Article

Not peer-reviewed version

Combined Compact Symplectic Schemes for Solving Good Boussinesq Equations

Zhenyu Lang , Xiuling Yin * , Yanqin Liu , Zhiguo Chen , Shuxia Kong

Posted Date: 21 June 2024

doi: 10.20944/preprints202406.1509.v1

Keywords: Hamiltonian system; good Boussinesq equation; symplectic scheme; combined compact scheme; conservation

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 June 2024 doi:10.20944/preprints202406.1509.v1

Article

Combined Compact Symplectic Schemes for Solving Good Boussinesq Equations

Zhenyu Lang [1], Xiuling Yin [2,*], Yanqin Liu [2], Zhiguo Chen [2] and Shuxia Kong [2]

[1] School of Mathematics and Statistics, Shandong University of Technology, Zibo, 255049, China; proflang@163.com; Tel.: +86-1855-362-7823 (Z.L.)
[2] School of Mathematics and Big Data, Dezhou University, Dezhou, 253023, China
[*] Correspondence: yinllmm@163.com; Tel.: +86-1831-591-6282

Abstract: Good Boussinesq equations will be considered in this work. First we apply three combined compact schemes to approximate spatial derivatives of good Boussinesq equations. Then three fully discrete schemes are developed based on symplectic scheme in time direction, which are sympletic-structure preserving. Meanwhile, the convergence and conservation of the fully discrete schemes are analyzed. Finally, we present numerical experiments to confirm our theoretical analysis. Both our analysis and numerical test indicate that the fully discrete schemes are efficient in solving the spatial derivative mixed equation.

Keywords: Hamiltonian system; good Boussinesq equation; symplectic scheme; combined compact scheme; conservation

---

1. Introduction

Boussinesq equations are important mathematical physical models in characterizing ocean mixing, atmospheric convection, and intra-Earth convection, which play a key role in fields such as earth sciences, meteorology, and oceanography fields. The study of Boussinesq equation is of great value because it helps us to better understand the hydrodynamic behavior, especially in terms of thermal convection, ocean currents, and atmospheric phenomena. In addition, the study of the Boussinesq equation is essential for the development of numerical models for weather forecasting, climate research, and oceanography. These studies have also helped uncover the fundamental principles that govern fluid motion and heat transfer, contributing to advances in fields such as engineering, environmental science, and geophysics. The GB equation and its various extensions have been extensively analyzed in the existing literature, such as a closed form solution for the two soliton interaction in [1],a highly complicated mechanism for the solitary waves interaction in [2],and the nonlinear stability and convergence of some simple finite difference schemes in [3]. In recent works concerning the numerical solution of PDE, a significant amount addresses the Schrödinger equation, see [4–9]. In [9],using a combined compact difference method to solve Schrödinger equation and this scheme is Structure-Preserving.This method originated from [10],it also can be found in[11–14]. In addition,many works related to GB equations could be found in [15–21].Higher order Boussinesq equations have been investigated by Z.L. Zou [22].

In solving PDEs numerically, high-order compact (HOC) schemes are often used to discretize spatial derivatives.For example,HOC schemes have been applied to solve steady convection-diffusion equation [12], nonlinear Schrödinger equation [9], Klein-Gordon-Schrö-dinger equation [23], and good Boussinesq equation [17].Compared with general finite difference schemes, HOC schemes have the advantages of smaller error and higher accuracy under the same calculation amount. However, for PDEs with multiple order spatial derivatives, such as good Boussinesq equation $u_{tt} = -u_{xxxx} + u_{xx} + (u^2)_{xx}$, the advantages of classical HOC schemes are often offset. If multiple HOC schemes are used to discretize multiple spatial derivatives simultaneously, it is necessary to perform multiple matrix inverse operations, which will reduce the computational efficiency and affect the accuracy. In [9], Then combined high-order compact (CHOC) scheme is used to approximate PDEs with multiple order spatial derivatives and achieve some discrete conservation laws.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 June 2024 doi:10.20944/preprints202406.1509.v1

In this paper, three CHOC schemes of good Boussinesq equation are derived. Applying Taylor analysis to an equality combining the solution u and its first derivative, second derivative, and third derivative yields the first three-point CHOC scheme. This scheme has 6th-order precision and has extensive application. Then similarly, we propose the second three-point 8th-order scheme by using a combination of the first,second,third derivatives of the solution. Since the two schemes have a large amount of matrix operations and complex formulation, the third scheme is designed finally by composing the solution and its second derivative, fourth derivative, which greatly simplifies the matrix operations and ensures certain accuracy. In this scheme, through simpler computations, the relationship between the solution and its fourth-order derivative, as well as the relationship between the solution and its second-order derivative can be directly obtained, which can't be done by first two schemes in this paper. Finally we will use the three schemes to simulate a motion invariant, and summarize the advantages and disadvantages of these schemes. At the same time, compared with a three-point compact scheme with sixth order accuracy derived by Chu and Fan in 1998 [10], our schemes are more accurate.

In this paper, we consider fully discrete schemes for linear good Boussinesq equation

$$\partial_t^2 u = \partial_x^2 u - \partial_x^4 u, \tag{1.1}$$

where $0 \leq x \leq L, t > 0$, L is a constant. The following nonlinear good Boussinesq equation is also numerically solved

$$\partial_t^2 u = \partial_x^2 u - \partial_x^4 u + \partial_x^2 (u^2). \tag{1.2}$$

We consider initial conditions and periodic boundary conditions as follows

$$u(t,0) = u(t,L),\ u(0,x) = f_1(x), u_t(0,x) = f_2(x), 0 \leq x \leq L. \tag{1.3}$$

2. Establishment of the CHOC Scheme

In this paper, we introduce three schemes for discretization of spatial derivatives. To detail the CHOC scheme, we introduce a uniform grid $x_0 < x_1 \cdots < x_N$ with $x_j = x_0 + jh$ and $h = \frac{x_L - x_0}{N}, j = 1, 2, \cdots, N$. First, we introduce the simplest scheme (2.1) and (2.2)

$$\alpha_1 \left(u'_{j+1} + u'_{j-1} \right) + u'_j + \beta_1 h \left(u''_{j+1} - u''_{j-1} \right) + \gamma_1 \frac{u_{j+1} - u_{j-1}}{h} = 0, \tag{2.1}$$

$$\alpha_2 \left(\frac{u'_{j+1} - u'_{j-1}}{h} \right) + u''_j + \beta_2 \left(u''_{j+1} + u''_{j-1} \right) + \gamma_2 \frac{u_{j+1} - 2u_j + u_{j+1}}{h^2} = 0, \tag{2.2}$$

where $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ are coefficients to be determined according to the accuracy of the approximation. The three-point CHOC scheme for the combination of first and second derivatives is to relate u_j, u'_j, u''_j to their neighbors $u_{j-1}, u'_{j-1}, u''_{j-1}$ and $u_{j+1}, u'_{j+1}, u''_{j+1}$. This scheme approximates first-order derivative and second-order derivative of u separately using above combinations by Wang and Kong et al [9].

By inserting Taylor expansion to equation (2.1)and(2.2), we can get the following Tables 1 and 2.

Table 1. Taylor series of scheme(2.1).

Term	$u_j h^{-1}$	u'_j	$u''_j h$	$u'''_j h^2$	$u_j^{(4)} h^3$	$u_j^{(5)} h^4$	$u_j^{(6)} h^5$	$u_j^{(7)} h^6$	$u_j^{(8)} h^7$
$\alpha_1 u'_{j+1}$	0	α_1	α_1	$\frac{\alpha_1}{2!}$	$\frac{\alpha_1}{3!}$	$\frac{\alpha_1}{4!}$	$\frac{\alpha_1}{5!}$	$\frac{\alpha_1}{6!}$	$\frac{\alpha_1}{7!}$
$\alpha_1 u'_{j-1}$	0	α_1	$-\alpha_1$	$\frac{\alpha_1}{2!}$	$-\frac{\alpha_1}{3!}$	$\frac{\alpha_1}{4!}$	$-\frac{\alpha_1}{5!}$	$\frac{\alpha_1}{6!}$	$-\frac{\alpha_1}{7!}$
u'_j	0	1	0	0	0	0	0	0	0
$\beta_1 u''_{j+1} h$	0	0	β_1	β_1	$\frac{\beta_1}{2!}$	$\frac{\beta_1}{3!}$	$\frac{\beta_1}{4!}$	$\frac{\beta_1}{5!}$	$\frac{\beta_1}{6!}$
$-\beta_1 u''_{j-1} h$	0	0	$-\beta_1$	β_1	$-\frac{\beta_1}{2!}$	$\frac{\beta_1}{3!}$	$-\frac{\beta_1}{4!}$	$\frac{\beta_1}{5!}$	$-\frac{\beta_1}{6!}$
$\frac{\gamma_1}{h} u_{j+1}$	γ_1	γ_1	$\frac{\gamma_1}{2!}$	$\frac{\gamma_1}{3!}$	$\frac{\gamma_1}{4!}$	$\frac{\gamma_1}{5!}$	$\frac{\gamma_1}{6!}$	$\frac{\gamma_1}{7!}$	$\frac{\gamma_1}{8!}$
$-\frac{\gamma_1}{h} u_{j+1}$	$-\gamma_1$	γ_1	$-\frac{\gamma_1}{2!}$	$\frac{\gamma_1}{3!}$	$-\frac{\gamma_1}{4!}$	$\frac{\gamma_1}{5!}$	$-\frac{\gamma_1}{6!}$	$\frac{\gamma_1}{7!}$	$-\frac{\gamma_1}{8!}$
Σ	0	y_{11}	0	y_{12}	0	y_{13}	0	y_{14}	0

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 June 2024 doi:10.20944/preprints202406.1509.v1

Table 2. Taylor series of scheme(2.2).

Term	$u_j h^{-2}$	$u_j' h^{-1}$	u_j''	$u_j''' h$	$u_j^{(4)} h^2$	$u_j^{(5)} h^3$	$u_j^{(6)} h^4$	$u_j^{(7)} h^5$	$u_j^{(8)} h^6$
$\frac{\alpha_2}{h} u_{j+1}'$	0	α_2	α_2	$\frac{\alpha_2}{2!}$	$\frac{\alpha_2}{3!}$	$\frac{\alpha_2}{4!}$	$\frac{\alpha_2}{5!}$	$\frac{\alpha_2}{6!}$	$\frac{\alpha_2}{7!}$
$-\frac{\alpha_2}{h} u_{j-1}'$	0	$-\alpha_2$	α_2	$-\frac{\alpha_2}{2!}$	$\frac{\alpha_2}{3!}$	$-\frac{\alpha_2}{4!}$	$\frac{\alpha_2}{5!}$	$-\frac{\alpha_2}{6!}$	$\frac{\alpha_1}{7!}$
u_j''	0	0	1	0	0	0	0	0	0
$\beta_2 u_{j+1}''$	0	0	β_2	β_2	$\frac{\beta_2}{2!}$	$\frac{\beta_2}{3!}$	$\frac{\beta_2}{4!}$	$\frac{\beta_2}{5!}$	$\frac{\beta_2}{6!}$
$-\beta_2 u_{j-1}''$	0	0	β_2	$-\beta_2$	$\frac{\beta_2}{2!}$	$-\frac{\beta_2}{3!}$	$\frac{\beta_2}{4!}$	$-\frac{\beta_2}{5!}$	$\frac{\beta_2}{6!}$
$\frac{\gamma_2}{h^2} u_{j+1}$	γ_2	γ_2	$\frac{\gamma_2}{2!}$	$\frac{\gamma_2}{3!}$	$\frac{\gamma_2}{4!}$	$\frac{\gamma_2}{5!}$	$\frac{\gamma_2}{6!}$	$\frac{\gamma_2}{7!}$	$\frac{\gamma_2}{8!}$
$-2\frac{\gamma_2}{h^2} u_j$	$-2\gamma_2$	0	0	0	0	0	0	0	0
$\frac{\gamma_2}{h^2} u_{j-1}$	γ_2	$-\gamma_2$	$\frac{\gamma_2}{2!}$	$-\frac{\gamma_2}{3!}$	$\frac{\gamma_2}{4!}$	$-\frac{\gamma_2}{5!}$	$\frac{\gamma_2}{6!}$	$-\frac{\gamma_2}{7!}$	$\frac{\gamma_2}{8!}$
Σ	0	0	y_{21}	0	y_{22}	0	y_{23}	0	y_{24}

To make this scheme with sixth order convergence, above coefficients must satisfy the following algebraic equations:

$$\begin{cases} y_{11} = 2(\alpha_1 + \gamma_1) + 1 = 0, \\ y_{12} = \alpha_1 + 2(\beta_1 + \frac{\gamma_1}{3!}) = 0, \\ y_{13} = 2\left(\frac{\alpha_1}{4!} + \frac{\beta_1}{3!} + \frac{\gamma_1}{5!}\right) = 0, \end{cases} \tag{2.3}$$

and

$$\begin{cases} y_{21} = 2(\alpha_2 + \beta_2) + \gamma_2 + 1 = 0, \\ y_{22} = \frac{\alpha_2}{3} + \beta_2 + \frac{\gamma_2}{12} = 0, \\ y_{23} = \frac{\alpha_2}{5!} + \frac{\beta_2}{4!} + \frac{\gamma_2}{6!} = 0. \end{cases} \tag{2.4}$$

The solutions of above equations are

$$\alpha_1 = \frac{7}{16}, \beta_1 = -\frac{1}{16}, \gamma_1 = \frac{15}{16},$$

and

$$\alpha_2 = \frac{9}{8}, \beta_2 = -\frac{1}{8}, \gamma_2 = -3.$$

Therefore, schemes(2.1) and (2.2) are in the spacific forms

$$\frac{1}{16}\left(7u_{j+1}' + 16u_j' + 7u_{j-1}'\right) - \frac{h}{16}\left(u_{j+1}'' - u_{j-1}''\right) = \frac{15}{16h}(u_{j+1} - u_{j-1}), \tag{2.5}$$

$$\frac{9}{8h}\left(u_{j+1}' - u_{j-1}'\right) - \frac{1}{8}\left(u_{j+1}'' - 8u_j'' + u_{j-1}''\right) = \frac{3}{h^2}(u_{j+1} - 2u_j + u_{j-1}). \tag{2.6}$$

After conducting a thorough analysis, it is determined that this scheme has a relatively limited applicability. Its usage often necessitates complex matrix operations, and it is insufficient for differential equations involving certain high-order derivatives. For Good Boussinesq equation under study in this paper, a fourth-order spatial derivative is involved. To get the numerical solutions of Good Boussinesq equations, we need the discretization of $\partial_x^4 u$ and $\partial_x^2 u$. Here, we adopt the combination of function values of u and its first-order derivative, second-order derivative to represent fourth-order spatial derivative

$$u_j^{(4)} = -\frac{36}{h^4}(u_{j+1} - 2u_j + u_{j-1}) + \frac{21}{h^3}\left(u_{j+1}' + u_{j-1}'\right) - \frac{3}{h^2}\left(u_{j+1}'' + u_{j-1}''\right). \tag{2.7}$$

Under periodic boundary conditions, by combining (2.5) and (2.6) we have

$$\frac{1}{16}\begin{bmatrix} 16 & 7 & & & 7\\ 7 & 16 & 7 & & \\ & \ddots & \ddots & \ddots & \\ & & 7 & 16 & 7\\ 7 & & & 7 & 16\end{bmatrix}\begin{bmatrix}u_1'\\u_2'\\ \vdots\\ u_{N-1}'\\u_N'\end{bmatrix} - \frac{h}{16}\begin{bmatrix}0&1&&&-1\\-1&0&1&&\\&\ddots&\ddots&\ddots&\\&&-1&0&1\\1&&&-1&0\end{bmatrix}\begin{bmatrix}u_1''\\u_2''\\ \vdots\\ u_{N-1}''\\u_N''\end{bmatrix} = \frac{15}{16h}\begin{bmatrix}0&1&&&-1\\-1&0&1&&\\&\ddots&\ddots&\ddots&\\&&-1&0&1\\1&&&-1&0\end{bmatrix}\begin{bmatrix}u_1\\u_2\\ \vdots\\ u_{N-1}\\u_N\end{bmatrix}$$

$$\frac{9}{8h}\begin{bmatrix}0&1&&&-1\\-1&0&1&&\\&\ddots&\ddots&\ddots&\\&&-1&0&1\\1&&&-1&0\end{bmatrix}\begin{bmatrix}u_1'\\u_2'\\ \vdots\\ u_{N-1}'\\u_N'\end{bmatrix} - \frac{1}{8}\begin{bmatrix}-8&1&&&1\\1&-8&1&&\\&\ddots&\ddots&\ddots&\\&&1&-8&1\\1&&&1&-8\end{bmatrix}\begin{bmatrix}u_1''\\u_2''\\ \vdots\\ u_{N-1}''\\u_N''\end{bmatrix} = \frac{3}{h^2}\begin{bmatrix}-2&1&&&1\\1&-2&1&&\\&\ddots&\ddots&\ddots&\\&&1&-2&1\\1&&&1&-2\end{bmatrix}\begin{bmatrix}u_1\\u_2\\ \vdots\\ u_{N-1}\\u_N\end{bmatrix}$$

where

$$\mathrm{A}_{11} = \frac{1}{16}\begin{bmatrix} 16 & 7 & & & 7\\ 7 & 16 & 7 & & \\ & \ddots & \ddots & \ddots & \\ & & 7 & 16 & 7\\ 7 & & & 7 & 16\end{bmatrix}, \mathrm{A}_{12} = -\frac{\mathrm{h}}{16}\begin{bmatrix}0&1&&&-1\\-1&0&1&&\\&\ddots&\ddots&\ddots&\\&&-1&0&1\\1&&&-1&0\end{bmatrix}, \mathrm{A}_{13} = \frac{15}{16\mathrm{h}}\begin{bmatrix}0&1&&&-1\\-1&0&1&&\\&\ddots&\ddots&\ddots&\\&&-1&0&1\\1&&&-1&0\end{bmatrix}$$

$$\mathrm{A}_{21} = \frac{9}{8\mathrm{h}}\begin{bmatrix}0&1&&&-1\\-1&0&1&&\\&\ddots&\ddots&\ddots&\\&&-1&0&1\\1&&&-1&0\end{bmatrix}, \mathrm{A}_{22} = -\frac{1}{8}\begin{bmatrix}-8&1&&&1\\1&-8&1&&\\&\ddots&\ddots&\ddots&\\&&1&-8&1\\1&&&1&-8\end{bmatrix}, \mathrm{A}_{23} = \frac{3}{\mathrm{h}^2}\begin{bmatrix}-2&1&&&1\\1&-2&1&&\\&\ddots&\ddots&\ddots&\\&&1&-2&1\\1&&&1&-2\end{bmatrix}$$

Therefore, we can represent it in the following form:

$$\begin{cases}\mathrm{A}_{11}U_x + \mathrm{A}_{12}U_{xx} = \mathrm{A}_{13}U\\ \mathrm{A}_{21}U_x + \mathrm{A}_{22}U_{xx} = \mathrm{A}_{23}U\end{cases} \tag{2.8}$$

By solving (2.8), we can obtain: $U_x = \mathrm{G}\cdot U, U_{xx} = \mathrm{H}\cdot U, \mathrm{G} = \mathrm{A}^{-1}\mathrm{B}, \mathrm{H} = \mathrm{A}^{-1}\mathrm{C}$, where $\mathrm{A} = \mathrm{A}_{11}\mathrm{A}_{22} - \mathrm{A}_{12}\mathrm{A}_{21}, \mathrm{B} = \mathrm{A}_{22}\mathrm{A}_{13} - \mathrm{A}_{12}\mathrm{A}_{23}, \mathrm{C} = \mathrm{A}_{11}\mathrm{A}_{23} - \mathrm{A}_{21}\mathrm{A}_{13}$. For (2.7) we have

$$\begin{bmatrix}\mathrm{u}_1^{(4)}\\\mathrm{u}_2^{(4)}\\ \vdots\\ \mathrm{u}_{N-1}^{(4)}\\\mathrm{u}_N^{(4)}\end{bmatrix} = -\frac{36}{\mathrm{h}^4}\begin{bmatrix}-2&1&&&1\\1&-2&1&&\\&\ddots&\ddots&\ddots&\\&&1&-2&1\\1&&&1&-2\end{bmatrix}\begin{bmatrix}\mathrm{u}_1\\\mathrm{u}_2\\ \vdots\\ \mathrm{u}_{N-1}\\\mathrm{u}_N\end{bmatrix} + \frac{21}{\mathrm{h}^3}\begin{bmatrix}0&1&&&-1\\-1&0&1&&\\&\ddots&\ddots&\ddots&\\&&-1&0&1\\1&&&-1&0\end{bmatrix}\begin{bmatrix}\mathrm{u}_1'\\\mathrm{u}_2'\\ \vdots\\ \mathrm{u}_{N-1}'\\\mathrm{u}_N'\end{bmatrix} - \frac{3}{\mathrm{h}^2}\begin{bmatrix}0&1&&&1\\1&0&1&&\\&\ddots&\ddots&\ddots&\\&&1&0&1\\1&&&1&0\end{bmatrix}\begin{bmatrix}\mathrm{u}_1''\\\mathrm{u}_2''\\ \vdots\\ \mathrm{u}_{N-1}''\\\mathrm{u}_N''\end{bmatrix},$$

where

$$B_1 = -\frac{3}{h^2}\begin{bmatrix}0&1&&&1\\1&0&1&&\\&\ddots&\ddots&\ddots&\\&&1&0&1\\1&&&1&0\end{bmatrix}.$$

By substituting $U_x = G\cdot U, U_{xx} = H\cdot U$ into it, above expression can be represented as follows

$$\partial_x^4 U = -\frac{12}{\mathrm{h}^2}\mathrm{A}_{23}\cdot U + \frac{56}{3h^2}\mathrm{A}_{21}\mathrm{G}\cdot U + \mathrm{BH}\cdot U. \tag{2.9}$$

Let $\mathrm{M} = \mathrm{B}\cdot\mathrm{H} + \frac{56}{3\mathrm{h}^2}\mathrm{A}_{21}\cdot\mathrm{G} - \frac{12}{\mathrm{h}^2}\mathrm{A}_{23}$. We will have the following schemes to the spatial derivatives

$$\begin{cases}\partial_x^4 U = \mathrm{M}\cdot U,\\ \partial_x^2 U = \mathrm{H}\cdot U.\end{cases} \tag{2.10}$$

Next, we will give second CHOC scheme with eighth order accuracy with the combination of first, second and third derivatives relating $u_j, u_j^{'}, u_j^{''}, u_j^{'''}$ to their neighbors $u_{j-1}, u_{j-1}^{'}, u_{j-1}^{''}, u_{j-1}^{'''}$ and $u_{j+1}, u_{j+1}^{'}, u_{j+1}^{''}, u_{j+1}^{'''}$. Generalization of (2.1) and (2.2) to the case of three derivatives jields similarly the next CHOC scheme

$$\alpha_1\left(u_{j+1}^{'}+u_{j-1}^{'}\right)+u_j^{'}+\beta_1 h\left(u_{j+1}^{''}-u_{j-1}^{''}\right)+\omega_1 h^2\left(u_{j+1}^{'''}+u_{j-1}^{'''}\right)+\gamma_1\frac{u_{j+1}-u_{j-1}}{h}=0, \tag{2.11}$$

$$\alpha_2\left(\frac{u_{j+1}^{'}-u_{j-1}^{'}}{h}\right)+u_j^{''}+\beta_2\left(u_{j+1}^{''}+u_{j-1}^{''}\right)+\omega_2 h\left(u_{j+1}^{'''}-u_{j-1}^{'''}\right)+\gamma_2\frac{u_{j+1}-2u_j+u_{j+1}}{h^2}=0, \tag{2.12}$$

$$\alpha_3\left(\frac{u_{j+1}^{'}+u_{j-1}^{'}}{h}\right)+u_j^{'''}+\beta_3\left(\frac{u_{j+1}^{''}-u_{j-1}^{''}}{h}\right)+\omega_3\left(u_{j+1}^{'''}+u_{j-1}^{'''}\right)+\gamma_3\frac{u_{j+1}-u_{j-1}}{h^3}=0, \tag{2.13}$$

where $\alpha_1, \beta_1, \omega_1, \gamma_1$ and $\alpha_2, \beta_2, \omega_2, \gamma_2$ and $\alpha_3, \beta_3, \omega_3, \gamma_3$ are coefficients to be determined according to the accuracy of the approximation. By Taylor expansion of equation(2.11),(2.12) and (2.13) we can get Tables 3–5.

Table 3. Taylor series of scheme(2.11).

Term	$u_j h^{-1}$	$u_j^{'}$	$u_j^{''}h$	$u_j^{'''}h^2$	$u_j^{(4)}h^3$	$u_j^{(5)}h^4$	$u_j^{(6)}h^5$	$u_j^{(7)}h^6$	$u_j^{(8)}h^7$
$\alpha_1 u_{j+1}^{'}$	0	α_1	α_1	$\frac{\alpha_1}{2!}$	$\frac{\alpha_1}{3!}$	$\frac{\alpha_1}{4!}$	$\frac{\alpha_1}{5!}$	$\frac{\alpha_1}{6!}$	$\frac{\alpha_1}{7!}$
$\alpha_1 u_{j-1}^{'}$	0	α_1	$-\alpha_1$	$\frac{\alpha_1}{2!}$	$-\frac{\alpha_1}{3!}$	$\frac{\alpha_1}{4!}$	$-\frac{\alpha_1}{5!}$	$\frac{\alpha_1}{6!}$	$-\frac{\alpha_1}{7!}$
$u_j^{'}$	0	1	0	0	0	0	0	0	0
$\beta_1 u_{j+1}^{''}h$	0	0	β_1	β_1	$\frac{\beta_1}{2!}$	$\frac{\beta_1}{3!}$	$\frac{\beta_1}{4!}$	$\frac{\beta_1}{5!}$	$\frac{\beta_1}{6!}$
$-\beta_1 u_{j-1}^{''}h$	0	0	$-\beta_1$	β_1	$-\frac{\beta_1}{2!}$	$\frac{\beta_1}{3!}$	$-\frac{\beta_1}{4!}$	$\frac{\beta_1}{5!}$	$-\frac{\beta_1}{6!}$
$\omega_1 u_{j+1}^{'''}h^2$	0	0	0	ω_1	ω_1	$\frac{\omega_1}{2!}$	$\frac{\omega_1}{3!}$	$\frac{\omega_1}{4!}$	$\frac{\omega_1}{5!}$
$\omega_1 u_{j-1}^{'''}h^2$	0	0	0	ω_1	$-\omega_1$	$\frac{\omega_1}{2!}$	$-\frac{\omega_1}{3!}$	$\frac{\omega_1}{4!}$	$-\frac{\omega_1}{5!}$
$\frac{\gamma_1}{h}u_{j+1}$	γ_1	γ_1	$\frac{\gamma_1}{2!}$	$\frac{\gamma_1}{3!}$	$\frac{\gamma_1}{4!}$	$\frac{\gamma_1}{5!}$	$\frac{\gamma_1}{6!}$	$\frac{\gamma_1}{7!}$	$\frac{\gamma_1}{8!}$
$-\frac{\gamma_1}{h}u_{j+1}$	$-\gamma_1$	γ_1	$-\frac{\gamma_1}{2!}$	$\frac{\gamma_1}{3!}$	$-\frac{\gamma_1}{4!}$	$\frac{\gamma_1}{5!}$	$-\frac{\gamma_1}{6!}$	$\frac{\gamma_1}{7!}$	$-\frac{\gamma_1}{8!}$
Σ	0	y_{11}	0	y_{12}	0	y_{13}	0	y_{14}	0

Table 4. Taylor series of scheme(2.12).

Term	$u_j h^{-2}$	$u_j^{'}h$	$u_j^{''}$	$u_j^{'''}h$	$u_j^{(4)}h^2$	$u_j^{(5)}h^3$	$u_j^{(6)}h^4$	$u_j^{(7)}h^5$	$u_j^{(8)}h^6$
$\frac{\alpha_2}{h}u_{j+1}^{'}$	0	α_2	α_2	$\frac{\alpha_2}{2!}$	$\frac{\alpha_2}{3!}$	$\frac{\alpha_2}{4!}$	$\frac{\alpha_2}{5!}$	$\frac{\alpha_2}{6!}$	$\frac{\alpha_2}{7!}$
$-\frac{\alpha_2}{h}u_{j-1}^{'}$	0	$-\alpha_2$	α_2	$-\frac{\alpha_2}{2!}$	$\frac{\alpha_2}{3!}$	$-\frac{\alpha_2}{4!}$	$\frac{\alpha_2}{5!}$	$-\frac{\alpha_2}{6!}$	$\frac{\alpha_2}{7!}$
$u_j^{''}$	0	0	1	0	0	0	0	0	0
$\beta_2 u_{j+1}^{''}$	0	0	β_2	β_2	$\frac{\beta_2}{2!}$	$\frac{\beta_2}{3!}$	$\frac{\beta_2}{4!}$	$\frac{\beta_2}{5!}$	$\frac{\beta_2}{6!}$
$\beta_2 u_{j-1}^{''}$	0	0	β_2	$-\beta_2$	$\frac{\beta_2}{2!}$	$-\frac{\beta_2}{3!}$	$\frac{\beta_2}{4!}$	$-\frac{\beta_2}{5!}$	$\frac{\beta_2}{6!}$
$\omega_2 u_{j+1}^{'''}h$	0	0	0	ω_2	ω_2	$\frac{\omega_2}{2!}$	$\frac{\omega_2}{3!}$	$\frac{\omega_2}{4!}$	$\frac{\omega_2}{5!}$
$-\omega_2 u_{j-1}^{'''}h$	0	0	0	$-\omega_2$	ω_2	$-\frac{\omega_2}{2!}$	$\frac{\omega_2}{3!}$	$-\frac{\omega_2}{4!}$	$\frac{\omega_2}{5!}$
$\frac{\gamma_2}{h^2}u_{j+1}$	γ_2	γ_2	$\frac{\gamma_2}{2!}$	$\frac{\gamma_2}{3!}$	$\frac{\gamma_2}{4!}$	$\frac{\gamma_2}{5!}$	$\frac{\gamma_2}{6!}$	$\frac{\gamma_2}{7!}$	$\frac{\gamma_2}{8!}$
$-\frac{2\gamma_2}{h^2}u_j$	$-2\gamma_2$	0	0	0	0	0	0	0	0
$\frac{\gamma_2}{h^2}u_{j-1}$	γ_2	$-\gamma_2$	$\frac{\gamma_2}{2!}$	$-\frac{\gamma_2}{3!}$	$\frac{\gamma_2}{4!}$	$-\frac{\gamma_2}{5!}$	$\frac{\gamma_2}{6!}$	$-\frac{\gamma_2}{7!}$	$\frac{\gamma_2}{8!}$
Σ	0	0	y_{21}	0	y_{22}	0	y_{23}	0	y_{24}

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 June 2024 doi:10.20944/preprints202406.1509.v1

Table 5. Taylor series of scheme(2.13).

Term	$u_j h^{-3}$	$u_j' h^{-2}$	$u_j'' h^{-1}$	u_j'''	$u_j^{(4)} h$	$u_j^{(5)} h^2$	$u_j^{(6)} h^3$	$u_j^{(7)} h^4$	$u_j^{(8)} h^5$
$\frac{\alpha_3}{h^2} u_{j+1}'$	0	α_3	α_3	$\frac{\alpha_3}{2!}$	$\frac{\alpha_3}{3!}$	$\frac{\alpha_3}{4!}$	$\frac{\alpha_3}{5!}$	$\frac{\alpha_3}{6!}$	$\frac{\alpha_3}{7!}$
$\frac{\alpha_3}{h^2} u_{j-1}'$	0	α_3	$-\alpha_3$	$\frac{\alpha_3}{2!}$	$-\frac{\alpha_3}{3!}$	$\frac{\alpha_3}{4!}$	$-\frac{\alpha_3}{5!}$	$\frac{\alpha_3}{6!}$	$-\frac{\alpha_3}{7!}$
u_j'''	0	0	0	1	0	0	0	0	0
$\beta_3 u_{j+1}'' h^{-1}$	0	0	β_3	β_3	$\frac{\beta_3}{2!}$	$\frac{\beta_3}{3!}$	$\frac{\beta_3}{4!}$	$\frac{\beta_3}{5!}$	$\frac{\beta_3}{6!}$
$-\beta_3 u_{j-1}'' h^{-1}$	0	0	$-\beta_3$	β_3	$-\frac{\beta_3}{2!}$	$\frac{\beta_3}{3!}$	$-\frac{\beta_3}{4!}$	$\frac{\beta_3}{5!}$	$-\frac{\beta_3}{6!}$
$\omega_3 u_{j+1}'''$	0	0	0	ω_3	ω_3	$\frac{\omega_3}{2!}$	$\frac{\omega_3}{3!}$	$\frac{\omega_3}{4!}$	$\frac{\omega_3}{5!}$
$\omega_3 u_{j-1}'''$	0	0	0	ω_3	$-\omega_3$	$\frac{\omega_3}{2!}$	$-\frac{\omega_3}{3!}$	$\frac{\omega_3}{4!}$	$-\frac{\omega_3}{5!}$
$\frac{\gamma_3}{h^3} u_{j+1}$	γ_3	γ_3	$\frac{\gamma_3}{2!}$	$\frac{\gamma_3}{3!}$	$\frac{\gamma_3}{4!}$	$\frac{\gamma_3}{5!}$	$\frac{\gamma_3}{6!}$	$\frac{\gamma_3}{7!}$	$\frac{\gamma_3}{8!}$
$-\frac{\gamma_3}{h^3} u_{j-1}$	$-\gamma_3$	γ_3	$-\frac{\gamma_3}{2!}$	$\frac{\gamma_3}{3!}$	$-\frac{\gamma_3}{4!}$	$\frac{\gamma_3}{5!}$	$-\frac{\gamma_3}{6!}$	$\frac{\gamma_3}{7!}$	$-\frac{\gamma_3}{8!}$
Σ	0	y_{31}	0	y_{32}	0	y_{33}	0	y_{34}	0

To make these schemes of eighth order, they must satisfy the algebraic equations:

$$\begin{cases} y_{11} = 2(\alpha_1 + \gamma_1) + 1 = 0 \\ y_{12} = \alpha_1 + 2\left(\beta_1 + \omega_1 + \frac{\gamma_1}{3!}\right) = 0 \\ y_{13} = 2\left(\frac{\alpha_1}{4!} + \frac{\beta_1}{3!} + \frac{\omega_1}{2!} + \frac{\gamma_1}{5!}\right) = 0 \\ y_{14} = 2\left(\frac{\alpha_1}{6!} + \frac{\beta_1}{5!} + \frac{\omega_1}{4!} + \frac{\gamma_1}{7!}\right) = 0 \end{cases} \tag{2.14}$$

and

$$\begin{cases} y_{21} = 2\left(\alpha_2 + \beta_2 + \frac{\gamma_2}{2!}\right) + 1 = 0 \\ y_{22} = 2\left(\frac{\alpha_2}{3!} + \frac{\beta_2}{2!} + \omega_2 + \frac{\gamma_2}{4!}\right) = 0 \\ y_{23} = 2\left(\frac{\alpha_2}{5!} + \frac{\beta_2}{4!} + \frac{\omega_2}{3!} + \frac{\gamma_2}{6!}\right) = 0 \\ y_{24} = 2\left(\frac{\alpha_2}{7!} + \frac{\beta_2}{6!} + \frac{\omega_2}{5!} + \frac{\gamma_2}{8!}\right) = 0 \end{cases} \tag{2.15}$$

and

$$\begin{cases} y_{31} = 2(\alpha_3 + \gamma_3) = 0 \\ y_{32} = 2\left(\frac{\alpha_3}{2!} + \beta_3 + \omega_3 + \frac{\gamma_3}{3!}\right) + 1 = 0 \\ y_{33} = 2\left(\frac{\alpha_3}{4!} + \frac{\beta_3}{3!} + \frac{\omega_3}{2!} + \frac{\gamma_3}{5!}\right) = 0 \\ y_{34} = 2\left(\frac{\alpha_3}{6!} + \frac{\beta_3}{5!} + \frac{\omega_3}{4!} + \frac{\gamma_3}{7!}\right) = 0 \end{cases} \tag{2.16}$$

Their unique solutions are

$$\alpha_1 = \frac{19}{32}, \beta_1 = -\frac{1}{8}, \omega_1 = \frac{1}{96}, \gamma_1 = -\frac{35}{32}$$

and

$$\alpha_2 = \frac{29}{16}, \beta_2 = -\frac{5}{16}, \omega_2 = \frac{1}{48}, \gamma_2 = -4$$

and

$$\alpha_3 = -\frac{105}{16}, \beta_3 = \frac{15}{8}, \omega_3 = -\frac{3}{16}, \gamma_3 = \frac{105}{16}$$

respectively. Therefore, the scheme (2.11),(2.12) and (2.13) has in the following specific form:

$$\frac{1}{32}\left(19u_{j+1}' + 32u_j' + 19u_{j-1}'\right) - \frac{h}{8}\left(u_{j+1}'' - u_{j-1}''\right) + \frac{h^2}{96}\left(u_{j+1}''' + u_{j-1}'''\right) = \frac{35}{32h}\left(u_{j+1} - u_{j-1}\right) \tag{2.17}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 June 2024 doi:10.20944/preprints202406.1509.v1

$$\frac{\left(29u'_{j+1}-29u'_{j-1}\right)}{16h}-\frac{\left(5u''_{j+1}-16u''_{j}+5u''_{j-1}\right)}{16}+\frac{h\left(u'''_{j+1}-u'''_{j-1}\right)}{48}=\frac{4\left(u_{j+1}-2u_j+u_{j-1}\right)}{h^2} \tag{2.18}$$

$$\frac{\left(105u'_{j+1}+105u'_{j-1}\right)}{-16h^2}+\frac{\left(15u''_{j+1}-15u''_{j-1}\right)}{8h}-\frac{\left(3u'''_{j+1}-16u'''_{j}+3u'''_{j-1}\right)}{16}=\frac{105\left(u_{j+1}-u_{j-1}\right)}{-16h^3} \tag{2.19}$$

This three-point scheme possesses eighth-order accuracy and involve three derivatives, so it is more applicable and allows for greater accuracy in comparison to (2.1) and (2.2).

Next, we adopt the combination of function values of u and its first three derivatives to represent fourth-order spatial derivative

$$u_j^{(4)}=-\frac{72}{h^4}\left(u_{j+1}-2u_j+u_{j-1}\right)+\frac{183}{4h^3}\left(u'_{j+1}-u'_{j-1}\right)-\frac{39}{4h^2}\left(u''_{j+1}+u''_{j-1}\right)+\frac{3}{4h}\left(u'''_{j+1}-u'''_{j-1}\right) \tag{2.20}$$

By Combining (2.17), (2.18), (2.19) and (2.20) we obtain

$$\begin{bmatrix} u_1^{(4)} \\ u_2^{(4)} \\ \vdots \\ u_{N-1}^{(4)} \\ u_N^{(4)} \end{bmatrix}=-\frac{72}{h^4}\begin{bmatrix} -2 & 1 & 0 & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}+\frac{183}{4h^3}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}\begin{bmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_{N-1} \\ u'_N \end{bmatrix}-\frac{39}{4h^2}\begin{bmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ 1 & & & 1 & 0 \end{bmatrix}\begin{bmatrix} u''_1 \\ u''_2 \\ \vdots \\ u''_{N-1} \\ u''_N \end{bmatrix}$$
$$+\frac{3}{4h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}\begin{bmatrix} u'''_1 \\ u'''_2 \\ \vdots \\ u'''_{N-1} \\ u'''_N \end{bmatrix} \tag{2.21}$$

$$\frac{19}{32}\begin{bmatrix} \frac{32}{19} & 1 & & & 1 \\ 1 & \frac{32}{19} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \frac{32}{19} & 1 \\ 1 & & & 1 & \frac{32}{19} \end{bmatrix}\begin{bmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_{N-1} \\ u'_N \end{bmatrix}-\frac{h}{8}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}\begin{bmatrix} u''_1 \\ u''_2 \\ \vdots \\ u''_{N-1} \\ u''_N \end{bmatrix}+\frac{h^2}{96}\begin{bmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ 1 & & & 1 & 0 \end{bmatrix}\begin{bmatrix} u'''_1 \\ u'''_2 \\ \vdots \\ u'''_{N-1} \\ u'''_N \end{bmatrix}$$
$$=\frac{35}{32h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} \tag{2.17*}$$

$$\frac{29}{16h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}\begin{bmatrix} u'_1 \\ u'_2 \\ \vdots \\ u'_{N-1} \\ u'_N \end{bmatrix}-\frac{5}{16}\begin{bmatrix} -\frac{16}{5} & 1 & & & 1 \\ 1 & -\frac{16}{5} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -\frac{16}{5} & 1 \\ 1 & & & 1 & -\frac{16}{5} \end{bmatrix}\begin{bmatrix} u''_1 \\ u''_2 \\ \vdots \\ u''_{N-1} \\ u''_N \end{bmatrix}+\frac{h}{48}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}\begin{bmatrix} u'''_1 \\ u'''_2 \\ \vdots \\ u'''_{N-1} \\ u'''_N \end{bmatrix}$$
$$=\frac{4}{h^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} \tag{2.18*}$$

$$\frac{-105}{16h^2}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \\ \vdots \\ u_{N-1}' \\ u_N' \end{bmatrix}+\frac{15}{8h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}\begin{bmatrix} u_1'' \\ u_2'' \\ \vdots \\ u_{N-1}'' \\ u_N'' \end{bmatrix}-\frac{3}{16}\begin{bmatrix} -\frac{16}{3} & 1 & & & 1 \\ 1 & -\frac{16}{3} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -\frac{16}{3} & 1 \\ 1 & & & 1 & -\frac{16}{3} \end{bmatrix}\begin{bmatrix} u_1''' \\ u_2''' \\ \vdots \\ u_{N-1}''' \\ u_N''' \end{bmatrix}$$

$$= -\frac{105}{16h^3}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} \quad (2.19^*)$$

where

$$B_{11} = \frac{19}{32}\begin{bmatrix} \frac{32}{19} & 1 & & & 1 \\ 1 & \frac{32}{19} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \frac{32}{19} & 1 \\ 1 & & & 1 & \frac{32}{19} \end{bmatrix}, B_{12} = -\frac{h}{8}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}, B_{13} = \frac{h^2}{96}\begin{bmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ 1 & & & 1 & 0 \end{bmatrix}$$

$$B_{14} = \frac{35}{32h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}, B_{21} = \frac{29}{16h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}, B_{22} = -\frac{5}{16}\begin{bmatrix} -\frac{16}{5} & 1 & & & 1 \\ 1 & -\frac{16}{5} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -\frac{16}{5} & 1 \\ 1 & & & 1 & -\frac{16}{5} \end{bmatrix}$$

$$B_{23} = \frac{h}{48}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}, B_{24} = \frac{4}{h^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}, B_{31} = -\frac{105}{16h^2}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}$$

$$B_{32} = \frac{15}{8h}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}, B_{33} = -\frac{3}{16}\begin{bmatrix} -\frac{16}{3} & 1 & & & 1 \\ 1 & -\frac{16}{3} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -\frac{16}{3} & 1 \\ 1 & & & 1 & -\frac{16}{3} \end{bmatrix}, B_{34} = -\frac{105}{16h^3}\begin{bmatrix} 0 & 1 & & & -1 \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ 1 & & & -1 & 0 \end{bmatrix}$$

Therefore, we can represent it as follows

$$\begin{cases} B_{11}U_x + B_{12}U_{xx} + B_{13}U_{xxx} = B_{14}U \\ B_{21}U_x + B_{22}U_{xx} + B_{23}U_{xxx} = B_{24}U \\ B_{31}U_x + B_{32}U_{xx} + B_{33}U_{xxx} = B_{34}U \end{cases} \quad (2.22)$$

In light of solving (2.22), we can obtain: $U_x = G^* \cdot U, U_{xx} = H^* \cdot U, U_{xxx} = I \cdot U,\ G^* = D^{-1}D_1, H^* = D^{-1}D_2, I = D^{-1}D_3$. $D = B_{11}B_{22}B_{33} + B_{12}B_{23}B_{31} + B_{13}B_{21}B_{32} - B_{11}B_{23}B_{32} - B_{12}B_{21}B_{33} - B_{13}B_{22}B_{31}$, $D_1 = B_{14}B_{22}B_{33} + B_{14}B_{23}B_{31} + B_{13}B_{21}B_{34} - B_{11}B_{23}B_{34} - B_{14}B_{21}B_{33} - B_{13}B_{24}B_{31}$, $D_2 = B_{11}B_{22}B_{34} + B_{12}B_{24}B_{31} + B_{14}B_{21}B_{32} - B_{13}B_{24}B_{31} - B_{14}B_{21}B_{33} - B_{11}B_{23}B_{34}$,
$D_3 = B_{11}B_{22}B_{34} + B_{12}B_{24}B_{31} + B_{14}B_{21}B_{32} - B_{11}B_{24}B_{32} - B_{12}B_{21}B_{34} - B_{14}B_{22}B_{31}$. substituting $U_x = G^* \cdot U, U_{xx} = H^* \cdot U, U_{xxx} = I \cdot U$ into (2.21) gives that

$$\partial_x^4 U = -\frac{18}{h^2}B_{24} \cdot U - \frac{366}{h^4}B_{12}G^* \cdot U + \frac{936}{h^4}B_{13}H^* \cdot U - \frac{6}{h^2}B_{12}I \cdot U \quad (2.23)$$

let $\mathrm{M}^* = \frac{936}{h^4}\mathrm{B}_{13}\mathrm{H}^* - \frac{366}{h^4}\mathrm{B}_{12}\mathrm{G}^* - \frac{18}{h^2}\mathrm{B}_{24} - \frac{6}{h^2}\mathrm{B}_{12}\mathrm{I}$. we will get the following discrete schemes of the spatial derivative:

$$\begin{cases} \partial_x^4 U = \mathrm{M}^* \cdot U \\ \partial_x^2 U = \mathrm{H}^* \cdot U \end{cases} \tag{2.24}$$

For above scheme, we find that matrix operation becomes complicated. To get the discrete form of $\partial_x^2 u$ and $\partial_x^4 u$ according to the (2.11), (2.12) and (2.13), it requires many matrix operations, and subsequent simulation of numerical solution will be more difficult. So we consider constructing a direct combination of $\partial_x^2 u, \partial_x^4 u$ and u to look for third CHOC scheme. This scheme will maintain a 6th-order precision and can easily obtain the discrete forms of $\partial_x^2 u$ and $\partial_x^4 u$, which will be more pertinent and accurate This scheme has the following formulation

$$\alpha_1\left(\frac{u''_{j+1}+u''_{j-1}}{2}\right) + u''_j + \beta_1 h^2\left(\frac{u^{(4)}_{j+1}+u^{(4)}_{j-1}}{2}\right) = \gamma_1\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}, \tag{2.25}$$

$$\alpha_2\left(\frac{u''_{j+1}-2u''_j+u''_{j-1}}{h^2}\right) + u^{(4)}_j + \beta_2\left(\frac{u^{(4)}_{j+1}+u^{(4)}_{j-1}}{2}\right) = \gamma_2\frac{u_{j+2}-4u_{j+1}+6u_j-4u_{j-1}+u_{j-2}}{h^4}. \tag{2.26}$$

We insert Taylor expansions to (2.25) to obtain

$$\begin{aligned} LHS &= u''_j + \alpha_1\left(\frac{u''_{j+1}+u''_{j-1}}{2}\right) + \beta_1 h^2\left(\frac{u^{(4)}_{j+1}+u^{(4)}_{j-1}}{2}\right) \\ &= u''_j + \alpha_1\left(u''_j + \frac{h^2}{2!}u^{(4)}_j + \frac{h^4}{4!}u^{(6)}_j + o\left(h^6\right)\right) + \beta_1 h^2\left(u^{(4)}_j + \frac{h^2}{2!}u^{(6)}_j + \frac{h^4}{4!}u^{(8)}_j + o\left(h^6\right)\right) \\ &= (1+\alpha_1)u''_j + \left(\frac{\alpha_1}{2!}+\beta_1\right)h^2 u^{(4)}_j + \left(\frac{\alpha_1}{4!}+\frac{\beta_1}{2!}\right)h^4 u^{(6)}_j + o\left(h^6\right), \\ RHS &= \gamma_1\left(\frac{u_{j+1}-2u_j+u_{j-1}}{h^2}\right) \\ &= \gamma_1\left(u''_j + \frac{u^{(4)}_j}{12}h^2 + \frac{u^{(6)}_j}{360}h^4\right) + o\left(h^6\right). \end{aligned}$$

To make the scheme with sixth order accuracy, the coefficients must satisfy the algebraic equations

$$\begin{cases} 1+\alpha_1 = \frac{\gamma_1}{2!} \\ \frac{\alpha_1}{2!}+\beta_1 = \frac{\gamma_1}{4!} \\ \frac{\alpha_1}{4!}+\frac{\beta_1}{2!} = \frac{\gamma_1}{6!} \end{cases} \Rightarrow \begin{cases} \alpha_1 = \frac{14}{61} \\ \beta_1 = -\frac{3}{244} \\ \gamma_1 = \frac{75}{61} \end{cases}$$

Similarly, for(2.26)we have:

$$\begin{aligned} LHS &= u^{(4)}_j + \alpha_2\left(u^{(4)}_j + \frac{u^{(6)}_j}{12}h^2 + \frac{u^{(8)}_j}{360}h^4\right) + \beta_2\left(u^{(4)}_j + \frac{u^{(6)}_j h^2}{2!} + \frac{u^{(8)}_j h^4}{4!}\right) + o\left(h^6\right) \\ &= (1+\alpha_2+\beta_2)u^{(4)}_j + \left(\frac{\alpha_2}{12}+\frac{\beta_2}{2!}\right)h^2 u^{(6)}_j + \left(\frac{\alpha_2}{360}+\frac{\beta_2}{4!}\right)h^4 u^{(8)}_j + o\left(h^6\right) \\ RHS &= \gamma_2\frac{u_{j+2}-4u_{j+1}+6u_j-4u_{j-1}+u_{j-2}}{h^4} + o\left(h^6\right) \\ &= \gamma_2\left(\frac{2^5-2^3}{4!}u^{(4)}_j + \frac{2^7-2^3}{6!}h^2 u^{(6)}_j + \frac{2^9-2^3}{8!}h^4 u^{(8)}_j\right) + o\left(h^6\right) \end{aligned}$$

 doi:10.20944/preprints202406.1509.v1

To make the scheme with sixth order accuracy, coefficients must satisfy the algebraic equations

$$\begin{cases} 1+\alpha_2+\beta_2=\frac{2^5-2^3}{4!}\gamma_2 \\ \frac{\alpha_2}{12}+\frac{\beta_2}{2!}=\frac{2^7-2^3}{6!}\gamma_2 \\ \frac{\alpha_2}{360}+\frac{\beta_2}{4!}=\frac{2^9-2^3}{8!}\gamma_2 \end{cases} \Longrightarrow \begin{cases} \alpha_2=\frac{6}{7} \\ \beta_2=\frac{5}{7} \\ \gamma_2=\frac{18}{7} \end{cases}$$

Under periodic boundary conditions, for (2.25) and (2.26), we obtain

$$\frac{7}{61}\begin{bmatrix} \frac{61}{7} & 1 & & & 1 \\ 1 & \frac{61}{7} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \frac{61}{7} & 1 \\ 1 & & & 1 & \frac{61}{7} \end{bmatrix}\begin{bmatrix} u_1'' \\ u_2'' \\ \vdots \\ u_{N-1}'' \\ u_N'' \end{bmatrix}-\frac{3h^2}{488}\begin{bmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ 1 & & & 1 & 0 \end{bmatrix}\begin{bmatrix} u_1^{(4)} \\ u_2^{(4)} \\ \vdots \\ u_{N-1}^{(4)} \\ u_N^{(4)} \end{bmatrix}=\frac{75}{61h^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}$$

$$\frac{6}{7h^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}\begin{bmatrix} u_1'' \\ u_2'' \\ \vdots \\ u_{N-1}'' \\ u_N'' \end{bmatrix}+\frac{5}{14}\begin{bmatrix} \frac{14}{5} & 1 & & & 1 \\ 1 & \frac{14}{5} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \frac{14}{5} & 1 \\ 1 & & & 1 & \frac{14}{5} \end{bmatrix}\begin{bmatrix} u_1^{(4)} \\ u_2^{(4)} \\ \vdots \\ u_{N-1}^{(4)} \\ u_N^{(4)} \end{bmatrix}=\frac{18}{7h^4}\begin{bmatrix} 6 & -4 & & & -4 \\ -4 & 6 & -4 & & \\ & \ddots & \ddots & \ddots & \\ & & -4 & 6 & -4 \\ -4 & & & -4 & 6 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}$$

where let

$$A_{11}^*=\frac{7}{61}\begin{bmatrix} \frac{61}{7} & 1 & & & 1 \\ 1 & \frac{61}{7} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \frac{61}{7} & 1 \\ 1 & & & 1 & \frac{61}{7} \end{bmatrix}, A_{12}^*=-\frac{3h^2}{488}\begin{bmatrix} 0 & 1 & & & 1 \\ 1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & 1 \\ 1 & & & 1 & 0 \end{bmatrix}, A_{13}^*=\frac{75}{61h^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}$$

$$A_{21}^*=\frac{6}{7h^2}\begin{bmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{bmatrix}, A_{22}^*=\frac{5}{14}\begin{bmatrix} \frac{14}{5} & 1 & & & 1 \\ 1 & \frac{14}{5} & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & \frac{14}{5} & 1 \\ 1 & & & 1 & \frac{14}{5} \end{bmatrix}, A_{23}^*=\frac{18}{7h^4}\begin{bmatrix} 6 & -4 & 1 & & & & 1 & -4 \\ -4 & 6 & -4 & 1 & & & & 1 \\ 1 & -4 & 6 & -4 & 1 & & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots & \ddots & 1 \\ 1 & & & & 1 & -4 & 6 & -4 \\ -4 & 1 & & & & 1 & -4 & 6 \end{bmatrix}$$

Therefore, we can represent it in the following form:

$$\begin{cases} A_{11}^*U_{xx}+A_{12}^*U_{xxxx}=A_{13}^*U \\ A_{21}^*U_{xx}+A_{22}^*U_{xxxx}=A_{23}^*U \end{cases} \tag{2.27}$$

With (2.27), we can readily derive U_{xx} and U_{xxxx} by expressions of U, respectively. This significantly streamlines the matrix operations. By solving (2.27), we can obtain $U_{xx}=G_1\cdot U, U_{xxxx}=H_1\cdot U$, where $G_1=A_1^{-1}B, H_1=A_1^{-1}C$, where $A_1=A_{11}^*A_{22}^*-A_{12}^*A_{21}^*, B_1=A_{22}^*A_{13}^*-A_{12}^*A_{23}^*, C_1=A_{11}^*A_{23}^*-A_{21}^*A_{13}^*$. For good Boussiensq equations, above scheme has higher spatial accuracy compared to the scheme given in [11].

3. Establishment of the Full Discrete Schemes

Let τ be temporal stepsize and $t_n=n\tau, n=0,1,2,\cdots,M$, where $M=T/\tau$. Denote the approximation of $u(x,t_n)$ by u^n. Define the following operators

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 June 2024 doi:10.20944/preprints202406.1509.v1

$$\delta_t u^{n+\frac{1}{2}} = \frac{u^{n+1} - u^n}{\tau}, \quad u^{n+\frac{1}{2}} = \frac{u^n + u^{n+1}}{2}.$$

Let $v = u_t$, the considered good Boussinesq equation can be written as

$$\partial_t v = \partial_x^2 u - \partial_x^4 u + \partial_x^2 (u^2).$$

Applying CHOC scheme (2.10) to the spatial derivatives of above equation gives that

$$\begin{aligned} v_t &= -M \cdot u + H \cdot u + H \cdot u^2 \\ &= N \cdot u + H \cdot u^2, \end{aligned}$$

where $N = H - M$. By adopting symplectic midpoint scheme with second-order accuracy to above equation, we have the following full discrete scheme

$$u^{n+1} = u^n + \frac{\tau}{2}\left(v^n + v^{n+1}\right), \tag{3.1}$$

$$-\frac{\tau}{2} N \cdot u^{n+1} + v^{n+1} = \frac{\tau}{2} N \cdot u^n + v^n + \frac{\tau}{2} H \cdot \left\{ \left(u^2\right)^n + \left(u^2\right)^{n+1} \right\}. \tag{3.2}$$

Similarly, using CHOC scheme (2.24) for the spatial derivatives yields that

$$\begin{aligned} v_t &= -M^* \cdot u + H^* \cdot u + H^* \cdot u^2 \\ &= N^* \cdot u + H^* \cdot u^2, \end{aligned}$$

where $N^* = H^* - M^*$. we obtain the corresponding full discrete scheme

$$-\frac{\tau}{2} N^* \cdot u^{n+1} + v^{n+1} = \frac{\tau}{2} N^* \cdot u^n + v^n + \frac{\tau}{2} H^* \cdot \left\{ \left(u^2\right)^n + \left(u^2\right)^{n+1} \right\}. \tag{3.3}$$

Applying CHOC scheme (2.27) for the spatial derivatives gives that

$$\begin{aligned} v_t &= -H_1 \cdot u + G_1 \cdot u + G_1 \cdot u^2 \\ &= N_1 \cdot u + G_1 \cdot u^2, \end{aligned}$$

where $N_1 = G_1 - H_1$. we get the following full discrete scheme similarly

$$-\frac{\tau}{2} N_1 \cdot u^{n+1} + v^{n+1} = \frac{\tau}{2} N_1 \cdot u^n + v^n + \frac{\tau}{2} G_1 \cdot \left\{ \left(u^2\right)^n + \left(u^2\right)^{n+1} \right\}. \tag{3.4}$$

By combining (3.2), (3.3), (3.4) with (3.1), respectively, we always obtain the algebraic equation as follows

$$A \cdot T^{n+1} = B \cdot T^n + F\left(T^{n+1}, T^n\right),$$

where A and B are some invertible tridiagonal matrices depending on the corresponding schemes, $T^n = [u^n \ , v^n]^\top$, and F is the corresponding nonlinear term.

For simplicity, we will denote the schemes corresponding to (3.2), (3.3) and (3.4) by CHOC-A, CHOC-B and CHOC-C, respectively.

4. Conservation Laws of CHOC Schemes

Under periodic boundary condition (1.3) with $L = 1$, some good Boussinesq equations have certain conservation laws. Below, we consider periodic domain $[0, 1]$.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 June 2024 doi:10.20944/preprints202406.1509.v1

Theorem 4.1. *Let $\|\cdot\|$ denote the standard L^2-norm for 1-periodic functions. Then along with $\partial_t^2 u = -\partial_x^4 u$, the quadratic functional*

$$\|u_t\|^2 + \|u_{xx}\|^2 \tag{4.1}$$

is an invariant of motion.

Proof. According to (1.3), by multiplying both sides of $\partial_t^2 u = -\partial_x^4 u$ by u_t and integrating it by parts, we have

$$\int_0^1 u_t \cdot u_{tt} dx = -\int_0^1 u_{xx} \cdot u_{txx} dx. \tag{4.2}$$

For (4.2), we take integration with respect to t to get

$$\int_0^t \int_0^1 u_t \cdot u_{tt}\, dx\, dt = -\int_0^t \int_0^1 u_{xx} u_{txx}\, dx\, dt,$$
$$\Rightarrow \int_0^1 u_t^2(x,t)\, dx + \int_0^1 u_{xx}^2(x,t)\, dx = \int_0^1 u_t^2(x,0)\, dx + \int_0^1 u_{xx}^2(x,0)\, dx.$$

Therefore we get (4.1).

Theorem 4.2. *For nonlinear Boussinesq equation (1.2), the following conservation law is satisfied*

$$\int_0^1 u_t(x,t)\, dx = \int_0^1 v^0(x)\, dx. \tag{4.3}$$

If $\int_0^1 v^0(x)\, dx = 0$, then we get the conservation law as follows

$$\int_0^1 u(x,t)\, dx = \int_0^1 u^0(x)\, dx. \tag{4.4}$$

Proof. For $t \in [0,T]$, integrate with respect to x on both sides of (1.2), we obtain

$$\int_0^1 u_{tt} dx = \int_0^1 u_{xx} - \partial_x^4 u + (u^2)_{xx} dx.$$

According to (1.3), the integration on right side of above equation is 0, i.e.

$$\int_0^1 u_{tt} dx = 0.$$

Taking integration with respect to t yields that $\int_0^1 u_t(x,t)dx = \int_0^1 v^0(x)dx$, that is (4.3). If $\int_0^1 v^0(x)\, dx = 0$, then $\int_0^1 u_t(x,t)dx = 0$, which gives (4.4) by integration with respect to t.

First, we are interested in its discrete version of Theorem 4.1 and Theorem 4.2 under numerical analysis for CHOC schemes. To this goal, we list some important properties of circulant matrices [9]. A matrix written in the form

$$Circ(c_0, c_1, c_2 \cdots c_{N-1}) = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\ \cdots & \ddots & \ddots & \ddots & \cdots \\ c_2 & c_3 & \ddots & \ddots & c_1 \\ c_1 & c_2 & \cdots & c_{N-1} & c_0 \end{bmatrix}$$

is said to be circulant matrix.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 June 2024 doi:10.20944/preprints202406.1509.v1

All of the matrices $A_{ij}(i,j=1,2.)$, $B_{ij}(i,j=1,2,3.)$, $A^*_{ij}(i,j=1,2.)$ are circulant. There are a lot of favourable characters of this kind of matrices.

We list some of them in the following which are useful in analyzing our schemes.

Proposition 4.3 ([9,23]). *If A, B are circulant matrices with the same number of rows and columns, then we have*
(i) $A+B$, $A-B$, AB are circulant matrices.
(ii) If A^{-1} is well defined, then A^{-1} is also a circulant matrix.
(iii) A, B are commutators, that is AB = BA.
(iv) If A, B are symmetric and positive definite matrices, then AB = BA is a symmetric and positive definite matrix.
(v) The eigenvalues of the circulant matrix above is $\lambda_j = \sum_{k=0}^{N-1} c_k e^{-ik\theta_j}$ with the corresponding eigenvector $w_j = \frac{1}{N}\left[1, e^{-i\theta_j}, \cdots, e^{-i(N-1)\theta_j}\right]^\intercal$.

By Proposition 4.3, we can get the following proposition.

Proposition 4.4. *For the matrices in the CHOC solvers (2.10), (2.24), (2.27), we have:*
(i) A_{11}, A_{22}, $-A_{23}$, B_{22}, B_{33}, A^*_{11}, A^*_{22}, A^*_{23} are symmetric and positive definite.
(ii) A_{12}, A_{13}, A_{21}, B_{12}, B_{14}, B_{21}, B_{23}, B_{34} are skew-symmetric.
(iii) B_{11}, B_{13}, B_{24}, B_{31}, B_{32}, A^*_{12}, A^*_{13}, A^*_{21} are symmetric.
(iv) A, A^{-1}, D, A_1, C_1 are symmetric and positive definite, and D_1, D_3 are skew-symmetric, D_2, B_1 are symmetric.
(v) M^*, H_1 are symmetric and positive definite, and G, G^*, I, are skew-symmetric, H, M, H^*, G_1, N, N^*, N_1 are symmetric.
(vi) All of them are circulant.

Proof. The conclusions(i)–(iv) and (vi) can be observed or be verified from Proposition 4.3. The result (v) can be derived from the last conclusion in Proposition 4.3 by finding their eigenvalues.

Theorem 4.5. *Define*

$$w^n = (v^n, v^n) + (u^n, R\cdot u^n),$$

where (u,v) is the standard unitary inner product in discrete level for finite dimensional sequence vectors, $v^n = [v^n_1, v^n_2, \cdots, v^n_{NX}]^\intercal, u^n = [u^n_1, u^n_2, \cdots, u^n_{NX}]^\intercal$. R is the matrix coefficient in approximating fourth-order spatial derivative expressed by M(2.10), M^*(2.24), H_1(2.27). Then for CHOC solvers (2.10), (2.24), (2.27) to solve $\partial_t^2 u = -\partial_x^4 u$, the numerical solutions satisfy that

$$\omega^{n+1} = w^n. \tag{4.5}$$

Moreover, to M^*(2.24), H_1(2.27), there exists C such that $C^\intercal \cdot C = R$, therefore

$$w^n = (v^n, v^n) + (C\cdot u^n, C\cdot u^n),$$

Proof. Since $v = u_t$, we have $v_t = -u_{xxxx}$, then its discrete scheme is

$$\frac{v^{n+1}-v^n}{\tau} = -\frac{Ru^{n+1}+Ru^n}{2}.$$

By multiplying $\frac{u^{n+1}-u^n}{\tau} = \frac{v^{n+1}+v^n}{2}$ on both sides of above formula, we have

$$\frac{(v^{n+1}+v^n)^\intercal(v^{n+1}-v^n)}{2\tau} = -\frac{(v^{n+1}+v^n)^\intercal}{2}\cdot\frac{Ru^{n+1}+Ru^n}{2}.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 June 2024 doi:10.20944/preprints202406.1509.v1

Therefore we obtain that

$$\left(v^{n+1}\right)^{\mathsf{T}}\left(v^{n+1}\right)-(v^n)^{\mathsf{T}}(v^n)=-\left(u^{n+1}\right)^{\mathsf{T}}R\cdot\left(u^{n+1}\right)-(u^n)^{\mathsf{T}}R\cdot(u^n),$$
$$\left(v^{n+1}\right)^{\mathsf{T}}\left(v^{n+1}\right)+\left(u^{n+1}\right)^{\mathsf{T}}\left(R\cdot u^{n+1}\right)=(v^n)^{\mathsf{T}}(v^n)+(u^n)^{\mathsf{T}}(R\cdot u^n),$$

that is $\omega^{n+1}=\omega^n$. Taking into symmetric positivity of M^*(2.24), H_1(2.27), there exists C such that $C^{\mathsf{T}}\cdot C=R$, which yields that

$$(u^n,R\cdot u^n)=(C\cdot u^n)^{\mathsf{T}}(C\cdot u^n).$$

Therefore, we obtain that

$$w^n=(v^n,v^n)+(C\cdot u^n,C\cdot u^n).$$

Theorem 4.6. *To nonlinear good Boussinesq equation (1.2) with periodic boundary condition, the schemes CHOC-A, CHOC-B and CHOC-C satisfy the following discrete conservation law*

$$\overline{u^n}=h\sum_{j=1}^{NX}u_j^n\equiv\overline{u^0} \tag{4.6}$$

provided that $\overline{v^0}=0$.

Proof. Firstly, we consider schemes CHOC-A to solve nonlinear system (1.2) and have the following formula

$$u^{n+1}=u^n+\tau\left(\frac{v^n+v^{n+1}}{2}\right),v^{n+1}=v^n+\frac{\tau}{2}N\left(u^{n+1}+u^n\right)+\frac{\tau}{2}H\left[(u^n)^2+\left(u^{n+1}\right)^2\right]. \tag{4.7}$$

By calculation, we obtain that

$$v^{n+1}=v^n+\frac{\tau}{2}\left(E-\frac{\tau^2}{4}N\right)^{-1}\left\{\tau Nv^n+2Nu^n+H\left[(u^n)^2+\left(u^{n+1}\right)^2\right]\right\}. \tag{4.8}$$

Construct the following iterative algorithm

$$\begin{aligned}v_{(k+1)}^{n+1}&=v^n+\frac{\tau}{2}\left(E-\frac{\tau^2}{4}N\right)^{-1}\left\{\tau Nv^n+2Nu^n+H\left[(u^n)^2+\left(u_{(k)}^{n+1}\right)^2\right]\right\},\\ u_{(k+1)}^{n+1}&=u^n+\tau\left(\frac{v^n+v_{(k+1)}^{n+1}}{2}\right),\end{aligned} \tag{4.9}$$

where $k=0,1,...$ and $u_{(0)}^{n+1}=u^n$. Then we get that

$$\lim_{k\to\infty}u_{(k)}^{n+1}=u^{n+1},\lim_{k\to\infty}v_{(k)}^{n+1}=v^{n+1}. \tag{4.10}$$

Considering $\overline{v^0}=0$ and the symmetry of N,H, we can obtain that $\overline{v_{(k+1)}^{n+1}}=\overline{v^n}=0,\overline{u_{(k+1)}^{n+1}}=\overline{u^n}$. The limit (4.10) yields that $\overline{v^{n+1}}=0,\overline{u^{n+1}}=\overline{u^n}$. The conservation identity (4.6) for the schemes CHOC-B and CHOC-C can be derived similarity.

For linear good Boussinesq equation (1.1), we have the following equivalent Hamiltonian system

$$\begin{cases}u_t=v,\\ v_t=u_{xx}-u_{xxxx},\end{cases} \tag{4.11}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 June 2024 doi:10.20944/preprints202406.1509.v1

with the Hamiltonian function $H = -\frac{1}{2}\int \left(v^2 + u_x^2 + u_{xx}^2 + V^2(u)\right)dx$. Thus we obtain the symlecitic conservation law as follows

$$w(t) = \int du \wedge dv dx = w(0). \tag{4.12}$$

Symplectic schemes for Hamiltonian systems are proven to be more efficient than non-symplectic schemes for long-time numerical computations and are widely applied to practical problems arising in many fields of science and engineering which involves celestial mechanics, quantum physics, statistics and so on (see [13,24–26]). Next, we will derive that the considered CHOC schemes are symplectic.

Theorem 4.7. *To linear good Boussinesq equation (1.1), the schemes CHOC-A, CHOC-B and CHOC-C are symplectic with the following conservation law*

$$w^n = h\sum_{j=1}^{NX} du_j^n \wedge dv_j^n = w^0. \tag{4.13}$$

Proof. To (3.2), scheme $CHOC - A$ has the formula as follows

$$\begin{bmatrix} -\frac{\tau}{2}N & E \\ E & -\frac{\tau}{2}E \end{bmatrix}\begin{bmatrix} U^{n+1} \\ V^{n+1} \end{bmatrix} = \begin{bmatrix} \frac{\tau}{2}N & E \\ E & \frac{\tau}{2}E \end{bmatrix}\begin{bmatrix} U^n \\ V^n \end{bmatrix}$$

Therefore through tedious calculations, scheme CHOC-A is symplectic. Similarity, the scheme CHOC-B and CHOC-C are also symplectic.

5. Numerical Experiments

In this section, we present some numerical results to illustrate above theoretical analysis about the CHOC schemes, mainly focusing on the convergence and discrete conservation laws for numerical solutions of the good Boussinesq equation.

First, for the linear good Boussinesq equation, we take the initial value $f(x) = sin(x)$ and exact solution $u(x,t) = \sin(x)\cos\left(\sqrt{2}t\right)$. Here, we focus on issues within a limited space-time domain $[0, 2\pi] \times [0, T]$. The L_2 and L_∞ norm of the errors between numerical solution and exact solution are defined respectively as

$$\|e^n(h,\tau)\|_2 = \sqrt{h\sum_j\left(U_j^n - u_j^n\right)^2}, \quad \|e^n(h,\tau)\|_\infty = \max_j\left|U_j^n - u_j^n\right|,$$

where $U_j^n = u(x_j, t_n)$ is the exact solution and u_j^n is the numerical solution. The convergence order in the space and time directions is defined as $order1$ and $order2$ respectively

$$order1 = \frac{\ln(\|e(h_1,\tau)\|/\|e(h_2,\tau)\|)}{\ln(h_1/h_2)}, \quad order2 = \frac{\ln(\|e(h,\tau_1)\|/\|e(h,\tau_2)\|)}{\ln(\tau_1/\tau_2)}.$$

First, we test the convergence order of $CHOC-A, CHOC-B, CHOC-C$ and take different step sizes in the direction to be considered, and takes very small step size in other direction.

Table 6 lists the error of numerical solution and exact solution under L_2 and L_∞ norm, and the spatial convergence order calculated by $order1$ for the three $CHOC$ schemes by taking different spatial step sizes. In order to make the error in the time direction relatively negligible, we take the time step size $\tau = 10^{-4}$.

Table 8 lists the error of numerical solution and exact solution under L_2 and L_∞ norm, and the time convergence order calculated by $order2$ when the three $CHOC$ schemes take different time step

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 June 2024 doi:10.20944/preprints202406.1509.v1

size. In order to make the error in the spatial direction relatively negligible, takes the spatial step size $h = \frac{2\pi}{80}$. Next, Table 7 shows the ratio of numerical error in Table 6 calculated by

$$\frac{\text{Numerical error by } CHOC-B \text{ } or \text{ } CHOC-C}{\text{Numerical error by } CHOA \text{ scheme}}.$$

Table 6. Numerical error of u_j^n with $\tau = 10^{-4}$.

h	*scheme*	$\|e^n\|_2$	order	$\|e^n\|_\infty$	**order**	
$\frac{2\pi}{5}$	$CHOC-A$	7.629×10^{-3}	−	4.1841×10^{-3}	−	
	$CHOC-B$	7.4229×10^{-4}	−	3.9829×10^{-4}	−	
	$CHOC-C$	9.4958×10^{-4}	−	5.0952×10^{-4}	−	
$\frac{2\pi}{10}$	$CHOC-A$	4.203×10^{-4}	4.1816	2.3058×10^{-4}	4.1816	
	$CHOC-B$	9.4353×10^{-6}	6.2978	5.0628×10^{-6}	6.2977	
	$CHOC-C$	1.348×10^{-5}	6.1384	7.2331×10^{-6}	6.1384	
$\frac{2\pi}{15}$	$CHOC-A$	8.0998×10^{-5}	4.0609	4.5682×10^{-5}	3.9927	
	$CHOC-B$	7.9367×10^{-7}	6.1054	4.4533×10^{-7}	5.9952	
	$CHOC-C$	1.1673×10^{-6}	6.0338	6.5498×10^{-7}	5.9236	
$\frac{2\pi}{20}$	$CHOC-A$	2.5403×10^{-5}	4.0307	1.4288×10^{-5}	4.0402	
	$CHOC-B$	1.3599×10^{-7}	6.1321	7.6726×10^{-8}	6.1129	
	$CHOC-C$	2.0994×10^{-7}	5.9636	1.1844×10^{-7}	5.9448	
$\frac{2\pi}{25}$	$CHOC-A$	1.0361×10^{-5}	4.0190	5.8447×10^{-6}	4.0059	
	$CHOC-B$	3.2376×10^{-8}	6.4316	1.823×10^{-8}	6.4406	
	$CHOC-C$	5.7932×10^{-8}	5.7701	3.262×10^{-8}	5.7784	

Table 7. The ratio of numerical error among different schemes of u_j^n with $\tau = 10^{-4}$.

h	$CHOC-B$		$CHOC-C$	
	rate e_2^u	*rate* e_∞^u	*rate* e_2^u	*rate* e_∞^u
$\frac{2\pi}{5}$	10.3	10.5	8.03	8.2
$\frac{2\pi}{10}$	44.5	45.5	31.2	31.9
$\frac{2\pi}{15}$	102.1	102.6	69.4	69.7
$\frac{2\pi}{20}$	186.8	186.2	121	120.6
$\frac{2\pi}{25}$	320	320.6	178.8	179.2

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 21 June 2024 doi:10.20944/preprints202406.1509.v1

Table 8. Verification of temporal convergence rate with $h = \frac{2\pi}{80}$.

τ	*scheme*	$\|e^n\|_2$	order	$\|e^n\|_\infty$	order
$\frac{1}{40}$	$CHOC-A$	5.5718×10^{-6}	–	1.1056×10^{-5}	–
	$CHOC-B$	5.574×10^{-6}	–	1.106×10^{-5}	–
	$CHOC-C$	5.574×10^{-6}	–	1.106×10^{-5}	–
$\frac{1}{80}$	$CHOC-A$	1.4094×10^{-6}	1.9831	2.8058×10^{-6}	1.9783
	$CHOC-B$	1.4116×10^{-6}	1.9814	2.8101×10^{-6}	1.9764
	$CHOC-C$	1.4116×10^{-6}	1.9814	2.8101×10^{-6}	1.9764
$\frac{1}{160}$	$CHOC-A$	3.5297×10^{-7}	1.9975	7.0373×10^{-7}	1.9953
	$CHOC-B$	3.5515×10^{-7}	1.9908	7.0808×10^{-7}	1.9886
	$CHOC-C$	3.5514×10^{-7}	1.9909	7.0806×10^{-7}	1.9887
$\frac{1}{320}$	$CHOC-A$	8.689×10^{-8}	2.0223	1.7336×10^{-7}	2.0213
	$CHOC-B$	8.907×10^{-8}	1.9954	1.7772×10^{-7}	1.9943
	$CHOC-C$	8.9065×10^{-8}	1.9955	1.777×10^{-7}	1.9944
$\frac{1}{640}$	$CHOC-A$	2.0124×10^{-8}	2.1103	4.0162×10^{-8}	2.1099
	$CHOC-B$	2.2307×10^{-8}	1.9974	4.4534×10^{-8}	1.9966
	$CHOC-C$	2.2301×10^{-8}	1.9978	4.4511×10^{-8}	1.9972

Secondly, for the linear principal part of the good Boussinesq equation, we simulate the discrete conservation law (4.5) in time interval $[0, 15]$, which is measured by the following approximate motion invariant error $\omega^n - \omega^0$.

Thirdly, consider nonlinear good Boussinesq equation (1.2) with exact solitary wave solution as follows

$$u(x,t) = -A\,\mathrm{sech}^2\left[\left(\frac{P}{2}\right)(\xi - \xi_0)\right], \quad \xi = x - ct, \quad (x,t) \in [-50, 50] \times [0, 1],$$

where $0 < P < 1,\ A = \frac{3P^2}{2},\ \xi_0 = \left(1 - P^2\right)^{1/2}$. Below we take a moderate amplitude $A = 0.5\,, \xi_0 = 0$, and take step sizes$h = 0.5\,, \tau = 0.01$. We simulate the discrete conservation law (4.6) in time interval $[0, 15]$, which is measured by the following approximate motion invariant error $\overline{u^n} - \overline{u^0}$.

Finally, we give three-dimensional waveform diagrams of exact solution and numerical solution of three schemes. We also give a comparison between the numerical solution and the exact solution.

6. Conclusions

In this paper, for a kind of good Boussinesq equation, we construct three combined high-order compact symplectic schemes, that is $CHOC-A$,$CHOC-B$, and $CHOC-C$. The schemes satisfy discrete conservation laws corresponding to structure-preserving property of good Boussinesq equation.

In numerical experiment, we first test the convergence order of $CHOC-A, CHOC-B, CHOC-C$ by taking different step sizes in the direction to be considered, and taking very small step size in other direction. In Table 6, we test the spatial convergence order of $CHOC-A, CHOC-B, CHOC-C$.

We can observe that the scheme $CHOC-A$ is of four order accuracy in space. The schemes $CHOC-B$ and $CHOC-C$ are of six order accuracy in space. From Table 7 we can observe that the ratio of numerical error of $CHOC-B$ is bigger than the scheme $CHOC-C$. In Table 8, we test the time convergence order of $CHOC-A$, $CHOC-B$ and $CHOC-C$. We can observe that the schemes are of two order in time.

Second, from the approximate motion invariant error simulation in Figures 1–3, we can see that the three schemes satisfy the discrete conservation law (4.5) for linear good Boussinesq equation.

Next, observe Figures 4–6. We find that the errors are small enough, in other words, the three schemes satisfy the conservation law (4.6) for nonlinear good Boussinesq equation(1.2).

Finally, we use the three schemes to numerically simulate the solitary wave solutions of nonlinear good Boussinesq equation(1.2). From Figures 7–10, we can observe that the numerical solutions fit the waveform of exact solution well.

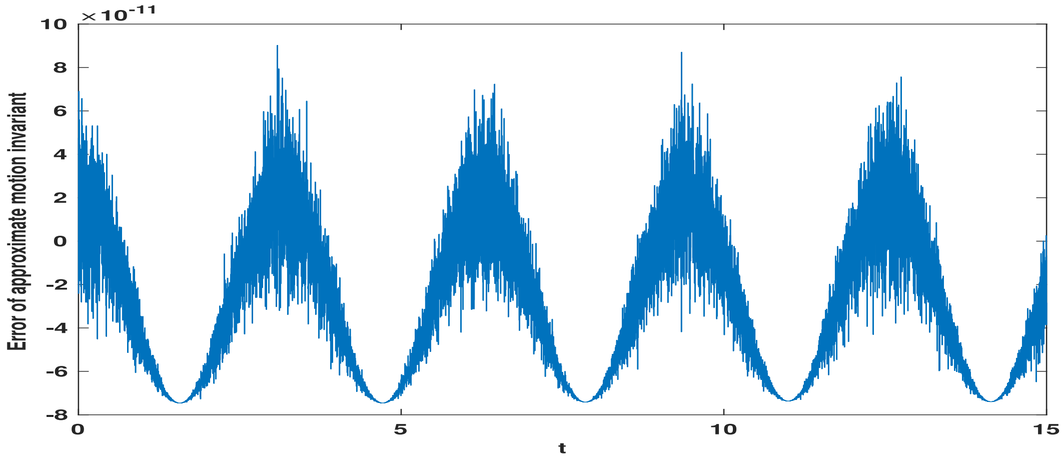

Figure 1. Approximate motion invariant error diagram for $CHOC-A$ when $h=\frac{2\pi}{80}, \tau=0.001$.

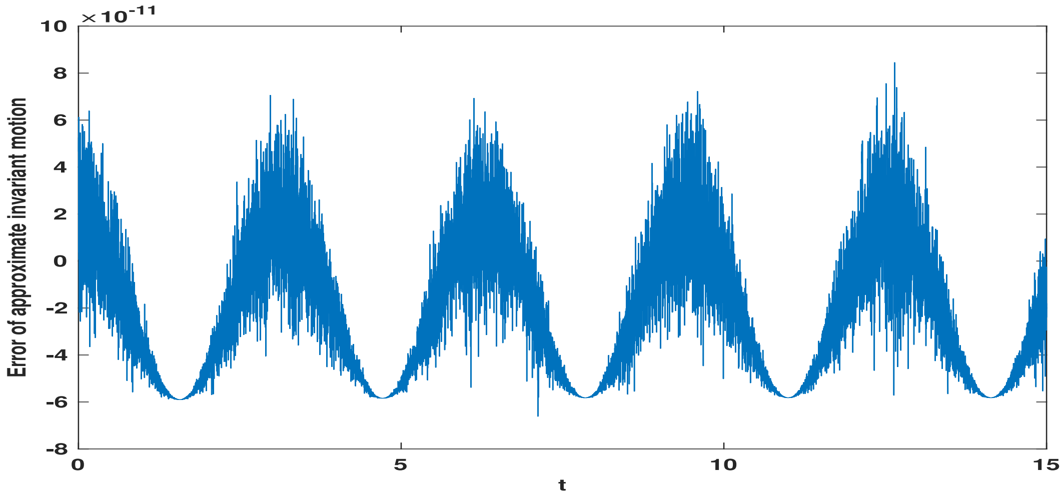

Figure 2. Approximate motion invariant error diagram for $CHOC-B$ when $h=\frac{2\pi}{80}, \tau=0.001$.

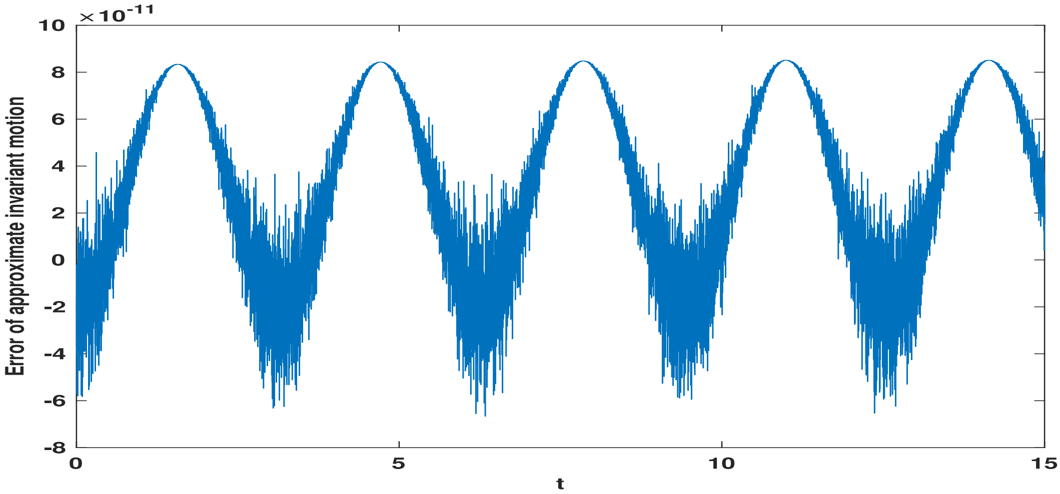

Figure 3. Approximate motion invariant error diagram for $CHOC - C$ when $h = \frac{2\pi}{80}, \tau = 0.001$.

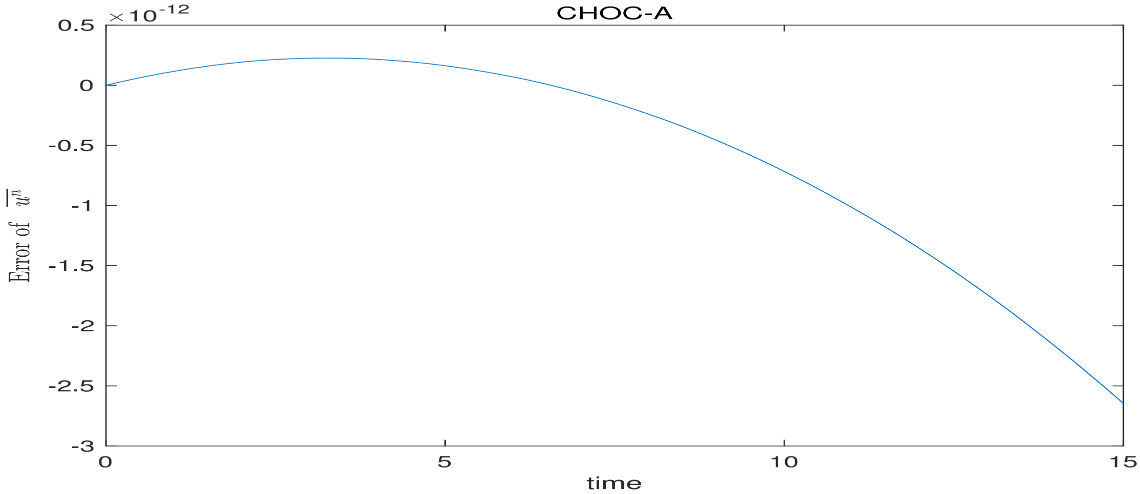

Figure 4. $\overline{u^n}$ error diagram for $CHOC - A$ when $h = 0.5, \tau = 0.01$.

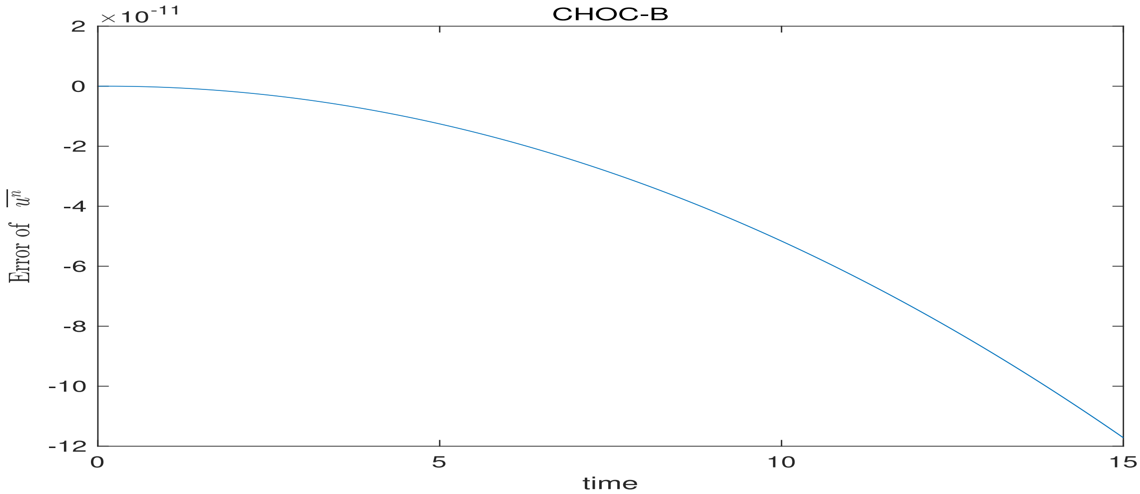

Figure 5. $\overline{u^n}$ error diagram for $CHOC - B$ when $h = 0.5, \tau = 0.01$.

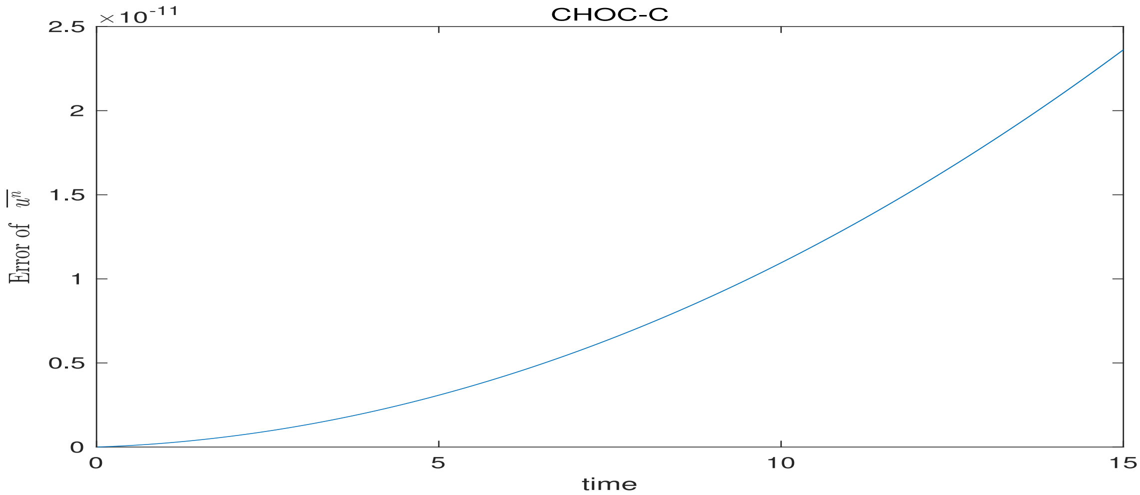

Figure 6. $\overline{u^n}$ error diagram for $CHOC-C$ when $h=0.5, \tau=0.01$.

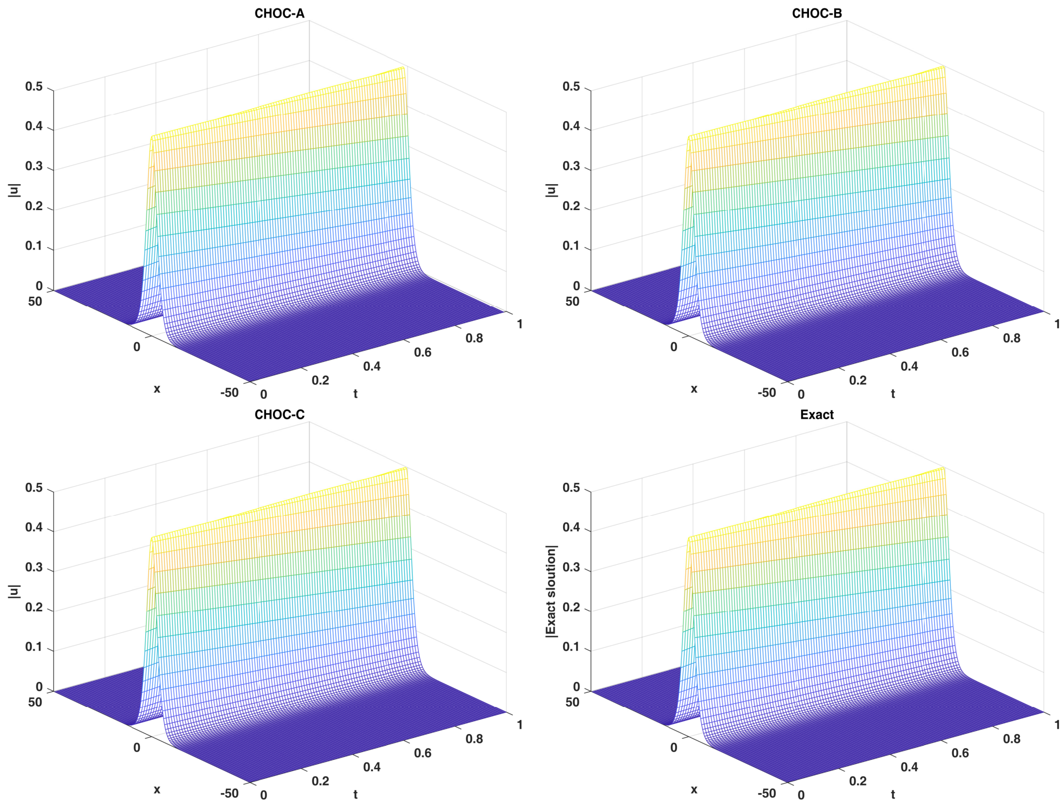

Figure 7. When $h=0.5\,,\tau=0.01$, three-dimensional waveform diagrams of the $CHOC-A$, $CHOC-B$ and $CHOC-C$ schemes, and three-dimensional waveform diagram of the exact solution.

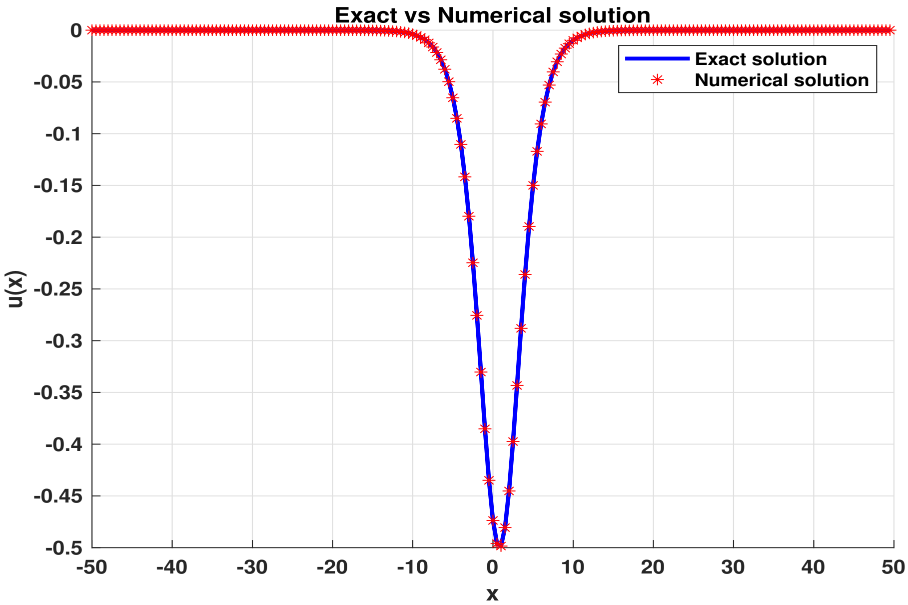

Figure 8. when $h = 0.5, at\ time\ T = 1$. The exact solution VS numerical solution of $CHOC - A$.

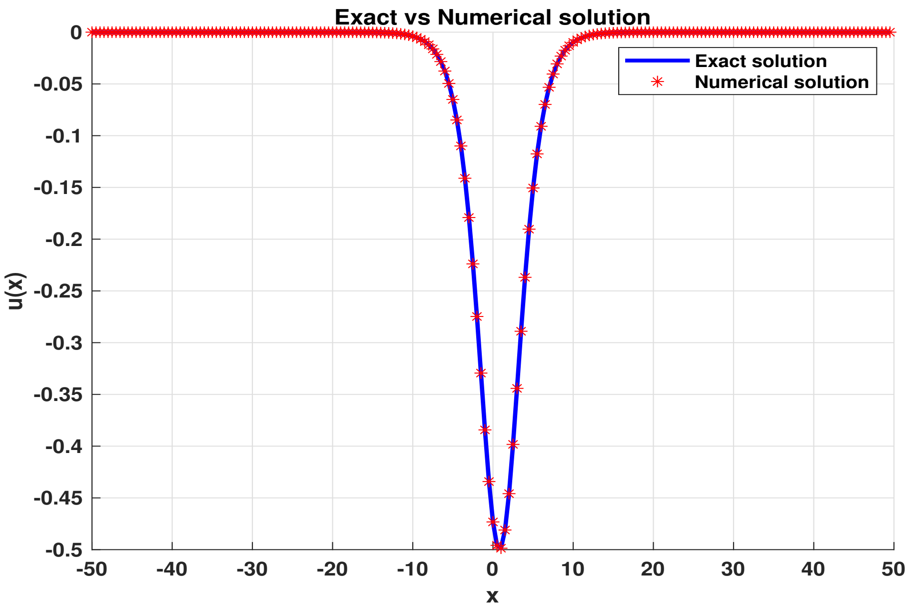

Figure 9. when $h = 0.5, at\ time\ T = 1$. The exact solution VS numerical solution of $CHOC - B$.

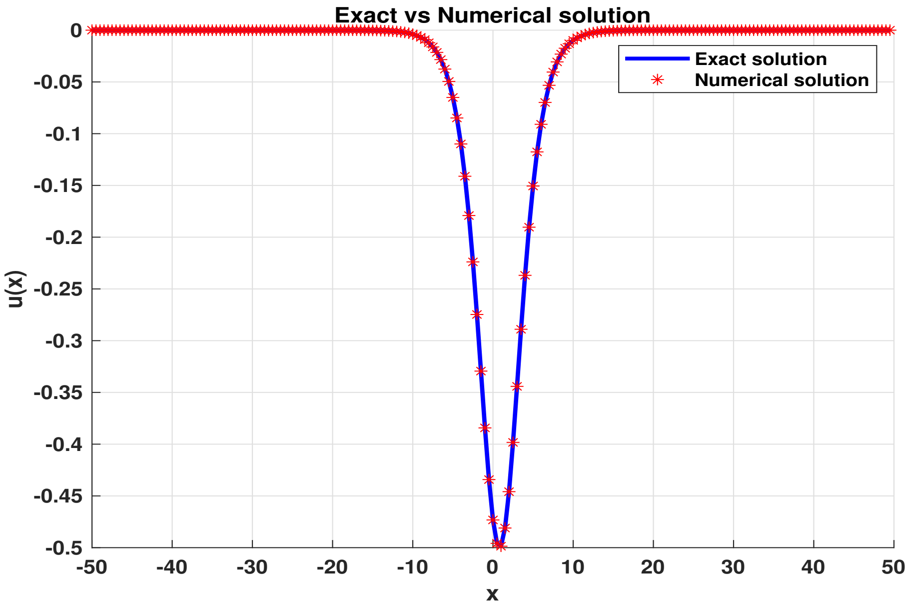

Figure 10. when $h = 0.5, at\ time\ T = 1$. The exact solution VS numerical solution of $CHOC - C$.

Author Contributions: Conceptualization, Z.L. and X.Y.; methodology, Z.L. and X.Y.; software, Z.L. and X.Y.; validation, Z.L., X.Y., Y.L., Z.C., and S.K.; formal analysis, Z.L. and X.L.; investigation, Z.L., X.Y., Y.L., Z.C., and S.K.; resources, X.Y.; data curation, Z.L. and X.Y.; writing—original draft preparation, Z.L. and X.L.; writing—review and editing, Z.L. and X.L.; visualization, Z.L., X.Y. ; supervision, X.Y.; project administration, X.Y.; funding acquisition, X.Y.

Funding: This work is supported by Natural Science Foundation of Shandong (No. ZR202204010001), Science and Technology Plan Project of Dezhou (No. 2021dzkj1638), Research Platform Project of Dezhou University (No. 2023XKZX024).

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.(Matlab codes can be provided if required).

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Manotanjan,V.S.;Mitchell,A.R.;Morris,J.L.Numerical solutions of the good Boussinesq equation. *Siam. J. Sci. Stat. Comput.* **1984**, *5*, 946–957.
2. Manotanjan,V.S.;Ortega,T.;Sanz-Serna,J.M. Soliton and antisoliton interactions in the "good" Boussinesq equation. *J. Math. Phys.* **1988** *29*, 1964–1968.
3. Ortega,T.;Sanz-Serna, J.M.Nonlinear stability and convergence of finite-difference methods for the "good" Boussinesq equation. *Numer. Math.* **1990**, *58*, 215–229.
4. Guo,Z.W.;Jiang,T.S.;Vasil'ev,V.I.;Wang,G.Complex structure-preserving method for Schrodinger equations in quaternionic quantum mechanics. *Numer. Algorithms* **2023**, *150*, 0893-9659.
5. Jiang,T.S.;Wang,G.;Guo,Z.W.;Zhang,D.Algebraic algorithms for a class of Schrodinger equations in split quaternionic mechanics. *Math. Math. Methods. Appl. Sci.* **2024**, *47*, 5349-5354.
6. Hu,H.Z.;Chen,Y.P.;Zhou,,J.W.Two-grid finite element method for time-fractional nonlinear schrodinger equation. *J. Comput. Math.* **2024**, *42*, 1124-1144.
7. Jiang,T.S.;Guo,Z.W.; Zhang,D.;Vasil'ev,V.I.A fast algorithm for the Schrödinger equation in quaternionic quantum mechanics. *Appl. Math. Letters.* **2024**, *150*, 108975.
8. Jiang,T.S.;Zhang,Z.Z.;Jiang,Z.W.Algebraic techniques for Schrodinger equations in split quaternionic mechanics. *Comput. Math. Appl.* **2018**, *75*, 2217-2222.
9. Wang,L.;Kong,L.H.;Chen,M.et al.Structure-Preserving Combined High-Order Compact Schemes for Multiple Order Spatial Derivatives Differential Equations. *J. Sci. Comput.* **2023**, *96*, 8.

10. Chu,P.C.;Fan,C.A three-point combined compact difference scheme. *J. Comput. Phys.* **1998**, *140*, 370-399.
11. Kaur,Deepti.;R.K.Mohanty.Highly accurate compact difference scheme for fourth order parabolic equation with Dirichlet and Neumann boundary conditions. *Appl. Numer. Math.* **2020**, *378*, 125202.
12. Wang.R.F.;Wang.W.H.;Yang,J.H.Containing High Order Compact Scheme Source of Steady Convection-Diffusion Equation. *Sci. Discovery.* **2016**.
13. Chen,B.;He,D.;Pan,K.A linearized high-order combined compact difference scheme for multidimensional coupled Burgers equations. *Numer. Math. Theor. Methods. Appl.* **2018**, *11*,299–320.
14. Yu,C.H.;Wang,D.et al.An optimized dispersion–relation-preserving combined compact difference scheme to solve advection equations. *J. Comput. Phys.* **2015**, *300*, 92-115.
15. Zhang,C.;Wang H.;Huang,J.F.;Wang,C.;Yue,X.Y. A second order operator splitting numerical scheme for the "good" Boussinesq equation. *Appl. Numer. Math.* **2017**, *119*, 179-193.
16. Chen.M;Kong.L.H.;Hong.Y.Efficient structure-preserving schemes for good Boussinesq equation. *Math. Meth. Appl. Sci.* **2018**.
17. Hu,W.P.;Deng,Z.C.Multi-symplectic method for generalized Boussinesq equation. *Appl. Math. Mech.* **2008**, *29*, 927-932.
18. Cienfuegos,R.;Barthélemy,E.;Bonneton,P.A fourth-order compact finite volume scheme for fully nonlinear and weakly dispersive Boussinesq-type equations. Part II: boundary conditions and validation. *Int. J. Numer. Methods Fluids.* **2007**,*53* 1423–1455.
19. Frutos,J.D.;Ortega,T.;Sanz-Serna,J.M.Pseudospectral method for the "good" Boussinesq equation. *Math. Comput.* **1991**,*57* 109–122.
20. Oh,S.;Stefanov,A.Improved local well-posedness for the periodic "good" Boussinesq equation. *J. Differ. Equ.* **2013**,*254* 4047–4065.
21. Farah,L.;Scialom,M.On the periodic "good" Boussinesq equation. *Proc. Am. Math. Soc.* **2010**,*138* 953–964.
22. Zou,Z.L.Higher order Boussinesq equations. *Ocean Engineering.* **1999**, *26*, 767-292.
23. Kong,L.H.;Chen,M.et al.A novel kind of efficient symplectic scheme for Klein-Gordon-Schrödinger equation. *Appl. Numer. Math.* **2019**, *135*, 481-496.
24. Bridges,T.J.;Reich.S.Multi-symplectic integrators: numerical schemes for Hamiltonian PDEs that conserve symplecticity. *Physics Letters A.* **2001**, *284*,184-193.
25. Shen,J.;Xu,J.;Yang,J. A new class of efficient and robust energy stable schemes for gradient flows. *SIAM. Rev.* **2019**, *61*,474–506.
26. Jiménez,S.;Vázquez.L.Some remarks on conservative and symplectic schemes. *World Scientific: Singapore.* **1991**, 151–162.