$$

In this case, $a_w^{\otimes} = a^{d,w}wqw^{-1}$.

Proof. \implies In view of Theorem 4.1, $a \in \mathcal{A}^{d,w}$ and there exists an idempotent $q \in \mathcal{A}$ such that

$$a^{d,w}\mathcal{A} = q\mathcal{A} \text{ and } (aw)^*(aw)q = q^*(aw)^*aw.$$

We infer that $\ell(a^{d,w}) = \ell(q)$, as desired.

\Leftarrow By hypothesis, there exists an idempotent $q \in \mathcal{A}$ such that $\ell(a^{d,w}) = \ell(q)$ and $(aw)^*(aw)q = q^*(aw)^*aw$. Clearly, $1 - awa^{d,w}w \in \ell(a^{d,w}) \subseteq \ell(q)$; hence, $q = awa^{d,w}wq$. This implies that $q\mathcal{A} \subseteq a^{d,w}\mathcal{A}$. Also we have $1 - q \in \ell(q) \subseteq \ell(a^{d,w})$, and then $a^{d,w} = qa^{d,w}$. We infer that $a^{d,w}\mathcal{A} \subseteq q\mathcal{A}$. Therefore $a^{d,w}\mathcal{A} = q\mathcal{A}$. In light of Theorem 4.1, $a \in \mathcal{A}_w^{\otimes}$. In this case, $a_w^{\otimes} = a^{d,w}wqw^{-1}$. \square

We are ready to prove:

Theorem 4.4. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}_w^{\otimes}$ if and only if*

- (1) $a \in \mathcal{A}^{d,w}$;
- (2) There exists some $x \in \mathcal{A}$ such that $(a^{d,w})^*a^{d,w}x = (a^{d,w})^*a$.

In this case, $a_w^{\otimes} = ((aw)^d)^4ax$.

Proof. \implies By virtue of Theorem 2.4, $a \in \mathcal{A}^{d,w}$ and there exists $z \in \mathcal{A}$ such that

$$z = a(wz)^2, ((aw)^d)^*(aw)^2z = ((aw)^d)^*a, \\ \lim_{n \rightarrow \infty} \|((aw)^n - zw(aw)^{n+1})\|^{\frac{1}{n}} = 0.$$

Here, $z = a_w^{\otimes}$. Let $y = (aw)^3z$. Obviously, $z = (aw)z(wz) = (aw)^2z(wz)^2 = (aw)^{k-2}z(wz)^{k-2}$. Then we verify that

$$\begin{aligned} & \|((aw)^d)^*a - ((aw)^d)^*(aw)^d y\| \\ &= \|((aw)^d)^*a - ((aw)^d)^*(aw)^d(aw)^3z\| \\ &= \|((aw)^d)^*a - ((aw)^d)^*(aw)^d(aw)^3[(aw)^{k-2}z(wz)^{k-2}]\| \\ &= \|((aw)^d)^*(aw)^2z - ((aw)^d)^*(aw)^d(aw)^{k+1}(zw)^{k-2}z\| \\ &= \|((aw)^d)^*(aw)^2[(aw)^{k-2}z(wz)^{k-2}] \\ &\quad - ((aw)^d)^*(aw)^d(aw)^{k+1}(zw)^{k-2}z\| \\ &= \|((aw)^d)^*(aw)^k(zw)^{k-2}z - ((aw)^d)^*(aw)^d(aw)^{k+1}(zw)^{k-2}z\| \\ &\leq \|((aw)^d)^* \| \|((aw)^k - (aw)^d(aw)^{k+1})\| \| (zw)^{k-2}z \|. \end{aligned}$$

Since $\lim_{k \rightarrow \infty} \|((aw)^k - (aw)^d(aw)^{k+1})\|^{\frac{1}{k}} = 0$, we derive that

$$\lim_{k \rightarrow \infty} \|((aw)^d)^*a - ((aw)^d)^*(aw)^d y\|^{\frac{1}{k}} = 0.$$

Hence $((aw)^d)^*(aw)^d y = ((aw)^d)^* a$.

Set $x = waw(aw)^d y$. Since $a^{d,w} = [(aw)^d]^2 a$, we see that

$$\begin{aligned} (a^{d,w})^* a^{d,w} x &= ([(aw)^d]^2 a)^* [(aw)^d]^2 ax \\ &= a^* ((aw)^d)^* ((aw)^d)^* [(aw)^d]^2 ax \\ &= a^* ((aw)^d)^* ((aw)^d)^* [(aw)^d]^2 a (waw(aw)^d y) \\ &= a^* ((aw)^d)^* [((aw)^d)^* (aw)^d y] \\ &= a^* ((aw)^d)^* ((aw)^d)^* a \\ &= ([(aw)^d]^2 a)^* a \\ &= (a^{d,w})^* a, \end{aligned}$$

as required.

\Leftarrow By hypothesis, $(a^{d,w})^* a^{d,w} x = (a^{d,w})^* a$ for some $x \in \mathcal{A}$. Set $z = (aw)^d ax$. Then $((aw)^d)^*(aw)^d z = ((aw)^d)^* a$. We verify that

$$\begin{aligned} [aw(aw)^d]^* aw(aw)^d &= (aw)^* [((aw)^d)^* a] w(aw)^d \\ &= (aw)^* [((aw)^d)^* (aw)^d z] w(aw)^d \\ &= [aw(aw)^d]^* (aw)^d zw(aw)^d. \end{aligned}$$

Since the involution $*$ is proper, we get $aw(aw)^d = (aw)^d zw(aw)^d$. Let $y = ((aw)^d)^3 z$. Then we verify that

$$\begin{aligned} a(aw)^2 &= aw((aw)^d)^3 zw((aw)^d)^3 z \\ &= ((aw)^d)^2 zw((aw)^d)^3 z \\ &= (aw)^d [(aw)^d zw(aw)^d] ((aw)^d)^2 z \\ &= (aw)^d aw(aw)^d ((aw)^d)^2 z \\ &= ((aw)^d)^3 z = y; \\ ((aw)^d)^* (aw)^2 y &= ((aw)^d)^* (aw)^2 ((aw)^d)^3 z \\ &= ((aw)^d)^* (aw)^d z \\ &= ((aw)^d)^* a. \end{aligned}$$

Moreover, we see that

$$\begin{aligned} &|| (aw)^n - yw(aw)^{n+1} || \\ &= || [(aw)^n - (aw)^d (aw)^{n+1}] + [(aw)^d (aw)^{n+1} - yw(aw)^{n+1}] || \\ &\leq || (aw)^n - (aw)^d (aw)^{n+1} || + || (aw)^d (aw)^{n+1} - ((aw)^d)^3 zw(aw)^{n+1} || \\ &\leq || (aw)^n - (aw)^d (aw)^{n+1} || + || ((aw)^d)^3 zw [1 - (aw)^d (aw)] (aw)^{n+1} || \\ &+ || (aw)^d (aw)^{n+1} - ((aw)^d)^3 zw(aw)^d (aw)^{n+2} || \\ &\leq || (aw)^n - (aw)^d (aw)^{n+1} || + || (aw)^d)^3 zw || \\ &+ || (aw)^n - (aw)^d (aw)^{n+1} || || (aw)^{n+1} || \\ &+ || (aw)^d (aw)^{n+1} - (aw)^d)^2 [(aw)^d zw(aw)^d] (aw)^{n+2} || \\ &\leq || (aw)^n - (aw)^d (aw)^{n+1} || + || (aw)^d)^3 zw || || (aw)^n - (aw)^d (aw)^{n+1} || \\ &+ || (aw)^{n+1} || + || (aw)^d (aw)^{n+1} - ((aw)^d)^2 aw(aw)^d (aw)^{n+2} || \\ &= [1 + || (aw)^d)^3 zw || || (aw)^{n+1} ||] || (aw)^n - (aw)^d (aw)^{n+1} ||. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} || (aw)^n - (aw)^d (aw)^{n+1} ||^{\frac{1}{n}} = 0$, we deduce that

$$\lim_{n \rightarrow \infty} || (aw)^n - yw(aw)^{n+1} ||^{\frac{1}{n}} = 0.$$

In light of Theorem 2.4, $a \in \mathcal{A}_w^{\otimes}$. In this case, $a_w^{\otimes} = y = ((aw)^d)^3 z = ((aw)^d)^3 (aw)^d ax = ((aw)^d)^4 ax$. This completes the proof. \square

Recall that $a \in R$ has weak w -group inverse provide that there exist $x \in R$ and $n \in \mathbb{N}$ such that

$$x = a(wx)^2, ((aw)^n)^*(aw)^2x = ((aw)^n)^*a, (aw)^n = xw(aw)^{n+1}.$$

If x exists, it is unique, and we denote it by a_w^{w} .

Lemma 4.5. Let $a \in \mathcal{A}, w \in \mathcal{A}^{-1}$. Then $a \in R_w^{\text{w}}$ if and only if $a \in R^{D,w} \cap R_w^{\text{g}}$. In this case, $a_w^{\text{w}} = a_w^{\text{g}}$.

Proof. \implies Obviously, $a \in R_w^{\text{g}}$. We easily check that $aw \in R_w^{\text{w}}$. Then $a \in R^{D,w}$ by ([19] Proposition 3.4).

\impliedby Let $x = a_w^{\text{g}}$. Then there exist $x \in R$ and $m \in \mathbb{N}$ such that

$$x = a(wx)^2, [(aw)^*(aw)^2xw]^* = (aw)^*(aw)^2xw, \\ \lim_{n \rightarrow \infty} \| |(aw)^n - xw(aw)^{n+1} | \|^{1/n} = 0.$$

Since $a \in R^{D,w}$, we can find some $y \in R$ such that

$$yw(aw)^{k+1} = (aw)^k, ywaww = y, awy = ywa$$

for some $k \in \mathbb{N}$. Set $z = (aw)^ky(wx)^k$. Then we verify that

$$awz = ((aw)^{k+1}yw(xw)^{k-1}x = (aw)^k(xw)^{k-1}xw \\ = (aw)^k(xw)^{k-1}x = (aw)^2xwx.$$

Claim 1. $z = a(wz)^2$.

$$a(wz)^2 = aw(aw)^ky(wx)^kw(aw)^ky(wx)^k \\ = (aw)^{k+1}yw(xw)^k(aw)^ky(wx)^k \\ = (aw)^k(xw)^k(aw)^ky(wx)^k = (aw)xw(aw)^ky(wx)^k \\ = (aw)xw(aw)^k(yw)(xw)^{k-1}x = (aw)^k(yw)(xw)^{k-1}x \\ = (aw)^ky(wx)^k = z.$$

Claim 2. $[(aw)^*(aw)^2zw]^* = (aw)^*(aw)^2zw$.

$$(aw)^*(aw)^2zw = (aw)^*(aw)(awz)w = (aw)^*(aw)(aw)^2xwxw \\ = (aw)^*(aw)^2a(wx)^2w = (aw)^*(aw)^2xw;$$

hence, $[(aw)^*(aw)^2zw]^* = (aw)^*(aw)^2zw$.

Claim 3. $(aw)^k = (zw)(aw)^{k+1}$.

Since $(aw)(yw) = (yw)(aw)$, we see that

$$(zw)(aw)^{k+n+1} = [(aw)^ky(wx)^kw](aw)^{k+n+1} \\ = (aw)^k(yw)(xw)^k(aw)^{k+n+1} \\ = (yw)[(aw)^k(xw)^k](aw)^{k+n+1} \\ = (yw)[(aw)(xw)](aw)^{k+n+1} \\ = (aw)(yw)(xw)(aw)^{k+n+1} \\ = (aw)(yw)[(aw)^k(xw)^{k+1}](aw)^{k+n+1} \\ = [(yw)(aw)^{k+1}](xw)^{k+1}(aw)^{k+n+1} \\ = (aw)^k(xw)^{k+1}(aw)^{k+n+1} \\ = (xw)(aw)^{k+n+1}.$$

Hence, we deduce that

$$(aw)^k - (zw)(aw)^{k+1} = (aw)^{k+n}(yw)^n - (zw)(aw)^{k+n+1}(yw)^n \\ = [(aw)^{k+n} - xw(aw)^{k+n+1}](yw)^n.$$

Then

$$\|(aw)^k - (zw)(aw)^{k+1}\| \leq \|(aw)^{k+n} - xw(aw)^{k+n+1}\| \|y\|^n.$$

Since $\lim_{n \rightarrow \infty} \|(aw)^{k+n} - xw(aw)^{k+n+1}\|^{\frac{1}{n}} = 0$, we see that

$$\lim_{n \rightarrow \infty} \|(aw)^k - (zw)(aw)^{k+1}\|^{\frac{1}{n}} = 0.$$

Hence, $(aw)^k = (zw)(aw)^{k+1}$. Therefore $a \in R_w^{\circledast}$. \square

Theorem 4.6. Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}_w^{\circledast}$ if and only if

- (1) $a \in \mathcal{A}^{D,w}$;
- (2) There exists some $x \in \mathcal{A}$ such that $(a^{D,w})^* a^{D,w} x = (a^{D,w})^* a$.

In this case, $a_w^{\circledast} = ((aw)^D)^4 ax$.

Proof. This is an immediate consequence of Theorem 4.4 and Lemma 4.5. \square

Corollary 4.7. Let $A, W \in \mathbb{C}^{n \times n}$, with conjugate transpose $*$ as the involution. Then the following are equivalent:

- (1) A has weak W -group inverse.
- (2) There exists some $X \in \mathbb{C}^{n \times n}$ such that $(A^{D,W})^* A^{D,W} X = (A^{D,W})^* A$.

In this case, $A_W^{\circledast} = ((AW)^D)^4 AX$.

Proof. This is an immediate consequence of Theorem 4.6. \square

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