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Application of the Transmutation Operator Method to the Solution of an Initial Boundary Value Problem for the Equation of Multidimensional Free Transverse Vibration of a Thin Elastic Plate Containing the Bessel Operator

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Abstract: In this paper, the transmutation operator method is used to construct an exact solution to the initial boundary value problem for the equation of multidimensional free transverse vibration of a thin elastic plate with a singular Bessel operator acting on geometric variables. We emphasize that multidimensional Erdélyi–Kober operators of fractional order have the property of a transmutation operator, allowing one to transform more complex multidimensional partial differential equations with singular coefficients acting over all variables into simpler ones. If formulas for solutions are known for a simple equation, then we also obtain representations for solutions to the first complex partial differential equation with singular coefficients. In particular, it is successfully applied to the singular differential equations, especially with Bessel-type operators. Using this operator, the considered problem is reduced to a similar problem without the Bessel operator. Based on the solution to the auxiliary problem, an exact solution to the original problem is constructed and analyzed.

Keywords: fourth-order equation; plate vibration equation; Bessel operator; transmutation operator; Erdélyi–Kober operator

MSC: 35A22, 35G05

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1. Introduction. Formulation of the Problem

Boundary value problems for the partial differential equations with singular coefficients have been the subject of study by many mathematicians. The study of more complex equations with singular coefficients represents a natural further stage on the path of theoretical generalizations. The value of the theoretical results obtained in this case increases significantly due to the fact that similar equations or their special cases are encountered in applications.

The importance of equations from these classes is also determined by their use in applications to problems in the theory of axisymmetric potential [1], Euler-Poisson-Darboux equations [2], Radon transform and tomography [3], gas dynamics and acoustics [4], jet theory in hydrodynamics [5], linearized Maxwell-Einstein equations[6], mechanics, theory of elasticity and plasticity [7] and many others.

In this direction, a special place is occupied by initial and boundary value problems for partial differential equations with singularities in the coefficients, typical representatives of which are equations with Bessel operators of the form

$$B_\eta^x = x^{-2\eta-1}\frac{d}{dx}x^{2\eta+1}\frac{d}{dx} = \frac{d^2}{dx^2} + \frac{2\eta+1}{x}\frac{d}{dx}$$

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For equations of elliptic, hyperbolic and parabolic types with the Bessel operator for each or several variables I.A. Kipriyanov [8] introduced, respectively, the terminology B-elliptic, B-hyperbolic and B-parabolic equations.

The entire range of questions for equations with Bessel operators was studied most fully by I.A. Kipriyanov and his students. More detailed information about this direction can be found in the monographs of V.V. Katrakhov and S.M. Sitnik [9], S.M. Sitnik and E.L. Shishkina [10].

It is known that degenerate and singular equations of the second order have the peculiarity that the correctness of classical problems does not always apply to them. The formulation of the problem is significantly affected by the lower coefficients. Such questions for high-order equations with singular coefficients have hardly been studied.

Let $x = (x_1, x_2, ..., x_n)$ be a point of the n-dimensional Euclidean space R^n, $R^n_+ = \{x \in R^n : \ x_k > 0, \ k = \overline{1, n}\}$, $\Omega = \{(x,t) : x \in R^n_+, \ t \in R^1_+\}$.

Problem. In the domain Ω we need to find a solution $u(x,t) \in C^4(\Omega)$

$$u_{tt} + \Delta_B^2 u = 0, (x,t) \in \Omega, \tag{1}$$

equation satisfying the initial

$$u(x,0) = \varphi(x), u_t(x,0) = 0, x \in R^n_+, \tag{2}$$

and the following boundary conditions

$$\left.\frac{\partial u}{\partial x_k}\right|_{x_k=0} = 0, \left.\frac{\partial^3 u}{\partial x_k^3}\right|_{x_k=0} = 0, t > 0, k = \overline{1,n}, \tag{3}$$

where $\Delta_B^2 = \Delta_B \Delta_B$, $\Delta_B \equiv \sum\limits_{k=1}^{n} B_{\gamma_k}^{(x_k)}$, $B_{\gamma_k}^{(x_k)} = \dfrac{\partial^2}{\partial x_k^2} + \dfrac{2\gamma_k + 1}{x_k}\dfrac{\partial}{\partial x_k}$, $\gamma_k = \alpha_k - 1/2$, $\alpha_k > 0$, $k = \overline{1,n}$, $\varphi(x)$ is a given function.

Thr equation (1) when $\gamma_k = -1/2, \ k = \overline{1,n}$, transforms into the equation of multidimensional free transverse vibration of a thin elastic plate $u_{yy} + \Delta^2 u = 0$, where $\Delta^2 = \Delta\Delta$ is the biharmonic operator, and Δ is the multidimensional Laplace operator.

Many problems about vibrations of rods, beams and plates, which are important in structural mechanics, the theory of stability of rotating shafts and vibration of ships, lead to differential equations of the fourth or higher order [11], [12].

Note that in problems of the general theory of partial differential equations containing the Bessel operator in one or more variables, the main research apparatus is the corresponding integral Fourier or Fourier - Bessel transform. In this work, in contrast to traditional methods, to solve the problem, we will use a different approach. Namely, taking into account the specifics of equations with singular coefficients, we use the Erdélyi–Kober transmutation operator.

Definition 1 ([9], [13]). Let a pair of operators be given (A, B). Non-null operator T is called a transformation operator (OP, Transmutation) if the relation

$$TA = BT \tag{4}$$

is satisfied.

In order for (4) to be a strict definition, it is necessary to specify the spaces or sets of functions on which the operators A, B, and, therefore, T act. The monographs [9,10,13–15] are devoted to the presentation of the theory of OP and their applications. Erdélyi–Kober operators, with a certain choice of parameters, are a generalization of the classical Sonin and Poisson transmutation operators. Therefore, first we will consider some properties of this operator.

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2. Multidimensional Erdélyi–Kober Transmutation Operator

Various modifications and generalizations of classical fractional order integration and differentiation operators are widely used in theory and applications. Such modifications include, in particular, the Erdélyi–Kober operators [16].

In [17], the multidimensional generalized Erdélyi–Kober operator was introduced in the form

$$J_\lambda\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x)=J_{\lambda_1,\lambda_2,\ldots,\lambda_n}\begin{pmatrix}\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n\\ \eta_1,\ \eta_2,\ \ldots,\ \eta_n\end{pmatrix}f(x)$$
$$=J^{x_1}_{\lambda_1}(\eta_1,\alpha_1)J^{x_2}_{\lambda_2}(\eta_2,\alpha_2)\cdots J^{x_n}_{\lambda_n}(\eta_n,\alpha_n)f(x)$$
$$=\left[\prod_{k=1}^{n}\frac{2x_k^{-2(\alpha_k+\eta_k)}}{\Gamma(\alpha_k)}\right]\int\limits_0^{x_1}\int\limits_0^{x_2}\ldots\int\limits_0^{x_n}\prod_{k=1}^{n}\left[t_k^{2\eta_k+1}\left(x_k^2-t_k^2\right)^{\alpha_k-1}\right.$$
$$\left.\times\bar{J}_{\alpha_k-1}\left(\lambda\sqrt{x_k^2-t_k^2}\right)\right]f(t_1,\ t_2,\ \ldots,\ t_n)dt_1dt_2\ldots dt_n,\tag{5}$$

where $\lambda,\alpha,\eta\in R^n$, $\alpha_k>0$, $\eta_k\geq-1/2$, $k=\overline{1,n}$; $\Gamma(\alpha)$ is the Euler gamma function; $\bar{J}_\nu(z)$ is the Bessel–Clifford function expressed through the Bessel function $J_\nu(z)$, by the formulas $\bar{J}_\nu(z)=\Gamma(\nu+1)(z/2)^{-\nu}J_\nu(z)$ and $J^{x_k}_{\lambda_k}(\eta_k,\alpha_k)$ is a particular Erdélyi–Kober integral of α_k-order of $k-$th variable

$$J^{x_k}_{\lambda_k}(\eta_k,\alpha_k)f(x)=\frac{2x_k^{-2(\alpha_k+\eta_k)}}{\Gamma(\alpha_k)}\int\limits_0^{x_k}(x_k^2-t^2)^{\alpha_k-1}\bar{J}_{\alpha_k-1}\left(\lambda_k\sqrt{x_k^2-t^2}\right)$$
$$\times t^{2\eta_k+1}f(x_1,x_2,\ldots,x_{k-1},t,x_{k+1},\ldots,x_n)dt.$$

In this work we also study the basic properties of the operator (5) and show that the inverse operator has the form

$$J_\lambda^{-1}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x)=J_{i\lambda}\begin{pmatrix}-\alpha\\ \eta+\alpha\end{pmatrix}f(x)$$
$$=2^{n-|m|}\left[\prod_{k=1}^{n}\frac{x_k^{-2\eta_k}}{\Gamma(m_k-\alpha_k)}\left(\frac{1}{x_k}\frac{\partial}{\partial x_k}\right)^{m_k}\right]\int\limits_0^{x_1}\int\limits_0^{x_2}\ldots\int\limits_0^{x_n}\prod_{k=1}^{n}\left[t_k^{2(\eta_k+\alpha_k)+1}\right.$$
$$\left.\times\left(x_k^2-t_k^2\right)^{m_k-1-\alpha_k}\bar{I}_{m_k-1-\alpha_k}\left(\lambda\sqrt{x_k^2-t_k^2}\right)\right]f(t_1,\ t_2,\ \ldots,\ t_n)dt_1dt_2\ldots dt_n,\tag{6}$$

where $\alpha_k>0$, $m_k=[\alpha_k]+1$, $\eta_k\geq-1/2$, $k=\overline{1,n}$, $\bar{I}_\nu(z)=\Gamma(\nu+1)(z/2)^{-\nu}I_\nu(z)$, $I_\nu(z)$ is the modified Bessel function. $m=(m_1,\ m_2,\ \ldots,\ m_n)$ is a multi-index, and $|m|=m_1+m_2+\ldots+m_n$ its length.

Taking into account $\bar{J}_\nu(0)=1$, in the limit for $\lambda_k\to0$, $k=\overline{1,n}$, we obtain

$$J_0\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x)=J_{0,0,\ldots,0}\begin{pmatrix}\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n\\ \eta_1,\ \eta_2,\ \ldots,\ \eta_n\end{pmatrix}f(x)$$
$$=\prod_{k=1}^{n}\left[\frac{2x_k^{-2(\alpha_k+\eta_k)}}{\Gamma(\alpha_k)}\right]\int\limits_0^{x_1}\int\limits_0^{x_2}\ldots\int\limits_0^{x_n}\prod_{k=1}^{n}\left[t_k^{2\eta_k+1}\left(x_k^2-t_k^2\right)^{\alpha_k-1}\right]f(t)dt_1\ldots dt_n,\tag{7}$$

This operator is a multidimensional analog of the ordinary (not generalized) Erdélyi–Kober operator. In this case, the inverse operator (7) has the form:

$$J_0^{-1}\begin{pmatrix}\alpha\\ \eta\end{pmatrix}f(x)=2^{n-|m|}\left[\prod_{k=1}^{n}\frac{x_k^{-2\eta_k}}{\Gamma(m_k-\alpha_k)}\left(\frac{1}{x_k}\frac{\partial}{\partial x_k}\right)^{m_k}\right]$$

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$$\int_0^{x_1}\int_0^{x_2}...\int_0^{x_n}\prod_{k=1}^{n}\left[t_k^{2(\eta_k+\alpha_k)+1}\left(x_k^2-t_k^2\right)^{m_k-1-\alpha_k}\right]f(t_1,\,t_2,\,...,\,t_n)dt_1dt_2\,...\,dt_n. \tag{8}$$

In addition, the following theorems were proved in [17–19]:

Let $\left[B_{\eta_k}^{x_k}\right]^0=E$, where E is the unit operator, $\left[B_{\eta_k}^{x_k}\right]^{m_k}=\left[B_{\eta_k}^{x_k}\right]^{m_k-1}\left[B_{\eta_k}^{x_k}\right]$ is the m_k th power of the operator $B_{\eta_k}^{x_k}$, $k=\overline{0,n}$.

Theorem 1. *([18,19]) Let $\alpha_k>0$, $\eta_k\geq-1/2$; $f(x)\in C^{2m_0}(\Omega^n)$; $x_k^{2\eta_k+1}\left[B_{\eta_k}^{x_k}\right]^{p_k+1}f(x)$ functions are integrable in a neighborhood of the origin and $\lim\limits_{x_k\to0}x_k^{2\eta_k+1}(\partial/\partial x_k)\left[B_{\eta_k}^{x_k}\right]^{p_k}f(x)=0$, $p_k=\overline{0,m_k-1}$, $k=\overline{1,n}$. Then*

$$\left[B_{\eta_k+\alpha_k}^{x_k}+\lambda_k^2\right]^{m_k}J_\lambda\binom{\alpha}{\eta}f(x)=J_\lambda\binom{\alpha}{\eta}\left[B_{\eta_k}^{x_k}\right]^{m_k}f(x),\ k=\overline{1,n},$$

where $m_0=\max\{m_1,m_2,\ldots,m_n\}$.

We note that Theorem 1 is also true in the case when some or all of the $\lambda_k=0$, $k=\overline{1,n}$.

Corollary 1. *Suppose that the conditions of Theorem 1 are satisfied. Then*

$$\sum_{k=1}^{n}\left[B_{\eta_k+\alpha_k}^{x_k}+\lambda_k^2\right]^{m_k}J_\lambda\binom{\alpha}{\eta}f(x)=J_\lambda\binom{\alpha}{\eta}\sum_{k=1}^{n}\left[B_{\eta_k}^{x_k}\right]^{m_k}f(x),$$

in addition, if $f(x)\in C^{2|m|}(\Omega^n)$, then

$$\prod_{k=1}^{n}\left[B_{\eta_k+\alpha_k}^{x_k}+\lambda_k^2\right]^{m_k}J_\lambda\binom{\alpha}{\eta}f(x)=J_\lambda\binom{\alpha}{\eta}\prod_{k=1}^{n}\left[B_{\eta_k}^{x_k}\right]^{m_k}f(x).$$

Theorem 2. *([18,19]) Let $\alpha_k>0$, $\eta_k\geq-1/2$, $q\in N$; $f(x)\in C^{2q}(\Omega^n)$; the functions $x_k^{2\eta_k+1}\left[B_{\eta_k}^{x_k}\right]^{l+1}f(x)$ are integrable in a neighborhood of the origin and*

$$\lim_{x_k\to0}x_k^{2\eta_k+1}(\partial/\partial x_k)\left[B_{\eta_k}^{x_k}\right]^{l}f(x)=0,\ l=\overline{0,q-1},\ k=\overline{1,n}.$$

Then

$$\left[\sum_{k=1}^{n}\left(B_{\eta_k+\alpha_k}^{x_k}+\lambda_k^2\right)\right]^{q}J_\lambda\binom{\alpha}{\eta}f(x)=J_\lambda\binom{\alpha}{\eta}\left[\sum_{k=1}^{n}B_{\eta_k}^{x_k}\right]^{q}f(x).$$

Corollary 2. *Suppose that the conditions of Theorem 2 are satisfied. Then for $\eta_k=-1/2$, $k=\overline{1,n}$,*

$$\left[\sum_{k=1}^{n}\left(B_{\alpha_k-1/2}^{x_k}+\lambda_k^2\right)\right]^{q}J_\lambda\binom{\alpha}{-1/2}f(x)=J_\lambda\binom{\alpha}{-1/2}\Delta^q f(x),$$

in particular, for $\lambda_k=0$, we have the equality

$$\Delta_B^q J_0\binom{\alpha}{-1/2}f(x)=J_0\binom{\alpha}{-1/2}\Delta^q f(x). \tag{9}$$

Note that in the works [20–24] the Erdélyi–Kober operators were used as a transmutation operator when solving initial and boundary value problems for hyperbolic type equations with the Bessel operator, and in the works [18,19,25] they were used for parabolic type equations with the Bessel operator.

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3. Solving the Problem

We will seek a solution to problem (1)–(3) in the form

$$u(x,t) = J_0^{\alpha} v(x,t) \tag{10}$$

where $v(x,t)$ is an unknown four times continuously differentiable function, $J_0^{\alpha} = J_0\begin{pmatrix}\alpha \\ -1/2\end{pmatrix}$ is the multidimensional Erdélyi–Kober operator (7).

Let us substitute (10) into equation (1), the initial conditions (2) and boundary conditions (3), taking into account Corollary 2 (see (9)), we obtain the problem of finding a solution $v(x,t)$ of the following equation

$$v_{tt} + \Delta^2 v = 0, (x,t) \in \Omega \tag{11}$$

satisfying initial

$$v(x,0) = \Phi(x), v_t(x,0) = 0, x \in R_+^n \tag{12}$$

and boundary conditions

$$\left.\frac{\partial v}{\partial x_k}\right|_{x_k=0} = 0, \left.\frac{\partial^3 v}{\partial x_k^3}\right|_{x_k=0} = 0, t \in R_+^1 \tag{13}$$

where

$$\Phi(x) = [J_0^{\alpha}]^{-1}\varphi(x) = \prod_{j=1}^{n}\left[\frac{1}{\Gamma(1-\alpha_j)}\right]\frac{\partial^n}{\partial x_1 \partial x_2 ... \partial x_n}$$

$$\times \int_0^{x_1}\int_0^{x_2}...\int_0^{x_n}\prod_{k=1}^{n}\left[s_k^{2\alpha_k}\left(x_k^2 - s_k^2\right)^{-\alpha_k}\right]\varphi(s_1, s_2, ..., s_n)ds_1 ds_2 ... ds_n. \tag{14}$$

Considering boundary conditions (13), we continue the function $\Phi(x)$ on $x_k \leqslant 0, k = \overline{1,n}$ evenly and denoted by $\Phi_1(x)$ continued function.

Then in the half-space $\Omega^+ = \{(x,t) : x \in R^n, t \in R_+^1\}$ we obtain the problem of finding a solution to equation (11) that satisfies the initial conditions

$$v(x,0) = \Phi_1(x), \ v_t(x,0) = 0, \ x \in R^n. \tag{15}$$

Let

$$\Phi_1(x) \in C^{n+3}(R^n), |x|^{n+5}|\Phi_1(x)| < M, |x|^{n+1}\left|D^{\beta}\Phi_1(x)\right| < M, |\beta| \leqslant n+3, \tag{16}$$

where $M = const > 0$, β is a multi index and $|\beta|$ its length. Then the solution to problem (11), (15) has the form [26]:

$$v(x,t) = \frac{1}{\left(2\sqrt{\pi t}\right)^n}\int_{R^n}\Phi_1(\xi)\cos\left(\frac{|x-\xi|^2}{4t} - \frac{\pi n}{4}\right)d\xi. \tag{17}$$

Considering

$$\mathrm{Re}\left\{e^{i\left(\frac{|x-\xi|^2}{4t} - \frac{\pi n}{4}\right)}\right\} = \cos\left(\frac{|x-\xi|^2}{4t} - \frac{\pi n}{4}\right)$$

we rewrite equality (17) in the form

$$v(x,t) = \mathrm{Re}\, w(x,t)$$

where

$$w(x,t) = \frac{e^{-i\frac{\pi n}{4}}}{\left(2\sqrt{\pi t}\right)^n}\int_{R^n}\Phi_1(\xi)\exp\left[i\frac{|x-\xi|^2}{4t}\right]d\xi. \tag{18}$$

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We rewrite equality (14) in the form

$$\Phi(\xi_1,\xi_2,...,\xi_n)=\frac{\partial^n\Phi_0(\xi_1,\xi_2,...,\xi_n)}{\partial\xi_1\partial\xi_2...\partial\xi_n}, \tag{19}$$

where $0<\alpha_k<1,\ k=\overline{1,n}$,

$$\Phi_0(\xi)=\prod_{j=1}^{n}\left[\frac{1}{\Gamma(1-\alpha_j)}\right]\int_0^{\xi_1}\int_0^{\xi_2}...\int_0^{\xi_n}\prod_{k=1}^{n}\left[s_k^{2\alpha_k}\left(\xi_k^2-s_k^2\right)^{-\alpha_k}\right]\varphi(s)ds \tag{20}$$

Changing the variables in the last integral by $s_k=\xi_k\mu_k, k=\overline{1,n}$, we obtain

$$\Phi_0(\xi)=\prod_{j=1}^{n}\left[\frac{\xi_j}{\Gamma(1-\alpha_j)}\right]\int_0^{1}\int_0^{1}...\int_0^{1}\prod_{k=1}^{n}\left[\mu_k^{2\alpha_k}\left(1-\mu_k^2\right)^{-\alpha_k}\right]\varphi(\xi_1\mu_1,\ ...,\xi_n\mu_n)d\mu_1...d\mu_n$$

Hense it follows thet $\Phi_0(\xi)|_{\xi_j=0}=0,\ j=\overline{1,n}$.

Let

$$\varphi(x)\in C^{2n+3}(R^n), |x|^{2n+5}|\varphi(x)|<M_1, |x|^{2n+1}\left|D^{\beta}\varphi(x)\right|<M_1, |\beta|\leqslant 2n+3, \tag{21}$$

$M_1=const>0$, then, by virtue of (19) and (20), the conditions (16) hold.

Taking into account $|x-\xi|^2=\sum\limits_{k=1}^{n}(x_k-\xi_k)^2$, we rewrite formula (18) in the form

$$w(x,t)=\frac{e^{-i\frac{\pi n}{4}}}{\left(2\sqrt{\pi t}\right)^n}\int_{R^n}\Phi_1(\xi)\prod_{j=1}^{n}\exp\left[i\frac{(x_j-\xi_j)^2}{4t}\right]d\xi$$

$$=\frac{e^{-i\frac{\pi n}{4}}}{\left(2\sqrt{\pi t}\right)^n}\prod_{j=1}^{n}\left\{\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\exp\left[i\frac{(x_j-\xi_j)^2}{4t}\right]d\xi_j\right\}$$

$$\times\left[\frac{1}{2\sqrt{\pi t}}\int_{-\infty}^{+\infty}\exp\left[i\frac{(x_n-\xi_n)^2}{4t}\right]\Phi_1(\xi_1,\xi_2,...,\xi_n)d\xi_n\right]$$

Considering parity of the function $\Phi_1(\xi)$, after simple transformations, we have

$$w(x,t)=e^{-i\frac{\pi n}{4}}\prod_{j=1}^{n-1}\left\{\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\left[e^{-\frac{(x_j-\xi_j)^2}{4it}}+e^{-\frac{(x_j+\xi_j)^2}{4it}}\right]d\xi_j\right\}$$

$$\times\left\{\frac{1}{2\sqrt{\pi t}}\int_0^{+\infty}\left[e^{-\frac{(x_n-\xi_n)^2}{4it}}+e^{-\frac{(x_n+\xi_n)^2}{4it}}\right]\Phi(\xi_1,\xi_2,...,\xi_n)d\xi_n\right\}.$$

Let

$$G_\varepsilon(x_k,\xi_k,t)=\frac{1}{2\sqrt{\pi t}}\left[e^{-\frac{(x_k-\xi_k)^2}{4a_\varepsilon}}+e^{-\frac{(x_k+\xi_k)^2}{4a_\varepsilon}}\right],\ a_\varepsilon=\varepsilon+it,\ \varepsilon>0.$$

Then $w(x,t)=\lim\limits_{\varepsilon\to+0}w_\varepsilon(x,t)$, where

$$w_\varepsilon(x,t)=e^{-i\frac{\pi n}{4}}\prod_{j=1}^{n-1}\left\{\int_0^{+\infty}G_\varepsilon(x_j,\xi_j,t)d\xi_j\right\}\int_0^{+\infty}G_\varepsilon(x_n,\xi_n,t)\frac{\partial^n\Phi_0(\xi)}{\partial\xi_1\partial\xi_2...\partial\xi_n}d\xi_n \tag{22}$$

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Appleying the integrating by parts rule to the last integral of (22), and taking into account $\lim_{\xi_j\to\infty} \Phi_0(\xi) = 0$, we obtain

$$w_\varepsilon(x,t) = e^{-i\frac{\pi n}{4}} \prod_{j=1}^{n-1} \left\{ \int_0^{+\infty} G_{\varepsilon\xi_j}(x_j,\xi_j,t)d\xi_j \right\} \int_0^{+\infty} G_{\varepsilon\xi_n}(x_n,\xi_n,t)\Phi_0(\xi_1,\xi_2,...,\xi_n)d\xi_n \tag{23}$$

Next, applying the formula [27]

$$\int_0^{+\infty} e^{-a\lambda^2}\cos(b\lambda)d\lambda = \sqrt{\frac{\pi}{4a}}e^{-\frac{b^2}{4a}}, \mathrm{Re}\, a > 0,$$

at $b = x_j - \xi_j, a_\varepsilon = \varepsilon + it,\ \varepsilon > 0, t > 0$, we have

$$G_\varepsilon(x_j,\xi_j,t) = \frac{2}{\pi}\frac{\sqrt{a_\varepsilon}}{\sqrt{t}} \int_0^{+\infty} e^{-a_\varepsilon\lambda^2}\cos(x_j\lambda)\cos(\xi_j\lambda)d\lambda,$$

Computing the derivative of this function, we get

$$G_{\varepsilon\xi_j}(x_j,\xi_j,t) = -\frac{2}{\pi}\frac{\sqrt{a_\varepsilon}}{\sqrt{t}} \int_0^{+\infty} e^{-a_\varepsilon\lambda^2}\cos(x_j\lambda)\sin(\xi_j\lambda)\lambda d\lambda \tag{24}$$

Substituting the (20) expression of the function $\Phi_0(\xi)$ into (23) and taking (24) into account, we have

$$w_\varepsilon(x,t) = e^{-i\frac{\pi n}{4}} \prod_{j=1}^{n}\left[\frac{1}{\Gamma(1-\alpha_j)}\right] \prod_{j=1}^{n}\left\{ \int_0^{+\infty} G_{\varepsilon\xi_j}(x_j,\xi_j,t)d\xi_j \int_0^{\xi_j} s_j^{2\alpha_j}\left(\xi_j^2 - s_j^2\right)^{-\alpha_j} ds_j \right\}$$

$$\times \int_0^{+\infty} G_{\varepsilon\xi_n}(x_n,\xi_n,t)d\xi_n \int_0^{\xi_n} s_n^{2\alpha_n}\left(\xi_n^2 - s_n^2\right)^{-\alpha_n}\varphi(s)ds_n.$$

In the last equality, through successively changing the order of integration, we obtain

$$w_\varepsilon(x,t) = e^{-i\frac{\pi n}{4}} \prod_{j=1}^{n}\left[\frac{1}{\Gamma(1-\alpha_j)}\right] \int_0^{\infty}\int_0^{\infty}...\int_0^{\infty} \varphi(s)\prod_{j=1}^{n}\left[s_j^{2\alpha_j}P_\varepsilon(x_j,s_j,t)\right]ds, \tag{25}$$

where

$$P_\varepsilon(x_j,s_j,t) = \int_{s_j}^{+\infty} G_{\varepsilon\xi_j}(x_j,\xi_j,t)\left(\xi_j^2 - s_j^2\right)^{-\alpha_j} d\xi_j$$

$$= -\frac{2}{\pi}\frac{\sqrt{a_\varepsilon}}{\sqrt{t}} \int_0^{+\infty} e^{-a_\varepsilon\lambda^2}\cos(x_j\lambda)\lambda \left\{ \int_{s_j}^{\infty}\left(\xi_j^2 - s_j^2\right)^{-\alpha_j}\sin(\xi_j\lambda)\lambda d\xi_j \right\} d\lambda.$$

Applying the Mehler-Sonin formula to the internal integral [28, p.93], we obtain

$$P_\varepsilon(x_j,s_j,t) = -\frac{2^{\frac{1}{2}-\alpha_j}}{\sqrt{\pi}}\frac{\sqrt{a_\varepsilon}}{\sqrt{t}}\Gamma(1-\alpha_j)s_j^{\frac{1}{2}-\alpha_j} \int_0^{\infty} e^{-a_\varepsilon\lambda^2}\lambda^{\alpha_j+\frac{1}{2}}\cos(x_j\lambda)J_{\alpha_j-\frac{1}{2}}(s_j\lambda)d\lambda, \tag{26}$$

where $J_\nu(z)$ is the Bessel function of the first type.

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Substituting (26) into (25), we get

$$w_\varepsilon(x,t) = (-1)^n e^{-i\frac{\pi n}{4}} \frac{2^{\frac{n}{2}-|\alpha|}}{(\sqrt{\pi})^n} \left(\frac{\sqrt{a_\varepsilon}}{\sqrt{t}}\right)^n \int_0^\infty \int_0^\infty \dots \int_0^\infty \varphi(s) \prod_{j=1}^n \left(s_j^{1/2+\alpha_j} P_{1\varepsilon}(x_j, s_j, t)\right) ds_n \tag{27}$$

where

$$P_{1\varepsilon}(x_j, s_j, t) = \int_0^\infty e^{-a_\varepsilon \lambda^2} \lambda^{\alpha_j+1/2} J_{\alpha_j-1/2}(s_j\lambda) \cos(x_j\lambda) d\lambda.$$

Applying the theorem on passage to the limit under the improper integral sign and taking into account $\lim\limits_{\varepsilon\to+0} \sqrt{a_\varepsilon} = \sqrt{it} = e^{i\frac{\pi}{4}}\sqrt{t}, \quad t>0$, we obtain

$$v(x,t) = \mathrm{Re} \lim_{\varepsilon\to+0} w_\varepsilon(x,t) = (-1)^n \frac{2^{\frac{3n}{2}-|\alpha|}}{(\sqrt{\pi})^n} \int_0^\infty \int_0^\infty \dots \int_0^\infty \varphi(s) \prod_{j=1}^n \left(s_j^{1/2+\alpha_j} P_2(x_j, s_j, t)\right) ds, \tag{28}$$

where $P_2(x_j, s_j, t) = \int\limits_0^\infty \cos(t\lambda^2) \lambda^{\alpha_j+1/2} J_{\alpha_j-1/2}(s_j\lambda) \cos(x_j\lambda) d\lambda$.

Substituting equality (28) into (10), we obtain

$$u(x,t) = J_0^\alpha v(x,t) = (-1)^n \prod_{j=1}^n \left[\frac{2x_j^{1-2\alpha_j}}{\Gamma(\alpha_j)}\right] \frac{2^{\frac{3n}{2}-|\alpha|}}{(\sqrt{\pi})^n} \int_0^{x_1} \int_0^{x_2} \dots \int_0^{x_n} \prod_{k=1}^n \left[\left(x_k^2 - \xi_k^2\right)^{\alpha_k-1}\right]$$

$$\times \left\{ \int_0^\infty \int_0^\infty \dots \int_0^\infty \varphi(s) \prod_{k=1}^n \left[s_j^{1/2-\alpha_j} P_2(x_j, s_j, t)\right] ds \right\} d\xi.$$

Changing the order of integration, we have

$$u(x,t) = (-1)^n \frac{2^{\frac{3n}{2}-|\alpha|}}{(\sqrt{\pi})^n} \prod_{j=1}^n \left[\frac{x_j^{1-2\alpha_j}}{\Gamma(\alpha_j)}\right] \int_0^\infty \int_0^\infty \dots \int_0^\infty \varphi(s) \prod_{k=1}^n \left[s_k^{1/2+\alpha_k}\right]$$

$$\times \left\{ \prod_{k=1}^n \left[\int_0^{x_k} \left(x_k^2 - \xi_k^2\right)^{\alpha_k-1} P_2(x_j, s_j, t) \right] d\xi \right\} ds.$$

Now, we will compute the internal integral

$$\int_0^{x_k} \left(x_k^2 - \xi_k^2\right)^{\alpha_k-1} P_2(x_j, s_j, t) d\xi_k$$

$$= \int_0^\infty \cos\left(t\lambda^2\right) \lambda^{\alpha_j+1/2} J_{\alpha_j-1/2}(s_j\lambda) d\lambda \int_0^{x_k} \left(x_k^2 - \xi_k^2\right)^{\alpha_k-1} \cos(\xi_k\lambda) d\xi_k.$$

Hence applying the Poisson formula [28, p.93], we get

$$u(x,t) = \prod_{k=1}^n \left[x_k^{1/2-\alpha_k}\right] \int_0^\infty \int_0^\infty \dots \int_0^\infty \varphi(s)$$

$$\times \prod_{j=1}^n \left[s_j^{1/2+\alpha_j} \int_0^\infty \cos\left(t\lambda^2\right) \lambda J_{\alpha_j-1/2}(s_j\lambda) J_{\alpha_j-1/2}(x_j\lambda) d\lambda \right] ds.$$

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Applying the following formula [27, p.201]

$$\int_0^\infty \lambda \cos\left(a\lambda^2\right) J_\nu(b\lambda) J_\nu(c\lambda) d\lambda = \frac{1}{2a} J_\nu\left(\frac{bc}{2a}\right) \sin\left[\frac{b^2+c^2}{4a} - \frac{\nu\pi}{2}\right], [\mathrm{Re}\,\nu > -1,\ \ a, b, c > 0],$$

for $a = t > 0,\ b = s_j > 0,\ c = x_j > 0,\ \nu = \alpha_j - 1/2 > -1; \alpha_j > -1/2$, we have

$$\int_0^\infty \lambda \cos\left(t\lambda^2\right) J_{\alpha_j-1/2}(s_j\lambda) J_{\alpha_j-1/2}(x_j\lambda) d\lambda$$
$$= \frac{1}{2t} J_{\alpha_j-1/2}\left(\frac{x_j s_j}{2t}\right) \sin\left[\frac{x_k^2+s_j^2}{4t} - \frac{\pi}{2}(\alpha_j - 1/2)\right].$$

Thus, we have obtained the representation of the solution of the problem as follows

$$u(x,t) = \frac{1}{(2t)^n} \prod_{j=1}^n \left[x_j^{\frac{1}{2}-\alpha_j}\right] \int_0^\infty \int_0^\infty \ldots \int_0^\infty \varphi(s)$$
$$\times \prod_{j=1}^n \left[s_j^{\frac{1}{2}+\alpha_j} J_{\alpha_j - \frac{1}{2}}\left(\frac{x_j s_j}{2t}\right) \sin\left[\frac{x_k^2+s_j^2}{4t} - \frac{\pi}{2}\left(\alpha_j - \frac{1}{2}\right)\right]\right] ds. \qquad (29)$$

The following theorem is true

Theorem 3. *Let conditions (21) be fulfilled. Then the function $u(x,t)$ defined by (29) will be a solution to the problem (1)–(3).*

We should note that the formula (29), when $n = 1$, coincides with the formula obtained in [29].

4. Conclusions

Using the Erdélyi–Kober transmutation operator, an exact solution of the problem is constructed. Despite the development of modern computer technology, the construction of exact solutions to boundary value problems for partial differential equations is still an important and urgent task. These solutions allow a deeper understanding of the qualitative features of the described processes and phenomena, the properties of mathematical models, and can also be used as test cases for asymptotic, approximate and numerical methods.

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