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Article

Rainbow Step Barrier Options

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Abstract: This article provides exact analytical formulae for various kinds of rainbow step barrier options. These are highly flexible and sophisticated multi-asset barrier options based on the following principle : the option life is divided into several time intervals on which different barriers are monitored w.r.t. different underlying assets. From a mathematical point of view, new formulae are provided for the first passage time of a multidimensional geometric Brownian motion to a boundary defined as a step function. The article shows how to implement the obtained valuation formulae in a simple and very efficient manner. Numerical results highlight a strong sensitivity of rainbow step barrier options to the correlations between the underlying assets.

Keywords: rainbow step barrier option; rainbow option; step barrier option; barrier option; multiasset option; multiasset barrier option; first passage time; boundary crossing probability; multidimensional Brownian motion

Introduction

Barrier options are characterized by the introduction in the option contract of a parameter called the ‘barrier’, which, in the most standard form, is a predefined reference value of the underlying asset S that may be located above the spot value of the underlying (upward barrier or ‘up-barrier’) or below it (downward barrier or ‘down-barrier’). The barrier may be of ‘knock-out’ type, i.e. the option expires worthless if the barrier is hit by S at any time during the option life (in which case the barrier is called ‘continuous’) or at one or a few predefined times (in which case the barrier is called ‘discrete’). The first passage time of an underlying to a knock-out barrier before the option expiry triggers the ‘deactivation’ of the option. Alternatively, the barrier may be of ‘knock-in’ type, i.e. the option expires worthless unless the barrier has been hit at least once before expiry, an event called ‘activation’.

Barrier options are the oldest and the most widely traded non-vanilla options. They are embedded in a lot of popular structured derivatives in stock and interest rate markets (see, e.g., Bouzoubaa and Osseiran, 2010). They are also extensively used as analytical tools in financial modeling, for instance in the so-called “structural models” of default risk (see, e.g., Bielecki and Rutkowski 2004) or in the valuation of investments (theory of ‘real options’). Since their first appearance in the financial markets during the 1970’s, there have been a huge number of variations in their original payoff, leading to an extraordinary variety of non-standard barrier options. Among the most well-known of them are the step barrier options, which divide the option lifetime into several time intervals on which the barrier takes on different values. In its standard form, a step barrier option features a piecewise constant barrier, i.e. a barrier defined as a step function. This allows to modulate the barrier level during the option life, thus offering increased flexibility and enhanced risk management capabilities, relative to a traditional barrier option. For instance, up-and-out barrier levels can be raised and down-and-out barrier levels can be lowered during time intervals when more protection is required, thus reducing the risk of deactivation. Likewise, up-and-in barrier levels can be lowered and down-and-in barrier levels can be raised in time intervals during which the implicit volatility of the underlying rises, thus increasing the chances of activation.

Exact analytical valuation of step barrier functions was first achieved by Guillaume (2001) when the barrier is one-sided (i.e. either upward or downward) on each time interval and by Guillaume (2010) when the barrier is two-sided (i.e. both upward and downward, also known as a ‘double

barrier') on each time interval. More mathematical details on how closed form valuation can be achieved, as well as exact results for more general, deterministic, step barriers are provided in Guillaume (2015; 2016), while stochastic step barriers are handled by Guillaume (2022). In the last couple of years, there has been a renewed interest in step barrier options, as they stand out as an essential component of innovative forms of investment such as autocallable structured products and other equity-linked products. This has led to a new series of academic contributions such as Lee et al. (2019 a; 2019 b), Lee et al. (2021), Lee et al. (2022). Unlike previously cited references, these articles do not provide explicit formulae, except for a few simple and already known cases, nor do they handle outside step barriers as in Guillaume (2001), alternatively upward and downward steps as in Guillaume (2015) and exponentially moving step barriers as in Guillaume (2016). These recent contributions ignore previous results given in references they do not cite, such as Guillaume (2001), that actually solve the problems they discuss. They also claim to be able to value analytically a step barrier option with an arbitrary number of steps, but without explaining how they intend to solve the difficult problem known in numerical integration as the 'curse of dimensionality', nor even beginning to discuss the numerical implementation of their approach, which constitutes the main issue, though.

There are still a number of unsolved problems related to the valuation of step barrier options. In particular, multi-asset step barrier options are barely touched upon in the existing literature, apart from an isolated formula for an 'outside' step barrier option given in Guillaume (2001), i.e. a contract that takes one underlying asset w.r.t. which barrier crossing is monitored and another underlying asset w.r.t. which the moneyness of the option is measured at expiry (for more background on outside barrier options in general, see Heynen and Kat, 1994). Yet, multi-asset contracts with step barriers are actively traded in today's financial markets, as they allow investors to benefit from the advantages of diversification in terms of risk control and expansion of investment opportunities. In particular, the so-called 'rainbow' step barrier option is characterized by the property that, on each time interval, the barrier is monitored w.r.t. a different underlying asset. Metaphorically, every asset represents a different colour, hence the origin of the term 'rainbow' as the number of colours increases. A contract featuring a number $n \in \mathbb{N}$ of underlying assets associated with n time intervals $[t_0 = 0, t_1], \dots, [t_{n-1}, t_n]$, on which n steps of a piecewise constant barrier are monitored, is called an n - colour rainbow barrier option. Each time interval is matched with a specific step of the barrier and a specific underlying asset. In the standard form of the contract, it is the n - th asset, associated with the n - th last step of the barrier, that is used to determine the moneyness of the contract at expiry. Rainbow step barrier options are typically priced by Monte Carlo simulation, even in a standard Black-Scholes model, because of the difficulties of the entailed analytical calculations, and also because the dimension of the valuation problem quickly increases with the number of 'colours', leading to non-trivial issues of numerical evaluation of high-dimensional integrals. Due to these obstacles, the present article is restricted to two-colour rainbow step barrier options. Closed form valuation is achieved not only for standard two-colour contracts, but also for two-colour outside step barrier options involving a third correlated asset at expiry, and for two-colour contracts featuring a two-sided barrier (double barrier) on each time interval. Numerical evaluation of the obtained analytical solutions is dealt with, so that the valuation formulae derived in this paper can be immediately implemented and yield extremely accurate results in a few tenths of one second. Numerical results are provided, which reveal a strong and stable dependency of rainbow step barrier options on the correlations between the underlying assets, as well as the importance of the volatility of the asset used to measure moneyness when dealing with rainbow outside step barrier options. These findings suggest clear ideas of application to both traders and investors, whether it be for a hedging or a speculative purpose. They highlight the specificity of rainbow step barrier options as instruments highly sensitive to correlation, in contrast to standard step barrier options which are sensitive to volatility but, by construction, cannot be sensitive to correlation.

This article is organized as follows : Section 1 derives the main analytical results at the core of the valuation of two-colour step barrier options and then shows how to price options and discusses

numerical results; Section 2 and Section 3 do the same for two-colour outside step barrier options, and two-colour step double barrier options, respectively.

1. Standard two-colour step barrier option valuation

This section deals with the valuation of standard two-colour step barrier options, whether the steps of the barrier are on the same side in each time interval (both upward or both downward) or not (one upward, the other downward).

1.1. Probability distributions involving two successive upward or downward barriers

Let S_1 and S_2 be two GBMs (Geometric Brownian Motions) modeling two asset prices, whose differentials, under a given probability measure P , are given by :

$$dS_1(t) = v_1 S_1(t) dt + \sigma_1 S_1(t) dB_1(t) \quad (1)$$

$$dS_2(t) = v_2 S_2(t) dt + \sigma_2 S_2(t) dB_2(t) \quad (2)$$

where $v_1, v_2 \in \mathbb{R}$, $\sigma_1, \sigma_2 \in \mathbb{R}_+$, and B_1 and B_2 are two standard Brownian motions whose correlation coefficient is denoted by $\rho_{1,2}$

The measure P is characterized by the pair (v_1, v_2) or, equivalently, by the pair :

$$(\mu_1 = v_1 - \sigma_1^2 / 2, \mu_2 = v_2 - \sigma_2^2 / 2) \quad (3)$$

if we refer to the log-return processes $X_i(t) = \ln(S_i(t) / S_i(0))$, $i \in \{1, 2\}$, whose differentials under P are given by :

$$dX_i(t) = \mu_i dt + \sigma_i dB_i(t) \quad (4)$$

Let H_1, H_2, K_1, K_2, K_3 be positive real numbers. The numbers H_1, H_2 are the values of two knock-out continuous barriers. H_1 is monitored w.r.t. S_1 on the time interval $[t_0 = 0, t_1]$, while H_2 is monitored w.r.t. S_2 on the time interval $[t_1, t_2]$. The numbers K_1, K_2, K_3 are the values of three discrete knock-out barriers; K_1 is monitored w.r.t. S_1 at time t_1 , while K_2 and K_3 are monitored w.r.t. S_2 at times t_2 and t_3 , respectively.

Our first objective is to find the value of the joint cumulative distribution function $P_{[RUU]}(\mu_1, \mu_2)$ defined by :

$$P_{[RUU]}(\mu_1, \mu_2) \triangleq P \left(\sup_{0 \leq t \leq t_1} S_1(t) \leq H_1, S_1(t_1) \leq K_1, S_2(t_1) \leq K_2, \sup_{t_1 \leq t \leq t_2} S_2(t) \leq H_2, S_2(t_2) \leq K_3 \right) \quad (5)$$

where the acronym ' $[RUU]$ ' stands for 'Rainbow Up and Up'

Proposition 1

The exact value of $P_{[RUU]}(\mu_1, \mu_2)$ is given by :

$$P_{[RUU]}(\mu_1, \mu_2) = N_3 \left[\frac{\min(k_1, h_1) - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{\min(k_2, h_2) - \mu_2 t_1}{\sigma_2 \sqrt{t_1}}, \frac{\min(k_3, h_2) - \mu_2 t_2}{\sigma_2 \sqrt{t_2}}; \theta_{1,2}, \theta_{1,3}, \theta_{2,3} \right] \quad (6)$$

$$- \exp \left(\frac{2\mu_1 h_1}{\sigma_1^2} \right) \quad (7)$$

$$\times N_3 \left[\frac{\min(k_1, h_1) - 2h_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{\min(k_2, h_2) - \mu_2 t_1}{\sigma_2 \sqrt{t_1}} - \frac{2\theta_{1.2} h_1}{\sigma_1 \sqrt{t_1}}, \frac{\min(k_3, h_2) - \mu_2 t_2}{\sigma_2 \sqrt{t_2}} - \frac{2\theta_{1.2} h_1}{\sigma_1 \sqrt{t_2}}; \right. \\ \left. \theta_{1.2}, \theta_{1.3}, \theta_{2.3} \right] - \exp \left(\frac{2\mu_2 h_2}{\sigma_2^2} \right) \quad (8)$$

$$\times N_3 \left[\frac{\min(k_1, h_1) - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} + \frac{2\theta_{1.2} \mu_2 \sqrt{t_1}}{\sigma_2}, \frac{\min(k_2, h_2) + \mu_2 t_1}{\sigma_2 \sqrt{t_1}}, \frac{\min(k_3, h_2) - 2h_2 - \mu_2 t_2}{\sigma_2 \sqrt{t_2}}; \right. \\ \left. \theta_{1.2}, -\theta_{1.3}, -\theta_{2.3} \right] + \exp \left(\left(\frac{2\mu_1}{\sigma_1^2} - \frac{4\mu_2 \theta_{1.2}}{\sigma_1 \sigma_2} \right) h_1 + \frac{2\mu_2 h_2}{\sigma_2^2} \right) \quad (9)$$

$$\times N_3 \left[\frac{\min(k_1, h_1) - 2h_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} + \frac{2\theta_{1.2} \mu_2 \sqrt{t_1}}{\sigma_2}, \frac{\min(k_2, h_2) + \mu_2 t_1}{\sigma_2 \sqrt{t_1}} - \frac{2\theta_{1.2} h_1}{\sigma_1 \sqrt{t_1}}, \right. \\ \left. \frac{\min(k_3, h_2) - 2h_2 - \mu_2 t_2}{\sigma_2 \sqrt{t_2}} + \frac{2\theta_{1.2} h_1}{\sigma_1 \sqrt{t_2}}; \theta_{1.2}, -\theta_{1.3}, -\theta_{2.3} \right]$$

where the μ_i 's are given by (3) and :

- $N_3[b_1, b_2, b_3; c_1, c_2, c_3]$ is the trivariate standard normal cumulative distribution function with correlation coefficients c_1, c_2, c_3

$$- h_1 = \ln \left(\frac{H_1}{S_1(0)} \right), h_2 = \ln \left(\frac{H_2}{S_2(0)} \right), k_1 = \ln \left(\frac{K_1}{S_1(0)} \right), k_2 = \ln \left(\frac{K_2}{S_2(0)} \right), k_3 = \ln \left(\frac{K_3}{S_2(0)} \right) \quad (10)$$

$$- \theta_{1.2} = \rho_{1.2}, \theta_{1.3} = \sqrt{\frac{t_1}{t_2}} \rho_{1.2}, \theta_{2.3} = \sqrt{\frac{t_1}{t_2}} \quad (11)$$

Corollary 1 : It suffices to multiply by (-1) all the first three arguments of each $N_3[., ., ., ., .]$ function and to substitute each $\min$ operator by a $\max$ operator in Proposition 1 to obtain an exact formula for $P_{[RDD]}(\mu_1, \mu_2)$ defined as :

$$P_{[RDD]}(\mu_1, \mu_2) \triangleq P \left(\inf_{0 \leq t \leq t_1} S_1(t) \geq H_1, S_1(t_1) \geq K_1, S_2(t_1) \geq K_2, \inf_{t_1 \leq t \leq t_2} S_2(t) \geq H_2, S_2(t_2) \geq K_3 \right) \quad (12)$$

where the acronym '[RDD]' stands for 'Rainbow Down and Down'

Corollary 2 : (9) gives the value of the corresponding up-and-in probability denoted by $P_{[RUU]}^{(i)}(\mu_1, \mu_2)$ and defined by :

$$P_{[RUU]}^{(i)}(\mu_1, \mu_2) \triangleq P \left(\sup_{0 \leq t \leq t_1} S_1(t) \geq H_1, S_1(t_1) \leq K_1, S_2(t_1) \leq K_2, \sup_{t_1 \leq t \leq t_2} S_2(t) \geq H_2, S_2(t_2) \leq K_3 \right) \quad (13)$$

Corollary 3 : It suffices to substitute each argument $\theta_{2,3}$ in each $N_3[\dots, \dots]$ function of Proposition 1 by $\theta_{2,3} = \sqrt{\frac{t_2}{t_3}}$, $\forall t_3 \geq t_2$, to obtain an exact formula for the early-ending variant $P_{[\text{EERUU}]}(\mu_1, \mu_2)$ defined by :

$$P_{[\text{EERUU}]}(\mu_1, \mu_2) \triangleq P\left(\sup_{0 \leq t \leq t_1} S_1(t) \leq H_1, S_1(t_1) \leq K_1, S_2(t_1) \leq K_2, \sup_{t_1 \leq t \leq t_2} S_2(t) \leq H_2, S_2(t_2) \leq K_3\right) \quad (14)$$

Corollary 4 : Let $\hat{p}$ be the value of $P_{[\text{RUU}]}(\mu_1, \mu_2)$ when the size of K_3 becomes 'very large', i.e. large enough for the probability $P(S_2(t_2) \leq K_3)$ to tend to zero; then, the difference $\hat{p} - P_{[\text{RUU}]}(\mu_1, \mu_2)$ provides the value of the following minor variant :

$$\hat{p} - P_{[\text{RUU}]}(\mu_1, \mu_2) = P\left(\sup_{0 \leq t \leq t_1} S_1(t) \leq H_1, S_1(t_1) \leq K_1, S_2(t_1) \leq K_2, \sup_{t_1 \leq t \leq t_2} S_2(t) \leq H_2, S_2(t_2) > K_3\right) \quad (15)$$

End of Proposition 1.

Proof of Proposition 1

Since the log function is strictly increasing, we have :

$$P_{[\text{RUU}]}(\mu_1, \mu_2) = P\left(\sup_{0 \leq t \leq t_1} X_1(t) \leq h_1, X_1(t_1) \leq k_1, X_2(t_1) \leq k_2, \sup_{t_1 \leq t \leq t_2} X_2(t) \leq h_2, X_2(t_2) \leq k_3\right) \quad (16)$$

Next, it can be noticed that, despite the non-zero correlation between $X_1(t)$ and $X_2(t)$, the law of $\sup_{0 \leq t \leq t_1} X_1(t)$ conditional on $X_1(t_1)$ and $X_2(t_1)$ is equal to the law of $\sup_{0 \leq t \leq t_1} X_1(t)$ conditional on $X_1(t_1)$

Indeed, denoting the density function as $f(\cdot)$ and making use of the Markov property of $X_2(t)$, we have :

$$f\left(\sup_{0 \leq t \leq t_1} X_1(t) \mid X_1(t_1), X_2(t_1)\right) = \frac{f\left(X_2(t_1) \mid \sup_{0 \leq t \leq t_1} X_1(t), X_1(t_1)\right) f\left(\sup_{0 \leq t \leq t_1} X_1(t), X_1(t_1)\right)}{f(X_1(t_1), X_2(t_1))} \quad (17)$$

$$= \frac{f(X_2(t_1) \mid X_1(t_1)) f\left(\sup_{0 \leq t \leq t_1} X_1(t), X_1(t_1)\right)}{f(X_1(t_1), X_2(t_1))} = \frac{f(X_1(t_1), X_2(t_1))}{f(X_1(t_1))} \frac{f\left(\sup_{0 \leq t \leq t_1} X_1(t), X_1(t_1)\right)}{f(X_1(t_1), X_2(t_1))} \quad (18)$$

$$= \frac{f\left(\sup_{0 \leq t \leq t_1} X_1(t), X_1(t_1)\right)}{f(X_1(t_1))} = f\left(\sup_{0 \leq t \leq t_1} X_1(t) \mid X_1(t_1)\right) \quad (19)$$

A translation from the time interval $[t_0 = 0, t_1]$ to the time interval $[t_1, t_2]$, through the substitution of $X_1(0)$ with $X_2(t_1)$, of $X_1(t_1)$ with $X_2(t_2)$ and of $X_2(t_1)$ with $X_1(t_2)$, shows similarly that the law of $\sup_{t_1 \leq t \leq t_2} X_2(t)$ conditional on $X_2(t_1)$, $X_2(t_2)$ and $X_1(t_2)$ is equal to the law of $\sup_{t_1 \leq t \leq t_2} X_2(t)$ conditional on $X_2(t_1)$ and $X_2(t_2)$

Thus, by conditioning w.r.t. the absolutely continuous random variables $X_1(t_1)$, $X_2(t_1)$ and $X_2(t_2)$, we can express the problem as the following integral :

$$P_{[RUU]}(\mu_1, \mu_2) \quad (20)$$

$$= \int_{-\infty}^{\min(k_1, h_1)} \int_{-\infty}^{\min(k_2, h_2)} \int_{-\infty}^{\min(k_3, h_3)} \varphi_1(x_1, x_2, x_3) \varphi_2(x_1) \varphi_3(x_2, x_3) dx_3 dx_2 dx_1$$

where :

$$\varphi_1(x_1, x_2, x_3) = P(X_1(t_1) \in dx_1, X_2(t_1) \in dx_2, X_2(t_2) \in dx_3) dx_3 dx_2 dx_1 \quad (21)$$

$$\varphi_2(x_1) = P\left(\sup_{0 \leq t \leq t_1} X_1(t) \leq h_1 \mid X_1(t_1) \in dx_1\right) dx_1 \quad (22)$$

$$\varphi_3(x_2, x_3) = P\left(\sup_{t_1 \leq t \leq t_2} X_2(t) \leq h_2 \mid X_2(t_1) \in dx_2, X_2(t_2) \in dx_3\right) dx_2 dx_3 \quad (23)$$

The functions $\varphi_2(x_1)$ and $\varphi_3(x_2, x_3)$ in (22) - (23) can be expanded by applying known formulae that can be found in Wang and Pötzelberger (1997) :

$$\varphi_2(x_1) = 1 - \exp\left(\frac{2h_1(x_1 - h_1)}{\sigma_1^2 t_1}\right) \quad (24)$$

$$\varphi_3(x_2, x_3) = 1 - \exp\left(\frac{2(h_2 - x_2)(x_3 - h_2)}{\sigma_2^2(t_2 - t_1)}\right) \quad (25)$$

The function $\varphi_1(x_1, x_2, x_3)$ derives from the trivariate normality of the triple $(X_1(t_1), X_2(t_1), X_2(t_2))$. It is elementary to obtain the marginal distributions :

$$X_1(t_1) \sim \mathcal{N}(\mu_1 t_1, \sigma_1^2 t_1), X_2(t_1) \sim \mathcal{N}(\mu_2 t_1, \sigma_2^2 t_1), X_2(t_2) \sim \mathcal{N}(\mu_2 t_1, \sigma_2^2 t_1) \quad (26)$$

where $\mathcal{N}(a, b^2)$ refers to the normal distribution with expectation a and variance b^2

Denoting by Z_1, Z_2, Z_3 three independent standard normal random variables, the pairwise covariances can be written as follows :

$$\text{cov}[X_1(t_1), X_2(t_1)] = \text{cov}\left[\mu_1 t_1 + \sigma_1 \sqrt{t_1} Z_1, \mu_2 t_1 + \sigma_2 \sqrt{t_1} (\rho_{1,2} Z_1 + \sqrt{1 - \rho_{1,2}^2} Z_2)\right] = \sigma_1 \sigma_2 \rho_{1,2} t_1 \quad (27)$$

$$\text{cov}[X_2(t_1), X_2(t_2)] = \text{cov}\left[\mu_2 t_1 + \sigma_2 \sqrt{t_1} (\rho_{1,2} Z_1 + \sqrt{1 - \rho_{1,2}^2} Z_2), \mu_2 t_2 + \sigma_2 \sqrt{t_1} (\rho_{1,2} Z_1 + \sqrt{1 - \rho_{1,2}^2} Z_2) + \sigma_2 \sqrt{t_2 - t_1} Z_3\right] = \sigma_2^2 t_1 \quad (28)$$

$$\text{cov}[X_1(t_1), X_2(t_2)] = \text{cov}\left[\mu_1 t_1 + \sigma_1 \sqrt{t_1} Z_1, \mu_2 t_2 + \sigma_2 \sqrt{t_1} (\rho_{1,2} Z_1 + \sqrt{1 - \rho_{1,2}^2} Z_2) + \sigma_2 \sqrt{t_2 - t_1} Z_3\right] \quad (29)$$

$$= \sigma_1 \sigma_2 \rho_{1,2} t_1$$

where we have applied the bilinearity of the covariance operator, the independence of increments of Brownian motion and the orthogonal decomposition of two-dimensional correlated Brownian motion. The correlation coefficients $\theta_{1,2}, \theta_{1,3}, \theta_{2,3}$ in Proposition 1 ensue. Expanding the trivariate normal density function $\varphi_1(x_1, x_2, x_3)$ as a product of normal conditional densities (Guillaume, 2018), we get:

$$\varphi_1(x_1, x_2, x_3) \quad (30)$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{x_1-\mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right)^2 - \frac{1}{2\sigma_{2|1}^2}\left(\frac{x_2-\mu_2 t_1}{\sigma_2 \sqrt{t_1}} - \theta_{1.2}\left(\frac{x_1-\mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right)\right)^2 - \frac{1}{2\sigma_{3|1.2}^2}\left(\frac{x_3-\mu_3 t_2}{\sigma_3 \sqrt{t_2}} - \theta_{1.3}\left(\frac{x_1-\mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right) - \theta_{2.3|1}\left(\frac{x_2-\mu_2 t_1}{\sigma_2 \sqrt{t_1}} - \theta_{1.2}\left(\frac{x_1-\mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right)\right)\right)^2}}{(2\pi)^{3/2} \sigma_{2|1} \sigma_{3|1.2} \sigma_2^2 \sigma_1 t_1 \sqrt{t_2}}$$

where :

$$- \sigma_{2|1} = \sqrt{1 - \theta_{1.2}^2}, \theta_{2.3|1} = \frac{\theta_{2.3} - \theta_{1.2}\theta_{1.3}}{\sqrt{1 - \theta_{1.2}^2}}, \sigma_{3|1.2} = \sqrt{1 - \theta_{1.3}^2 - \theta_{2.3|1}^2} \quad (31)$$

The terms $\sigma_{2|1}$, $\theta_{2.3|1}$ and $\sigma_{3|1.2}$ have the following precise meanings :

- $\sigma_{2|1}$ is the conditional standard deviation of $X_2(t_1)$ given $X_1(t_1)$
- $\theta_{2.3|1}$ is the conditional correlation between $X_2(t_1)$ and $X_3(t_2)$ given $X_1(t_1)$
- $\sigma_{3|1.2}$ is the conditional standard deviation of $X_3(t_2)$ given $X_1(t_1)$ and $X_2(t_1)$

The rest of the proof, whose cumbersome details are omitted, then consists in solving the four integrals implied by (20). The final result takes the form of the linear combination of four $N_3[\dots, \dots, \dots]$ functions written in Proposition 1.

Corollary 1 comes from the property of symmetry of Brownian paths.

Corollary 2 is a consequence of the fact that :

$$P_{[RUU]}^{(i)}(\mu_1, \mu_2) \quad (32)$$

$$= \int_{-\infty}^{\min(k_1, h_1)} \int_{-\infty}^{\min(k_2, h_2)} \int_{-\infty}^{\min(k_3, h_2)} \varphi_1(x_1, x_2, x_3) \exp\left(\frac{2h_1(x_1 - h_1)}{\sigma_1^2 t_1}\right) \exp\left(\frac{2(h_2 - x_1)(x_2 - h_2)}{\sigma_2^2(t_2 - t_1)}\right) dx_3 dx_2 dx_1$$

Corollary 3 comes from the fact that the correlation coefficient between the random variables $S_2(t_2)$ and $S_2(t_3)$ is equal to $\sqrt{\frac{t_2}{t_3}}$.

Corollary 4 is a straightforward application of the law of total probability.

□

1.2. Distributions involving one upward barrier and one downward barrier

The next result deals with the case when the steps of the barrier are not on the same side in each time interval, i.e. either first downward then upward, or first upward then downward.

Proposition 2

Let $P_{[RUD]}(\mu_1, \mu_2)$ denote the joint cumulative distribution function defined by :

$$P_{[RUD]}(\mu_1, \mu_2) \triangleq P\left(\sup_{0 \leq t \leq t_1} S_1(t) \leq H_1, S_1(t_1) \leq K_1, S_2(t_1) \geq K_2, \inf_{t_1 \leq t \leq t_2} S_2(t) \geq H_2, S_2(t_2) \geq K_3\right) \quad (33)$$

where the acronym ' $[RUD]$ ' stands for 'Rainbow Up and Down'

Then, the exact value of $P_{[RUD]}(\mu_1, \mu_2)$ is given by :

$$P_{[RUD]}(\mu_1, \mu_2) \quad (34)$$

$$= N_3 \left[\frac{\min(k_1, h_1) - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{-\max(k_2, h_2) + \mu_2 t_1}{\sigma_2 \sqrt{t_1}}, \frac{-\max(k_3, h_2) + \mu_2 t_2}{\sigma_2 \sqrt{t_2}}; -\theta_{1.2}, -\theta_{1.3}, \theta_{2.3} \right] \\ - \exp \left(\frac{2\mu_1 h_1}{\sigma_1^2} \right) \quad (35)$$

$$\times N_3 \left[\frac{\min(k_1, h_1) - 2h_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{-\max(k_2, h_2) + \mu_2 t_1}{\sigma_2 \sqrt{t_1}} + \frac{2\theta_{1.2} h_1}{\sigma_1 \sqrt{t_1}}, \right. \\ \left. \frac{-\max(k_3, h_2) + \mu_2 t_2}{\sigma_2 \sqrt{t_2}} + \frac{2\theta_{1.2} h_1}{\sigma_1 \sqrt{t_2}}; -\theta_{1.2}, -\theta_{1.3}, \theta_{2.3} \right] \\ - \exp \left(\frac{2\mu_2 h_2}{\sigma_2^2} \right) \quad (36)$$

$$\times N_3 \left[\frac{\min(k_1, h_1) - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} + \frac{2\theta_{1.2} \mu_2 \sqrt{t_1}}{\sigma_2}, \frac{-\max(k_2, h_2) - \mu_2 t_1}{\sigma_2 \sqrt{t_1}}, \frac{-\max(k_3, h_2) + 2h_2 + \mu_2 t_2}{\sigma_2 \sqrt{t_2}}; \right. \\ \left. -\theta_{1.2}, \theta_{1.3}, -\theta_{2.3} \right] \\ + \exp \left(\left(\frac{2\mu_1}{\sigma_1^2} - \frac{4\mu_2 \theta_{1.2}}{\sigma_1 \sigma_2} \right) h_1 + \frac{2\mu_2 h_2}{\sigma_2^2} \right) \quad (37)$$

$$\times N_3 \left[\frac{\min(k_1, h_1) - 2h_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} + \frac{2\theta_{1.2} \mu_2 \sqrt{t_1}}{\sigma_2}, \frac{-\max(k_2, h_2) - \mu_2 t_1}{\sigma_2 \sqrt{t_1}} + \frac{2\theta_{1.2} h_1}{\sigma_1 \sqrt{t_1}}, \right. \\ \left. \frac{-\max(k_3, h_2) + 2h_2 + \mu_2 t_2}{\sigma_2 \sqrt{t_2}} - \frac{2\theta_{1.2} h_1}{\sigma_1 \sqrt{t_2}}; \theta_{1.2}, -\theta_{1.3}, -\theta_{2.3} \right]$$

where all the notations are identical as in Proposition 1

Corollary 1 : It suffices to multiply by (-1) all the first three arguments of each $N_3[\cdot, \cdot, \cdot, \cdot, \cdot, \cdot]$ function and substitute each $\min$ operator by a $\max$ operator as well as each $\max$ operator by a $\min$ operator in Proposition 2 to obtain an exact formula for $P_{[\text{RDU}]}(\mu_1, \mu_2)$ defined as :

$$P_{[\text{RDU}]}(\mu_1, \mu_2) \triangleq P \left(\inf_{0 \leq t \leq t_1} S_1(t) \geq H_1, S_1(t_1) \geq K_1, S_2(t_1) \leq K_2, \sup_{t_1 \leq t \leq t_2} S_2(t) \leq H_2, S_2(t_2) \leq K_3 \right) \quad (38)$$

Corollary 2 : Eq. (37) provides the value of the corresponding up-and-in, then down-and-in probability, denoted as $P_{[\text{RUD}]}^{(I)}(\mu_1, \mu_2)$ and defined by :

$$P_{[\text{RUD}]}^{(I)}(\mu_1, \mu_2) \triangleq P \left(\sup_{0 \leq t \leq t_1} S_1(t) \geq H_1, S_1(t_1) \leq K_1, S_2(t_1) \geq K_2, \inf_{t_1 \leq t \leq t_2} S_2(t) \leq H_2, S_2(t_2) \geq K_3 \right) \quad (39)$$

End of Proposition 2.

Proof of Proposition 2 :

Using similar steps as in the proof of Proposition 1, one can express the problem at hand as the following integral :

$$P_{[\text{RUD}]}(\mu_1, \mu_2) = \int_{-\infty}^{\min(k_1, h_1)} \int_{\max(k_2, h_2)}^{\infty} \int_{\max(k_3, h_2)}^{\infty} \varphi_1(x_1, x_2, x_3) \varphi_2(x_1) \varphi_4(x_2, x_3) dx_3 dx_2 dx_1 \quad (40)$$

where the functions φ_1 and φ_2 are given by eq. (30) and eq. (24), respectively, and :

$$\varphi_4(x_2, x_3) = P\left(\inf_{t_1 \leq t \leq t_2} X_2(t) \geq h_2 \mid X_2(t_1) \in dx_2, X_2(t_2) \in dx_3\right) dx_2 dx_3 = \varphi_3(x_2, x_3) \quad (41)$$

where the function φ_3 is given by eq. (25)

Performing the necessary calculations, one can obtain the linear combination of four $N_3[., ., ., ., ., .]$ functions given in Proposition 2.

As in Proposition 1, Corollary 1 comes from the property of symmetry of Brownian paths.

Corollary 2 is a consequence of the fact that :

$$P_{[\text{RUD}]}^{(t)}(\mu_1, \mu_2) = \int_{-\infty}^{\min(k_1, h_1)} \int_{\max(k_2, h_2)}^{\infty} \int_{\max(k_3, h_2)}^{\infty} \varphi_1(x_1, x_2, x_3) e^{\frac{2\mu_1 h_1}{\sigma_1^2}} e^{\frac{-\frac{1}{2}\left(\frac{x_1 - 2h_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right)^2}{\sigma_1 \sqrt{2\pi t_1}}} e^{\frac{2\mu_2(h_2 - x_2)}{\sigma_2^2}} e^{\frac{-\frac{1}{2}\left(\frac{-x_3 - x_2 + 2h_2 + \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right)^2}{\sigma_2 \sqrt{2\pi(t_2 - t_1)}}} dx_1 dx_2 dx_3 \quad (42)$$

□

1.3. Reverse two-colour rainbow distributions

A two-colour rainbow barrier option is said to be reverse when the moneyness of the option is defined w.r.t. the first and former 'colour' (i.e. asset S_1) instead of the second and last one (asset S_2) : the option, so to speak, reverts back to asset one at expiry, hence the denomination. From a computational standpoint, this is not a trivial difference since it adds an additional dimension to the integral formulation of the problem. Let us define as $P_{[\text{RRUU}]}(\mu_1, \mu_2)$ the following cumulative joint distribution at the core of reverse rainbow option valuation :

$$P_{[\text{RRUU}]}^{(\text{Rev})}(\mu_1, \mu_2) = P\left(\sup_{0 \leq t \leq t_1} S_1(t) \leq H_1, S_1(t_1) \leq K_1, S_2(t_1) \leq K_2, \sup_{t_1 \leq t \leq t_2} S_2(t) \leq H_2, S_1(t_2) \leq K_3\right) \quad (43)$$

where the acronym ' $[\text{RRUU}]$ ' stands for 'Reverse Rainbow Up and Up'

Then, Proposition 3 provides the exact value of $P_{[\text{RRUU}]}(\mu_1, \mu_2)$ in the form of a triple integral.

Proposition 3

$$P_{[\text{RRUU}]}(\mu_1, \mu_2) = \frac{1}{(2\pi)^{3/2} \sigma_{2|1} \sigma_{3|1.2} \sigma_1^2 \sigma_2 t_1 \sqrt{t_2}} \int_{-\infty}^{\min(k_1, h_1)} \int_{-\infty}^{h_2} \int_{-\infty}^{\min(k_2, h_2)} \varphi_2(x_1) \varphi_3(x_2, x_3) \quad (44)$$

$$e^{-\frac{1}{2}\left(\frac{x_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right)^2 - \frac{1}{2\sigma_{2|1}^2}\left(\frac{x_2 - \mu_2 t_1}{\sigma_2 \sqrt{t_1}} - \theta_{1.2}\left(\frac{x_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right)\right)^2 - \frac{1}{2\sigma_{3|1.2}^2}\left(\frac{x_3 - \mu_2 t_2}{\sigma_2 \sqrt{t_2}} - \theta_{1.3}\left(\frac{x_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right) - \theta_{2.3|1}\left(\frac{x_2 - \mu_2 t_1}{\sigma_2 \sqrt{t_1}} - \theta_{1.2}\left(\frac{x_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right)\right)\right)^2}$$

$$N\left[\frac{1}{\sigma_{4|1.2.3}} \left[\frac{k_3 - \mu_1 t_2}{\sigma_1 \sqrt{t_2}} - \theta_{1.4} \frac{x_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} - \frac{\theta_{2.4|1}}{\sigma_{2|1}} \left(\frac{x_2 - \mu_2 t_1}{\sigma_2 \sqrt{t_1}} - \theta_{1.2} \frac{x_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} \right) \right. \right. \\ \left. \left. - \frac{\theta_{3.4|1.2}}{\sigma_{3|1.2}} \left(\frac{x_3 - \mu_2 t_2}{\sigma_2 \sqrt{t_2}} - \theta_{1.3} \frac{x_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} - \frac{\theta_{2.3|1}}{\sigma_{2|1}} \left(\frac{x_2 - \mu_2 t_1}{\sigma_2 \sqrt{t_1}} - \theta_{1.2} \frac{x_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} \right) \right) \right] \right] dx_3 dx_2 dx_1$$

where :

$$- \theta_{2.4|1} = \frac{\theta_{2.4} - \theta_{1.2}\theta_{1.4}}{\sqrt{1 - \theta_{1.2}^2}} \cdot \theta_{3.4|1.2} = \frac{\theta_{3.4} - \theta_{1.3}\theta_{1.4} - \theta_{2.3|1}\theta_{2.4|1}}{\sigma_{3|1.2}}, \sigma_{4|1.2.3} = \sqrt{1 - \theta_{1.4}^2 - \theta_{2.4|1}^2 - \theta_{3.4|1.2}^2} \quad (45)$$

- $N[\cdot]$ is the univariate standard normal cumulative distribution function

and the other notations have been previously defined.

Remark 1 : Other types of reverse two-colour knock-out or knock-in barrier probability distributions are handled similarly by modifying the upper bounds of the integral and, possibly, the φ_i functions, according to the considered combination of events.

Remark 2 : $\theta_{2.4|1}$ is the partial correlation between $X_2(t_1)$ and $X_1(t_2)$ conditional on $X_1(t_1)$, while $\theta_{3.4|1.2}$ is the partial correlation between $X_2(t_2)$ and $X_1(t_2)$ conditional on $X_1(t_1)$ and $X_2(t_1)$, and $\sigma_{4|1.2.3}$ is the conditional standard deviation of $X_1(t_2)$ given $X_1(t_1)$, $X_2(t_1)$ and $X_2(t_2)$.

End of Proposition 3.

Proof of Proposition 3

One can express the problem at hand as the following integral :

$$P_{[RRUU]}(\mu_1, \mu_2) \quad (46)$$

$$= \int_{-\infty}^{\min(k_1, h_1)} \int_{-\infty}^{h_2} \int_{-\infty}^{\min(k_2, h_2)} \int_{-\infty}^{k_3} \varphi_2(x_1) \varphi_3(x_2, x_3) \varphi_5(x_1, x_2, x_3, x_4) dx_4 dx_3 dx_2 dx_1$$

where

$$\varphi_5(x_1, x_2, x_3, x_4) = P(X_1(t_1) \in dx_1, X_2(t_1) \in dx_2, X_2(t_2) \in dx_3, X_1(t_2) \in dx_4) dx_4 dx_3 dx_2 dx_1 \quad (47)$$

Plugging the quadrivariate normal joint density function of the set of random variables $X_1(t_1)$, $X_2(t_1)$, $X_1(t_2)$ and $X_2(t_2)$, as a product of conditional density functions as explained in Guillaume (2018), and then factoring in the conditional cumulative distribution function of $X_1(t_2)$ given the triple $(X_1(t_1), X_2(t_1), X_2(t_2))$, Proposition 3 ensues.

□

1.4. Option valuation and numerical implementation

Applying the theory of non-arbitrage pricing in a complete market (Harrison and Kreps, 1979; Harrison and Pliska, 1981), the value of a two-colour up-and-up knock-out put, denoted by $V_{[RUU]}$, is given by :

$$V_{[RUU]} = e^{-rt_2} \left(E_Q \left[K_3 \mathbf{1}_{\{A\}} - S_2(t_2) \mathbf{1}_{\{A\}} \right] \right) \quad (48)$$

where r is the riskless interest rate assumed to be constant, $\mathbf{1}_{\{A\}}$ is the indicator function and A is the set constructed by the intersection of elements of the σ -algebra generated by the pair of processes $(S_1(t), S_2(t))$ that characterizes the probability $P_{[RUU]}(\mu_1, \mu_2)$ as given by the arguments of the probability operator in (5).

A simple application of the Cameron-Martin-Girsanov theorem yields :

$$V_{[RUU]} = e^{-rt_2} K_3 P_{[RUU]} \left(\mu_1^{(Q)}, \mu_2^{(Q)} \right) - S_2(0) P_{[RUU]} \left(\mu_1^{(P_2)}, \mu_2^{(P_2)} \right) \quad (49)$$

where :

$$\mu_i^{(Q)} = r - \frac{\sigma_i^2}{2}, \mu_1^{(P_2)} = r - \frac{\sigma_1^2}{2} + \sigma_1 \sigma_2 \rho_{1,2}, \mu_2^{(P_2)} = r + \frac{\sigma_2^2}{2} \tag{50}$$

Q is the measure under which $\left\{B_i(t) + \frac{\mu_i - r}{\sigma_i}t, t \geq 0\right\}$ is a standard Brownian motion (the classical so-called risk-neutral measure), while P_2 is the measure under which $\left\{B_1(t) - \sigma_2 \rho_{1,2}t, t \geq 0\right\}$ and $\left\{B_2(t) - \sigma_2 \sqrt{1 - \rho_{1,2}^2}t, t \geq 0\right\}$ are two independent standard Brownian motions.

To factor in a continuous dividend rate δ_i associated with each asset S_i , simply replace r by $r - \delta_i$. All the other two-colour rainbow barrier options mentioned in Section 1, whether they be knock-in or whether they feature a mixture of a downward and an upward barrier, are identically valued, by taking the relevant $P_{[]}$ probability along with the pairs $\left(\mu_1^{(Q)}, \mu_2^{(Q)}\right)$ and $\left(\mu_1^{(P_2)}, \mu_2^{(P_2)}\right)$.

The numerical implementation of Proposition 1 and Proposition 2 is easy. Using Genz’s algorithm (2004) to evaluate the trivariate standard normal cumulative distribution function, the accuracy and efficiency required for all practical purposes can be achieved in computational times in the order of 0.1 second. Table 1 provides the prices of a few two-colour up-and-up knock-out put options, for various levels of the volatility and correlation parameters of the underlying assets S_1 and S_2 , and different values of the knock-out barriers. All the initial values of the underlying assets $S_i(0)$ and the strike prices K_i are set at 100. Expiry is 1 year. The two time intervals $[t_0, t_1]$ and $[t_1, t_2]$ have equal length, i.e. $t_1 = 6$ months, but unequal time lengths can be handled just as well by the formulae. The riskless interest rate is assumed to be 2.5%.

In each cell, four prices are reported : the first one is the exact analytical value as obtained by implementing Proposition 1, while the prices in brackets are three successive approximations obtained by performing increasingly large Monte Carlo simulations. More specifically, these approximations rely on the conditional Monte Carlo method, which is well known for its accuracy and efficiency (Glasserman, 2003). The number of simulations performed is 500,000 for the first approximation, 2,000,000 for the second approximation, and 10,000,000 for the third approximation. The pseudo-random numbers are drawn from the reliable Mersenne-Twister generator.

Table 1. : Two-colour up-and-up knock-out put.

	$\rho_{1,2} = -0.6$	$\rho_{1,2} = -0.2$	$\rho_{1,2} = 0.2$	$\rho_{1,2} = 0.6$
$\sigma_1 = \sigma_2 = 20\%$	1.189	2.163	3.134	4.141
	(1.123, 1.168,	(2.237, 2.141,	(3.082, 3.116,	(4.174, 4.127,
$H_1 = H_2 = 115$	1.182)	2.161)	3.132)	4.140)
$\sigma_1 = \sigma_2 = 60\%$	4.065	6.440	8.769	11.213
	(4.109, 4.072,	(6.392, 6.449,	(8.734, 8.755,	(11.278, 11.196,
$H_1 = H_2 = 125$	4.061)	6.442)	8.763)	11.218)
$\sigma_1 = 20\%, \sigma_2 = 60\%$	4.454	7.260	9.979	12.844
	(4.411, 4.443,	(7.355, 7.301,	(9.912, 9.101,	(12.957, 12.892,

$H_1 = 115, H_2 = 125$	4.454)	7.264)	9.975)	12.848)
$\sigma_1 = 60\%, \sigma_2 = 20\%$	1.096	1.921	2.753	3.617
	(1.082, 1.108,	(1.107, 1.953,	(2.796, 2.745,	(3.692, 3.599,
$H_1 = 125, H_2 = 115$	1.091)	1.928)	2.752)	3.614)

In purely numerical terms, it can be clearly observed that the conditional Monte Carlo approximations steadily converge to the analytical values as more and more simulations are performed. A minimum of 10,000,000 simulations are necessary to guarantee a modest 10^{-3} convergence. This requires a computational time of approximately 35 seconds on a computer equipped with a Core i7 CPU. Much more accurate values can be obtained by means of Proposition 1 in only two tenths of a second. This gap in accuracy and efficiency makes a particularly valuable difference when pricing large portfolios of options.

From a financial point of view, the most striking phenomenon observed in Table 1 is that the option price regularly and significantly increases with the value of the correlation coefficient between assets S_1 and S_2 , whatever the volatilities and the levels of the barriers. Roughly speaking, the price of an at-the-money two-colour up-and-up knock-out put option when $\rho_{1,2} = 0.6$ is three times greater than when $\rho_{1,2} = -0.6$. This property can be exploited by traders who take positions on correlation, as the prices of these options will substantially increase if implicit correlation turns out to be underestimated by the markets. This property can also be harnessed by traders to construct hedges on sold derivatives that are sensitive to pairwise correlation. From an investor’s perspective, the observed phenomenon allows to define effective strategies to reduce the cost of hedging by tapping into negative correlation. Such a significant functional relation w.r.t. correlation is a major attraction of rainbow step barrier options relative to non-rainbow step barrier options, as the latter can only handle volatility effects.

Another noticeable fact in Table 1 is that lowering the up-and-out barriers seems much more effective in reducing the option’s price than lowering the volatilities of assets S_1 and S_2 , regardless of the sign and the magnitude of correlation. Indeed, looking at row 1 in Table 1, one can see that the options are relatively cheap, although the volatilities of both assets S_1 and S_2 are low, because the knock-out barriers are located quite near the spot prices of the underlying assets; and looking at row 2 in Table 2, one can see that the options are relatively expensive, although the volatilities of both assets S_1 and S_2 are high, because the knock-out barriers are more distant. This shows that the barrier effect, which drives prices down as up-and-out barriers get lower, and conversely drive prices up as up-and-out barrier get higher, prevails over the volatility effect, which exerts its influence in the opposite direction, i.e. a lower volatility pushes prices up by decreasing the probability of knocking out before expiry and a higher volatility pushes prices down by increasing the latter probability. This phenomenon can be explained by the ambivalent nature of volatility : on the one hand, less volatility means less risk of being deactivated before expiry, but on the other hand it also means fewer chances of ending in-the-money at expiry; whichever of this positive and this negative effect weighs more on the option price depends on the relative values of barrier, strike, volatility and expiry parameters in a complex manner.

Similarly, Table 2 reports the prices of a few down-and-up two-colour knock-out put options by implementing Proposition 2 to obtain exact analytical values and by computing three successive conditional Monte Carlo approximations in the same way as in Table 1.

Table 2. : Two-colour down-and-up knock-out put.

	$\rho_{1.2} = -0.6$	$\rho_{1.2} = -0.2$	$\rho_{1.2} = 0.2$	$\rho_{1.2} = 0.6$
$\sigma_1 = \sigma_2 = 20\%$	4.299	3.293	2.307	1.303
$H_1 = 85, H_2 = 115$	(4.287, 4.305, 4.297)	(3.318, 3.286, 3.292)	(2.282, 2.298, 2.305)	(1.284, 1.309, 1.304)
$\sigma_1 = \sigma_2 = 60\%$	10.221	7.610	5.212	2.862
$H_1 = 75, H_2 = 125$	(10.142, 10.228, 10.224)	(7.563, 7.597, 7.612)	(5.255, 5.204, 5.214)	(2.834, 2.854, 2.865)
$\sigma_1 = 20\%, \sigma_2 = 60\%$	13.377	10.496	7.735	4.857
$H_1 = 85, H_2 = 125$	(13.385, 13.391, 13.376)	(10.472, 10.482, 10.948)	(7.783, 7.717, 7.731)	(4.894, 4.866, 4.857)
$\sigma_1 = 60\%, \sigma_2 = 20\%$	3.344	2.401	1.545	0.748
$H_1 = 75, H_2 = 115$	(3.387, 3.358, 3.346)	(2.383, 2.413, 2.402)	(1.596, 1.530, 1.542)	(0.884, 0.787, 0.752)

In Table 2, the most salient feature is still the functional dependency of the option's price on the correlation between assets S_1 and S_2 , but, this time, the direction is opposite to that in Table 1, i.e. the two-colour down-and-up knock-out put prices steadily decrease as $\rho_{1.2}$ goes from -60% to 60% . In a trader's perspective, one could sum up the argument by saying that two-colour rainbow barrier options are a bet on positive correlation when both barriers are on the same side (upward or downward), while they are a bet on negative correlation when the barriers stand on opposite sides (up-and-down or down-and-up).

The barrier effect also prevails over the volatility effect in Table 2. Overall, two-colour down-and-up knock-out puts display maximum values that are a little higher, and minimum values that are a little lower, than two-colour up-and-up knock-out puts, although up-and-out barriers and down-and-out barriers are designed with the exact same distance to the spot prices of S_1 and S_2 .

Eventually, the application of Proposition 3 is briefly discussed. The no-arbitrage price of a reverse two-colour rainbow up-and-up knock-out put, denoted by $V_{[\text{RRUU}]}$, is given by :

$$V_{[\text{RRUU}]}^{(R)} = e^{-rt_2} \left(E_Q \left[K_3 \mathbf{1}_{\{A\}} - S_1(t_2) \mathbf{1}_{\{A\}} \right] \right) = e^{-rt_2} K_3 P_{[\text{RRUU}]}^{(R)} \left(\mu_1^{(Q)}, \mu_2^{(Q)} \right) - S_2(0) P_{[\text{RRUU}]}^{(R)} \left(\mu_1^{(P_1)}, \mu_2^{(P_1)} \right) \quad (51)$$

where :

$$- \mu_1^{(P_1)} = r + \frac{\sigma_1^2}{2}, \quad \mu_2^{(P_1)} = r - \frac{\sigma_2^2}{2} + \sigma_1 \sigma_2 \rho_{1.2} \quad (52)$$

- A is the set constructed by the intersection of elements of the σ -algebra generated by the pair of processes $(S_1(t), S_2(t))$ that characterizes the probability $P_{[\text{RRUU}]}(\mu_1, \mu_2)$ as given by the arguments of the probability operator in (43)

- P_1 is the measure under which $B_1(t) - \sigma_1 t$ is a standard Brownian motion

However, it is less easy to evaluate Proposition 3. The problem at hand has two 'nice' features from the standpoint of numerical integration : first, the dimension, equal to 3, is moderate; second,

the integrand is continuous. The snag is the large number of parameters in each evaluation of the integrand in a quadrature process, especially the various conditional standard deviations at the denominators of the fractions, that may hinder fast convergence when they take on absolute values that get smaller and smaller. That is why it is recommended to use a subregion adaptive algorithm of numerical integration, as explained by Bernstein, Espelid & Genz (1991), that adapts the number of integrand evaluations in each subregion according to the rate of change of the integrand. Although more time-consuming than a fixed degree rule, it is more accurate to control the approximation error, as the subdivision of the integration domain stops only when the sum of the local error deterministic estimates becomes smaller than some prespecified requested accuracy. Adaptive integration can be enhanced by a Kronrod rule to reduce the number of required iterations (see, e.g., Davis and Rabinowitz 2007). These techniques are widely used in numerical integration and it is easy to find available code or built-in functions in the usual scientific computing software.

2. Two-colour outside step barrier option valuation

In this Section, a third correlated asset S_3 is introduced, w.r.t. which the option's moneyness is measured at expiry, while the assets S_1 and S_2 serve exclusively as the underlyings w.r.t. which barrier crossing is monitored. This is an important extension, as outside barrier options allow to manage volatility more consistently than standard (non-outside) barrier options, as explained, e.g., by Das (2006).

2.1. Statement of Proposition 4

Let us consider a third asset S_3 with the following differential :

$$dS_3(t) = v_3 S_3(t) dt + \sigma_3 S_3(t) dB_3(t) \quad (53)$$

The instantaneous pairwise correlations between the Brownian motions B_i 's are denoted as $\rho_{i,j}$.

The objective is to compute the probabilities $p_m(\mu_1, \mu_2, \mu_3), m \in \{1, 2, 3, 4\}$ defined by :

$$p_1(\mu_1, \mu_2, \mu_3) = P \left(\begin{array}{l} \sup_{0 \leq t \leq t_1} S_1(t) \leq H_1, S_1(t_1) \leq K_1, S_2(t_1) \leq H_2, \\ \sup_{t_1 \leq t \leq t_2} S_2(t) \leq H_2, S_2(t_2) \leq K_2, S_3(t_2) \leq K_3 \end{array} \right) \quad (54)$$

$$p_2(\mu_1, \mu_2, \mu_3) = P \left(\begin{array}{l} \inf_{0 \leq t \leq t_1} S_1(t) \geq H_1, S_1(t_1) \geq K_1, S_2(t_1) \geq H_2, \\ \inf_{t_1 \leq t \leq t_2} S_2(t) \geq H_2, S_2(t_2) \geq K_2, S_3(t_2) \geq K_3 \end{array} \right) \quad (55)$$

$$p_3(\mu_1, \mu_2, \mu_3) = P \left(\begin{array}{l} \sup_{0 \leq t \leq t_1} S_1(t) \leq H_1, S_1(t_1) \leq K_1, S_2(t_1) \geq H_2, \\ \inf_{t_1 \leq t \leq t_2} S_2(t) \geq H_2, S_2(t_2) \geq K_2, S_3(t_2) \geq K_3 \end{array} \right) \quad (56)$$

$$p_4(\mu_1, \mu_2, \mu_3) = P \left(\begin{array}{l} \inf_{0 \leq t \leq t_1} S_1(t) \geq H_1, S_1(t_1) \geq K_1, S_2(t_1) \leq H_2, \\ \sup_{t_1 \leq t \leq t_2} S_2(t) \leq H_2, S_2(t_2) \leq K_2, S_3(t_2) \leq K_3 \end{array} \right) \quad (57)$$

Let the function $\Psi_4[b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4, c_5], \forall b_1, b_2, b_3, b_4 \in \mathbb{R}, \forall c_1, c_2, c_3, c_4, c_5 \in]-1, 1[$, be defined by :

$$\begin{aligned}
& \Psi_4 \left[b_1, b_2, b_3, b_4; c_1, c_2, c_3, c_4, c_5 \right] \\
&= \int_{x_1=-\infty}^{b_1} \int_{x_2=-\infty}^{\frac{b_2-c_1x_1}{\sqrt{1-c_1^2}}} \int_{x_3=-\infty}^{\frac{b_3-c_4x_2\sqrt{1-c_1^2}-c_4c_1x_1}{\sqrt{1-c_2^2}}} \frac{1}{(2\pi)^{3/2}} \exp \left(-\frac{x_1^2}{2} - \frac{x_2^2}{2} - \frac{x_3^2}{2} \right) \\
& N \left[\frac{b_4 - \frac{c_5-c_2c_3}{1-c_2^2} \left(x_3\sqrt{1-c_4^2} + x_2c_4\sqrt{1-c_1^2} \right) - x_1 \left(c_3 + \frac{c_5-c_2c_3}{1-c_2^2} (c_1c_4 - c_2) \right)}{\sqrt{1-c_3^2 - \left(\frac{c_5-c_2c_3}{1-c_2^2} \right)^2}} \right] dx_3 dx_2 dx_1
\end{aligned} \tag{58}$$

The following Proposition 4 combines all the probabilities defined by eq. (54) – (56) into a single formula.

Proposition 4

The exact values of the probabilities $p_m(\mu_1, \mu_2, \mu_3)$, $m \in \{1, 2, 3, 4\}$, written in shorter notation as p_m , are given by :

$$p_m = \Psi_4 \left[\delta_1 \left(\frac{G_1(k_1, h_1) - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} \right), \delta_2 \left(\frac{h_2 - \mu_2 t_1}{\sigma_2 \sqrt{t_1}} \right), \delta_2 \left(\frac{G_2(k_2, h_2) - \mu_2 t_2}{\sigma_2 \sqrt{t_2}} \right), \delta_2 \left(\frac{k_3 - \mu_3 t_2}{\sigma_3 \sqrt{t_2}} \right); x_1 \right] \tag{58}$$

$$\begin{aligned}
& - \exp \left(\frac{2\mu_1 h_1}{\sigma_1^2} \right) \times \Psi_4 \left[\delta_1 \left(\frac{G_1(k_1, h_1) - 2h_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} \right), \delta_2 \left(\frac{h_2 - \mu_2 t_1}{\sigma_2 \sqrt{t_1}} - \theta_{1.2} \frac{2h_1}{\sigma_1 \sqrt{t_1}} \right), \right. \\
& \left. \delta_2 \left(\frac{G_2(k_2, h_2) - \mu_2 t_2}{\sigma_2 \sqrt{t_2}} - \theta_{1.2} \frac{2h_1}{\sigma_1 \sqrt{t_2}} \right), \right. \\
& \left. \delta_2 \left(\frac{k_3 - \mu_3 t_2}{\sigma_3 \sqrt{t_2}} - \theta_{1.4} \frac{2h_1}{\sigma_1 \sqrt{t_1}} - \theta_{3.4|1} \left(\theta_{1.2} \frac{2h_1}{\sigma_1 \sqrt{t_2}} - \theta_{1.3} \frac{2h_1}{\sigma_1 \sqrt{t_1}} \right) \right); x_1 \right]
\end{aligned} \tag{59}$$

$$\begin{aligned}
& - \exp \left(\frac{2\mu_2 h_2}{\sigma_2^2} \right) \times \Psi_4 \left[\delta_1 \left(\frac{G_1(k_1, h_1) - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} + \theta_{1.2} \frac{2\mu_2 t_1}{\sigma_2 \sqrt{t_1}} \right), \delta_2 \left(\frac{h_2 + \mu_2 t_1}{\sigma_2 \sqrt{t_1}} \right), \right. \\
& \left. \delta_2 \left(\frac{G_2(k_2, h_2) - 2h_2 - \mu_2 t_2}{\sigma_2 \sqrt{t_2}} \right), \right. \\
& \left. \delta_2 \left(\frac{k_3 - \mu_3 t_2}{\sigma_3 \sqrt{t_2}} - \theta_{3.4|1} \left(\frac{2h_2}{\sigma_2 \sqrt{t_2}} + \theta_{1.2} \theta_{1.3} \frac{2\mu_2 t_1}{\sigma_2 \sqrt{t_1}} \right) + \theta_{1.2} \theta_{1.4} \frac{2\mu_2 t_1}{\sigma_2 \sqrt{t_1}} \right); x_2 \right]
\end{aligned} \tag{60}$$

$$+ \exp \left(\left(\frac{2\mu_1}{\sigma_1^2} - \frac{4\mu_2 \rho_{1.2}}{\sigma_1 \sigma_2} \right) h_1 + \frac{2\mu_2 h_2}{\sigma_2^2} \right) \tag{61}$$

$$\times \Psi_4 \begin{bmatrix} \delta_1 \left(\frac{G_1(k_1, h_1) - 2h_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} + \theta_{1.2} \frac{2\mu_2 t_1}{\sigma_2 \sqrt{t_1}} \right), \\ \delta_2 \left(\frac{h_2 + \mu_2 t_1}{\sigma_2 \sqrt{t_1}} - \theta_{1.2} \frac{2h_1}{\sigma_1 \sqrt{t_1}} \right), \\ \delta_2 \left(\frac{G_2(k_2, h_2) - 2h_2 - \mu_2 t_2}{\sigma_2 \sqrt{t_2}} + \theta_{1.2} \frac{2h_1}{\sigma_1 \sqrt{t_2}} \right), \\ \delta_2 \left(\frac{k_3 - \mu_3 t_2}{\sigma_3 \sqrt{t_2}} + \theta_{1.4} \left(\theta_{1.2} \frac{2\mu_2 \sqrt{t_1}}{\sigma_2} - \frac{2h_1}{\sigma_1 \sqrt{t_1}} \right) \right. \\ \left. - \theta_{3.4|1} \left(\frac{2h_2}{\sigma_2 \sqrt{t_2}} - \theta_{1.2} \frac{2h_1}{\sigma_1 \sqrt{t_2}} + \theta_{1.3} \left(\theta_{1.2} \frac{2\mu_2 \sqrt{t_1}}{\sigma_2} - \frac{2h_1}{\sigma_1 \sqrt{t_1}} \right) \right) \right) \right] ; \mathbf{x}_2$$

where k_1, k_2, h_1, h_2 are as in Proposition 2, $k_3 = \ln \left(\frac{K_3}{S_3(0)} \right)$, and we have :

$$- \theta_{1.2} = \rho_{1.2}, \theta_{1.3} = \sqrt{\frac{t_1}{t_2}} \rho_{1.2}, \theta_{1.4} = \sqrt{\frac{t_1}{t_2}} \rho_{1.3}, \theta_{2.3} = \sqrt{\frac{t_1}{t_2}}, \theta_{3.4} = \rho_{2.3}, \theta_{3.4|1} = \frac{\theta_{3.4} - \theta_{1.3} \theta_{1.4}}{\sqrt{1 - \theta_{1.3}^2}} \quad (62)$$

$$- \delta_1 = \begin{cases} 1 & \text{if } p_m = p_1 \text{ or } p_m = p_3 \\ -1 & \text{if } p_m = p_2 \text{ or } p_m = p_4 \end{cases}, \delta_2 = \begin{cases} 1 & \text{if } p_m = p_1 \text{ or } p_m = p_4 \\ -1 & \text{if } p_m = p_2 \text{ or } p_m = p_3 \end{cases} \quad (63)$$

$$- G_1(\cdot, \cdot) = \begin{cases} \min(\cdot, \cdot) & \text{if } p_m = p_1 \text{ or } p_m = p_3 \\ \max(\cdot, \cdot) & \text{if } p_m = p_2 \text{ or } p_m = p_4 \end{cases}, \quad (64)$$

$$G_2(\cdot, \cdot) = \begin{cases} \max(\cdot, \cdot) & \text{if } p_m = p_1 \text{ or } p_m = p_3 \\ \min(\cdot, \cdot) & \text{if } p_m = p_2 \text{ or } p_m = p_4 \end{cases}$$

$$- \mathbf{x}_1 = \begin{cases} [\theta_{1.2}, \theta_{1.3}, \theta_{1.4}, \theta_{2.3}, \theta_{3.4}] & \text{if } p_m = p_1 \text{ or } p_m = p_2 \\ [-\theta_{1.2}, -\theta_{1.3}, -\theta_{1.4}, \theta_{2.3}, \theta_{3.4}] & \text{if } p_m = p_3 \text{ or } p_m = p_4 \end{cases} \quad (65)$$

$$- \mathbf{x}_2 = \begin{cases} [\theta_{1.2}, \theta_{1.3}, \theta_{1.4}, -\theta_{2.3}, \theta_{3.4}] & \text{if } p_m = p_1 \text{ or } p_m = p_2 \\ [-\theta_{1.2}, -\theta_{1.3}, -\theta_{1.4}, -\theta_{2.3}, \theta_{3.4}] & \text{if } p_m = p_3 \text{ or } p_m = p_4 \end{cases} \quad (66)$$

Corollary 1 : The corresponding knock-in probabilities can be inferred in the same way as in Proposition 1 and Proposition 2. Let the probabilities $p_m^{(I)}(\mu_1, \mu_2, \mu_3), m \in \{1, 2, 3, 4\}$ be defined by :

$$p_1^{(I)}(\mu_1, \mu_2, \mu_3) = P \left(\begin{array}{l} \sup_{0 \leq t \leq t_1} S_1(t) \geq H_1, S_1(t_1) \leq K_1, S_2(t_1) \leq H_2, \\ \sup_{t_1 \leq t \leq t_2} S_2(t) \geq H_2, S_2(t_2) \leq K_2, S_3(t_2) \leq K_3 \end{array} \right) \quad (67)$$

$$p_2^{(I)}(\mu_1, \mu_2, \mu_3) = P \left(\begin{array}{l} \inf_{0 \leq t \leq t_1} S_1(t) \leq H_1, S_1(t_1) \geq K_1, S_2(t_1) \geq H_2, \\ \inf_{t_1 \leq t \leq t_2} S_2(t) \leq H_2, S_2(t_2) \geq K_2, S_3(t_2) \geq K_3 \end{array} \right) \quad (68)$$

$$p_3^{(I)}(\mu_1, \mu_2, \mu_3) = P \left(\begin{array}{l} \sup_{0 \leq t \leq t_1} S_1(t) \geq H_1, S_1(t_1) \leq K_1, S_2(t_1) \geq H_2, \\ \inf_{t_1 \leq t \leq t_2} S_2(t) \leq H_2, S_2(t_2) \geq K_2, S_3(t_2) \geq K_3 \end{array} \right) \quad (69)$$

$$p_4^{(I)}(\mu_1, \mu_2, \mu_3) = P \left(\begin{array}{l} \inf_{0 \leq t \leq t_1} S_1(t) \leq H_1, S_1(t_1) \geq K_1, S_2(t_1) \leq H_2, \\ \sup_{t_1 \leq t \leq t_2} S_2(t) \geq H_2, S_2(t_2) \leq K_2, S_3(t_2) \leq K_3 \end{array} \right) \quad (70)$$

Then, $p_m^{(I)}(\mu_1, \mu_2, \mu_3)$ is given by (61).

Corollary 2 : It suffices to substitute each argument $\theta_{3,4}$ in each $\Psi_4[\dots, \dots, \dots]$ function of Proposition 4 by $\theta_{3,4} = \rho_{2,3} \sqrt{\frac{t_2}{t_3}}$, $\forall t_3 \geq t_2$, to obtain an exact formula for the early-ending variant of $p_m(\mu_1, \mu_2, \mu_3)$

End of Proposition 4

2.2. Proof of Proposition 4

Proof is given only for $p_1(\mu_1, \mu_2, \mu_3)$ and $p_3(\mu_1, \mu_2, \mu_3)$, as $p_2(\mu_1, \mu_2, \mu_3)$ and $p_4(\mu_1, \mu_2, \mu_3)$ can then be deduced by the same symmetry argument as that already used in Corollary 1 of Proposition 1.

Following steps similar to the beginning of the proof of Proposition 1, one can express the problem at hand as the following two integrals :

$$p_1(\mu_1, \mu_2, \mu_3) = \int_{x_1=-\infty}^{\min(k_1, h_1)} \int_{x_2=-\infty}^{\min(k_2, h_2)} \int_{x_3=-\infty}^{\min(k_3, h_2)} \int_{x_4=-\infty}^{k_4} \varphi_6(x_1) \varphi_7(x_1, x_2) \varphi_8(x_2, x_3) \varphi_9(x_1, x_3, x_4) dx_4 dx_3 dx_2 dx_1 \quad (71)$$

$$p_3(\mu_1, \mu_2, \mu_3) = \int_{x_1=-\infty}^{\min(k_1, h_1)} \int_{x_2=\max(k_2, h_2)}^{\infty} \int_{x_3=\max(k_3, h_2)}^{\infty} \int_{x_4=k_4}^{\infty} \varphi_6(x_1) \varphi_7(x_1, x_2) \varphi_{10}(x_2, x_3) \varphi_9(x_1, x_3, x_4) dx_4 dx_3 dx_2 dx_1 \quad (72)$$

where:

$$\varphi_6(x_1) = P \left(\sup_{0 \leq t \leq t_1} X_1(t) \leq h_1, X_1(t_1) \in dx_1 \right) dx_1 \quad (73)$$

$$\varphi_7(x_1, x_2) = P \left(X_2(t_1) \in dx_2 \mid X_1(t_1) \in dx_1 \right) dx_1 dx_2 \quad (74)$$

$$\varphi_8(x_2, x_3) = P \left(\sup_{t_1 \leq t \leq t_2} X_2(t) \leq h_2, X_2(t_2) \in dx_3 \mid X_2(t_1) \in dx_2 \right) dx_2 dx_3 \quad (75)$$

$$\varphi_{10}(x_2, x_3) = P \left(\inf_{t_1 \leq t \leq t_2} X_2(t) \geq h_2, X_2(t_2) \in dx_3 \mid X_2(t_1) \in dx_2 \right) dx_2 dx_3 \quad (76)$$

$$\varphi_9(x_1, x_3, x_4) = P \left(X_3(t_2) \in dx_4 \mid X_2(t_2) \in dx_3, X_1(t_1) \in dx_1 \right) \quad (77)$$

The function $\varphi_6(x_1)$ is obtained by differentiating the classical formula for the joint cumulative distribution of the maximum of a Brownian motion with drift and its endpoint over a closed time interval (see, e.g., Karatzas and Shreve, 2000) :

$$\varphi_6(x_1) = \frac{e^{-\frac{1}{2}\left(\frac{x_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right)^2}}{\sigma_1 \sqrt{2\pi t_1}} - e^{\frac{2\mu_1}{\sigma_1^2} h_1} \frac{e^{-\frac{1}{2}\left(\frac{x_1 - 2h_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right)^2}}{\sigma_1 \sqrt{2\pi t_1}} dx_1 \quad (78)$$

The function $\varphi_7(x_1, x_2)$ is easily derived from the bivariate normality of the pair $(X_1(t_1), X_2(t_1))$:

$$\varphi_7(x_1, x_2) = \frac{e^{-\frac{1}{2(1-\rho_{1,2}^2)}\left(\frac{x_2 - \mu_2 t_1}{\sigma_2 \sqrt{t_1}} - \rho_{1,2} \frac{x_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right)^2}}{\sigma_2 \sqrt{2\pi t_1 (1 - \rho_{1,2}^2)}} dx_1 dx_2 \quad (79)$$

To handle the function $\varphi_8(x_2, x_3)$, we notice that, by conditioning w.r.t. the filtration at time t_1 the same classical formula as the one used to derive $\varphi_6(x_1)$, we can obtain :

$$\begin{aligned} & P\left(\sup_{t_1 \leq t \leq t_2} S_2(t) \leq H_2, S_2(t_2) \leq S_2(0)e^{x_3} \mid S_2(t_1) = S_2(0)e^{x_2}\right) \\ &= N\left[\frac{\ln\left(\frac{S_2(0)e^{x_3}}{S_2(0)e^{x_2}}\right) - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right] - \left(\frac{H_2}{S_2(0)e^{x_2}}\right)^{\frac{2\mu_2}{\sigma_2^2}} N\left[\frac{\ln\left(\frac{S_2(0)e^{x_3}}{S_2(0)e^{x_2}}\right) - 2\ln\left(\frac{H_2}{S_2(0)e^{x_2}}\right) - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right] \end{aligned} \quad (80)$$

for any given $(x_2, x_3) \in \mathbb{R}^2$ and $H_2 > S_2(0)e^{x_3}$

Eq. (80) can be rewritten as follows :

$$\begin{aligned} & P\left(\sup_{t_1 \leq t \leq t_2} X_2(t) \leq h_2, X_2(t_2) \leq x_3 \mid X_2(t_1) \in dx_2\right) \\ &= N\left[\frac{x_3 - x_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right] - \exp\left(\frac{2\mu_2}{\sigma_2^2}(h_2 - x_2)\right) N\left[\frac{x_3 - x_2 - 2(h_2 - x_2) - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right] \end{aligned} \quad (81)$$

Therefore, by differentiating (81) w.r.t. x_3 , we obtain :

$$\varphi_8(x_2, x_3) = \frac{e^{-\frac{1}{2}\left(\frac{x_3 - x_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right)^2}}{\sigma_2 \sqrt{2\pi(t_2 - t_1)}} - e^{\frac{2\mu_2}{\sigma_2^2}(h_2 - x_2)} \frac{e^{-\frac{1}{2}\left(\frac{x_3 + x_2 - 2h_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right)^2}}{\sigma_2 \sqrt{2\pi(t_2 - t_1)}} dx_2 dx_3 \quad (82)$$

By the symmetry of paths of Brownian motion, we have :

$$P\left(\inf_{t_1 \leq t \leq t_2} X_2(t) \geq h_2, X_2(t_2) \geq x_3 \mid X_2(t_1) \in dx_2\right) \quad (83)$$

$$= N \left[\frac{-x_3 + x_2 + \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}} \right] - \exp \left(\frac{2\mu_2}{\sigma_2^2} (h_2 - x_2) \right) N \left[\frac{-x_3 + x_2 + 2(h_2 - x_2) + \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}} \right]$$

Hence,

$$\varphi_{10}(x_2, x_3) = \varphi_8(x_2, x_3) \quad (84)$$

The function φ_9 derives from the joint trivariate normality of the triple $(X_1(t_1), X_2(t_2), X_3(t_2))$. The marginal distributions of the elements of this triple come from the known marginal distributions of $S_1(t_1)$, $S_2(t_2)$ and $S_3(t_2)$. The pairwise correlations, as given by $\theta_{1,3}, \theta_{1,4}$ and $\theta_{3,4}$ in Proposition 4 can be easily determined using the same method as in the proof of Proposition 3. We get :

$$\varphi_9(x_1, x_3, x_4) = \frac{e^{-\frac{1}{2\phi_{4|1,3}^2} \left(\frac{x_4 - \mu_{3,2}}{\sigma_3 \sqrt{t_2}} - \theta_{1,4} \left(\frac{x_1 - \mu_{1,1}}{\sigma_1 \sqrt{t_1}} \right) - \theta_{3,4} \left(\frac{x_3 - \mu_{3,2}}{\sigma_2 \sqrt{t_2}} - \theta_{1,3} \left(\frac{x_1 - \mu_{1,1}}{\sigma_1 \sqrt{t_1}} \right) \right) \right)^2}}{\phi_{4|1,3} \sigma_3 \sqrt{2\pi t_2}} \quad (85)$$

where $\sigma_{3|1} = \sqrt{1 - \theta_{1,3}^2}$

The rest of the proof, whose cumbersome details are omitted, then consists in solving the integrals implied by (71) – (72). The final result can be expressed as the linear combination of four $\Psi_4[.,.,.,.;.,.,.,.]$ functions written in Proposition 4. The origin of the function $\Psi_4[.,.,.,.;.,.,.,.]$, which is a special form of quadrivariate normal cumulative distribution, lies in the FDD (Finite Dimensional Distribution) of the quadruple $[S_1(t_1), S_2(t_1), S_2(t_2), S_3(t_2)]$. Indeed, a little algebra shows that, $\forall D_1, D_2, D_3, D_4 \in \mathbb{R}^+$, we have :

$$\begin{aligned} & P(S_1(t_1) \leq D_1, S_2(t_1) \leq D_2, S_2(t_2) \leq D_3, S_3(t_2) \leq D_4) \\ &= \Psi_4 \left[\frac{\ln \left(\frac{D_1}{S_1(0)} \right) - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{\ln \left(\frac{D_2}{S_2(0)} \right) - \mu_2 t_1}{\sigma_2 \sqrt{t_1}}, \frac{\ln \left(\frac{D_3}{S_2(0)} \right) - \mu_2 t_2}{\sigma_2 \sqrt{t_2}}, \frac{\ln \left(\frac{D_4}{S_3(0)} \right) - \mu_3 t_2}{\sigma_3 \sqrt{t_2}}; \right. \\ & \quad \left. \theta_{1,2}, \theta_{1,3}, \theta_{1,4}, \theta_{2,3}, \theta_{3,4} \right] \end{aligned} \quad (86)$$

Corollary 1 is a consequence of the fact that :

$$\begin{aligned} p_1^{(I)}(\mu_1, \mu_2) &= \int_{x_1=-\infty}^{\min(k_1, h_1)} \int_{x_2=-\infty}^{\min(k_2, h_2)} \int_{x_3=-\infty}^{\min(k_3, h_2)} \int_{x_4=-\infty}^{k_4} e^{\frac{2\mu_1 h_1}{\sigma_1^2}} e^{\frac{-1}{2} \left(\frac{x_1 - 2h_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} \right)^2} \frac{1}{\sigma_1 \sqrt{2\pi t_1}} \varphi_7(x_1, x_2) \\ & \quad e^{\frac{2\mu_2}{\sigma_2^2} (h_2 - x_2)} e^{\frac{-1}{2} \left(\frac{x_3 + x_2 - 2h_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}} \right)^2} \frac{1}{\sigma_2 \sqrt{2\pi(t_2 - t_1)}} \varphi_9(x_1, x_3, x_4) dx_4 dx_3 dx_2 dx_1 \end{aligned} \quad (87)$$

and :

$$p_3^{(I)}(\mu_1, \mu_2) = \int_{x_1=-\infty}^{\min(k_1, h_1)} \int_{x_2=\max(k_2, h_2)}^{\infty} \int_{x_3=\max(k_3, h_2)}^{\infty} \int_{x_4=-\infty}^{k_4} e^{\frac{2\mu_1}{\sigma_1^2} h_1} \frac{e^{-\frac{1}{2} \left(\frac{x_1 - 2h_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} \right)^2}}{\sigma_1 \sqrt{2\pi t_1}} \varphi_7(x_1, x_2) \frac{e^{\frac{2\mu_2}{\sigma_2^2} (h_2 - x_2)}}{\sigma_2 \sqrt{2\pi(t_2 - t_1)}} \varphi_9(x_1, x_3, x_4) dx_4 dx_3 dx_2 dx_1 \quad (88)$$

Corollary 2 comes from the fact that the correlation coefficient between the random variables $S_2(t_2)$ and $S_3(t_3)$ is equal to $\rho_{2,3} \sqrt{\frac{t_2}{t_3}}$.

□

2.3. Option valuation and numerical implementation

The value of a two-colour outside up-and-out put, denoted by $V_{[\text{ORUU}]}$, is given by :

$$V_{[\text{ORUU}]} = e^{-rt_2} \left(E_Q \left[K_3 \mathbf{1}_{\{A\}} - S_3(t_2) \mathbf{1}_{\{A\}} \right] \right) \quad (89)$$

where A is the set constructed by the intersection of elements of the σ -algebra generated by the pair of processes $(S_1(t), S_2(t))$ that characterizes the probability $p_1(\mu_1, \mu_2)$ as given by the arguments of the probability operator in (54), and the acronym ' $[\text{ORUU}]$ ' stands for 'Outside Rainbow Up and Up'

Using the following orthogonal decomposition of Brownian motion $B_3(t)$:

$$B_3(t) = \rho_{1,3} W_1(t) + \rho_{2,3|1} W_2(t) + \sigma_{3|1,2} W_3(t) \quad (90)$$

where :

$$\rho_{2,3|1} = \frac{\rho_{2,3} - \rho_{1,2}\rho_{1,3}}{\sqrt{1 - \rho_{1,2}^2}}, \quad \sigma_{3|1,2} = \sqrt{1 - \rho_{1,3}^2 - \rho_{2,3|1}^2} \quad (91)$$

and $(W_1(t), W_2(t), W_3(t))$ is a basis of three independent Brownian motions (Guillaume, 2018), the multidimensional Cameron-Martin-Girsanov theorem yields :

$$V_{[\text{ORUU}]} = e^{-rt_2} K_3 p_1(\mu_1^{(Q)}, \mu_2^{(Q)}, \mu_3^{(Q)}) - S_3(0) p_1(\mu_1^{(P_3)}, \mu_2^{(P_3)}, \mu_3^{(P_3)}) \quad (92)$$

where :

$$\mu_1^{(P_3)} = r - \frac{\sigma_1^2}{2} + \sigma_1 \sigma_3 \rho_{1,3}, \quad \mu_2^{(P_3)} = r - \frac{\sigma_2^2}{2} + \sigma_2 \sigma_3 \rho_{2,3|1}, \quad \mu_3^{(P_3)} = r + \frac{\sigma_3^2}{2} \quad (93)$$

The measure P_3 is the measure under which $B_1(t) - \sigma_3 \rho_{1,3} t$, $B_2(t) - \sigma_3 \rho_{2,3|1} t$ and $B_3(t) - \sigma_3 \sigma_{3|1,2} t$ are three independent standard Brownian motions.

A simple and robust numerical evaluation of the function $\Psi_4[\dots, \dots, \dots]$ in Proposition 4 consists in selecting an appropriate cutoff value for the negative infinity lower bounds and then applying a fixed-degree quadrature rule. Given the smoothness of the integrand, even a low-degree rule will perform well. Table 3 provides the prices of a few two-colour outside up-and-down knock-out call options for various levels of the volatility and correlation parameters of the underlying assets

S_1 , S_2 and S_3 , and different values of the knock-out barriers. The parameters $S_i(0)$, K_i , t_1 , t_2 and r are identical as those as in Tables 1 and 2. In each cell, the first reported value is the exact analytical price as obtained by implementing Proposition 4 by means of a classical 16-point Gauss-Legendre quadrature, while the numbers in the brackets are three successive Monte Carlo approximations as explained in Section 1, subsection 1.4.

Table 3. Outside two-colour up-and-down knock-out call.

	$\rho_{1,2} = -0.6$	$\rho_{1,2} = -0.6$	$\rho_{1,2} = 0.6$	$\rho_{1,2} = 0.6$
	$\rho_{1,3} = \rho_{2,3}$ $= -0.4$	$\rho_{1,3} = \rho_{2,3}$ $= 0.4$	$\rho_{1,3} = \rho_{2,3}$ $= 0.4$	$\rho_{1,3} = \rho_{2,3}$ $= -0.4$
$\sigma_1 = \sigma_2 = 20\%$	2.378	2.772	1.717	1.182
$\sigma_3 = 20\%$	(2.452, 2.361, 2.375)	(2.914, 2.812, 2.779)	(1.585, 1.731, 1.720)	(1.193, 1.178, 1.180)
$H_1 = 115, H_2 = 85$				
$\sigma_1 = \sigma_2 = 20\%$	5.769	7.053	4.522	2.897
$\sigma_3 = 60\%$	(5.728, 5.781, 5.764)	(7.137, 7.036, 7.054)	(4.534, 4.541, 4.524)	(2.852, 2.923, 2.893)
$H_1 = 115, H_2 = 85$				
$\sigma_1 = \sigma_2 = 60\%$	1.627	2.783	1.351	0.849
$\sigma_3 = 20\%$	(1.592, 1.614, 1.628)	(2.848, 2.767, 2.788)	(1.320, 1.365, 1.353)	(0.915, 0.828, 0.842)
$H_1 = 125, H_2 = 75$				
$\sigma_1 = \sigma_2 = 60\%$	3.864	7.554	3.823	2.076
$\sigma_3 = 60\%$	(3.814, 3.872, 3.865)	(7.518, 7.535, 7.558)	(3.856, 3.829, 3.827)	
$H_1 = 125, H_2 = 75$				
$\sigma_1 = 20\%, \sigma_2 = 60\%$	1.534	2.573	1.188	0.621
$\sigma_3 = 20\%$	(1.502, 1.526, 1.535)	(2.495, 2.556, 2.577)	(1.207, 1.179, 1.185)	(0.774, 0.684, 0.613)
$H_1 = 115, H_2 = 75$				
$\sigma_1 = 20\%, \sigma_2 = 60\%$	3.629	6.697	3.217	1.518
	(3.787, 3.662, 3.629)	(6.724, 6.684, 6.697)	(3.051, 3.252, 3.217)	(1.586, 1.476, 1.518)

$\sigma_3 = 60\%$	3.621)	6.692)	3.221)	1.511)
$H_1 = 115, H_2 = 75$				
$\sigma_1 = 60\%, \sigma_2 = 20\%$	2.572	2.989	1.931	1.496
	(2.734, 2.548,	(3.125, 3.016,	(2.071, 1.965,	(1.634, 1.454,
$\sigma_3 = 20\%$	2.567)	2.994)	1.938)	1.489)
$H_1 = 125, H_2 = 85$				
$\sigma_1 = 60\%, \sigma_2 = 20\%$	6.248	7.953	5.323	3.707
	(6.304, 6.237,	(8.060, 7.984,	(5.212, 5.348,	(3.569, 3.726,
$\sigma_3 = 60\%$	6.241)	7.961)	5.324)	3.702)
$H_1 = 125, H_2 = 85$				

From a purely numerical standpoint, the pattern of convergence of conditional Monte Carlo approximations to the analytical values is as clear in Table 3 as in Table 1 and Table 2. This illustrates the robustness of our numerical integration scheme for $\Psi_4[\cdot, \cdot, \cdot, \cdot; \cdot, \cdot, \cdot, \cdot]$ functions. The efficiency gap between Monte Carlo pricing and analytical pricing is even more pronounced than for non-outside rainbow step barrier options, due to the presence of an additional stochastic process to simulate : the average computational time required by simulation is 42 seconds, whereas the evaluation of the analytical formula based on Proposition 4 only takes a few tenths of a second.

From a financial point of view, the prices in Table 3 display a very different pattern from those in Table 1 and Table 2. With regard to correlation, the highest option values attained are when the correlation between S_1 and S_2 is negative and the correlation between S_3 and both S_1 and S_2 is positive. The lowest option values are when the correlation between S_1 and S_2 is positive and the the correlation between S_3 and both S_1 and S_2 is negative. On average across all volatilities and barrier levels in Table 3, options are approximately 3 times more expensive under the former correlation structure than under the latter one. In terms of volatility, the highest option values attained are when the volatility of asset S_3 is high. This remains true under very different combinations of values for all the other parameters (volatilities of S_1 and S_2 , barrier levels and correlation structure). Such an observation highlights the prominent role of the volatility of the asset chosen to determine the moneyness of the option at expiry. In particular, the value of a rainbow outside step barrier option is a monotonically increasing function of σ_3 , whereas the value of a rainbow step barrier option is not a monotonically increasing function of σ_2 , just like the value of a reverse rainbow step barrier option is not a monotonically increasing function of σ_1 . This is because a rainbow outside step barrier option allows to make a clear distinction between the functions of each underlying asset : two of them, S_1 and S_2 , are only concerned with barrier crossing during the option life, and the third one, S_3 , is only concerned with moneyness testing at the option expiry. That distinction is impossible to make when it comes to non-outside rainbow step barrier options, so that the impact of volatility becomes ambiguous and difficult to handle. It should be emphasized that, for the vast majority of parameters, the sensitivities of rainbow outside step knock-out barrier options to σ_1 and σ_2 is negative, reflecting an increased risk of being deactivated before expiry. Only for quite specific correlation structures between the underlying assets and quite specific

combinations of barrier values can these sensitivities be positive. A major advantage of closed form formulae such as those derived in this article is precisely to allow to measure such sensitivities with high precision by mere differentiation of the formulae w.r.t. the relevant parameters.

One more noticeable difference in the reported numerical results between outside and non-outside rainbow step barrier options is that the volatility effect prevails over the barrier effect in Table 3, in contrast to Table 1 and Table 2. Indeed, in row 2 of Table 3, tight barrier levels do not preclude relatively high option prices thanks to the volatility of asset S_3 set at 60%. Likewise, in row 3 of Table 3, wider barrier levels do not preclude relatively low option prices due to the volatility of asset S_3 set at only 20%.

3. Two-sided two-colour rainbow barrier option valuation

In this final Section, a two-sided barrier is introduced in each time interval, i.e. the valuation of rainbow step double barrier options is handled.

3.1. Statement of Proposition 5

Let H_1 and H_2 denote an upward and a downward barrier, respectively, on the time interval $[t_0 = 0, t_1]$. Similarly, H_3 and H_4 represent an upward and a downward barrier, respectively, on the time interval $[t_1, t_2]$. As in the previous sections, barrier crossing is monitored w.r.t. process S_1 following eq. (1) on $[t_0 = 0, t_1]$ and w.r.t. process S_2 following eq. (2) on $[t_1, t_2]$. Our objective now is to find the value of the joint cumulative distribution function $P_{[\text{RDKO}]}(\mu_1, \mu_2)$ defined by :

$$P_{[\text{RDKO}]}(\mu_1, \mu_2) = P \left(\begin{array}{l} \sup_{0 \leq t \leq t_1} S_1(t) \leq H_1, \inf_{0 \leq t \leq t_1} S_1(t) \geq H_2, S_1(t_1) \leq \min(K_1, H_1), S_1(t_1) \geq H_2, S_2(t_1) \leq H_3, \\ S_2(t_1) \geq H_4, \sup_{t_1 \leq t \leq t_2} S_2(t) \leq H_3, \inf_{t_1 \leq t \leq t_2} S_2(t) \geq H_4, S_2(t_2) \leq \min(H_3, K_2) \end{array} \right) \quad (94)$$

where the acronym ' $[\text{RDKO}]$ ' stands for 'Rainbow Double Knock Out'

Proposition 5

The exact value of $P_{[\text{RDKO}]}(\mu_1, \mu_2)$ is given by :

$$P_{[\text{RDKO}]}(\mu_1, \mu_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \exp \left(\frac{2\mu_1}{\sigma_1^2} n_1 a_1 + \frac{2\mu_2}{\sigma_2^2} n_2 a_2 \right) \quad (95)$$

$$\left\{ N_3 \left[A_1(\min(k_1, h_1)), A_2(h_3), A_3(\min(h_3, k_2)); x_1 \right] - N_3 \left[\begin{array}{l} A_1(h_2), A_2(h_3), \\ A_3(\min(h_3, k_2)); x_1 \end{array} \right] \right.$$

$$\left. - N_3 \left[A_1(\min(k_1, h_1)), A_2(h_4), A_3(\min(h_3, k_2)); x_1 \right] + N_3 \left[\begin{array}{l} A_1(h_2), A_2(h_4), \\ A_3(\min(h_3, k_2)); x_1 \end{array} \right] \right.$$

$$\left. - N_3 \left[A_1(\min(k_1, h_1)), A_2(h_3), A_3(h_4); x_1 \right] + N_3 \left[A_1(h_2), A_2(h_3), A_3(h_4); x_1 \right] \right.$$

$$\left. + N_3 \left[A_1(\min(k_1, h_1)), A_2(h_4), A_3(h_4); x_1 \right] - N_3 \left[A_1(h_2), A_2(h_4), A_3(h_4); x_1 \right] \right\}$$

$$- \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \exp \left(n_1 a_1 \left(\frac{2\mu_1}{\sigma_1^2} - \frac{4\rho_{1,2}\mu_2}{\sigma_1\sigma_2} \right) + \frac{2\mu_2}{\sigma_2^2} (h_4 - n_2 a_2) \right) \quad (96)$$

$$\begin{aligned} & \left\{ N_3 \left[A_4 \left(\min(k_1, h_1) \right), A_5(h_3), A_6 \left(\min(h_3, k_2) \right); x_2 \right] - N_3 \left[\begin{matrix} A_4(h_2), A_5(h_3), \\ A_6 \left(\min(h_3, k_2) \right); x_2 \end{matrix} \right] \right. \\ & - N_3 \left[A_4 \left(\min(k_1, h_1) \right), A_5(h_4), A_6 \left(\min(h_3, k_2) \right); x_2 \right] + N_3 \left[\begin{matrix} A_4(h_2), A_5(h_4), \\ A_6 \left(\min(h_3, k_2) \right); x_2 \end{matrix} \right] \\ & - N_3 \left[A_4 \left(\min(k_1, h_1) \right), A_5(h_3), A_6(h_4); x_2 \right] + N_3 \left[A_4(h_2), A_5(h_3), A_6(h_4); x_2 \right] \\ & \left. + N_3 \left[A_4 \left(\min(k_1, h_1) \right), A_5(h_4), A_6(h_4); x_2 \right] - N_3 \left[A_4(h_2), A_5(h_4), A_6(h_4); x_2 \right] \right\} \\ & - \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \exp \left(\frac{2\mu_1}{\sigma_1^2} (h_2 - n_1 a_1) + \frac{2\mu_2}{\sigma_2^2} n_2 (h_3 - a_2) \right) \quad (97) \end{aligned}$$

$$\begin{aligned} & \left\{ N_3 \left[A_7 \left(\min(k_1, h_1) \right), A_8(h_3), A_9 \left(\min(h_3, k_2) \right); x_1 \right] - N_3 \left[\begin{matrix} A_7(h_2), A_8(h_3), \\ A_9 \left(\min(h_3, k_2) \right); x_1 \end{matrix} \right] \right. \\ & - N_3 \left[A_7 \left(\min(k_1, h_1) \right), A_8(h_4), A_9 \left(\min(h_3, k_2) \right); x_1 \right] + N_3 \left[\begin{matrix} A_7(h_2), A_8(h_4), \\ A_9 \left(\min(h_3, k_2) \right); x_1 \end{matrix} \right] \\ & - N_3 \left[A_7 \left(\min(k_1, h_1) \right), A_8(h_3), A_9(h_4); x_1 \right] + N_3 \left[A_7(h_2), A_8(h_3), A_9(h_4); x_1 \right] \\ & \left. + N_3 \left[A_7 \left(\min(k_1, h_1) \right), A_8(h_4), A_9(h_4); x_1 \right] - N_3 \left[A_7(h_2), A_8(h_4), A_9(h_4); x_1 \right] \right\} \\ & + \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \exp \left((h_2 - n_1 a_1) \left(\frac{2\mu_1}{\sigma_1^2} - \frac{4\rho_{1,2}\mu_2}{\sigma_1\sigma_2} \right) + \frac{2\mu_2}{\sigma_2^2} (h_4 - n_2 a_2) \right) \quad (98) \end{aligned}$$

$$\begin{aligned} & \left\{ N_3 \left[A_{10} \left(\min(k_1, h_1) \right), A_{11}(h_3), A_{12} \left(\min(h_3, k_2) \right); x_2 \right] - N_3 \left[\begin{matrix} A_{10}(h_2), A_{11}(h_3), \\ A_{12} \left(\min(h_3, k_2) \right); x_2 \end{matrix} \right] \right. \\ & - N_3 \left[A_{10} \left(\min(k_1, h_1) \right), A_{11}(h_4), A_{12} \left(\min(h_3, k_2) \right); x_2 \right] + N_3 \left[\begin{matrix} A_{10}(h_2), A_{11}(h_4), \\ A_{12} \left(\min(h_3, k_2) \right); x_2 \end{matrix} \right] \\ & - N_3 \left[A_{10} \left(\min(k_1, h_1) \right), A_{11}(h_3), A_{12}(h_4); x_2 \right] + N_3 \left[A_{10}(h_2), A_{11}(h_3), A_{12}(h_4); x_2 \right] \\ & \left. + N_3 \left[A_{10} \left(\min(k_1, h_1) \right), A_{11}(h_4), A_{12}(h_4); x_2 \right] - N_3 \left[A_{10}(h_2), A_{11}(h_4), A_{12}(h_4); x_2 \right] \right\} \end{aligned}$$

where :

$$-h_1 = \ln\left(\frac{H_1}{S_1(0)}\right) > 0, h_2 = \ln\left(\frac{H_2}{S_1(0)}\right) < 0, h_3 = \ln\left(\frac{H_3}{S_2(0)}\right), h_4 = \ln\left(\frac{H_4}{S_2(0)}\right) \quad (99)$$

$$-a_1 = h_1 - h_2, a_2 = h_3 - h_4 \quad (100)$$

$$-A_1(x) = \frac{x - 2n_1a_1 - \mu_1t_1}{\sigma_1\sqrt{t_1}}, A_2(x) = \frac{x - \mu_2t_1}{\sigma_2\sqrt{t_1}} - \frac{2\rho_{1.2}n_1a_1}{\sigma_1\sqrt{t_1}} \quad (101)$$

$$-A_3(x) = \frac{x - 2n_2a_2 - \mu_2t_2}{\sigma_2\sqrt{t_2}} - \frac{2\rho_{1.2}n_1a_1}{\sigma_1\sqrt{t_2}} \quad (102)$$

$$-A_4(x) = A_1(x) - \frac{2\rho_{1.2}\mu_2\sqrt{t_1}}{\sigma_2}, A_5(x) = \frac{x + \mu_2t_1}{\sigma_2\sqrt{t_1}} - \frac{2\rho_{1.2}n_1a_1}{\sigma_1\sqrt{t_1}} \quad (103)$$

$$-A_6(x) = \frac{x - 2h_4 + 2n_2a_2 - \mu_2t_2}{\sigma_2\sqrt{t_2}} + \frac{2\rho_{1.2}n_1a_1}{\sigma_1\sqrt{t_2}} \quad (104)$$

$$-A_7(x) = \frac{x - 2h_2 + 2n_1a_1 - \mu_1t_1}{\sigma_1\sqrt{t_1}}, A_8(x) = \frac{x - \mu_2t_1}{\sigma_2\sqrt{t_1}} - \frac{2\rho_{1.2}(h_2 - n_1a_1)}{\sigma_1\sqrt{t_1}} \quad (105)$$

$$-A_9(x) = \frac{x - 2n_2a_2 - \mu_2t_2}{\sigma_2\sqrt{t_2}} - \frac{2\rho_{1.2}(h_2 - n_1a_1)}{\sigma_1\sqrt{t_2}} \quad (106)$$

$$-A_{10}(x) = A_7(x) - \frac{2\rho_{1.2}\mu_2\sqrt{t_1}}{\sigma_2}, A_{11}(x) = \frac{x + \mu_2t_1}{\sigma_2\sqrt{t_1}} - \frac{2\rho_{1.2}(h_2 - n_1a_1)}{\sigma_1\sqrt{t_1}} \quad (107)$$

$$-A_{12}(x) = \frac{x - 2h_4 + 2n_2a_2 - \mu_2t_2}{\sigma_2\sqrt{t_2}} + \frac{2\rho_{1.2}(h_2 - n_1a_1)}{\sigma_1\sqrt{t_2}} \quad (108)$$

$$-x_1 = \left\{ \rho_{1.2}, \rho_{1.2}\sqrt{\frac{t_1}{t_2}}, \sqrt{\frac{t_1}{t_2}} \right\}, x_2 = \left\{ \rho_{1.2}, -\rho_{1.2}\sqrt{\frac{t_1}{t_2}}, -\sqrt{\frac{t_1}{t_2}} \right\} \quad (109)$$

All other notations have been defined in the previous sections.

End of Proposition 5.

3.2. Proof of Proposition 5

Following steps similar to the beginnings of the previous proofs, one can express the problem at hand as the following integral :

$$P_{[\text{RDKO}]}(\mu_1, \mu_2) \quad (110)$$

$$= \int_{x_1=h_2}^{\min(k_1, h_1)} \int_{x_2=h_4}^{h_3} \int_{x_3=h_4}^{\min(k_2, h_3)} \varphi_1(x_1, x_2, x_3) \varphi_{11}(x_1, x_2) \varphi_{12}(x_2, x_3) dx_3 dx_2 dx_1$$

where $\varphi_1(x_1, x_2, x_3)$ is given by eq. (30) and :

$$\varphi_{11}(x_1, x_2) = P \left(\sup_{0 \leq t \leq t_1} X_1(t) \leq h_1, \inf_{0 \leq t \leq t_1} X_1(t) \geq h_2 \mid X_1(t_1) \in dx_1 \right) dx_2 dx_1 \quad (111)$$

$$\varphi_{12}(x_2, x_3) \quad (112)$$

$$= P \left(\sup_{t_1 \leq t \leq t_2} X_2(t) \leq h_3, \inf_{t_1 \leq t \leq t_2} X_2(t) \geq h_4 \mid X_2(t_1) \in dx_2, X_2(t_2) \in dx_3 \right) dx_3 dx_2$$

From Pötzelberger and Wang (2001), one can plug :

$$\varphi_{11}(x_1, x_2) = \sum_{n=-\infty}^{\infty} e^{\frac{2na_1(x_1-na_1)}{\sigma_1^2 t_1}} - e^{\frac{2(h_1-na_1)(x_1-h_1+na_1)}{\sigma_1^2 t_1}} \quad (113)$$

$$\varphi_{12}(x_2, x_3) = \sum_{n=-\infty}^{\infty} e^{\frac{2na_2(x_3-x_2-na_2)}{\sigma_2^2(t_2-t_1)}} - e^{\frac{2(h_3-x_2-na_2)(x_3-h_3+na_2)}{\sigma_2^2(t_2-t_1)}} \quad (114)$$

The bulk of the proof, whose cumbersome details are omitted, then consists in solving the sixteen integrals implied by (110). The final result takes the form of the double sum of $N_3[\cdot, \cdot, \cdot, \cdot, \cdot, \cdot]$ functions in Proposition 5.

An elementary adjustment identical to the one in Corollary 3 of Proposition 1 allows to value an early-ending variant of $P_{[\text{RDKO}]}(\mu_1, \mu_2)$.

Alternatively, one can also expand the problem as the following integral :

$$P_{[\text{RDKO}]}(\mu_1, \mu_2) = \int_{x_1=h_2}^{\min(k_1, h_1)} \int_{x_2=h_4}^{h_3} \int_{x_3=h_4}^{\min(k_2, h_3)} \varphi_{13}(x_1) \varphi_7(x_1, x_2) \varphi_{14}(x_2, x_3) dx_3 dx_2 dx_1 \quad (115)$$

where $\varphi_7(x_1, x_2)$ is given by eq. (79) and :

$$\varphi_{13}(x_1) = P \left(\sup_{0 \leq t \leq t_1} X_1(t) \leq h_1, \inf_{0 \leq t \leq t_1} X_1(t) \geq h_2, X_1(t_1) \in dx_1 \right) dx_1 \quad (116)$$

$$\varphi_{14}(x_2, x_3) \quad (117)$$

$$= P \left(\sup_{t_1 \leq t \leq t_2} X_2(t) \leq h_3, \inf_{t_1 \leq t \leq t_2} X_2(t) \geq h_4, X_2(t_2) \in dx_3 \mid X_2(t_1) \in dx_2 \right) dx_3 dx_2$$

According to the classical formula for the distribution of the maximum, the minimum and the endpoint of a Brownian motion with drift over a closed time interval, which can be traced back to Anderson (1960), we have :

$$\begin{aligned} & P \left(\sup_{0 \leq t \leq t_1} S_1(t) < H_1, \inf_{0 \leq t \leq t_1} S_1(t) > H_2, S_1(t_1) < S_1(0) e^{x_1} \mid S_1(0) \right) \\ &= \sum_{n_1=-\infty}^{\infty} \exp \left(\frac{2\mu_1}{\sigma_1^2} n_1 a_1 \right) \left\{ N \left[\frac{x_1 - 2n_1 a_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} \right] - N \left[\frac{h_2 - 2n_1 a_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}} \right] \right\} \end{aligned} \quad (118)$$

$$- \sum_{n_1=-\infty}^{\infty} \exp\left(\frac{2\mu_1}{\sigma_1^2}(h_2 - n_1 a_1)\right) \left\{ N\left[\frac{x_1 - 2h_2 + 2n_1 a_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right] - N\left[\frac{-h_2 + 2n_1 a_1 - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}\right] \right\}$$

for a given $x_1 \in \mathbb{R}$, and $\forall H_1 > S_1(0)e^{x_1}$

Mere differentiation of (118) w.r.t. x_1 yields $\varphi_{13}(x_1)$:

$$\begin{aligned} & \varphi_{13}(x_1) \\ &= \sum_{n_1=-\infty}^{\infty} \frac{e^{\frac{2\mu_1}{\sigma_1^2} n_1 a_1 - \frac{1}{2\sigma_1^2 t_1} (x_1 - \mu_1 t_1 - 2n_1 a_1)^2}}{\sigma_1 \sqrt{2\pi t_1}} - \sum_{n_1=-\infty}^{\infty} \frac{e^{\frac{2\mu_1}{\sigma_1^2} (h_2 - n_1 a_1) - \frac{1}{2\sigma_1^2 t_1} (x_1 - 2h_2 - \mu_1 t_1 + 2n_1 a_1)^2}}{\sigma_1 \sqrt{2\pi t_1}} \end{aligned} \quad (119)$$

To handle the function $\varphi_{14}(x_2, x_3)$, we notice that, by conditioning w.r.t. the filtration at time t_1 the same classical formula as the one used to derive $\varphi_{13}(x_1)$, we can obtain :

$$\begin{aligned} & P\left(\sup_{t_1 \leq t \leq t_2} S_2(t) < H_3, \inf_{t_1 \leq t \leq t_2} S_2(t) > H_4, S_2(t_2) \leq S_2(0)e^{x_3} \mid S_2(t_1) = S_2(0)e^{x_2}\right) \\ &= \sum_{n_2=-\infty}^{\infty} \exp\left(\frac{2\mu_2}{\sigma_2^2} n_2 a_2\right) \left\{ N\left[\frac{\ln\left(\frac{S_2(0)e^{x_3}}{S_2(0)e^{x_2}}\right) - 2n_2 a_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right] \right. \\ & \quad \left. - N\left[\frac{\ln\left(\frac{H_4}{S_2(0)e^{x_2}}\right) - 2n_2 a_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right] \right\} \end{aligned} \quad (120)$$

$$- \sum_{n_2=-\infty}^{\infty} \exp\left(\frac{2\mu_2}{\sigma_2^2} \left(\ln\left(\frac{H_4}{S_2(0)e^{x_2}}\right) - n_2 a_2\right)\right) \quad (121)$$

$$\left\{ N\left[\frac{\ln\left(\frac{S_2(0)e^{x_3}}{S_2(0)e^{x_2}}\right) - 2\ln\left(\frac{H_4}{S_2(0)e^{x_2}}\right) + 2n_2 a_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right] \right. \\ \left. - N\left[\frac{-\ln\left(\frac{H_4}{S_2(0)e^{x_2}}\right) + 2n_2 a_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right] \right\}$$

for any given $(x_2, x_3) \in \mathbb{R}^2$ and $H_3 > S_2(0)e^{x_3}$

Eq. (120) – (121) can be rewritten as follows :

$$P\left(\sup_{t_1 \leq t \leq t_2} X_2(t) < h_3, \inf_{t_1 \leq t \leq t_2} X_2(t) > h_4, X_2(t_2) \leq x_3 \mid X_2(t_1) \in dx_2\right)$$

$$= \sum_{n_2=-\infty}^{\infty} \exp\left(\frac{2\mu_2}{\sigma_2^2} n_2 a_2\right) \left\{ \frac{N\left[\frac{x_3 - x_2 - 2n_2 a_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right]}{-N\left[\frac{h_4 - x_2 - 2n_2 a_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right]} \right\} \quad (122)$$

$$- \sum_{n_2=-\infty}^{\infty} \exp\left(\frac{2\mu_2}{\sigma_2^2} ((h_4 - x_2) - n_2 a_2)\right) \left\{ \frac{N\left[\frac{x_3 - x_2 - 2(h_4 - x_2) + 2n_2 a_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right]}{-N\left[\frac{-(h_4 - x_2) + 2n_2 a_2 - \mu_2(t_2 - t_1)}{\sigma_2 \sqrt{t_2 - t_1}}\right]} \right\} \quad (123)$$

Therefore, by differentiating (122) – (123) w.r.t. x_3 , we obtain :

$$\varphi_{14}(x_2, x_3) = \sum_{n_2=-\infty}^{\infty} e^{\left(\frac{2\mu_2}{\sigma_2^2} n_2 a_2\right)} \frac{e^{-\frac{1}{2\sigma_2^2(t_2-t_1)}(x_3-x_2-2n_2a_2-\mu_2(t_2-t_1))^2}}{\sigma_2 \sqrt{2\pi(t_2-t_1)}} \quad (124)$$

$$- \sum_{n_2=-\infty}^{\infty} e^{\frac{2\mu_2}{\sigma_2^2}(h_4-x_2-n_2a_2)} \frac{e^{-\frac{1}{2\sigma_2^2(t_2-t_1)}(x_3+x_2-2h_4+2n_2a_2-\mu_2(t_2-t_1))^2}}{\sigma_2 \sqrt{2\pi(t_2-t_1)}} \quad (125)$$

This second formulation leads to a formula identical to Proposition 5 except for the fact that the $N_3[\dots, \dots, \dots]$ functions are replaced by $\Phi_3[\dots, \dots, \dots]$ functions defined as follows :

$$\Phi_3[b_1, b_2, b_3; c_1, c_2] = \int_{x=-\infty}^{b_2} \frac{\exp(-x^2/2)}{\sqrt{2\pi}} N\left[\frac{b_1 - c_1 x}{\sqrt{1 - c_1^2}}\right] N\left[\frac{b_3 - c_2 x}{\sqrt{1 - c_2^2}}\right] dx \quad (126)$$

$$\forall b_1, b_2, b_3 \in \mathbb{R}^3, \forall c_1, c_2 \in]-1, 1[$$

The vectors of correlation coefficients x_1 and x_2 become :

$$x_1 = \left\{ \rho_{1.2}, \sqrt{\frac{t_1}{t_2}} \right\}, x_2 = \left\{ \rho_{1.2}, -\sqrt{\frac{t_1}{t_2}} \right\} \quad (127)$$

The origin of the function $\Phi_3[\dots, \dots, \dots]$, which is a special form of trivariate normal cumulative distribution, lies in the FDD (Finite Dimensional Distribution) of the triple $[S_1(t_1), S_2(t_1), S_2(t_2)]$.

Indeed, a little algebra shows that, $\forall D_1, D_2, D_3 \in \mathbb{R}^+$, we have :

$$P(S_1(t_1) \leq D_1, S_2(t_1) \leq D_2, S_2(t_2) \leq D_3)$$

$$= \Phi_3 \left[\frac{\ln \left(\frac{D_1}{S_1(0)} \right) - \mu_1 t_1}{\sigma_1 \sqrt{t_1}}, \frac{\ln \left(\frac{D_2}{S_2(0)} \right) - \mu_2 t_1}{\sigma_2 \sqrt{t_1}}, \frac{\ln \left(\frac{D_3}{S_2(0)} \right) - \mu_2 t_2}{\sigma_2 \sqrt{t_2}}; \theta_{1,2}, \theta_{2,3} \right] \tag{128}$$

Notice that the two-colour probability distributions of Section 1 can also be written as linear combinations of functions $\Phi_3[\dots,\dots,\dots]$.

□

3.3. Option valuation and numerical implementation

Pricing two-colour double knock-out barrier options can be achieved through the same changes of probability measures as those applicable to two-colour single knock-out barrier options, i.e.the value of a two-colour double knock-out put, denoted as $V_{[RDKO]}$, is given by :

$$V_{[RDKO]} = e^{-rt_2} K_3 P_{[RDKO]} \left(\mu_1^{(Q)}, \mu_2^{(Q)} \right) - S_2(0) P_{[RDKO]} \left(\mu_1^{(P_2)}, \mu_2^{(P_2)} \right) \tag{129}$$

where the parameters $\mu_i^{(Q)}$ and $\mu_i^{(P_2)}$ are given by eq. (50)

Table 4 provides the prices of a few two-colour knock-out double barrier puts for various levels of the volatility and correlation parameters of the underlying assets S_1 and S_2 , and different values of the knock-out barriers. Expiry is 6 months and t_1 is one quarter of a year. The parameters $S_i(0)$, K_i , and r are identical as those as in Tables 1, 2 and 3. In each cell, the first number is the exact analytical value as derived from (129), while the numbers in the brackets are three successive Monte Carlo approximations as explained in Section 1, subsection 1.4.

Table 4. : Two-colour rainbow double knock-out put.

	$\rho_{1,2} = -0.6$	$\rho_{1,2} = -0.2$	$\rho_{1,2} = 0.2$	$\rho_{1,2} = 0.6$
$\sigma_1 = \sigma_2 = 15\%$	2.919	2.938	2.936	3.027
$H_1 = H_3 = 120$	(2.985, 2.956,	(2.792, 2.883,	(2.974, 2.918,	(3.191, 3.088,
$H_2 = H_4 = 80$	2.916)	2.932)	2.931)	3.032)
$\sigma_1 = \sigma_2 = 30\%$	4.370	5.791	4.427	5.175
$H_1 = H_3 = 130$	(4.529, 4.281,	(5.978, 5.697,	(4.196, 4.443,	(5.329, 5.092,
$H_2 = H_4 = 70$	4.365)	5.786)	4.426)	5.158)
$\sigma_1 = 15\%, \sigma_2 = 30\%$	4.726	4.744	4.724	5.042
$H_1 = 120, H_3 = 130$	(4.594, 4.752,	(4.868, 4.785,	(4.574, 4.771,	(5.226, 5.018,
	4.728)	4.749)	4.728)	

$H_2 = 80, H_4 = 70$	5.045)			
$\sigma_1 = 30\%, \sigma_2 = 15\%$	2.614	2.683	2.771	2.942
$H_1 = 130, H_3 = 120$	(2.429, 2.576,	(2.872, 2.612,	(2.602, 2.742,	(3.165, 3.036,
	2.610)	2.679)	2.773)	2.953)
$H_2 = 70, H_4 = 80$				

Thanks to the rapidly decaying exponential functions in the integrands, a level of 10^{-7} convergence is attained by stopping at 8 the number of iterations controlled by the absolute values of n_1 and n_2 in the double sum operators, which results in a total computational time of less than 1 second. For higher values of the volatility parameters than those in Table 4, however, the uniform convergence of the double sums in (95) – (98) may require a greater number of iterations and thus take more time. The implementation of Proposition 5 using the $\Phi_3[\dots,\dots]$ function is slightly faster than the one using the trivariate standard normal cumulative distribution function $N_3[\dots,\dots,\dots]$, although the difference is relatively negligible for most practical purposes. Both methods of implementation yield prices equal to at least 4 decimals.

From a financial standpoint, a striking contrast between the numerical results in Table 4 and those of the previous sections is the much weaker dependency of the option value on the correlation structure, as illustrated by the smaller differences between the four option prices associated with each combination of volatility and barrier parameters. It seems that, the more volatility, the more dependency on the correlation structure, as suggested by the comparison between row 1 and row 2. Another noticeable difference is that the functional relation with the correlation structure is not monotonic. This is particularly clear in row 2 where a relatively significant increase in value from $\rho_{1.2} = -0.6$ to $\rho_{1.2} = -0.2$ is followed by a relatively significant decrease in value from $\rho_{1.2} = -0.2$ to $\rho_{1.2} = 0.2$, before a new increase in value from $\rho_{1.2} = 0.2$ to $\rho_{1.2} = 0.6$. This more complex and unstable dependency on correlation structure suggests that two-colour knock-out double barrier options are a less suitable instrument for correlation trading than two-colour knock-out single barrier options. However, one should remain wary of drawing hasty conclusions from the comparison of the results in Table 4 and those in the previous sections, as the option parameters are not identical, especially as regards volatility and expiry.

Conclusion

This article has shown how to value in closed form an important kind of multi-asset step barrier option known as a rainbow step barrier option, under the condition that the number of ‘colours’ is restricted to two, along with widespread variants such as a two-colour outside step barrier and a two-colour step double barrier. It may be feasible, albeit tedious, to find an analytical solution to an extended valuation problem with three or four colours, but the expected benefits, compared with a conditional Monte Carlo approximation method, would greatly depend on the degree of the quadrature required to numerically evaluate the resulting multidimensional integrals. It should be emphasized that, even if more sophisticated models (allowing, e.g., for stochastic volatility) or a greater number of underlying assets are needed, closed form solutions obtained in low-dimensional Black-Scholes framework remain useful as fast and accurate benchmarks that can : (i) serve as control variates in a simulation; (ii) speed up the calibration process; (iii) facilitate the analysis and the understanding of the interactions between the variables, as well as of the sensitivities of the option value w.r.t. its main parameters, which is instrumental in devising appropriate hedging techniques or trading strategies.

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