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Not peer-reviewed version

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Posted Date: 18 June 2024

doi: 10.20944/preprints202406.1246.v1

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Article

Generalized Right Core Inverse in *-Banach Algebras

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Abstract: We introduce a new generalized inverse which is a natural generalization of (pseudo) right core inverse in a Banach *-algebra. We characterize this generalized inverse by using right core and quasi-nilpotent decomposition. We then present its polar-like characterization and investigate related algebraic properties. Finally, the right core-EP inverse is characterized by certain new ways.

Keywords: core inverse; right core inverse; core-EP inverse; right core-EP inverse; generalized right core inverse; polar-like property; *-Banach algebra

MSC: 15A09, 16U90, 46H05

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1. Introduction

A Banach algebra $\mathcal{A}$ is called a Banach *-algebra if there exists an involution $* : x \to x^*$ satisfying $(x+y)^* = x^* + y^*, (\lambda x)^* = \overline{\lambda} x^*, (xy)^* = y^* x^*, (x^*)^* = x$. An element a in a Banach *-algebra $\mathcal{A}$ has core inverse if and only if there exist $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, xa^2 = a.$$

If such x exists, it is unique, and denote it by $a^{\text{\textcircled{\#}}}$ (see[1,9,16]).

Wang et al. generalized the core inverse to the right core inverse (see [14]). An element $a \in \mathcal{A}$ has right core inverse if there exist $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, axa = a.$$

If such x exists, it is unique, and denote it by $a_r^{\text{\textcircled{\#}}}$. Let $\mathcal{A}_r^{\text{\textcircled{\#}}}$ denote the set of all right core invertible elements in $\mathcal{A}$. Here we list some characterizations of right core inverse.

Theorem 1. *(see [14]) Let $\mathcal{A}$ be a Banach *-algebra, and let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\text{\textcircled{\#}}}$.
(2) *There exists some $x \in \mathcal{A}$ such that*

$$axa = a, x = xax, ax^2 = x, (ax)^* = ax.$$

(3) $a \in \mathcal{A}^{(1,3)}$ *and* $a\mathcal{A} = a^2\mathcal{A}$.
(4) $a \in \mathcal{A}$ *is right* (a, a^*)*-invertible.*
(5) $\mathcal{A}a = \mathcal{A}(a^*)^2 a$.
(6) *There exists unique an idempotent $p \in \mathcal{A}$ such that*

$$pa = 0, a + p \in \mathcal{A} \textit{ is right invertible.}$$

In [6], Gao and Chen extended the concept of the core inverse and introduced the notion of core-EP inverse (i.e., pseudo core inverse). An element $a \in \mathcal{A}$ has core-EP inverse if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$ax^2 = x, (ax)^* = ax, xa^{k+1} = a^k.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{\dagger}}$. Many authors have investigated core-EP inverses from many different views, e.g., [6,7,10–13].

The motivation of this paper is to introduce and study a new kind of generalized inverse as a natural generalization of generalized inverses mentioned above. Let

$$\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid \lim_{n\to\infty} \parallel x^n \parallel^{\frac{1}{n}} = 0\}.$$

Evidently, $x \in \mathcal{A}^{qnil}$ if and only if $1 + \lambda x \in \mathcal{A}$ is invertible for any $\lambda \in \mathbb{C}$.

Definition 1. *An element $a \in \mathcal{A}$ has generalized right core decomposition if there exist $x, y \in \mathcal{A}$ such that*

$$a = x + y, x^*y = yx = 0, x \in \mathcal{A}_r^{\textcircled{\#}},\ y \in \mathcal{A}^{qnil}.$$

In Section 2, we prove that $a \in \mathcal{A}$ has generalized right core decomposition if and only if there exist unique a $x \in \mathcal{A}$ such that

$$x = ax^2, (ax)^* = ax, \lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0.$$

The preceding x is called generalized right core inverse of a and we denote it by $a_r^{\textcircled{d}}$.

Recall that $a \in \mathcal{A}$ has generalized Drazin inverse if there exists $x \in \mathcal{A}$ such that $ax = xa, ax^2 = x, a - a^2x \in \mathcal{A}^{qnil}$. Such x is unique, if it exists, and denote it by a^d. As it is well known, a has generalized Drazin inverse if and only if it has quasi-polar property, i.e., there exists an idempotent $p \in \mathcal{A}$ such that $a + p \in \mathcal{A}^{-1}$ and $ap = pa \in \mathcal{A}^{qnil}$ (see [2, Theorem 6.4.8]). In Section 3, we characterize generalized right core inverse by using a polar-like property. We prove that $a \in \mathcal{A}$ has generalized right core inverse if and only if there exists a projection $p \in \mathcal{A}$ (i.e., $p^2 = p = p^*$)such that

$$a + p \in \mathcal{A} \text{ is right invertible, } ap \in \mathcal{A}^{qnil}, (1 - p)\mathcal{A} = a(1 - p)\mathcal{A}.$$

Related equivalent characterizations are given.

In Section 4, we are concerned with algebraic properties of generalized right core inverse. The necessary and sufficient conditions under which the sum of two generalized right core invertible elements has generalized right core inverse are established.

Following Wang et al.(see [14]), an element a in $\mathcal{A}$ has right core-EP (i.e., right pseudo core) inverse if there exists $x \in \mathcal{A}$ such that $x = ax^2, (ax)^* = ax, a^n = axa^n$. Such x is unique, if exists, and denote it by $a_r^{\textcircled{\dagger}}$. Finally, in Section 5, the right core-EP inverse is characterized by certain new ways.

Throughout the paper, all Banach *-algebras are complex with an identity. We use $\mathcal{A}_r^{-1}, \mathcal{A}_r^{\textcircled{d}}, \mathcal{A}^{\textcircled{\dagger}}$ and $\mathcal{A}_r^{\textcircled{\dagger}}$ to denote the sets of all right invertible, generalized right core invertible, core-EP invertible and right core-EP invertible elements in $\mathcal{A}$, respectively. Let $\mathcal{A}^{nil}$ denote the set of all nilpotents in $\mathcal{A}$. If a and x satisfy the equations $a = axa$ and $(ax)^* = eax$, then x is called $(1,3)$-inverse of a and is denoted by $a^{(1,3)}$. We use $\mathcal{A}^{(1,3)}$ to stand for the set of all $(1,3)$-invertible elements in $\mathcal{A}$.

2. generalized right core inverse

In this section, we introduce generalized right core inverse by using a kind of right core and quasi-nilpotent decomposition. We begin with

Theorem 2. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized right core decomposition.
(2) There exist $x \in \mathcal{A}$ such that

$$x = ax^2, (ax)^* = ax, \lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0.$$

 doi:10.20944/preprints202406.1246.v1

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exist $z, y \in \mathcal{A}$ such that

$$a = z + y, z^*y = yz = 0, z \in \mathcal{A}_r^{\#}, \ y \in \mathcal{A}^{qnil}.$$

Set $x = z_r^{\#}$. Then we check that

$$\begin{aligned} ax &= (z+y)z_r^{\#} = zz_r^{\#}, \\ ax^2 &= (ax)x = z(z_r^{\#})^2 = z_r^{\#} = x, \\ (ax)^* &= (zz_r^{\#})^* = zz_r^{\#} = ax, \\ a - axa &= a - zz_r^{\#}(z+y) = a - z = y \in \mathcal{A}^{qnil}. \end{aligned}$$

By using Cline's formula (see [2, Theorem 6.4.11]), we have

$$a - xa^2 \in \mathcal{A}^{qnil}.$$

Moreover, we see that

$$(1-ax)a^n = (a-axa)a^{n-1} = y(z+y)a^{n-2} = y^2a^{n-2} = \cdots = y^n,$$

and therefore

$$\lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = \lim_{n\to\infty} ||y^n||^{\frac{1}{n}} = 0.$$

(2) $\Rightarrow$ (1) By hypotheses, there exist $t \in \mathcal{A}$ such that

$$t = at^2, (at)^* = at, \lim_{n\to\infty} ||a^n - ata^n||^{\frac{1}{n}} = 0.$$

For any $n \in \mathbb{N}$, we have

$$\begin{aligned} at &= a(at^2) = a^2t^2 = a^2(at^2)z = a^3t^3 \\ &= \cdots = a^nt^n = \cdots = a^{n+1}t^{n+1}. \end{aligned}$$

Let $z = tat$. One directly verifies that

$$\begin{aligned} ||at - az|| &= ||a^nt^n - ata^nt^n|| = ||[a^n - ata^n]t^n|| \\ &\le ||a^n - ata^n||||t^n||, \end{aligned}$$

Then

$$\lim_{n\to\infty} ||at - az||^{\frac{1}{n}} = 0;$$

hence, $az = at$, and so $(az)^* = (at)^* = at = az$.

$$\begin{aligned} z - az^2 &= tat - (at)z = tat - at^2at \\ &= (tat - at^2) + at(1 - ta)t \\ &= (ta - 1)t + at(1 - ta)t \\ &= (at - 1)(1 - ta)t = (at - 1)[1 - a^nt^{n+1}a]t. \end{aligned}$$

Since $t = at^2$, by induction, we have $t = a^nt^{n+1}$, and so $z - az^2 = (at-1)[1 - a^nt^{n+1}a]a^nt^{n+1}$; hence,

$$||z - az^2|| \le ||a^n - ata^n||||1 - t^{n+1}a^{n+1}||||t||^{n+1}.$$

Since $\lim_{n\to\infty} ||a^n - ata^n||^{\frac{1}{n}} = 0$, we deduce that

$$\lim_{n\to\infty} ||z - az^2||^{\frac{1}{n}} = 0.$$

This implies that $az^2 = z$.

Moreover, we have $zaz = z(at) = (tat)at = t(ata)a^{n-1}t^n = t(ata^n)t^n$; hence,

$$\begin{array}{rcl} \|z - zaz\| & = & \|ta^nt^n - t(ata^n)t^n\| \\ & \le & \|t\|\|a^n - ata^n\|\|t^n\| \end{array}$$

This implies that

$$\lim_{n\to\infty} \|zaz - z\|^{\frac{1}{n}} = 0;$$

whence, $z = zaz$.

Since $at = az$, we see that $\|a^n - aza^n\|^{\frac{1}{n}} = \|a^n - ata^n\|^{\frac{1}{n}}$, and so

$$\lim_{n\to\infty} \|a^n - aza^n\|^{\frac{1}{n}} = 0.$$

Set $x = aza$ and $y = a - aza$. Then $a = x + y$. We check that

$$\begin{array}{rcl} (1 - ax)a(ax) & = & (a - axa)a^{n-1}x^{n-1} \\ & = & (a^n - axa^n)x^{n-1}. \end{array}$$

Therefore

$$\|(1 - ax)a(ax)\|^{\frac{1}{n}} \le \|a^n - axa^n|^{\frac{1}{n}}\|\|x^{n-1}\|^{\frac{1}{n}}.$$

Since

$$\lim_{n\to\infty} \|a^n - axa^n\|^{\frac{1}{n}} = 0,$$

we prove that

$$\lim_{n\to\infty} \|(1 - ax)a(ax)\|^{\frac{1}{n}} = 0.$$

This implies that $(1 - ax)a(ax) = 0$.

We claim that x has right core inverse. Evidently, we verify that

$$\begin{array}{rcl} xz & = & azaz = az, \\ xz^2 & = & (xz)z = az^2 = z, \\ xzx & = & (xz)x = azaza = aza = x, \\ (xz)^* & = & (az)^* = az = xz. \end{array}$$

Therefore $x \in \mathcal{A}_r^{\#}$ and $z = x_r^{\#}$.

We verify that

$$\begin{array}{rcl} \|(a - xa^2)^{n+2}\|^{\frac{1}{n+2}} & = & \|(1 - xa)a(a - xa^2)^n(a - xa^2)\|^{\frac{1}{n+2}} \\ & = & \|(1 - xa)a(a - xa^2)^{n-1}(a - xa^2)a\|^{\frac{1}{n+2}} \\ & = & \|(1 - xa)a(a - xa^2)^{n-1}a^2\|^{\frac{1}{n+2}} \\ & \vdots & \\ & = & \|(1 - xa)a(a - xa^2)a^n\|^{\frac{1}{n+2}} \\ & \le & \|1 - xa\|^{\frac{1}{n+2}}\left[\|a^n - axa^n\|^{\frac{1}{n}}\right]^{\frac{n}{n+2}}\|a\|^{\frac{1}{n+2}}. \end{array}$$

Accordingly,

$$\lim_{n\to\infty} \|(a - xa^2)^{n+2}\|^{\frac{1}{n+2}} = 0.$$

This implies that $a - xa^2 \in \mathcal{A}^{qnil}$. By using Cline's formula, $y = a - aza \in \mathcal{A}^{qnil}$.

Moreover, we see that

$$\begin{aligned} x^*y &= (axa)^*(1-ax)a = a^*(ax)^*(1-ax)a \\ &= a^*(ax)(1-ax)a = 0, \\ yx &= (a-axa)axa = (1-ax)a(ax)a = 0. \end{aligned}$$

Then we have a generalized right core decomposition $a = x + y$, as required. □

Corollary 1. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized right core decomposition.
(2) There exists unique $x \in \mathcal{A}$ such that

$$x = ax^2, (ax)^* = ax, \lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0.$$

Proof. (2) ⇒ (1) This is obvious by Theorem 2.1.

(1) ⇒ (2) In light of Theorem 2.1, there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (ax)^* = ax, \lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0.$$

Assume that there exists $z \in \mathcal{A}$ such that

$$z = az^2, (az)^* = az, \lim_{n\to\infty} ||a^n - aza^n||^{\frac{1}{n}} = 0.$$

Let $a_1 = axa, a_2 = a - a_1$ and $b_1 = aza, b_2 = a - b_1$. As in the proof of Theorem 2.1, we prove that

$$\begin{aligned} a_1^*a_2 = a_2a_1 = 0, a_2 \in \mathcal{A}^{qnil}, \\ b_1^*b_2 = b_2b_1 = 0, b_2 \in \mathcal{A}^{qnil}. \end{aligned}$$

For every $n \in \mathbb{N}$, $a^n = \sum\limits_{i=0}^{n} b_1^i b_2^{n-i}$, and then $(a^n)^*b_2 = (b_2^n)^*b_2$. Since $b_2b_1 = 0$, we have $a^n b_1 (b_1^n)^\# = (b_1)^n b_1 (b_1^n)^\# = b_1$. Since $ax^2 = x$, we have $a^n x^n = ax$. Then

$$\begin{aligned} ||b_1 - a_1||^2 &= ||b_1 - axa||^2 \\ &= ||b_1 - axb_1 - axb_2||^2 \\ &= ||b_1 - axb_1 - (ax)^*b_2||^2 \\ &= ||b_1 - a^nx^nb_1 - (a^nx^n)^*b_2||^2 \\ &= ||b_1 - a^nx^nb_1 - (x^n)^*(a^n)^*b_2||^2 \\ &= ||b_1 - a^nx^na^nb_1(b_1^n)^\# - (x^n)^*(a^n)^*b_2||^2 \\ &= ||(a^n - axa^n)b_1(b_1^n)^\# - (x^n)^*(b_2^*)^n b_2||^2 \\ &\leq ||a^n - axa^n||^2||b_1(b_1^n)^\#||^2 + ||(x^n)^*||^2||b_2||^2||(b_2^*)^n||^2 \\ &+ 2||a^n - axa^n||||(b_2^*)^n||||b_1(b_1^n)^\#||||(x^n)^*||||b_2||. \end{aligned}$$

Therefore

$$\begin{aligned} & ||b_1 - a_1||^{\frac{2}{n}} \\ \leq\; & ||a^n - axa^n||^{\frac{2}{n}}||b_1(b_1^n)^\#||^{\frac{2}{n}} + ||(x^n)^*||^{\frac{2}{n}}||b_2||^{\frac{2}{n}}||(b_2^*)^n||^{\frac{2}{n}} \\ +\; & 2||a^n - axa^n||^{\frac{1}{n}}||(b_2^*)^n||^{\frac{1}{n}}||b_1(b_1^n)^\#||^{\frac{2}{n}}||e^{-1}(x^n)^*||^{\frac{2}{n}}||b_2||^{\frac{2}{n}} \\ \leq\; & ||1 - ax||^{\frac{2}{n}}||||a||^2||b_1(b_1)^\#||^{\frac{2}{n}} + ||x^*||^2||b_2||^{\frac{2}{n}}||(b_2^*)^n||^{\frac{2}{n}} \\ +\; & 2||1 - ax||^{\frac{1}{n}}||a||^2||(b_2^*)^n||^{\frac{2}{n}}||b_1(b_1)^\#||^{\frac{2}{n}}||x^*||^2||b_2||^{\frac{2}{n}}. \end{aligned}$$

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Since $b_2 \in \mathcal{A}^{qnil}$, then $1-\overline{\lambda}b_2 \in \mathcal{A}^{-1}$; whence, $1-\lambda b_2^* \in \mathcal{A}^{-1}$. Then $b_2^* \in \mathcal{A}^{qnil}$, and so

$$\lim_{n\to\infty} ||(b_2^*)^n||^{\frac{1}{n}} = 0.$$

Accordingly,

$$\lim_{n\to\infty} ||b_1 - a_1||^{\frac{2}{n}} = 0.$$

Therefore $a_1 = b_1$.

As in the proof of Theorem 2.1, we check that $x = (axa)_r^{\textcircled{\#}} = (a_1)_r^{\textcircled{\#}} = (b_1)_r^{\textcircled{\#}} = (aza)_r^{\textcircled{\#}} = z$. Therefore $x = z$, as required. □

We denote such a x in Corollary 2.2 by $a_r^{\textcircled{d}}$, and call it the generalized right core inverse of a. Let $\mathcal{A}_r^{\textcircled{d}}$ denote the sets of all generalized right core invertible elements in $\mathcal{A}$.

Corollary 2. *Let $a \in \mathcal{A}_r^{\textcircled{d}}$. Then the following hold:*

(1) $a_r^{\textcircled{d}} = a_r^{\textcircled{d}} a a_r^{\textcircled{d}}$.
(2) $a a_r^{\textcircled{d}} = a^m (a_r^{\textcircled{d}})^m$ for any $m \in \mathbb{N}$.

Proof. (1) By hypothesis, there exist $z, y \in \mathcal{A}$ such that

$$a = z + y, z^*y = yz = 0, z \in \mathcal{A}_r^{\textcircled{\#}},\ y \in \mathcal{A}^{qnil}.$$

As in the proof of Theorem 2.1, we have $a_r^{\textcircled{d}} = z_r^{\textcircled{\#}}$. We directly verify that

$$xax = z_r^{\textcircled{\#}} z z_r^{\textcircled{\#}} = z_r^{\textcircled{\#}} = x,$$

as required.

(2) This is obvious by the proof of Theorem 2.1. □

An element $a \in \mathcal{A}$ has generalized core inverse if there exist $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{d}}$.

Corollary 3. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{\textcircled{d}}$ if and only if*

(1) $a \in \mathcal{A}^d$;
(2) $a \in \mathcal{A}_r^{\textcircled{d}}$.

Proof. This is obvious by Theorem 2.1 and [3, Theorem 2.5]. □

Corollary 4. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{\textcircled{D}}$ if and only if*

(1) $a \in \mathcal{A}^D$;
(2) $a \in \mathcal{A}_r^{\textcircled{d}}$.

Proof. This is obvious by Corollary 2.4 and [3, Corollary 3.4]. □

Recently, Zhu et al. extended right core inverse and introduced right w-core inverse (see [18]). An element $a \in \mathcal{A}$ has right w-core inverse if there exist $x \in \mathcal{A}$ such that

$$awx^2 = x, (awx)^* = awx, awxa = a.$$

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If such x exists, it is unique, and denote it by $a^{\#}_{r,w}$. Let $\mathcal{A}^{\#}_{r,w}$ denote the set of all right w-core invertible elements in $\mathcal{A}$.

Theorem 3. *Let $a, w \in \mathcal{A}$. Then the following are equivalent:*

(1) $aw \in \mathcal{A}^{\textcircled{d}}_{r}$.
(2) There exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}^{\#}_{r,w},\ yw \in \mathcal{A}^{qnil}.$$

Proof. (1) $\Rightarrow$ (2) By hypotheses, there exist $t \in \mathcal{A}$ such that

$$t = (aw)t^2, [(aw)t]^* = (aw)t, \lim_{n\to\infty} ||(aw)^n - (aw)t(aw)^n||^{\frac{1}{n}} = 0.$$

Let $z = t(aw)t, x = awza$ and $y = a - awza$.. As in the proof of Theorem 2.1, we prove that

$$(awz)^* = awz, awz^2 = z, z = zawz;$$
$$\lim_{n\to\infty} ||(aw)^n - (aw)z(aw)^n||^{\frac{1}{n}} = 0;$$
$$a = x + y, yw = (a - awza)w = aw - (aw)z(aw) \in \mathcal{A}^{qnil}.$$

We claim that x has right w-core inverse. Evidently, we verify that

$$\begin{aligned} xwz &= (awza)wz = awz, \\ xwz^2 &= (xwz)z = awz^2 = z, \\ xwzx &= (xwz)x = (awz)awza = (awz)^2a = (awz)a = x, \\ (xwz)^* &= (awz)^* = awz = xwz. \end{aligned}$$

Therefore $x \in \mathcal{A}^{\#}_{r,w}$ and $z = x^{\#}_{r,w}$.

Moreover, we see that

$$\begin{aligned} x^*y &= (awza)^*(1 - awz)a = a^*(awz)^*(1 - awz)a \\ &= a^*(awz)(1 - awz)a = 0. \end{aligned}$$

Since $z = awz^2$, we see that $z = (aw)^{n-1}z^n$. Then

$$\begin{aligned} ||ywx|| &= ||(a - awza)w(awza)|| \\ &= ||[(aw) - (aw)z(aw)]awza|| \\ &= ||[(aw) - (aw)z(aw)](aw)^{n-1}z^n a|| \\ &= ||[(aw)^n - (aw)z(aw)^n]z^n a||. \end{aligned}$$

Hence,

$$||ywx||^{\frac{1}{n}} \leq ||(aw)^n - (aw)z(aw)^n||^{\frac{1}{n}} ||z^n a||^{\frac{1}{n}}.$$

This implies that

$$\lim_{n\to\infty} ||ywx||^{\frac{1}{n}} = 0.$$

Thus $ywx = 0$, as required.

(2) $\Rightarrow$ (1) By hypothesis, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = ywx = 0, x \in \mathcal{A}^{\#}_{r,w},\ yw \in \mathcal{A}^{qnil}.$$

Hence, $aw = xw + yw, (xw)^*(yw) = w^*(x^*y)w = 0, (yw)(xw) = (ywx)w = 0$ and $yw \in \mathcal{A}^{qnil}$. Since $x \in \mathcal{A}^{\#}_{r,w}$, we have

$$xwz^2 = z, (xwz)^* = xwz, xwzx = x.$$

 doi:10.20944/preprints202406.1246.v1

Hence, $xwzxw = xw$, and so $xw \in \mathcal{A}^{\#}_{r,w}$. Therefore $aw \in \mathcal{A}^{\textcircled{d}}_r$, as asserted. □

3. equivalent characterizations

In this section, we present a polar-like property for generalized right core inverse in a Banach *-algebra. The related characterizations of generalized right core inverse are established.

Theorem 4. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{\textcircled{d}}_r$.
(2) *There exists a projection $p \in \mathcal{A}$ such that*

$$a + p \in \mathcal{A}^{-1}_r, ap \in \mathcal{A}^{qnil}, (1-p)\mathcal{A} = a(1-p)\mathcal{A}.$$

Proof. (1) ⇒ (2) Since $a \in \mathcal{A}^{\textcircled{d}}_r$, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = yx = 0, x \in \mathcal{A}^{\#}_r, y \in \mathcal{A}^{qnil}_r.$$

In view of [5, Lemma 4.3], we have

$$x^{\#}_r = x(x^{\#}_r)^2 = x^{\#}_r x x^{\#}_r, (xx^{\#}_r)^* = xx^{\#}_r, x = xx^{\#}_r x.$$

Let $p = 1 - xx^{\#}_r$. Then $p^2 = p = p^*$ and $px = 0$. We directly check that

$$(x + 1 - xx^{\#}_r)(x^{\#}_r + 1 - xx^{\#}_r) = 1 + x(1 - xx^{\#}_r).$$

Let $q = [x^{\#}_r + 1 - xx^{\#}_r][1 + x(1 - xx^{\#}_r)]^{-1}$. Then $(x + p)q = 1$.

$$\begin{aligned} 1 + yq &= 1 + (yx^{\#}_r + y - yxx^{\#}_r)[1 + x(1 - xx^{\#}_r)]^{-1} \\ &= 1 + [y - yxx^{\#}_r][1 + x(1 - xx^{\#}_r)]. \end{aligned}$$

We check that

$$\begin{aligned} & 1 + [1 - xx^{\#}_r][1 + x(1 - xx^{\#}_r)]y \\ = & 1 + [1 - xx^{\#}_r][y + xy] \\ = & 1 + y + [1 - xx^{\#}_r]xy \\ = & 1 + y \in \mathcal{A}^{-1}. \end{aligned}$$

Hence, $1 + qy \in \mathcal{A}^{-1}$. Therefore we check that

$$\begin{aligned} pa &= p(x + y) = py = (1 - xx^{\#}_r)y = y \in \mathcal{A}^{qnil}, \\ pa(1-p) &= yxx^{\#}_r = 0, \\ a + p &= x + y + p = (x + p)[1 + qy] \in \mathcal{A} \text{ is right invertible.} \end{aligned}$$

Moreover, we see that $1 - p = xx^{\#}_r = [(x + y)xx^{\#}_r]x^{\#}_r \in a(1-p)\mathcal{A}$. On the other hand, $a(1-p) = (1-p)a(1-p) \in (1-p)\mathcal{A}$. Then

$$(1-p)\mathcal{A} = a(1-p)\mathcal{A}.$$

(2) ⇒ (1) By hypothesis, there exists a projection $p \in \mathcal{A}$ such that

$$a + p \in \mathcal{A}^{-1}_r, ap \in \mathcal{A}^{qnil}, (1-p)\mathcal{A} = a(1-p)\mathcal{A}.$$

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Set $x=(1-p)a$ and $y=pa$. Then

$$\begin{aligned} x^*y &= [a^*(1-p)^*]pa=0,\\ yx &= pa(1-p)a=0,\\ y &= pa\in \mathcal{A}^{qnil}. \end{aligned}$$

Write $(a+p)q=1$ for some $q\in\mathcal{A}$. Then $(1-p)aq=(1-p)(a+p)q=1-p$, and so $(1-p)aq(1-p)a=(1-p)a$ and $[(1-p)aq]^*=(1-p)aq$. Hence, $(1-p)a\in\mathcal{A}^{(1,3)}$.

Since $(1-p)\mathcal{A}=a(1-p)\mathcal{A}$, we have $pa(1-p)=0$. Write $1-p=a(1-p)r$ for some $r\in\mathcal{A}$. Then $1-p=(1-p)a(1-p)r$; hence,

$$\begin{aligned} 1-p &= (1-p)[a(1-p)r]=(1-p)a[(1-p)r]\\ &= [(1-p)a][(1-p)a(1-p)r]r\\ &\in [(1-p)a]^2\mathcal{A}. \end{aligned}$$

Then we have $(1-p)a\mathcal{A}=[(1-p)a]^2\mathcal{A}$. According to [14, Theorem 3.1], $(1-p)a\in\mathcal{A}_r^{\#}$. That is, $x\in\mathcal{A}_r^{\#}$. Therefore $a\in\mathcal{A}_r^{\textcircled{d}}$. □

Corollary 5. *Every generalized right core invertible element in a Banach *-algebra is the sum of two invertible and a right invertible elements.*

Proof. Let $a\in\mathcal{A}_r^{\textcircled{d}}$. In view of Theorem 3.1, we have $p^2=p=p^*\in\mathcal{A}$ such that $u:=a+p\in\mathcal{A}_r^{-1}$. Then $a=u-p$. Clearly, $-p=\frac{1-2p}{2}-\frac{1}{2}$. We easily check that

$$(\frac{1-2p}{2})^2=\frac{1}{4},$$

and so

$$(\frac{1-2p}{2})^{-1}=2(1-2p).$$

Accordingly, $a=u+\frac{1-2p}{2}-\frac{1}{2}$, as required. □

We are ready to prove:

Theorem 5. *Let $a\in\mathcal{A}$. Then the following are equivalent:*

(1) $a\in\mathcal{A}_r^{\textcircled{d}}$.
(2) There exists $b\in\mathcal{A}$ such that

$$bab=b,(ab)^*=ab,aba\mathcal{A}=a^2ba\mathcal{A},a-a^2b\in\mathcal{A}^{qnil}.$$

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exist $x,y\in\mathcal{A}$ such that

$$a=x+y,x^*y=yx=0,x\in\mathcal{A}_r^{\#},y\in\mathcal{A}^{qnil}.$$

It is easy to verify that

$$\begin{aligned} x_r^{\#}y &= x_r^{\#}xx_r^{\#}y\\ &= [x_r^{\#}][xx_r^{\#}]y\\ &= [x_r^{\#}](xx_r^{\#})^*y\\ &= [x_r^{\#}](x_r^{\#})^*(x^*y)\\ &= 0. \end{aligned}$$

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Set $b = x_r^{\circledast}$. Then $ab = (x+y)x_r^{\circledast} = xx_r^{\circledast} + yx(x_r^{\circledast})^2 = xx_r^{\circledast}$. Hence, $(ab)^* = (xx_r^{\circledast})^* = xx_r^{\circledast} = ab$. We easily verify that

$$\begin{aligned} ab^2 &= (ab)b = (xx_r^{\circledast})x_r^{\circledast} = x_r^{\circledast} = b, \\ b(1-ab) &= x_r^{\circledast}[1 - xx_r^{\circledast}] = 0, \\ a - a^2b &= a(1-ab) = a(1-xx^{\circledast}). \end{aligned}$$

Thus $b = bab$, and so $ab^2 = bab$.

Moreover, we see that

$$\begin{aligned} aba &= (xx_r^{\circledast})(x+y) \\ &= xx_r^{\circledast}x = x; \\ a^2ba &= a(aba) = (x+y)x = x^2. \end{aligned}$$

Since $x \in \mathcal{A}_r^{\circledast}$, it follows by [14, Theorem 3.1] that $x\mathcal{A} = x^2\mathcal{A}$. Thus, $aba\mathcal{A} = a^2ba\mathcal{A}$. Since $(1 - xx^{\circledast})a = (1-xx^{\circledast})(x+y) = y \in \mathcal{A}^{qnil}$, by using Cline's formula, $a - a^2b = a(1-xx^{\circledast}) \in \mathcal{A}^{qnil}$.

(2) $\Rightarrow$ (1) By hypothesis, there exists $b \in \mathcal{A}$ such that

$$bab = b, (ab)^* = ab, aba\mathcal{A} = a^2ba\mathcal{A}, a - a^2b \in \mathcal{A}^{qnil}.$$

Let $x = aba$ and $y = a - aba$. Then

$$\begin{aligned} a &= x+y, \\ x^*y &= (aba)^*(a-aba) = a^*(ab)^*(1-ab)a = 0, \\ yx &= (a-aba)aba = (1-ab)a^2ba = (1-ab)abar = 0 \text{ for a } r \in \mathcal{A}. \end{aligned}$$

Since $a - a^2b \in \mathcal{A}^{qnil}$. By using Cline's formula, we have $y = (1-ab)a \in \mathcal{A}^{qnil}$. Clearly, we have $xb = (aba)b = a(bab) = ab$, and so $xbx = ab(aba) = a(bab)a = aba = x$ and $(xb)^* = (ab)^* = ab = xb$. That is, $x \in \mathcal{A}^{(1,3)}$. Moreover, we check that

$$aba = ababa \in aba^2ba\mathcal{A} = (aba)^2\mathcal{A}.$$

Hence, $aba\mathcal{A} = (aba)^2\mathcal{A}$. By virtue of [14, Theorem 3.1], $x \in \mathcal{A}_r^{\circledast}$. This completes the proof by Theorem 2.1. □

Corollary 6. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\circledast}$.
(2) Tere exists $b \in \mathcal{A}$ such that

$$bab = b = ab^2, (ab)^* = ab, aba\mathcal{A} = a^2ba\mathcal{A}, a - a^2b \in \mathcal{A}^{qnil}.$$

Proof. (1) $\Rightarrow$ (2) By hypothesis, there exist $x, y \in \mathcal{A}$ such that

$$a = x+y, x^*y = yx = 0, x \in \mathcal{A}_r^{\circledast}, y \in \mathcal{A}^{qnil}.$$

Set $b = x_r^{\circledast}$. As in the proof of Theorem 3.3, we check that

$$bab = b, (ab)^* = ab, aba\mathcal{A} = a^2ba\mathcal{A}, a - a^2b \in \mathcal{A}^{qnil}.$$

Moreover, we verify that

$$\begin{aligned} ab &= (x+y)x_r^{\circledast} = xx_r^{\circledast} + yx(x_r^{\circledast})^2 = xx_r^{\circledast}, \\ ab^2 &= (ab)b = (xx_r^{\circledast})x_r^{\circledast} = x_r^{\circledast} = b, \end{aligned}$$

as required.

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(2) ⇒ (1) This is obvious by Theorem 3.3. □

Let $a \in \mathcal{A}$. Set
$$\{a_r^d\} = \{x \in \mathcal{A} \mid ax^2 = x, a - xa^2 \in \mathcal{A}^{qnil}\}.$$
We now derive the following.

Theorem 6. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\textcircled{d}}$.
(2) $\{a_r^d\} \bigcap \mathcal{A}_r^{\textcircled{\#}} \neq \emptyset$.

In this case, $a_r^{\textcircled{d}} = z^2 z_r^{\textcircled{\#}}$ for $z \in \{a_r^d\} \bigcap \mathcal{A}_r^{\textcircled{\#}}$.

Proof. (1) ⇒ (2) In view of Theorem 2.1, there exist $x, y \in \mathcal{A}$ such that
$$a = x + y, x^* y = yx = 0, x \in \mathcal{A}_r^{\textcircled{\#}}, y \in \mathcal{A}^{qnil}.$$
Let $z = x_r^{\textcircled{\#}}$. Then

Claim 1. $z \in \{a_r^d\}$. We directly verify that
$$\begin{array}{rcl} az & = & (x+y)x_r^{\textcircled{\#}} = (x+y)x(x_r^{\textcircled{\#}})^2 = xx_r^{\textcircled{\#}}, \\ az^2 & = & [xx_r^{\textcircled{\#}}]x_r^{\textcircled{\#}} = x(x_r^{\textcircled{\#}})^2 = x_r^{\textcircled{\#}} = z, \\ aza & = & xx_r^{\textcircled{\#}}(x+y) = xx_r^{\textcircled{\#}}x + (x_r^{\textcircled{\#}})^*(x^*y) = x, \\ a - aza & = & a - x = y \in \mathcal{A}^{qnil}. \end{array}$$
By using Cline's formula, we have $a - za^2 \in \mathcal{A}^{qnil}$. Therefore $z \in \{a_r^d\}$.

Claim 2. $z \in \mathcal{A}_r^{\textcircled{\#}}$. We verify that
$$\begin{array}{rcl} z[x^2z] & = & x_r^{\textcircled{\#}}[x^2x_r^{\textcircled{\#}}] = xx_r^{\textcircled{\#}}, \\ z[x^2z]^2 & = & [xx_r^{\textcircled{\#}}](x^2z) = x^2z, \\ (z(x^2z))^* & = & (xx_r^{\textcircled{\#}})^* = xx_r^{\textcircled{\#}} = z(x^2z), \\ z(x^2z)z & = & [xx_r^{\textcircled{\#}}]x_r^{\textcircled{\#}} = x_r^{\textcircled{\#}} = z. \end{array}$$
Accordingly, $z \in \mathcal{A}_r^{\textcircled{\#}}$ and $z_r^{\textcircled{\#}} = x^2z$. Therefore $\{a_r^d\} \bigcap \mathcal{A}_r^{\textcircled{\#}} \neq \emptyset$.

(2) ⇒ (1) Let $z \in \{a_r^d\} \bigcap \mathcal{A}_r^{\textcircled{\#}}$. Then
$$az^2 = z, a - za^2 \in \mathcal{A}^{qnil}.$$
Set $x = z^2 z_r^{\textcircled{\#}}$. Then we check that
$$(zz_r^{\textcircled{\#}}a)x = zz_r^{\textcircled{\#}}(az^2)z_r^{\textcircled{\#}} = [zz_r^{\textcircled{\#}}]^2 = zz_r^{\textcircled{\#}},$$
hence, we see that
$$\begin{array}{rcl} [(zz_r^{\textcircled{\#}}a)x]^* & = & [zz_r^{\textcircled{\#}}]^* = zz_r^{\textcircled{\#}} = (zz_r^{\textcircled{\#}}a)x, \\ zz_r^{\textcircled{\#}}ax^2 & = & [zz_r^{\textcircled{\#}}][z^2z_r^{\textcircled{\#}}] = z^2z_r^{\textcircled{\#}} = x, \\ (zz_r^{\textcircled{\#}}a)x(zz_r^{\textcircled{\#}}a) & = & (zz_r^{\textcircled{\#}})(zz_r^{\textcircled{\#}}a) = zz_r^{\textcircled{\#}}a. \end{array}$$
Then $zz_r^{\textcircled{\#}}a \in \mathcal{A}_r^{\textcircled{\#}}$ and $[zz_r^{\textcircled{\#}}a]_r^{\textcircled{\#}} = z^2z_r^{\textcircled{\#}}$.

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Write $a = a_1 + a_2$, where $a_1 = zz_r^{\#}a$ and $a_2 = a - zz_r^{\#}a$. It is easy to verify that

$$\begin{aligned} a_2a_1 &= [a - zz_r^{\#}a]zz_r^{\#}a \\ &= azz_r^{\#}a - zz^{\#}azz_r^{\#}a \\ &= azz_r^{\#}a - zz^{\#}(az^2)(z_r^{\#})^2a \\ &= azz_r^{\#}a - zz^{\#}z(z_r^{\#})^2a \\ &= (az^2)(z_r^{\#})^2a - z(z^{\#})^2a \\ &= 0, \\ a_1^*a_2 &= a^*(zz_r^{\#})^*[a - zz_r^{\#}a] \\ &= a^*(zz_r^{\#})[a - zz_r^{\#}a] \\ &= a^*zz_r^{\#}[1 - zz_r^{\#}]a = 0. \end{aligned}$$

Moreover, we check that

$$\begin{aligned} [1 - zz_r^{\#}]a &= [1 - zz_r^{\#}]a - [1 - zz_r^{\#}]za^2 \\ &= [1 - zz_r^{\#}](a - za^2). \end{aligned}$$

Clearly,

$$az = azz_r^{\#}z = az^2(z_r^{\#})^2z = z(z_r^{\#})^2z = z_r^{\#}z.$$

It is easy to verify that

$$\begin{aligned} (a - za^2)[1 - zz_r^{\#}] &= a - za^2 - (1 - za)(az)z_r^{\#} \\ &= a - za^2 - (1 - za)(z_r^{\#}z)z_r^{\#} \\ &= a - za^2 - (1 - za)z_r^{\#} \\ &= a - za^2 - (1 - za)z^2(z_r^{\#})^3 \\ &= a - za^2 - [z_r^{\#} - z(az^2)(z_r^{\#})^3] \\ &= a - za^2 - [z_r^{\#} - z^2(z_r^{\#})^3] \\ &= a - za^2 \in \mathcal{A}^{qnil}. \end{aligned}$$

By using Cline's formula again,

$$a_2 = [1 - zz_r^{\#}]a = [1 - zz_r^{\#}](a - za^2) \in \mathcal{A}^{qnil}.$$

Therefore $a = a_1 + a_2$ is the generalized right core decomposition of a. Therefore

$$a_r^{\textcircled{d}} = (a_1)_r^{\#} = z^2z_r^{\#},$$

as asserted. $\square$

4. algebraic properties

In this section, we are concerned with algebraic properties of generalized right core inverse. Let $a, p^2 = p \in \mathcal{A}$. Then a has the Pierce decomposition relative to p, and we denote it by $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_p$. Let $a \in \mathcal{A}_r^{\textcircled{d}}$. We use a_r^{π} to stands for $1 - aa_r^{\textcircled{d}}$. For further use, we now derive

Lemma 1. *Let p be a projection, $a \in \big(p\mathcal{A}p\big)_r^{\textcircled{d}}, d \in \big((1-p)\mathcal{A}(1-p)\big)_r^{\textcircled{d}}$ and $x = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}_p$. If $a_r^{\pi}b = 0$, then $x \in \mathcal{A}_r^{\textcircled{d}}$. In this case,*

$$x_r^{\textcircled{d}} = a_r^{\textcircled{d}} + d_r^{\textcircled{d}} - a_r^{\textcircled{d}}bd_r^{\textcircled{d}}.$$

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Proof. Set $y = \begin{pmatrix} a_r^{\text{\textcircled{d}}} & z \\ 0 & d_r^{\text{\textcircled{d}}} \end{pmatrix}_p$, where $z = -a_r^{\text{\textcircled{d}}} b d_r^{\text{\textcircled{d}}}$. Then we directly check that

$$xy = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}_p \begin{pmatrix} a_r^{\text{\textcircled{d}}} & z \\ 0 & d_r^{\text{\textcircled{d}}} \end{pmatrix}_p = \begin{pmatrix} aa_r^{\text{\textcircled{d}}} & az + bd_r^{\text{\textcircled{d}}} \\ 0 & dd_r^{\text{\textcircled{d}}} \end{pmatrix}_p .$$

We see that

$$\begin{aligned} az + bd_r^{\text{\textcircled{d}}} &= -aa_r^{\text{\textcircled{d}}} bd_r^{\text{\textcircled{d}}} + bd_r^{\text{\textcircled{d}}} \\ &= a_r^{\pi} bd_r^{\text{\textcircled{d}}} = 0. \end{aligned}$$

Then

$$\begin{aligned} (xy)^* &= xy, \\ xy^2 &= \begin{pmatrix} aa_r^{\text{\textcircled{d}}} & 0 \\ 0 & dd_r^{\text{\textcircled{d}}} \end{pmatrix}_p \begin{pmatrix} a_r^{\text{\textcircled{d}}} & z \\ 0 & d_r^{\text{\textcircled{d}}} \end{pmatrix}_p = y, \\ yxy &= \begin{pmatrix} a_r^{\text{\textcircled{d}}} & z \\ 0 & d_r^{\text{\textcircled{d}}} \end{pmatrix}_p \begin{pmatrix} aa_r^{\text{\textcircled{d}}} & 0 \\ 0 & dd_r^{\text{\textcircled{d}}} \end{pmatrix}_p = y. \end{aligned}$$

Moreover, we have

$$1 - xy = \begin{pmatrix} p - aa_r^{\text{\textcircled{d}}} & 0 \\ 0 & p^{\pi} - dd_r^{\text{\textcircled{d}}} \end{pmatrix}_p .$$

Then

$$(1 - xy)x^n = \begin{pmatrix} p(1 - aa_r^{\text{\textcircled{d}}})a^n & 0 \\ 0 & p^{\pi}(1 - dd_r^{\text{\textcircled{d}}})d^n \end{pmatrix}_p .$$

Therefore

$$\lim_{n\to\infty} ||x^n - xyx^n||^{\frac{1}{n}} = 0,$$

and then

$$x_r^{\text{\textcircled{d}}} = y = a_r^{\text{\textcircled{d}}} + d_r^{\text{\textcircled{d}}} - a_r^{\text{\textcircled{d}}} bd_r^{\text{\textcircled{d}}}.$$

□

We come now to the demonstration of the additive property of generalized right core inverse.

Theorem 7. *Let $a, b, a_r^{\pi} b, (a+b)aa_r^{\text{\textcircled{d}}} \in \mathcal{A}_r^{\text{\textcircled{d}}}$. If $a_r^{\pi} ab = 0, a_r^{\pi} ba = 0$ and $[(a+b)aa_r^{\text{\textcircled{d}}}]_r^{\pi} aa_r^{\text{\textcircled{d}}} ba_r^{\pi} = 0$, then $a + b \in \mathcal{A}_r^{\text{\textcircled{d}}}$. In this case,*

$$(a+b)_r^{\text{\textcircled{d}}} = [(a+b)aa_r^{\text{\textcircled{d}}}]_r^{\text{\textcircled{d}}} + [a_r^{\pi} b]_r^{\text{\textcircled{d}}} - [(a+b)aa_r^{\text{\textcircled{d}}}]_r^{\text{\textcircled{d}}} b[a_r^{\pi} b]_r^{\text{\textcircled{d}}}.$$

Proof. Let $p = aa_r^{\text{\textcircled{d}}}$. By hypothesis, $p^{\pi} bp = (1 - aa_r^{\text{\textcircled{d}}})baa_r^{\text{\textcircled{d}}} = (a_r^{\pi} ba)a_r^{\text{\textcircled{d}}} = 0$,

$$p^{\pi} ap = (1 - aa_r^{\text{\textcircled{d}}})a^2 a_r^{\text{\textcircled{d}}} = 0$$

and

$$pap^{\pi} = aa_r^{\text{\textcircled{d}}} a(1 - aa_r^{\text{\textcircled{d}}}) = aa_r^{\text{\textcircled{d}}}[a - aa_r^{\text{\textcircled{d}}} a]aa_r^{\text{\textcircled{d}}} = 0.$$

Then we have

$$a = \begin{pmatrix} a_1 & 0 \\ 0 & a_4 \end{pmatrix}_p , b = \begin{pmatrix} b_1 & b_2 \\ 0 & b_4 \end{pmatrix}_p .$$

Hence

$$a + b = \begin{pmatrix} a_1 + b_1 & b_2 \\ 0 & a_4 + b_4 \end{pmatrix}_p .$$

 doi:10.20944/preprints202406.1246.v1

Here, $a_1 = aa_r^{ⓓ}aaa_r^{ⓓ} = a^2a_r^{ⓓ}$ and $b_1 = aa_r^{ⓓ}baa_r^{ⓓ} = baa_r^{ⓓ}$. Then

$$a_1 + b_1 = (a+b)aa_r^{ⓓ} \in \mathcal{A}_r^{ⓓ}.$$

Also we have $a_4 = a_r^{\pi}aa_r^{\pi} = a_r^{\pi}a - a_r^{\pi}a^2a_r^{ⓓ} = a_r^{\pi}a$ and $b_4 = a_r^{\pi}ba_r^{\pi} = a_r^{\pi}b$, and so

$$a_4 + b_4 = a_r^{\pi}(a+b).$$

We claim that

$$(a_4)_r^{ⓓ} = (a_r^{\pi}b)_r^{ⓓ}.$$

One easily verifies that

$$\begin{aligned}
a_r^{\pi}(a+b)(a_r^{\pi}b)_r^{ⓓ} &= a_r^{\pi}a(a_r^{\pi}b)_r^{ⓓ} + a_r^{\pi}b(a_r^{\pi}b)_r^{ⓓ} = a_r^{\pi}b(a_r^{\pi}b)_r^{ⓓ},\\
\left(a_r^{\pi}(a+b)(a_r^{\pi}b)_r^{ⓓ}\right)^* &= a_r^{\pi}(a+b)(a_r^{\pi}b)_r^{ⓓ},\\
a_r^{\pi}(a+b)[(a_r^{\pi}b)_r^{ⓓ}]^2 &= a_r^{\pi}[b(a_r^{\pi}b)_r^{ⓓ}]^2 = (a_r^{\pi}b)_r^{ⓓ},\\
(a_r^{\pi}b)_r^{ⓓ}a_r^{\pi}(a+b)(a_r^{\pi}b)_r^{ⓓ} &= (a_r^{\pi}b)_r^{ⓓ}a_r^{\pi}b(a_r^{\pi}b)_r^{ⓓ} = (a_r^{\pi}b)_r^{ⓓ},\\
1 - a_r^{\pi}(a+b)(a_r^{\pi}b)_r^{ⓓ} &= 1 - a_r^{\pi}b(a_r^{\pi}b)_r^{ⓓ}.
\end{aligned}$$

Then

$$\begin{aligned}
& [1 - a_r^{\pi}(a+b)(a_r^{\pi}b)_r^{ⓓ}][a_r^{\pi}(a+b)]^n\\
= & [1 - a_r^{\pi}b(a_r^{\pi}b)_r^{ⓓ}][(a_r^{\pi}a)^n + (a_r^{\pi}b)^n]\\
= & [1 - a_r^{\pi}b(a_r^{\pi}b)_r^{ⓓ}][a_r^{\pi}a)^n] + [1 - a_r^{\pi}b(a_r^{\pi}b)_r^{ⓓ}][(a_r^{\pi}b)^n].
\end{aligned}$$

Hence,

$$\begin{aligned}
& ||[1 - a_r^{\pi}(a+b)(a_r^{\pi}b)_r^{ⓓ}][a_r^{\pi}(a+b)]^n||^{\frac{1}{n}}\\
\leq & ||1 - a_r^{\pi}b(a_r^{\pi}b)_r^{ⓓ}||^{\frac{1}{n}}||a_r^{\pi}a)^n||^{\frac{1}{n}} + ||[1 - a_r^{\pi}b(a_r^{\pi}b)_r^{ⓓ}][(a_r^{\pi}b)^n]||^{\frac{1}{n}}.
\end{aligned}$$

We infer that

$$\lim_{n\to\infty} ||[1 - a_r^{\pi}(a+b)(a_r^{\pi}b)_r^{ⓓ}][a_r^{\pi}(a+b)]^n||^{\frac{1}{n}} = 0.$$

Therefore $a_4 + b_4 \in \mathcal{A}_r^{ⓓ}$ and $(a_4 + b_4)_r^{ⓓ} = (a_r^{\pi}b)_r^{ⓓ}$.

Since $[(a+b)aa_r^{ⓓ}]_r^{\pi}aa_r^{ⓓ}ba_r^{\pi} = 0$, we have $(a_1+b_1)_r^{\pi}b_2 = 0$. By using Lemma 4.1, we have

$$\begin{aligned}
(a+b)_r^{ⓓ} &= (a_1+b_1)_r^{ⓓ} + (a_4+b_4)_r^{ⓓ}\\
&- (a_1+b_1)_r^{ⓓ}b_2(a_4+b_4)_r^{ⓓ}\\
&= [(a+b)aa_r^{ⓓ}]_r^{ⓓ} + [a_r^{\pi}b]_r^{ⓓ}\\
&- [(a+b)aa_r^{ⓓ}]_r^{ⓓ}b[a_r^{\pi}b]_r^{ⓓ},
\end{aligned}$$

as asserted. □

Corollary 7. *Let $a, b, a_r^{\pi}b, (a+b)aa_r^{ⓓ} \in \mathcal{A}_r^{ⓓ}$. If $a_r^{\pi}ab = 0, a_r^{\pi}ba = 0$ and $a_r^{\pi}b^*a = 0$, then $a+b \in \mathcal{A}_r^{ⓓ}$. In this case,*

$$(a+b)_r^{ⓓ} = [(a+b)aa_r^{ⓓ}]_r^{ⓓ} + [a_r^{\pi}b]_r^{ⓓ} - [(a+b)aa_r^{ⓓ}]_r^{ⓓ}b[a_r^{\pi}b]_r^{ⓓ}.$$

Proof. Since $a_r^{\pi}b^*a = 0$, we see that

$$\begin{aligned}
aa_r^{ⓓ}ba_r^{\pi} &= aa_r^{ⓓ}b(1 - aa_r^{ⓓ})\\
&= (aa_r^{ⓓ})^*b(1 - aa_r^{ⓓ})^*\\
&= (a_r^{ⓓ})^*a^*b(1 - aa_r^{ⓓ})^*\\
&= (a_r^{ⓓ})^*[a_r^{\pi}b^*a]^* = 0.
\end{aligned}$$

Therefore the proof is true by Theorem 4.2. □

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Corollary 8. *Let $a, b \in \mathcal{A}_r^{\textcircled{d}}$. If $ab = ba = a^*b = 0$, then $a + b \in \mathcal{A}_r^{\textcircled{d}}$. In this case,*

$$(a+b)_r^{\textcircled{d}} = a_r^{\textcircled{d}} + b_r^{\textcircled{d}}.$$

Proof. Since $a^*b = 0$, we have $b^*a = (a^*b)^* = 0$. Hence, $a_r^{\pi}ab = 0, a_r^{\pi}ba = 0$ and $a_r^{\pi}b^*a = 0$. According to Corollary 4.3, the result follows. □

We now establish the power property of generalized right core inverse.

Theorem 8. *Let $a \in \mathcal{A}$ and $n \in \mathbb{N}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\textcircled{d}}$.
(2) $a^n \in \mathcal{A}_r^{\textcircled{d}}$.

In this case, $a_r^{\textcircled{d}} = a^{n-1}(a^n)_r^{\textcircled{d}}$.

Proof. $(1) \Rightarrow (2)$ Let $x = a_r^{\textcircled{d}}$. Then we verify that

$$ax = a(ax^2) = a^2x^2 = \cdots = a^nx^n.$$

Hence,

$$\begin{array}{rcl} a^n(x^n)^2 & = & (a^nx^n)x^n = ax^{n+1} = (ax^2)x^{n-1} = x^n, \\ (a^nx^n)^* & = & (ax)^* = ax = a^nx^n, \\ (a^n)^m - a^nx^n(a^n)^m & = & a^{nm} - axa^{nm}. \end{array}$$

Hence,

$$\lim_{m\to\infty} ||(a^n)^m - a^nx^n(a^n)^m||^{\frac{1}{m}} = 0.$$

Therefore $(a^n)_r^{\textcircled{d}} = x^n$, as required.

$(2) \Rightarrow (1)$ Let $x = a^{n-1}(a^n)_r^{\textcircled{d}}$. Then

$$ax = a^n(a^n)_r^{\textcircled{d}}.$$

Hence we check that

$$\begin{array}{rcl} ax^2 & = & a^n(a^n)_r^{\textcircled{d}}a^{n-1}(a^n)_r^{\textcircled{d}} = x, \\ (ax)^* & = & (a^n(a^n)_r^{\textcircled{d}})^* = a^n(a^n)_r^{\textcircled{d}} = ax, \\ a^m - axa^m & = & a^m - a^n(a^n)_r^{\textcircled{d}}a^m \\ & = & (a^n - a^n(a^n)_r^{\textcircled{d}}a^n)a^{m-n}(m \geq n). \end{array}$$

This implies that

$$\lim_{m\to\infty} ||a^m - axa^m||^{\frac{1}{m}} = 0.$$

Accordingly, $(a^n)_r^{\textcircled{d}} = x$, as asserted. □

The next theorem provides a criteria for a triangular matrix has generalized right core inverse.

Theorem 9. *Let $\alpha = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathcal{A})$ with $a, d \in \mathcal{A}_r^{\textcircled{d}}$. If $a_r^{\pi}bd_r^{\textcircled{d}} = 0$, then $\alpha \in M_2(\mathcal{A})_r^{\textcircled{d}}$ and*

$$\alpha_r^{\textcircled{d}} = \begin{pmatrix} a_r^{\textcircled{d}} & -a_r^{\textcircled{d}}bd_r^{\textcircled{d}} \\ 0 & d_r^{\textcircled{d}} \end{pmatrix}.$$

Proof. Since $a, d \in \mathcal{A}_r^{\textcircled{d}}$, we have generalized right core decompositions:

$$a = x + y, d = s + t,$$

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where
$$x, s \in \mathcal{A}_r^{\#}, y, t \in \mathcal{A}^{qnil}$$
and
$$x^* y = 0, yx = 0; s^* t = 0, ts = 0.$$
Then we have $\alpha = \beta + \gamma$, where
$$\beta = \begin{pmatrix} x & xx_r^{\#} b \\ 0 & s \end{pmatrix}, \gamma = \begin{pmatrix} y & (1 - xx_r^{\#})b \\ 0 & t \end{pmatrix}.$$
By hypothesis, we have
$$x_r^{\pi}(xx_r^{\#} b) = 0.$$
Then $\beta \in M_2(\mathcal{A})_r^{\#}$. Since, $y, t \in \mathcal{A}^{qnil}$, we see that $\gamma \in M_2(\mathcal{A})^{qnil}$.

We directly check that
$$\begin{aligned} \beta^* \gamma &= \begin{pmatrix} x^* & 0 \\ b^* xx_r^{\#} & s^* \end{pmatrix} \begin{pmatrix} y & (1 - xx_r^{\#})b \\ 0 & t \end{pmatrix} \\ &= \begin{pmatrix} 0 & x_r^*(1 - xx_r^{\#}) \\ b^* xx_r^{\#} y & b^* xx_r^{\#}(1 - xx_r^{\#})b \end{pmatrix} = 0, \\ \gamma\beta &= \begin{pmatrix} y & (1 - xx_r^{\#})b \\ 0 & t \end{pmatrix} \begin{pmatrix} x & xx_r^{\#} b \\ 0 & s \end{pmatrix} \\ &= \begin{pmatrix} 0 & yxx_r^{\#} b + (1 - xx_r^{\#})bs \\ 0 & 0 \end{pmatrix} = 0. \end{aligned}$$
Then $\alpha \in M_2(\mathcal{A})_r^{\textcircled{d}}$. In light of Theorem 2.1, we have
$$\begin{aligned} \alpha_r^{\textcircled{d}} &= \beta_r^{\#} \\ &= \begin{pmatrix} x_r^{\#} & z \\ 0 & s_r^{\#} \end{pmatrix} \\ &= \begin{pmatrix} a_r^{\textcircled{d}} & z \\ 0 & d_r^{\textcircled{d}} \end{pmatrix}. \end{aligned}$$
where
$$z = -x_r^{\#}[xx_r^{\#} b]s_r^{\#} = -a_r^{\textcircled{d}} b d_r^{\textcircled{d}}.$$
This completes the proof. □

5. right core-EP inverses

In this section, we apply our main results and characterize right core-EP inverse by certain new ways. The following lemma is crucial.

Lemma 2. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}_r^{\textcircled{D}}$ if and only if*

(1) $a \in \mathcal{A}_r^{\textcircled{d}}$;
(2) there exists some $k \in \mathbb{N}$ such that $a^k \mathcal{A} = a^{k+1} \mathcal{A}$.

In this case, $a_r^{\textcircled{D}} = a_r^{\textcircled{d}}$.

Proof. $\Longrightarrow$ Obviously, $a \in \mathcal{A}_r^{\textcircled{d}}$. In view of [14, Theorem 4.9], there exists some $k \in \mathbb{N}$ such that $a^k \mathcal{A} = a^{k+1} \mathcal{A}$.

$\Longleftarrow$ Let $x = a_r^{\circledast}$. Then

$$xax = x = ax^2, (ax)^* = ax, \lim_{n\to\infty} ||a^n - axa^n||^{\frac{1}{n}} = 0.$$

Write $a^k = a^{k+1}y$ for a $y \in \mathcal{A}$. Set $z = a^k y x^k$. Then we verify that $az = (a^{k+1}y)x^k = a^k x^k = ax$, and so $(az)^* = (ax)^* = ax = az$. We observe that

$$\begin{array}{rcl} az^2 & = & (az)z = axa^k yx^k = axa^{k+n}y^{n+1}x^k \\ & = & a^{k+n}y^{n+1}x^k - [a^{k+n} - axa^{k+n}]y^{n+1}x^k \\ & = & a^k yx^k - [a^{k+n} - axa^{k+n}]y^{n+1}x^k \\ & = & z - [a^{k+n} - axa^{k+n}]y^{n+1}x^k, \end{array}$$

and then $\lim\limits_{n\to\infty} ||az^2 - z||^{\frac{1}{n}} = 0$. This implies that $az^2 = z$. Moreover, we have

$$a^k - aza^k = a^k - axa^k = a^{k+n}y^{n+1} - axa^{k+n}y^{n+1} = [a^{k+1} - axa^{k+n}]y^{n+1}.$$

Hence,

$$||a^k - aza^k|| \leq ||a^{k+1} - axa^{k+n}||||y^{n+1}||.$$

Since $\lim\limits_{n\to\infty} ||a^{k+1} - axa^{k+n}||^{\frac{1}{n}} = \lim\limits_{n\to\infty} [||a^{k+1} - axa^{k+n}||^{\frac{1}{k+n}}]^{1+\frac{k}{n}} = 0$, we deduce that $\lim\limits_{n\to\infty} ||a^k - aza^k||^{\frac{1}{n}} = 0$; and then $a^k = aza^k$. Therefore $a \in \mathcal{A}_r^{\circledcirc}$. By using the uniqueness of generalized right core inverse, we have $z = x$, as required. $\square$

We are ready to prove:

Theorem 10. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\circledcirc}$.
(2) $a^k \in \mathcal{A}_r^{\#}$ for some $k \in \mathbb{N}$.
(3) There exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = yx = 0, x \in \mathcal{A}_r^{\#}, \ y \in \mathcal{A}^{nil}.$$

Proof. (1) $\Leftrightarrow$ (2) See [14, Theorem 4.8].

(1) $\Rightarrow$ (3) By virtue of Lemma 5.1, $a \in \mathcal{A}_r^{\circledast}$ and $a_r^{\circledast} = a_r^{\circledcirc}$. In view of Theorem 2.1, we can find $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = yx = 0, x \in \mathcal{A}_r^{\#}, \ y \in \mathcal{A}^{qnil}.$$

Explicitly, we have $y = a - aa_r^{\circledast}a = a - aa_r^{\circledcirc}a$. Set $z = a_r^{\circledcirc}$. Then $az^2 = z, (az)^* = az$ and $a^k = aza^k$ for some $k \in \mathbb{N}$. It is easy to see that

$$(1 - az)a(az) = (1 - az)a(a^{k-1}z^{k-1}) = (1 - az)a^k z^{k-1} = 0.$$

Moreover, we verify that

$$\begin{array}{rcl} (a - za^2)^{k+2} & = & (1 - za)a(a - za^2)^k(a - za^2) \\ & = & (1 - za)a(a - za^2)^{k-1}(a - za^2)a \\ & = & (1 - za)a(a - za^2)^{k-1}a^2 \\ & \vdots & \\ & = & (1 - za)a(a - za^2)a^k \\ & = & (1 - za)(a^k - aza^k)a \\ & = & 0. \end{array}$$

This implies that $a - za^2 \in \mathcal{A}^{nil}$. Therefore $y = a - aza \in \mathcal{A}^{nil}$, as desired.

(3) $\Rightarrow$ (1) By hypothesis, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = yx = 0, x \in \mathcal{A}_r^{\#}, \ y \in \mathcal{A}^{nil}.$$

Since $\mathcal{A}^{nil} \subseteq \mathcal{A}^{qnil}$. In view of Theorem 2.1, $a \in \mathcal{A}_r^{\circledD}$ and $z := a_r^{\circledD} = x^{\#}$. Hence, $az^2 = z$ and $(az)^* = az$. Write $y^k = 0 (k \in \mathbb{N})$. Then

$$a^k = \sum_{i=0}^{k} x^i y^{n-i} = y^k + xy^{k-1} + \cdots + x^{k-1}y + x^k = xy^{k-1} + \cdots + x^{k-1}y + x^k.$$

Then

$$aza^k = (x+y)x_r^{\#}x[y^{k-1} + \cdots + x^{k-2}y + 1] = x[y^{k-1} + \cdots + x^{k-2}y + 1] = a^k.$$

Therefore $a \in \mathcal{A}_r^{\circledD}$, as asserted. $\square$

Corollary 9. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{\circledD}$ if and only if*

(1) $a \in \mathcal{A}^D$;
(2) there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = yx = 0, x \in \mathcal{A}_r^{\#}, \ y \in \mathcal{A}^{nil}.$$

Proof. $\Longrightarrow$ This is obvious by [6, Theorem 2.9].

$\Longleftarrow$ In view of Theorem 5.2, $a \in \mathcal{A}_r^{\circledD}$. Therefore we complete the proof by Corollary 2.5. $\square$

We are ready to prove:

Theorem 11. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\circledD}$.
(2) *There exists a projection $p \in \mathcal{A}$ such that*

$$a + p \in \mathcal{A}_r^{-1}, ap \in \mathcal{A}^{nil}, (1-p)\mathcal{A} = a(1-p)\mathcal{A}.$$

Proof. (1) $\Rightarrow$ (2) In light of Theorem 3.1, there exists a projection $p \in \mathcal{A}$ such that

$$a + p \in \mathcal{A}_r^{-1}, ap \in \mathcal{A}^{qnil}, (1-p)\mathcal{A} = a(1-p)\mathcal{A}.$$

Explicitly, $p = 1 - aa_r^{\circledD}$. As in the proof of Theorem 3.1, we check that $ap = a(1 - aa_r^{\circledD}) \in \mathcal{A}^{nil}$, as desired.

(2) $\Rightarrow$ (1) By hypothesis, there exists a projection $p \in \mathcal{A}$ such that

$$u := a + p \in \mathcal{A}_r^{-1}, ap \in \mathcal{A}^{nil}, (1-p)\mathcal{A} = a(1-p)\mathcal{A}.$$

Then $a = (1-p)a + pa$. Obviously, $[(1-p)a]^*pa = a^*(1-p)pa = 0, pa(1-p)a = p[a(1-p)]a = 0$ and $pa \in \mathcal{A}^{nil}$.

Write $uv = 1$ for a $v \in \mathcal{A}$. Then we check that

$$\begin{aligned} (1-p)av(1-p)a &= (1-p)uv(1-p)a = (1-p)a, \\ ((1-p)av)^* &= 1 - p = (1-p)av. \end{aligned}$$

Hence, $(1-p)a \in \mathcal{A}^{(1,3)}$. Moreover, we see that

$$a(1-p) = a(1-p)uv = a(1-p)av = (1-p)a(1-p)av = [(1-p)a]^2 v,$$

and then $(1-p)a\mathcal{A} = [(1-p)a]^2\mathcal{A}$. In light of [14, Theorem 3.1], $(1-p)a \in \mathcal{A}_r^{\#}$. According to Theorem 5.2, $a \in \mathcal{A}_r^{\textcircled{D}}$. □

Corollary 10. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{\textcircled{D}}$ if and only if*

(1) $a \in \mathcal{A}^D$;
(2) *there exists a projection $p \in \mathcal{A}$ such that*

$$a + p \in \mathcal{A}_r^{-1}, ap \in \mathcal{A}^{nil}, (1-p)\mathcal{A} = a(1-p)\mathcal{A}.$$

Proof. This is obvious by Theorem 5.4 and Corollary 2.5. □

Theorem 12. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}_r^{\textcircled{D}}$.
(2) There exists $b \in \mathcal{A}$ such that

$$bab = b, (ab)^* = ab, aba\mathcal{A} = a^2ba\mathcal{A}, a - a^2b \in \mathcal{A}^{nil}.$$

Proof. (1) ⇒ (2) In view of Theorem 5.2, there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*y = yx = 0, x \in \mathcal{A}_r^{\#}, y \in \mathcal{A}^{nil}.$$

Set $b = x_r^{\#}$. As in the proof of Theorem 3.3, we verify that

$$bab = b, (ab)^* = ab, aba\mathcal{A} = a^2ba\mathcal{A}.$$

Similarly to Theorem 3.3, we check that $a - a^2b \in \mathcal{A}^{nil}$, as desired.

(2) ⇒ (1) In view of Theorem 3.3, we see that $a \in \mathcal{A}_r^{\textcircled{D}}$. Since $a - a^2b \in \mathcal{A}$, we can find some $k \in \mathbb{N}$ such that $(a - aba)^k = 0$. As $aba\mathcal{A} = a^2ba\mathcal{A}$, by induction, we see that $aba \in a^m ba\mathcal{A}$ for all $m \in \mathbb{N}$. Then $a^k \in a^{k+1}\mathcal{A}$; hence, $a^k\mathcal{A} = a^{k+1}\mathcal{A}$. Therefore we complete the proof by Lemma 5.1. □

Corollary 11. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{\textcircled{D}}$ if and only if*

(1) $a \in \mathcal{A}^D$;
(2) There exists $b \in \mathcal{A}$ such that

$$bab = b, (ab)^* = ab, aba\mathcal{A} = a^2ba\mathcal{A}, a - a^2b \in \mathcal{A}^{nil}.$$

Proof. This is proof by Theorem 5.6 and Corollary 2.5. □

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