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Article

Analysis of Complex Entities in Algebra B

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Abstract: We present an exploration of algebra B, a recently published unital and non-associative algebra. Unique complex entities emerged from this algebra, distinct from both quaternion and complex number systems, which we termed treons. We defined the treonic number system and established an isomorphism between this system and the real vector space. Our findings revealed that the treonic representation maintained structural integrity under the defined operations. Based on this foundation, we proceed to determine Euler identity for algebra B within the treonic system. By presenting the fundamental definitions and properties of algebra B, we derived a generalized version of Euler identity applicable within this algebra. This formula revealed the emergence of hyperbolic trigonometric entities, extending the applicability of Euler identity beyond traditional complex numbers. Our results provide a theoretical foundation for a deeper understanding of the properties and behaviors of complex entities within this expanded algebraic framework, thus enabling new theoretical developments and practical applications in the realms of advanced mathematics and theoretical physics.

Keywords: isomorphism; treonic number; algebra B; Euler identity; hyperbolic

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Introduction

The algebra B is a recently described algebraic structure by Alejandro Bermejo [1]; it is defined as a non-associative and unital algebra with the potential to satisfy the definitional requirements of Lie and Malcev algebras when the respective products of these latter algebras are defined using the product of algebra B. However, an added quality of this algebra is the emergence of complex entities with a structure different from existing ones. They appear similar to quaternions [2] lacking one dimension, but they are not. They also resemble the set of complex numbers $\mathbb{C}^2$ [3] with an additional dimension, but they are not that either. Our study aims to investigate the structural properties of these entities, which we refer to as "treons", and to establish an isomorphism between algebra B and the treonic number system. We will demonstrate the mathematical correctness of representing elements of algebra B in the form $a_1 + a_2 i + a_3 j$, ultimately proving the existence of an isomorphism to $\mathbb{R}^3$. Based on this, that is, on the possibility of expressing the elements of algebra B in the form of treons, we seek to explore how Euler's identity manifests in the context of algebra B. Euler's identity (or Euler's formula), $e^{i\theta} = \cos\theta + i\sin\theta$, is a cornerstone of complex analysis, illustrating the profound relationship between exponential and trigonometric functions [4,5]. We find that the structure of this identity in algebra B has a similar form, but with its own unique characteristics and differences.

1. Products in Quaternion and $\mathbb{C}^2$ Algebras

The product of quaternions is defined as [2]:

$$(a_1, a_2, a_3, a_4)(b_1, b_2, b_3, b_4) \equiv$$

$$(a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4, a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3, a_1b_3 - a_2b_4 + a_3b_1 + a_4b_2, a_1b_4 + a_2b_3 - a_3b_2 + a_4b_1).$$

And the product in $\mathbb{C}^2$ is defined as [3,6]:

$$(a_1, a_2)(b_1, b_2) \equiv (a_1b_1 - a_2b_2, a_1b_2 + a_2b_1).$$

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2. Product in Algebra B

The product $\odot$ in algebra B is defined as [1]:

$$(a_1, a_2, a_3) \odot (b_1, b_2, b_3) \equiv (a_1 b_1 - a_2 b_2 - a_3 b_3, a_1 b_2 + a_2 b_1 + a_3 b_2, a_1 b_3 + a_3 b_2 + a_3 b_1).$$

Both the quaternionic product and the product in $\mathbb{C}^2$ are different from the product in B. It is not sufficient to nullify one or another component to derive algebra B from quaternions. Similarly, it does not make sense to impose an additional dimension to $\mathbb{C}^2$ to reach the product of algebra B. Therefore, from algebra B, the complex elements that arise must be different from the complex elements of the set $\mathbb{C}^2$ and the quaternions.

3. Representations in Quaternion and $\mathbb{C}^2$ Algebras

Quaternions can be represented as:

$$(a_1, a_2, a_3, a_4) = a_1 + a_2 i + a_3 j + a_4 k, \quad i^2 = j^2 = k^2 = -1.$$

And complex numbers in $\mathbb{C}^2$ as:

$$(a_1, a_2) = a_1 + a_2 i, \quad i^2 = -1.$$

In algebra B, Bermejo proposes the presence of the representation [1]:

$$(a_1, a_2, a_3) = a_1 + a_2 i + a_3 j,$$

though he does not define it strictly, i.e., from an algebra isomorphism.

4. Defining Treons

Motivated by investigating this equivalent representation, we seek to demonstrate that this representation is mathematically correct. This implies that there must exist an isomorphism between the structure (a_1, a_2, a_3) and $a_1 + a_2 i + a_3 j$; both referenced by Bermejo in his definition and analysis of algebra B [1]. We assume that the field over which the algebra is defined is the real field $\mathbb{R}$. Accordingly, we will seek an isomorphism between the real field $\mathbb{R}^3$ and the field of complex entities with structure $a_1 + a_2 i + a_3 j$ which we will call "treons" or "treonic numbers" to differentiate from the term "trions" used in various disciplines and from hypercomplex numbers called "ternions".

We define treons as:

$$a_1 + a_2 i + a_3 j,$$

such that $a_1, a_2, a_3 \in \mathbb{R}$ and $i^2 = j^2 = -1$.

We assumed that the treonic elements $a_1 + a_2 i + a_3 j$ are elements of an arbitrary algebra A.

5. Addition and Product in Treons

The addition $+$ in A is:

$$(a_1 + a_2 i + a_3 j) + (b_1 + b_2 i + b_3 j) = (a_1 + b_1) + (a_2 + b_2) i + (a_3 + b_3) j,$$

and the product $\otimes$ in A is:

$$(a_1 + a_2 i + a_3 j) \otimes (b_1 + b_2 i + b_3 j) = (a_1 b_1 - a_2 b_2 - a_3 b_3) + (a_1 b_2 + a_2 b_1) i + (a_1 b_3 + a_3 b_1) j + a_2 i b_3 j + a_3 j b_2 i.$$

Note that we have not imposed a definition, but have simply grouped the terms in the addition $+$ by factoring out i and j without altering their action on the right on the elements of the field. We have also considered the distributive property of the product $\otimes$ over $+$.

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6. Isomorphism with $\mathbb{R}^3$

Now, we analyze $\mathbb{R}^3$: The addition $+$ in algebra B is defined as:

$$(a_1, a_2, a_3) + (b_1, b_2, b_3) \equiv (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

While the product $\odot$ in B was previously defined as:

$$(a_1, a_2, a_3) \odot (b_1, b_2, b_3) \equiv (a_1 b_1 - a_2 b_2 - a_3 b_3, a_1 b_2 + a_2 b_1 + a_3 b_2, a_1 b_3 + a_3 b_2 + a_3 b_1).$$

Under this product operation, an expression of the type $a_3 j b_2 i$ can be expressed as [1]:

$$\begin{aligned}
a_3 j \odot b_2 i &= a_3 b_2 j \odot i \\
&= (a_3 b_2)((0,0,1) \odot (0,1,0)) \\
&= a_3 b_2 (0,1,1) \\
&= (0, a_3 b_2, a_3 b_2) \\
&= a_3 b_2 (0,1,0) + a_3 b_2 (0,0,1) \\
&= a_3 b_2 i + a_3 b_2 j.
\end{aligned}$$

Since in algebra B [1] we have the property of *orthomulearity*, $ai \odot bj = 0$, expressions of the type $a_2 i b_3 j$ would remain:

$$a_2 i b_3 j = a_2 b_3 i \odot j = 0.$$

Taking into account that under the product in B, $a_3 j b_2 i = a_3 b_2 i + a_3 b_2 j$ and $a_2 i b_3 j = 0$, then,

$$(a_1 + a_2 i + a_3 j) \otimes (b_1 + b_2 i + b_3 j) = (a_1 b_1 - a_2 b_2 - a_3 b_3) + (a_1 b_2 + a_2 b_1 + a_3 b_2) i + (a_1 b_3 + a_3 b_1 + a_3 b_2) j.$$

This has a preserved structure with respect to the product in B:

$$(a_1, a_2, a_3) \odot (b_1, b_2, b_3) = (a_1 b_1 - a_2 b_2 - a_3 b_3, a_1 b_2 + a_2 b_1 + a_3 b_2, a_1 b_3 + a_3 b_2 + a_3 b_1).$$

Thus, for $(c_1, c_2, c_3) = c_1 + c_2 i + c_3 j$, such that $(c_1, c_2, c_3) \in \mathbb{R}^3$ and $(c_1 + c_2 i + c_3 j) \in A$, we can consider:

$$c_1 = \text{Re}(c), \quad c_2 = \text{Im}_1(c), \quad c_3 = \text{Im}_2(c),$$

where $\text{Re}(c)$ is the real part of c, $\text{Im}_1(c)$ is the first imaginary part of c, and $\text{Im}_2(c)$ is the second imaginary part of c.

With this, we have sufficient data to define an isomorphism Φ such that:

$$\Phi : A \to \mathbb{R}^3,$$

$$\Phi(a_1 + a_2 i + a_3 j) = (a_1, a_2, a_3).$$

7. Verification of Isomorphism Properties

7.1. Preservation of Addition

$$\Phi((a_1 + a_2 i + a_3 j) + (b_1 + b_2 i + b_3 j)) =$$

$$\Phi(a_1 + b_1 + (a_2 + b_2) i + (a_3 + b_3) j) = (a_1 + b_1, a_2 + b_2, a_3 + b_3).$$

7.2. Preservation of Product

$$\Phi((a_1 + a_2 i + a_3 j) \otimes (b_1 + b_2 i + b_3 j)) =$$

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$$\Phi((a_1b_1 - a_2b_2 - a_3b_3) + (a_1b_2 + a_2b_1 + a_3b_2)i + (a_1b_3 + a_3b_2 + a_3b_1)j) =$$
$$(a_1b_1 - a_2b_2 - a_3b_3, a_1b_2 + a_2b_1 + a_3b_2, a_1b_3 + a_3b_2 + a_3b_1).$$

7.3. Isomorphism Verification

7.3.1. Morphism Verification

Φ is an algebra morphism (homomorphism) if [7–9]:

$$1) \quad \Phi : A \to B,$$

$$\Phi((a_1 + a_2i + a_3j) \otimes (b_1 + b_2i + b_3j)) = \Phi((a_1, a_2, a_3) \odot \Phi(b_1, b_2, b_3)),$$

and moreover:

$$2) \quad \Phi\mathrm{id}_A = \mathrm{id}_B,$$

where id_A is the identity under the product $\otimes$ in A, and id_B is the identity under the product $\odot$ in B.

The first condition holds:

$$\Phi((a_1 + a_2i + a_3j) \otimes (b_1 + b_2i + b_3j)) = (a_1b_1 - a_2b_2 - a_3b_3, a_1b_2 + a_2b_1 + a_3b_2, a_1b_3 + a_3b_1 + a_3b_2),$$

$$\Phi(a_1, a_2, a_3) \odot \Phi(b_1, b_2, b_3) = (a_1b_1 - a_2b_2 - a_3b_3, a_1b_2 + a_2b_1 + a_3b_2, a_1b_3 + a_3b_2 + a_3b_1),$$

where we used $\Phi(a_k)\Phi(b_l) = (a_k + 0i + 0j)(b_l + 0i + 0j) = a_kb_l$.

The second condition also holds:

$$\Phi\mathrm{id}_A = \mathrm{id}_B,$$

The identity in A is defined as $\mathrm{id}_A \equiv (1 + 0i + 0j)$:

$$(a_1 + a_2i + a_3j) \otimes (1 + 0i + 0j) = (1 + 0i + 0j) \otimes (a_1 + a_2i + a_3j) = (a_1 + a_2i + a_3j),$$

Therefore:

$$\Phi(1 + 0i + 0j) = (1, 0, 0).$$

Since in algebra B we have $\mathrm{id}_B = (1, 0, 0)$, it is verified that Φ is an algebra morphism between A and B.

For Φ to be an isomorphism, Φ must be both a monomorphism and an epimorphism [7,8,10].

7.3.2. Monomorphism Verification (Injectivity)

Assuming $\Phi(a) = \Phi(b)$. Then:

$$\Phi(a_1 + a_2i + a_3j) = \Phi(b_1 + b_2i + b_3j) \implies (a_1, a_2, a_3) = (b_1, b_2, b_3).$$

This implies that:

$$a_1 = b_1, \quad a_2 = b_2, \quad a_3 = b_3.$$

Therefore:

$$a_1 + a_2i + a_3j = b_1 + b_2i + b_3j.$$

Thus, Φ is an monomorphism.

Through the kernel, we can equally verify this. We define the kernel of Φ as the set of elements in A that map to the identity element of addition in B [7,8,10]. In our case:

$$\mathrm{Ker}(\Phi) = \{a \in A \mid \Phi(a) = 0\}.$$

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For $a = a_1 + a_2 i + a_3 j$, we have $\Phi(a) = (a_1, a_2, a_3)$. Therefore:

$$\Phi(a) = 0 \implies (a_1, a_2, a_3) = (0, 0, 0).$$

This implies that:

$$a_1 = 0, \quad a_2 = 0, \quad a_3 = 0.$$

Therefore:

$$a = 0 + 0i + 0j = 0.$$

Thus, the only element in the kernel of Φ is the identity element of addition in A.

The fact that $\ker(\Phi)$ is trivial (i.e., $\ker(\Phi) = \{0\}$) implies that the homomorphism is injective [7,8,10].

7.3.3. Epimorphism Verification (Surjectivity)

To verify that Φ is an epimorphism, we considered any $(a_1, a_2, a_3) \in \mathbb{R}^3$. There exists a treon $a_1 + a_2 i + a_3 j \in A$ such that:

$$\Phi(a_1 + a_2 i + a_3 j) = (a_1, a_2, a_3).$$

Thus, Φ is surjective, as by definition every element in $\mathbb{R}^3$ has a preimage in A. Thus, by definition, it is an epimorphism.

□

With all this, we have the necessary tools to tackle the search for Euler's identity in the context of algebra B. The isomorphism of algebras allows us to conduct a well-defined analysis of algebra B in the form of treons. With these, we proceed to deduce the form of Euler's identity in our algebra.

8. Analysis of Treons and Their Complex Conjugates

8.1. Definition of a Complex Entity

Considering Bermejo's work on algebra B [1], a complex entity can be described as:

$$b \equiv b_1 \cdot (1, 0, 0) + b_2 \cdot (0, 1, 0) + b_3 \cdot (0, 0, 1) = (b_1, b_2, b_3),$$

where $b_1 = \Re(b)$, $b_2 = \Im_1(b)$, and $b_3 = \Im_2(b)$. $\Re(b)$ is the real part of b, $\Im_1(b)$ is the first imaginary part of b, and $\Im_2(b)$ is the second imaginary part of b. It should be noted that the definition is equivalent to the one we articulated in Section 6: Isomorphism with $\mathbb{R}^3$.

8.2. Definition of the Complex Conjugate

The complex conjugate, b^*, was obtained by changing $i \equiv (0, 1, 0)$ or $j \equiv (0, 0, 1)$ [1] to their respective additive inverses in algebra B. Thus:

$$b^{(*_i)} = b_1 \cdot (1, 0, 0) - b_2 \cdot (0, 1, 0) + b_3 \cdot (0, 0, 1),$$
$$b^{(*_j)} = b_1 \cdot (1, 0, 0) + b_2 \cdot (0, 1, 0) - b_3 \cdot (0, 0, 1),$$
$$b^{(*_{i,j})} = b_1(1, 0, 0) - b_2(0, 1, 0) - b_3(0, 0, 1),$$

where $b^{(*_i)}$ denotes conjugation in i, and $b^{(*_j)}$ denotes conjugation in j.

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8.3. Powers of a Complex Entity

We define the product $b^2 \equiv b \odot b$ as:

$$b^2 = (b_1, b_2, b_3) \odot (b_1, b_2, b_3) = (b_1 \cdot b_1 - b_2 \cdot b_2 - b_3 \cdot b_3, b_1 \cdot b_2 + b_2 \cdot b_1 + b_3 \cdot b_2, b_1 \cdot b_3 + b_3 \cdot b_2 + b_3 \cdot b_1).$$

Hence, simplifying the notation:

$$b_1^2 = b_1 \cdot b_1 \equiv b_1 b_1.$$

Thus, b^2 becomes:

$$b^2 = (b_1^2 - b_2^2 - b_3^2, 2b_1 b_2 + b_2 b_3, 2b_1 b_3 + b_3 b_2),$$

where we use the notation $2b$ to denote $b + b$.

We define higher powers as follows:

$$b^3 \equiv (b \odot b) \odot b = (b_1^2 - b_2^2 - b_3^2, 2b_1 b_2 + b_2 b_3, 2b_1 b_3 + b_3 b_2) \odot b,$$
$$b^4 \equiv ((b \odot b) \odot b) \odot b,$$
$$b^5 \equiv (((b \odot b) \odot b) \odot b) \odot b,$$

and so on.

We also define the quantity $\langle b^2 \rangle \equiv b \odot b^{(*_{i,j})}$. Therefore:

$$\langle b^2 \rangle = (b_1, b_2, b_3) \odot (b_1, b_2, b_3)^{(*_{i,j})} = (|b|^2, 2b_1 b_2 + b_2 b_3, 2b_1 b_3 + b_3 b_2),$$

where $|b|^2 \equiv b_1^2 + b_2^2 + b_3^2$, which we call the "squared norm" of the vector. Note that $|b|^2$ results from the definition of the product of the vector b with its complex conjugate $(*_{i,j})$, as in the field of complex numbers [11]. However, in algebra B, this quantity appears in the first component (the real component) of the vector.

9. Derivation of Euler's Identity

9.1. Step 1: Power Series

We perform the following power series expansion [12]:

$$\sum_0^\infty \frac{b^n}{n!} \equiv \frac{b^0}{0!} + \frac{b^1}{1!} + \frac{b^2}{2!} + \frac{b^3}{3!} + \cdots + \frac{b^n}{n!}.$$

where ! denotes some kind of "factorial" in the field F of algebra B [1]; therefore, the multiplication of the factorial corresponds to the multiplication $\cdot$ defined for the elements of the field. On the other hand, we have: 0 is the identity element of addition in F, 1 is the identity element of multiplication in F, 2 is the successor of 1, 3 is the successor of 2, and so on. We also define: $0! \equiv 1$, $1! \equiv 1$, $b^0 \equiv (1, 0, 0)$, and $b^1 \equiv b$. Assuming that the field F is the real field $\mathbb{R}$, then $\sum_{n=0}^\infty \frac{b^n}{n!}$ represents the Taylor series expansion of e^b [12,13].

9.2. Step 2: Taylor Series for e^{ib_2}

We utilize the powers of the components of b multiplied by their corresponding imaginary units to perform a Taylor series expansion [13], considering algebra B over the real field $\mathbb{R}$. For the case of the first imaginary component, we have:

$$e^{ib_2} = \sum_0^\infty \frac{(ib_2)^n}{n!} = \frac{(ib_2)^0}{0!} + \frac{(ib_2)^1}{1!} + \frac{(ib_2)^2}{2!} + \frac{(ib_2)^3}{3!} + \cdots + \frac{(ib_2)^n}{n!},$$

$$= (1, 0, 0) + b_2(0, 1, 0) + \frac{(b_2(0, 1, 0))^2}{2!} + \frac{(b_2(0, 1, 0))^3}{3!} + \cdots + \frac{(b_2(0, 1, 0))^n}{n!}.$$

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Note that $(ib_2)^0 = (b_2(0,1,0))^0 \equiv (1,0,0)$ and $(ib_2)^1 = b_2(0,1,0) \equiv (ib_2)$.

9.2.1. Analyzing the Powers of $(0,1,0)$

$$\begin{aligned}
i^0 &\equiv (1,0,0) \equiv id_\odot \quad (id_\odot \text{ denotes the identity element of the product } \odot \text{ in algebra B}),\\
i^1 &\equiv (0,1,0),\\
i^2 &\equiv (0,1,0) \odot (0,1,0) = (-1,0,0) = -id_\odot,\\
i^3 &\equiv ((0,1,0) \odot (0,1,0)) \odot (0,1,0) = (-1,0,0) \odot (0,1,0) = (0,-1,0) = -i = i^*,\\
i^4 &= (0,-1,0) \odot (0,1,0) = (-1,0,0) = -id_\odot,\\
i^5 &= (-1,0,0) \odot (0,1,0) = (0,-1,0) = i^*,\\
i^6 &= (0,-1,0) \odot (0,1,0) = i^4 = (-1,0,0) = -id_\odot,\\
i^7 &= (-1,0,0) \odot (0,1,0) = i^5 = i^*.
\end{aligned}$$

The sequence is cyclic and has the form, from the second power: $-id_\odot, -i, -id_\odot, -i, -id_\odot, -i$. Therefore, the power series expansion becomes:

$$\begin{aligned}
e^{ib_2} = \sum_0^\infty \frac{(ib_2)^n}{n!} &= id_\odot + b_2 i - id_\odot\left(\frac{b_2^2}{2!} + \frac{b_2^4}{4!} + \frac{b_2^6}{6!} + \cdots\right) + i^*\left(\frac{b_2^3}{3!} + \frac{b_2^5}{5!} + \cdots\right),\\
&= id_\odot + b_2 i - id_\odot \cosh(b_2) - i \sinh(b_2),
\end{aligned}$$

where cosh() and sinh() denote *hyperbolic* cosines and sines [14], respectively.

9.2.2. Analyzing the Powers of $(0,0,1)$

$$\begin{aligned}
j^0 &\equiv (1,0,0) \equiv id_\odot,\\
j^1 &\equiv (0,0,1),\\
j^2 &\equiv (0,0,1) \odot (0,0,1) = (-1,0,0) = -id_\odot,\\
j^3 &\equiv ((0,0,1) \odot (0,0,1)) \odot (0,0,1) = (-1,0,0) \odot (0,0,1) = (0,0,-1) = -j = j^*,\\
j^4 &= (0,0,-1) \odot (0,0,1) = (-1,0,0) = -id_\odot,\\
j^5 &= (-1,0,0) \odot (0,0,1) = (0,0,-1) = j^*,\\
j^6 &= (0,-1,0) \odot (0,0,1) = i^4 = (-1,0,0) = -id_\odot,\\
j^7 &= (-1,0,0) \odot (0,0,1) = j^5 = j^*.
\end{aligned}$$

Thus:

$$e^{jb_3} = id_\odot + b_3 j - id_\odot \cosh(b_3) - j \sinh(b_3).$$

For the full vector, with its two imaginary components, we have:

$$e^{(b_1 + b_2 i + b_3 j)} = e^{b_1}(id_\odot + b_2 i - id_\odot \cosh(b_2) - i \sinh(b_2))(id_\odot + b_3 j - id_\odot \cosh(b_3) - j \sinh(b_3)).$$

□

This demonstrates that Euler's identity in algebra B, or equivalently in the algebra of treons, possesses a structure that preserves the connection between the exponential function and trigonometric functions. However, instead of the standard sine and cosine functions, *hyperbolic sine* and *hyperbolic cosine* functions appear.

Conclusions

We demonstrated that the treonic number system is a valid representation of algebra B and established an isomorphism between $\mathbb{R}^3$ and the algebra of treons. By preserving both the addition and product operations, our isomorphism confirmed the structural integrity of algebra B in this new representation.

Using the isomorphism between algebra B and treons, we have systematically extended the classical Euler's identity to the domain of algebra B. We demonstrated how the exponential function and its associated trigonometric identities can be formulated within this broader context. The significance of this work lies in its ability to expand the applicability of Euler's identity and the understanding of complex entities beyond traditional complex numbers and algebras, opening new avenues for exploring higher-dimensional algebraic structures with significant implications.

Furthermore, we demonstrated the emergence of hyperbolic trigonometric entities within this new Euler's identity applied to algebra B. This finding underscores the potential of Euler's identity to encompass a broader range of mathematical phenomena, which could lead to new theoretical developments in advanced mathematics.

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