Article

Not peer-reviewed version

Optimal Solutions for a Class of Impulsive Differential Problems with Feedback Controls and Volterra-Type Distributed Delay: A Topological Approach

Paola Rubbioni *

Posted Date: 11 June 2024

doi: 10.20944/preprints202406.0710.v1

Keywords: Optimal solutions; feedback controls; distributed delay; impulses; mild solutions; population dynamics; integro-differential inclusions

Article

Optimal Solutions for a Class of Impulsive Differential Problems with Feedback Controls and Volterra-Type Distributed Delay: A Topological Approach

Paola RUBBIONI

Department of Mathematics and Computer Science, University of Perugia, Italy; paola.rubbioni@unipg.it

Abstract: In this paper, the existence of optimal solutions for problems governed by differential equations involving feedback controls is established, when the problem must account for a Volterra-type distributed delay and is subject to the action of impulsive external forces. The problem is reformulated within the class of impulsive semilinear integro-differential inclusions in Banach spaces and is studied using topological methods and multivalued analysis. The paper concludes with an application to a population dynamics model.

Keywords: optimal solutions; feedback controls; distributed delay; impulses; integro-differential inclusions; mild solutions; population dynamics

MSC: 34G20; 34G25; 34A37; 93B24; 93B52; 92D25

---

1. Introduction

In this article, we aim to provide sufficient conditions for the existence of optimal solutions for differential problems that describe models of real phenomena where the state of the system depends on its past evolution according to a fading memory process. This means that the further back in time an event occurs, the less it influences the current state of the system. This fading memory process is well described by the type of distributed delay induced by the exponential probability distribution $\mathcal{k}(r) = \frac{e^{-r/\tau}}{\tau}$ used as a kernel in a Volterra-type integral involved in the system's equation. The formalization of the problem will thus be governed by an integro-differential equation. Examples of such problems include population dynamics models, where only fertile individuals are considered in the phenomenon under study. Often, in such cases, the time between an individual's birth and the moment it begins participating in the reproductive process is not negligible. Consequently, it is necessary to consider in the population development equation a term representing the delay with which an individual becomes an active part in the evolution of the considered process, linked to its maturation time. Other examples can be found in the context of beams fixed at one end with a mass at the free end, recently used to describe robotic arms of flexible robots. In these cases, depending on the material used, the deflections of the beam (or arm) can significantly affect the system's current state, and thus, in these cases as well, the problem is better defined if a delay term is included in the equation. To explore these topics further, we refer to the papers [1,5,10,19,22,24] and books [15,25], acknowledging that they do not represent an exhaustive bibliography on the problem. In this article, by way of example, we will demonstrate the application of our results to a population dynamics model.

The differential problems we address in this article are subject to feedback controls, so the optimal solutions will actually be trajectory-control pairs where the trajectory minimizes or maximizes the cost functional (depending on whether it is lower semicontinuous or upper semicontinuous) for a particular control. From our perspective, this leads to considering an integro-differential inclusion structure, as the evolution function

$$f\left(t, y(t), \int_{t_0}^{t} \frac{e^{-(t-s)/\tau}}{\tau} y(s)ds, \eta(s)\right),$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 June 2024 doi:10.20944/preprints202406.0710.v1

coupled with the feedback control condition

$$\eta(t) \in H\Big(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds\Big),$$

leads to the multimapping

$$F\Big(t, y(t), \int_{t_0}^{t} \frac{e^{-(t-s)/\tau}}{\tau} y(s)ds\Big) = f\Big(t, y(t), \int_{t_0}^{t} \frac{e^{-(t-s)/\tau}}{\tau} y(s)ds, H\Big(t, y(t), \int_{t_0}^{t} \frac{e^{-(t-s)/\tau}}{\tau} y(s)ds\Big)\Big).$$

Furthermore, we allow external forces to act on the system at fixed instants, acting instantaneously, assimilated to what are called "impulse functions". This type of phenomenon is quite common in the real world (think, for example, but not only, about the application of pesticides in plantations or the administration of antibiotics to patients with bacterial diseases, or the electrical stimulation of a limb), and has been and is the subject of study by the scientific community. In this regard, we refer, for example, to the works [16–18,23] and texts [3,8,11,12,21].

Our approach to the problem involves formalizing the equation describing the model and the additional conditions into a system of equations in function spaces. We demonstrate this process in Section 7, where, thanks to appropriate settings, the system provided by the differential equation

$$u_t(t,x) = -b(t,x)u(t,x) + g\Big(t, u(t,x), \int_{t_0}^{t} \frac{e^{-(t-s)/\tau}}{\tau} u(s,x)\, ds\Big) + \omega(t,x),$$

subject to feedback controls

$$\omega(t,\cdot) \in \Omega(u(t,\cdot)),$$

and conditions

$$u(t_0,x) = u_0(x), \text{ and } u\Big(t_j^+, x\Big) = u(t_j,x) + \mathcal{I}_j(u(t_j,x)) \,,\ j = 1,\dots,m,$$

is reduced to the system

$$y'(t) = A(t)y(t) + f\Big(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds, \eta(t)\Big),$$
$$\eta(t) \in H(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds),$$

under conditions

$$y(t_0) = y_0; \quad y(t_j^+) = y(t_j) + I_j(y(t_j)) \,,\ j = 1,\dots,m,$$

in the space $E = L^2([0,1])$.

Therefore, this system can be reinterpreted as a particular case of the broader class of impulsive semilinear integro-differential inclusions in Banach spaces,

$$y'(t) \in A(t)y(t) + F\Big(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds\Big).$$

It is within this framework that we develop our work, employing topological methods and multivalued analysis tools.

Our approach offers a twofold advantage. On one hand, it provides a novel contribution to the theory of integro-differential inclusions in abstract spaces, thanks to the new compactness theorems for the set of solutions, both in the case without impulses and in the impulsive case (cf. Theorem 4.1 and Theorem 5.1). On the other hand, it simultaneously establishes the existence results of optimal mild trajectory-control pairs for a wide range of real-world phenomena models as a consequence of those for the abstract case (cf. Theorem 6.1).

The article is structured as follows. In Section 2, we provide the definitions and preliminary results necessary for an easy understanding of the work. In Section 3, we position the problem in

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 June 2024 doi:10.20944/preprints202406.0710.v1

Banach spaces, introducing the space to which the mild trajectories belong, the assumptions on the problem data, and the definition of mild solution for our problem.

The entire Section 4 is dedicated to the compactness of the set of mild solutions in the non-impulsive case. As far as we know, indeed, our results are new in this case as well, and therefore, we have decided to isolate them so that they can be used in the future separately from the rest of the article.

Section 5 contains the main results of the work, namely the compactness of the set of mild solutions under the action of impulses and the existence of optimal solutions.

In Section 6, we apply the theorem on optimal solutions from the preceding section to an impulsive feedback control system in abstract spaces.

Finally, in Section 7, we demonstrate how this can be applied to a concrete model.

2. Notations and Reference Results

Let us recall some definitions and properties which will be used throughout the paper.

If $\mathcal{X}, \mathcal{Y}$ are topological spaces, a multifunction $\mathcal{F} : \mathcal{X} \multimap \mathcal{Y}$ is said to be *closed* if the set *graph* $\mathcal{F} := \{(x,y) \in X \times Y : y \in \mathcal{F}(x)\}$ is a closed subset of $X \times Y$.

The multimap $\mathcal{F}$ is said to be *upper semicontinuous* at a point $x \in \mathcal{X}$ if for every open $W \subset \mathcal{Y}$ such that $\mathcal{F}(x) \subset W$ there exists a neighborhood $V(x)$ of x such that $\mathcal{F}(V(x)) \subset W$.

If (Y, d) is a metric space and the multimap $\mathcal{F} : \mathcal{X} \multimap Y$ takes compact values, then $\mathcal{F}$ is upper semicontinuous at a point $x \in \mathcal{X}$ if and only if for every $\varepsilon > 0$ there exists a neighborhood $V(x)$ such that $\mathcal{F}(z) \subset W_\varepsilon(\mathcal{F}(x))$ for every $z \in V(x)$, where $W_\varepsilon(\mathcal{F}(x)) := \{y \in Y : d(y, \mathcal{F}(x)) < \varepsilon\}$ and $d(y, \mathcal{F}(x)) := \inf_{w \in \mathcal{F}(x)} d(y, w)$ (cf. [13, Theorem 1.1.8]).

In $\mathbb{R}^n$, let 0_n be the zero-element and $\preccurlyeq$ the partial ordering defined by

$$x := (x_1, \ldots, x_n) \preccurlyeq y := (y_1, \ldots, y_n) \text{ if and only if } y_k - x_k \geq 0 \text{ for every } k = 1, \ldots, n.$$

Of course, $x \prec y$ stands for $x \preccurlyeq y \wedge x \neq y$.

Let $\mathcal{E}$ be a Banach space and $\mathcal{P}_b(\mathcal{E})$ be the family of all the bounded subsets of $\mathcal{E}$.

A function $\beta : \mathcal{P}_b(\mathcal{E}) \to \mathbb{R}^n_{0,+}$ is said to be a *measure of noncompactness* in $\mathcal{E}$ if (cf. [6, Definition 3.1]):

(β_1) $\beta(\Omega) = 0_n$ if and only if $\overline{\Omega}$ is compact, $\Omega \in \mathcal{P}_b(\mathcal{E})$;
(β_2) $\beta(\overline{co}(\Omega)) = \beta(\Omega)$, $\Omega \in \mathcal{P}_b(\mathcal{E})$.

Moreover, β is said to be:

monotone if $\Omega_1 \subset \Omega_2$ implies $\beta(\Omega_1) \preccurlyeq \beta(\Omega_2)$, $\Omega_1, \Omega_2 \in \mathcal{P}_b(\mathcal{E})$;
nonsingular if $\beta(\{x\} \cup \Omega) = \beta(\Omega)$, for every $x \in \mathcal{E}, \Omega \in \mathcal{P}_b(\mathcal{E})$;
invariant under closure if $\beta(\overline{\Omega}) = \beta(\Omega)$, $\Omega \in \mathcal{P}_b(\mathcal{E})$;
invariant with respect to the union with compact sets if $\beta(\Omega \cup C) = \beta(\Omega)$, for every relatively compact set $C \subset \mathcal{E}, \Omega \in \mathcal{P}_b(\mathcal{E})$.

An example of measure of noncompactness satisfying all the above properties is the Hausdorff measures of noncompactness in $\mathcal{E}$,

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ can be covered by finitely many balls with radius } \varepsilon\}.$$

Throughout the paper, we will use also the next monotone vectorial measure of noncompactness which was introduced in [4] (cf. also [13]). Fixed $L > 0$, let $\nu_L : \mathcal{P}_b(C([a,b]; E)) \to \mathbb{R}^+_0 \times \mathbb{R}^+_0$ be given by

$$\nu_L(\Omega) := \max_{\{w_n\}_n \subset \Omega} (\gamma(\{w_n\}_n), \eta(\{w_n\}_n)), \tag{1}$$

where

$$\gamma(\{w_n\}_n) := \sup_{t \in [a,b]} e^{-Lt} \left[\chi(\{w_n(t)\}_n) + \chi\left(\left\{ \int_a^t k(t,s) w_n(s) ds \right\}_n \right) \right], \tag{2}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 June 2024 doi:10.20944/preprints202406.0710.v1

and

$$\eta(\{w_n\}_n) := mod_C(\{w_n\}_n) + mod_C\left(\left\{\int_a^{(\cdot)} k(\cdot,s)w_n(s)ds\right\}_n\right), \tag{3}$$

being mod_C the modulus of continuity in $C([a,b];E)$.

If D is a nonempty subset of $\mathcal{E}$, a multifunction $\mathcal{G} : D \multimap \mathcal{E}$ is said to be *condensing* with respect to a measure of noncompactness $\beta : \mathcal{P}_b(\mathcal{E}) \to \mathbb{R}^n_{0,+}$ (β-condensing, for short) if $\mathcal{G}(D)$ is bounded and for every $\Omega \in \mathcal{P}_b(D)$ the inequality $\beta(\Omega) \preccurlyeq \beta(\mathcal{G}(\Omega))$ implies that $\beta(\Omega) = 0_n$, where $\preccurlyeq$ denotes the partial ordering induced in $\mathbb{R}^n$ by its normal cone $\mathbb{R}^n_{0,+}$.

Equivalently, $\mathcal{G}$ is β-condensing if $\mathcal{G}(D)$ is bounded and for every $\Omega \in \mathcal{P}_b(D)$ the inequality $0_n \prec \beta(\Omega)$ implies that $\beta(\Omega) \not\preccurlyeq \beta(\mathcal{G}(\Omega))$, i.e., $\beta(\mathcal{G}(\Omega)) \prec \beta(\Omega)$ or $\beta(\mathcal{G}(\Omega))$, $\beta(\Omega)$ are not comparable.

Proposition 2.1. *[13, Proposition 3.5.1] Let X be a closed subset of $\mathcal{E}$ and $\mathcal{G} : X \multimap \mathcal{E}$ a closed multimap with compact values and β-condensing on every bounded subset of X, where β is a monotone measure of noncompactness in $\mathcal{E}$. If the set $Fix\, \mathcal{G} := \{x \in X : x \in \mathcal{G}(x)\}$ is bounded, then it is compact.*

Let E be a real Banach space endowed with the norm $\|\cdot\|$.

For $[a,b] \subset \mathbb{R}$ we denote $C([a,b];E)$ the space of all the E-valued continuous functions defined on $[a,b]$ endowed with the supremum norm $\|\cdot\|_{C([a,b];E)}$, and by $L^p([a,b];E)$ the space of all functions $v : [a,b] \to E$ with v^p is Bochner integrable equipped with the norm $\|v\|_{L^p([a,b];E)} = \left(\int_{[a,b]} \|v(z)\|^p\, dz\right)^{\frac{1}{p}}$ (if $E = \mathbb{R}$, $L^p([a,b])$ and $\|v\|_{L^p}$ respectively), $p \geq 1$.

A countable set $\mathcal{M} \subset L^1([a,b];E)$ is said to be *integrably bounded* if there exists $\omega \in L^1_+([a,b])$ such that for every $\mu \in \mathcal{M}$ it is

$$\|\mu(t)\| \leq \omega(t), \text{ a.e. } t \in [a,b].$$

Proposition 2.2. *[4, Proposition 3.1] If $\mathcal{M} \subset L^1([a,b],E)$ is a countable and integrably bounded set, then the function $\chi(\mathcal{M}(\cdot))$ belongs to $L^1_+([a,b])$ and satisfies the inequality*

$$\chi\left(\int_a^b \mathcal{M}(s)\, ds\right) \leq 4\int_a^b \chi(\mathcal{M}(s))\, ds\,.$$

A countable set $\{f_n\}_n \subset L^1([a,b];E)$ is said to be *semicompact* if

(i) it is *integrably bounded*;
(ii) the set $\{f_n(t)\}_n$ is relatively compact for a.e. $t \in [a,b]$.

Proposition 2.3. *(cf. [13, Proposition 4.2.1]) If $\{f_n\}_n \subset L^1([a,b];E)$ is a semicompact sequence, then it is weakly compact in $L^1([a,b];E)$.*

Let $T > 0$ and define $\Delta_0 := \{(t,s) : 0 \leq s \leq t \leq T\}$. Following [20], we say that a family $\{U(t,s)\}_{(t,s)\in\Delta_0}$ of bounded linear operators on E is a *strongly continuous evolution system* (evolution system for short) if

$U(s,s) = I, \quad U(t,r)U(r,s) = U(t,s)$ for $0 \leq s \leq r \leq t \leq T$;
for every $x \in E$, the map $\xi_x : (t,s) \in \Delta_0 \mapsto U(t,s)x$ is continuous.

Furthemore, a family of linear operators $\{A(t)\}_{t\in[0,T]}$, $A(t) : D(A) \subset E \to E$, $D(A)$ dense subset of E not depending on t, *generates* an evolution system $\{U(t,s)\}_{(t,s)\in\Delta_0}$ if (see, e.g. [14])

$$\frac{\partial U(t,s)}{\partial t} = A(t)U(t,s) \quad \text{and} \quad \frac{\partial U(t,s)}{\partial s} = -U(t,s)A(s), \quad (t,s) \in \Delta_0.$$

 doi:10.20944/preprints202406.0710.v1

Moreover, if $\mathcal{L}(E)$ is the space of all bounded linear operators from E to E furnished with the strong operator topology, then for an evolution system the next condition holds:

$$\exists D \geq 1 \ : \ \|U(t,s)\|_{\mathcal{L}(E)} \leq D, \ (t,s) \in \Delta_0. \tag{4}$$

Let us consider the generalized Cauchy operator $G : L^1([a,b];E) \to C([a,b];E)$,

$$Gf(t) = \int_a^t U(t,s)f(s)\,ds, \ t \in [a,b]. \tag{5}$$

By [7, Theorem 2] joined with, respectively, [13, Theorem 5.1.1] and [13, Theorem 4.2.2], we can claim that the next results hold:

Proposition 2.4. *For every semicompact set $\{f_n\}_{n=1}^{\infty} \subset L^1([a,b];E)$, the set $\{Gf_n\}_{n=1}^{\infty}$ is relatively compact in $C([a,b];E)$ and, moreover, if $f_n \rightharpoonup \bar{f}$ then $Gf_n \to G\bar{f}$.*

Proposition 2.5. *If $\{f_n\}_n \subset L^1([a,b];E)$ is integrably bounded and there exists $q \in L^1([a,b])$ such that*

$$\chi(\{f_n(t)\}_n) \leq q(t), \ a.e. \ t \in [a,b],$$

then

$$\chi(\{Gf_n(t)\}_n) \leq 2D \int_a^t q(s)ds,$$

where D is from (4).

3. the Impulsive Integro-Differential Problem in Banach Spaces

Let E be a real Banach space, $T > 0$, and $\{t_0, \dots, t_{m+1}\}$ with $m > 0$ be a set of fixed real numbers such that $0 \leq t_0 < t_1 < t_2 < ... < t_m < t_{m+1} = T$. By the symbol $\mathcal{PC}([t_0,T];E)$ we denote the Banach space

$$\mathcal{PC}([t_0,T];E) := \left\{ \begin{array}{ll} y : [t_0,T] \to E \ : & y_{|[t_0,t_1]} \text{ continuous;} \\ & y_{|]t_{j-1},t_j]} \text{ continuous, for all } j = 2,\dots,m+1; \\ & \exists\, y(t_j^+) = \lim\limits_{h \to 0^+} y(t_j + h) \in E, \text{ for all } j = 1,\dots,m \end{array} \right\}$$

endowed with the norm $\|y\|_\infty = \sup_{t\in[t_0,T]} \|y(t)\|$.

Let $y_0 \in E$ be fixed and consider the corresponding initial value problem driven by a semilinear integro-differential inclusion subject to impulses $I_j : E \to E$, $j = 1,\dots,m$ at the given times $\{t_j : j = 1,\dots,m\}$

$$(P) \begin{cases} y'(t) \in A(t)y(t) + F\left(t, y(t), \int_{t_0}^t k(t,s)y(s)ds\right), \ t \in [t_0,T], \ t \neq t_j, \ j = 1,\dots,m, \\ y(t_0) = y_0, \\ y(t_j^+) = y(t_j) + I_j(y(t_j)), \ j = 1,\dots,m, \end{cases}$$

where $\{A(t)\}_{t\in[0,T]}$ is a family of linear operators, $A(t) : D(A) \subset E \to E$, $D(A)$ dense subset of E not depending on t; $F : [t_0,T] \times E \times E \multimap E$ is a given multimap; $k : \Delta \to \mathbb{R}^+$, with $\Delta := \{(t,s) : t_0 \leq s \leq t \leq T\}$, is a given kernel.

On the functions involved in the problem we assume that:

(A) the family $\{A(t)\}_{t\in[0,T]}$ of densely defined linear operators generates an evolution system $\{U(t,s)\}_{(t,s)\in\Delta_0}$;

(k) the kernel k is continuous and we put

$$M := \max_{(t,s)\in\Delta} k(t,s); \tag{6}$$

(I) the impulse functions $I_1, \ldots, I_m$ are continuous.

Further, on the nonlinear multifunction F we will suppose that it satisfies the properties:

(F1) F takes compact and convex values;
(F2) for every $v, w \in E$, the multimap $F(\cdot, v, w)$ admits a strongly measurable selection;
(F3) for a.e. $t \in [t_0, T]$, the multimap $F(t, \cdot, \cdot)$ is upper semicontinuous;
(F4) there exists a nonnegative function $\alpha \in L^1([t_0, T])$ such that

$$\|F(t, v, w)\| \le \alpha(t)(1 + \|v\| + \|w\|), \tag{7}$$

for a.e. $t \in [t_0, T]$ and all $v, w \in E$;
(F5) there exists a nonnegative function $h \in L^1([t_0, T])$ such that

$$\chi(F(t, \Omega_1, \Omega_2)) \le h(t)[\chi(\Omega_1) + \chi(\Omega_2)], \tag{8}$$

for a.e. $t \in [t_0, T]$ and every bounded $\Omega_1, \Omega_2 \subset E$.

Definition 3.1. *A function $y \in \mathcal{PC}([t_0, T]; E)$ is said to be a* mild solution *to (P) if*

$$y(t) = U(t, t_0)y_0 + \sum_{t_0 < t_j < t} U(t, t_j)I_j(y(t_j)) + \int_{t_0}^{t} U(t, s)f(s)\, ds,\ t \in [t_0, T], \tag{9}$$

where $f : [t_0, T] \to E$ is a L^1-function on $[t_0, T]$ such that

$$f(s) \in F\Big(s, y(s), \textstyle\int_{t_0}^{s} k(s, r)y(r)dr\Big) \text{ for a.e. } s \in [t_0, T],$$

with the agreement that $\sum_{t_0 < t_j < t} U(t, t_j)I_j(y(t_j)) = 0$ *if $t \in [t_0, t_1]$.*

Note that every mild solution also satisfies the conditions

$y(t_0) = y_0$;
$y(t_j^+) = y(t_j) + I_j(y(t_j))$, $j = 1, \ldots, m$.

4. Compactness of the Mild Solutions Set in the Non-Impulsive Case

To obtain optimal solutions for problem (P), we aim to demonstrate the compactness of the set of its mild solutions. In the proof we will use the compactness of the set of mild solutions of non-impulsive problems. This result for integro-differential problems like the one we are studying is not already present in the literature, at least as far as we know. Clearly, it potentially has a relevance in itself, which is why we dedicate this paragraph to it.

Let us consider the following non-impulsive Cauchy problem

$$(P_v) \begin{cases} y'(t) \in A(t)y(t) + F\Big(t, y(t), \int_a^t k(t, s)y(s)ds\Big), & t \in [a, b], \\ y(a) = v, \end{cases}$$

where $[a, b]$ is a given subinterval of $[t_0, T]$, and $v \in E$.

A mild solution of (P_v) is a function $y \in C([a, b]; E)$ such that

$$y(t) = U(t, a)v + \int_a^t U(t, s)f(s)\, ds,\ t \in [a, b], \tag{10}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 June 2024 doi:10.20944/preprints202406.0710.v1

with $f \in L^1([a,b];E)$ and such that $f(s) \in F\left(s, y(s), \int_a^s k(s,r)y(r)dr\right)$ for a.e. $s \in [a,b]$.
We put

$$\mathcal{S}_v := \{y \in C([a,b];E) : y \text{ mild solution of } (P_v)\}. \tag{11}$$

We recall the following result on the weak closeness of the superposition operator for multifunctions involving the Volterra operator.

Lemma 4.1. *[4, Lemma 5.1] Assume that k satisfies (k) and that for F properties (F1)-(F5) hold.*
Then the operator $\mathcal{N}_F : C([a,b],E) \multimap L^1([a,b],E)$,

$$\mathcal{N}_F(y) := \{f \in L^1([a,b],E) : f(t) \in F(t, y(t), \textstyle\int_a^t k(t,s)y(s)\,ds),\ a.e.\ t \in [a,b]\}, \tag{12}$$

is correctly defined.
Moreover, if we consider sequences $(y_n)_n$, $y_n \in C([a,b],E)$, $(f_n)_n$, $f_n \in \mathcal{N}_F(y_n), n \in \mathbb{N}$, *such that* $y_n \to \bar{y}$, *and* $f_n \rightharpoonup \bar{f}$, *then* $\bar{f} \in \mathcal{N}_F(\bar{y})$.

We also need the next technical result inspired by [9, Lemma 2.1].

Lemma 4.2. *For every* $H, K > 0$ *and* $v, w \in L^1_+([a,b])$ *there exists* $N = N(H,K,v,w) \in \mathbb{N}$ *such that*

$$p_N := \max_{t\in[a,b]} e^{-Nt}\left[\int_a^t He^{Ns}v(s)ds + \int_a^t \int_a^s Ke^{Nr}w(r)drds\right] < 1.$$

Proof. For every $n \in \mathbb{N}$, we consider

$$p_n := \max_{t\in[a,b]} e^{-nt}\left[\int_a^t He^{ns}v(s)ds + \int_a^t \int_a^s Ke^{nr}w(r)drds\right].$$

By the properties of the supremum, there exists $t_n \in [a,b]$ such that

$$\begin{aligned} p_n - \frac{1}{n} &< e^{-nt_n}\left[\int_a^{t_n} He^{ns}v(s)ds + \int_a^{t_n}\int_a^s Ke^{nr}w(r)drds\right] \\ &= \int_a^b \kappa_{[a,t_n]}(s)He^{-n(t_n-s)}v(s)ds + \int_a^b \kappa_{[a,t_n]}(s)\left(\int_a^s Ke^{-n(t_n-r)}w(r)dr\right)ds, \end{aligned} \tag{13}$$

where $\kappa_{[a,t_n]}$ is the characteristic function of interval $[a,t_n]$.
Now, let us put

$$\phi_n(s) := \kappa_{[a,t_n]}(s)He^{-n(t_n-s)}v(s), \tag{14}$$

$$\psi_n(s) := \kappa_{[a,t_n]}(s)\int_a^s Ke^{-n(t_n-r)}w(r)dr, \tag{15}$$

a.e. $s \in [a,b]$.
Notice that both $(\phi_n)_n$ and $(\phi_n)_n$ a.e. pointwise converge to zero. Indeed, the sequence $\{t_n\}_n \subset [a,b]$, eventually passing to a subsequence, converges to an element $t^* \in [a,b]$. Clearly, if $s > t^*$, then definively $s > t_n$ as well, so $\kappa_{[a,t_n]}(s) = 0$ and it holds that

$$\lim_{n\to+\infty} \phi_n(s) = 0 \text{ and } \lim_{n\to+\infty} \psi_n(s) = 0.$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 June 2024 doi:10.20944/preprints202406.0710.v1

On the other hand, if $s < t^*$, then definitively $s < t_n$. In this case,

$$\lim_{n\to+\infty} \phi_n(s) = \lim_{n\to+\infty} He^{-n(t_n-s)}v(s) = 0$$

and, by the Lebesgue dominated convergence theorem,

$$\lim_{n\to+\infty} \psi_n(s) = \lim_{n\to+\infty} \int_a^s Ke^{-n(t_n-r)}w(r)dr = 0.$$

Now, by (14) and (15), we have respectively

$$\|\phi_n(s)\| \le Hv(s) \text{ and } \|\psi_n(s)\| \le \int_a^s Kw(r)dr, \text{ a.e. } s \in [a,b].$$

We can hence use the Lebesgue dominated convergence theorem and pass the limit under the integral sign in (13), so that

$$0 \le p_n \le \frac{1}{n} + \int_a^b \phi_n(s)ds + \int_a^b \psi_n(s)ds \to 0.$$

Thus $\lim_{n\to+\infty} p_n = 0$, from which the existence of $N \in \mathbb{N}$ such that $p_N < 1$. □

We can now state and prove the compactness result for the solutions set in the non-impulsive case. The proof is based on the use of Proposition 2.1.

Theorem 4.1. *Suppose that $\{A(t)\}_{t\in[0,T]}$, F and k respectively satisfy (A), (F1)-(F5), and (k). Then the set of all mild solutions of (P_v) is a nonempty compact subset of $C([a,b];E)$.*

Proof. Firstly we notice that the solutions set $\mathcal{S}_v$ is nonempty. Indeed, the existence of mild solutions to (P_v) can be deduced, albeit for different reasons, both from [24, Corollary 1] and from [4, Theorem 5.1].

Let us show that $\mathcal{S}_v$ is bounded in the Banach space $C([a,b];E)$.

Fixed any $y \in \mathcal{S}_v$, let $f \in L^1([a,b];E)$ be an a.e. selector of $F(\cdot, y(\cdot), \int_a^{(\cdot)} k(\cdot,r)y(r)dr)$ for which y has the representation (10). Then, for every $t \in [a,b]$, by (4) and (F4) we get

$$\|y(t)\| \le D\|v\| + \int_a^t D\alpha(s)\left(1 + \|y(s)\| + \textstyle\int_a^s k(s,r)\|y(r)\|dr\right) ds. \tag{16}$$

Let us define the real positive function $m : [a,b] \to \mathbb{R}$,

$$m(t) := \sup_{a\le s\le t} \|y(s)\|, \; t \in [a,b]. \tag{17}$$

Thus, by (16), (6), and (17), for every $s \in [a,t]$ we have the estimate

$$\begin{aligned} \|y(s)\| &\le D(\|v\| + \|\alpha\|_{L^1}) + \int_a^s D\alpha(r)\left(\|y(r)\| + M\textstyle\int_a^r \|y(\tau)\|d\tau\right) dr \\ &\le D(\|v\| + \|\alpha\|_{L^1}) + \int_a^s D\alpha(r)[1 + M(b-a)]m(r)\, dr. \end{aligned}$$

By the monotonicity of the supremum and using again (17), for every $t \in [a,b]$ we have

$$\begin{aligned} m(t) &\le \sup_{a\le s\le t}\left[D(\|v\| + \|\alpha\|_{L^1}) + \int_a^s D\alpha(r)[1 + M(b-a)]m(r)\, dr\right] \\ &\le D(\|v\| + \|\alpha\|_{L^1}) + \int_a^t D\alpha(r)[1 + M(b-a)]m(r)\, dr. \end{aligned}$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 June 2024 doi:10.20944/preprints202406.0710.v1

We can therefore apply the Gronwall inequality and obtain

$$m(t) \leq D(\|v\| + \|\alpha\|_{L^1})e^{D[1+M(b-a)]\|\alpha\|_{L^1}} := H. \tag{18}$$

It implies that $\|y(t)\| \leq H$ for every $t \in [a,b]$, hence

$$\|y\|_{C([a,b];E)} \leq H.$$

From the arbitrarieness of y, the boundedness of $\mathcal{S}_v$.

Now, in the Banach space $C([a,b];E)$ we consider the closed set

$$X := \{z \in C([a,b];E) : \|z\|_{C([a,b];E)} \leq H\}, \tag{19}$$

and define the multioperator $\Gamma : X \multimap C([a,b];E)$ as

$$\Gamma(y) = \left\{ \begin{array}{ll} z \in C([a,b];E) : & z(t) = U(t,a)v + \int_a^t U(t,s)f(s)ds,\ t \in [a,b], \\ & f \in L^1([a,b];E),\ f(s) \in F(s, y(s), \int_a^s k(s,r)y(r)dr)\ a.e.\ s \in [a,b] \end{array} \right\}. \tag{20}$$

The multimap Γ is actually the solution multioperator to (P_v), because $Fix\,\Gamma = \mathcal{S}_v$.

This identity and what shown above yield that $Fix\,\Gamma$ is a nonempty and bounded subset of $C([a,b];E)$.

We prove now that Γ takes compact values.

Let y be arbitrarily fixed in X and let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $C([a,b];E)$ such that $z_n \in \Gamma(y)$ for all $n \in \mathbb{N}$. Then, consider a sequence $(f_n)_{n\in\mathbb{N}}$ in $L^1([a,b];E)$ such that

$$z_n(t) = U(t,a)v + \int_a^t U(t,s)f_n(s)ds,\ t \in [a,b], \tag{21}$$

with

$$f_n(s) \in F(s, y(s), \textstyle\int_a^s k(s,r)y(r)dr),\ a.e.\ s \in [a,b].$$

The set $\{f_n\}_{n\in\mathbb{N}}$ is integrably bounded. Indeed, by (F4) and (6), and recalling that $y \in X$ (see (19)), for a.e. $t \in [a,b]$ we get

$$\begin{aligned} \|f_n(t)\| &\leq \alpha(t)(1 + \|y(t)\| + \textstyle\int_a^t M\|y(s)\|ds) \\ &\leq \alpha(t)(1 + H + MH(b-a)), \end{aligned} \tag{22}$$

from which the integrably boundedness of $\{f_n\}_{n\in\mathbb{N}}$. Further, $\{f_n(t)\}_{n\in\mathbb{N}}$ is relatively compact for a.e. $t \in [a,b]$, since by the monotonicity of the Hausdorff measure of noncompactness and (F5) it is

$$\chi(\{f_n(t)\}_{n\in\mathbb{N}}) \leq h(t)[\chi(\{y(t)\}) + \chi(\{\textstyle\int_a^t k(t,s)y(s)ds\})] = 0,\ a.e.\ t \in [a,b].$$

Hence we can apply Proposition 2.3, so that $\{f_n\}_{n\in\mathbb{N}}$ is weakly compact in $L^1([a,b];E)$. W.l.o.g. we can say that there exists $\tilde{f} \in L^1([a,b];E)$ such that $f_n \rightharpoonup \tilde{f}$ in $L^1([a,b];E)$.

Therefore, by Proposition 2.4 it follows that $Gf_n \to G\tilde{f}$ in $C([a,b];E)$ (see (5)). This implies that the sequence $(z_n)_{n\in\mathbb{N}}$ converges in $C([a,b];E)$ to the function

$$\tilde{z}(t) := U(t,a)v + \int_a^t U(t,s)\tilde{f}(s)ds,\ t \in [a,b].$$

By applying Lemma 4.1 to the sequence $\{f_n\}_{n\in\mathbb{N}}$, we have that

$$\tilde{f}(s) \in F(s, y(s), \textstyle\int_a^s k(s,r)y(r)dr),\ a.e.\ s \in [a,b],$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 June 2024 doi:10.20944/preprints202406.0710.v1

so that $\tilde{z} \in \Gamma(y)$ (see (20)). Thus $\Gamma(y)$ is compact.

Now, we prove that Γ is a closed multimap.

Let us consider the sequences $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ with $y_n \in X$ and $z_n \in \Gamma(y_n)$ for all $n \in \mathbb{N}$ such that $y_n \to \bar{y}$ and $z_n \to \bar{z}$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^1([a,b];E)$ such that for each z_n the representation (21) holds, where this time

$$f_n(s) \in F(s, y_n(s), \textstyle\int_a^s k(s,r)y_n(r)dr), \ a.e.\ s \in [a,b].$$

The estimate (22) holds also for this sequence $\{f_n\}_n$, so its integrably boundedness follows.

Further, by (F5) we have

$$\chi(\{f_n(t)\}_{n\in\mathbb{N}}) \le h(t)[\chi(\{y_n(t)\}_{n\in\mathbb{N}}) + \chi(\{\textstyle\int_a^t k(t,s)y_n(s)ds\}_{n\in\mathbb{N}})], \ a.e.\ t \in [a,b]. \tag{23}$$

The convergence $y_n \to \bar{y}$ allows to say that

$$y_n(t) \to \bar{y}(t) \quad \text{and} \quad \textstyle\int_a^t k(t,s)y_n(s)ds \to \int_a^t k(t,s)\bar{y}(s)ds, \quad \text{for all } t \in [a,b].$$

So both

$$\chi(\{y_n(t)\}_{n\in\mathbb{N}}) = 0 \quad \text{and} \quad \chi(\{\textstyle\int_a^t k(t,s)y_n(s)ds\}_{n\in\mathbb{N}}) = 0, \quad \text{for all } t \in [a,b],$$

and then by (23) it is $\chi(\{f_n(t)\}_{n\in\mathbb{N}}) = 0$ for a.e. $t \in [a,b]$, i.e. the relative compactness of the sets $\{f_n(t)\}_{n\in\mathbb{N}}$ for a.e. $t \in [a,b]$.

With the same reasoning as above, by Proposition 2.3 there exists $\bar{f} \in L^1([a,b];E)$ such that $f_n \rightharpoonup \bar{f}$ (eventually passing to a subsequence), and by Proposition 2.4, it holds that $Gf_n \to G\bar{f}$ in $C([a,b];E)$, from which

$$z_n(t) \to U(t,a)v + \int_a^t U(t,s)\bar{f}(s)ds, \ t \in [a,b].$$

Invoking the uniqueness of the limit, we have

$$\bar{z}(t) = U(t,a)v + \int_a^t U(t,s)\bar{f}(s)ds, \ t \in [a,b].$$

Also in this case we can use Lemma 4.1 and then deduce that $\bar{z} \in \Gamma(\bar{y})$.

Let us put

$$p_L := \max_{t\in[a,b]} e^{-Lt}\left[2D\int_a^t e^{Ls}h(s)ds + 8MD\int_a^t\int_a^s e^{Lr}h(r)drds\right], \ L > 0 \tag{24}$$

where D, h, M are from (4), (F5), and (6) respectively. By Lemma 4.2, there exists $L > 0$ large enaugh to have $p_L < 1$. For such an L, we consider the corresponding monotone measure of noncompactness ν_L on $C([a,b];E)$ (cf. (1)).

We are going to show that Γ is ν_L-condensing. Fixed an arbitrary bounded set $\Omega \subset X$ such that

$$\nu_L(\Omega) \preccurlyeq \nu_L(\Gamma(\Omega)), \tag{25}$$

we have to show that $\nu_L(\Omega) = (0,0)$.

To this aim, let $\{z_n\}_{n\in\mathbb{N}} \subset \Gamma(\Omega)$ be a countable set where the maximum $\nu_L(\Gamma(\Omega))$ is achieved, and $\{y_n\}_{n\in\mathbb{N}} \subset \Omega$ and $\{f_n\}_{n\in\mathbb{N}} \subset L^1([a,b];E)$ with

$$f_n(s) \in F(s, y_n(s), \textstyle\int_a^s k(s,r)y_n(r)dr), \ a.e.\ s \in [a,b],$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 June 2024 doi:10.20944/preprints202406.0710.v1

be such that the representation (21) of z_n holds for every $n \in \mathbb{N}$. Thus, bearing in mind the definition of ν_L, it is immediate from (25) that

$$(\gamma(\{y_n\}_n), \eta(\{y_n\}_n)) \preccurlyeq \nu_L(\Omega) \preccurlyeq \nu_L(\gamma(\Omega)) = (\gamma(\{z_n\}_n), \eta(\{z_n\}_n)). \tag{26}$$

Let us show that $\gamma(\{z_n\}_n) = 0$ (cf. (2) for the definition of γ). From (26), we can immediately say that

$$\gamma(\{y_n\}_n) \leq \gamma(\{z_n\}_n). \tag{27}$$

Of course, the estimate (22) holds for the sequence $\{f_n\}_n$ and hence it is integrably bounded. Further,

$$\begin{aligned}\chi(\{f_n(s)\}_{n\in\mathbb{N}}) &\leq h(s)[\chi(\{y_n(s)\}_n) + \chi(\{\textstyle\int_a^s k(s,r)y_n(r)dr\}_n)] \\ &\leq e^{Ls}h(s)\gamma(\{y_n\}_n),\ a.e.\ s \in [a,b],\end{aligned} \tag{28}$$

hence by Proposition 2.5 we can write

$$\chi\left(\left\{\int_a^t U(t,s)f_n(s)ds\right\}_n\right) \leq 2D\int_a^t e^{Ls}h(s)\gamma(\{y_n\}_n)ds,\ t \in [a,b],$$

implying the estimate

$$\chi(\{z_n(t)\}_n) \leq 2D\gamma(\{y_n\}_n)\int_a^t e^{Ls}h(s)ds,\ t \in [a,b].$$

On the other hand, in our setting we can apply Proposition 2.2 and obtain

$$\chi(\{\textstyle\int_a^t k(t,s)z_n(s)ds\}_n) \leq 8MD\gamma(\{y_n\}_n)\displaystyle\int_a^t\int_a^s e^{Lr}h(r)drds,\ t \in [a,b].$$

Therefore, recalling the defintions of function γ and number p_L (see (2) and (24) respectively), we have

$$\begin{aligned}\gamma(\{z_n\}_n) &\leq \gamma(\{y_n\}_n)\sup_{t\in[a,b]} e^{-Lt}\left[2D\int_a^t e^{Ls}h(s)ds + 8MD\int_a^t\int_a^s e^{Lr}h(r)drds\right] \\ &= \gamma(\{y_n\}_n)p_L.\end{aligned}$$

From this and by (27), we obtain $\gamma(\{y_n\}_n) \leq \gamma(\{z_n\}_n) \leq \gamma(\{y_n\}_n)p_L$. Since $p_L < 1$, it follows that

$$\gamma(\{y_n\}_n) = 0 \tag{29}$$

and, as a consequence of the same inequality, also

$$\gamma(\{z_n\}_n) = 0. \tag{30}$$

We prove now that $\eta(\{z_n\}_n) = 0$ (cf. (3) for the definition of η). First of all, from (29) we deduce that $\chi(\{y_n(t)\}_n) = 0$, for every $t \in [a,b]$. Moreover, we know that the set $\{f_n\}_n$ is integrably bounded (see above) and by (28) and (29) we have $\chi(\{f_n(t)\}_n) = 0$ for a.e. $t \in [a,b]$. So $\{f_n\}_n$ is semicompact. By Propositions 2.3 and 2.4, we have the convergence of the sequence $(Gf_n)_n$ (see (5)). Hence, the set $\{Gf_n\}_n$ is relatively compact in $C([a,b],E)$ and hence it is equicontinuous, so that

$$mod_C(\{z_n\}_n) = mod_C(\{Gf_n\}_n) = 0.$$

 doi:10.20944/preprints202406.0710.v1

About the other term of η, from the continuity of the Volterra operator we have that

$$mod_C(\{\int_a^{(\cdot)} k(\cdot,s)z_n(s)ds\}_n) = 0.$$

We hence achieve

$$\eta(\{z_n\}_n) = 0. \tag{31}$$

Therefore, by (30), (31), and (26), we obtain $\nu_L(\Omega) = (0,0)$ as desired.

We have shown that all the hypotheses of Proposition 2.1 are satisfied, allowing to the compactness of the set $\mathcal{S}_v = Fix\,\Gamma$. □

5. Existence of Optimal Solutions for Impulsive Integro-Differential Problems

We are here interested in the minimization or maximization of a cost functional of problem (P), say

$$\mathcal{J} : \mathcal{PC}([t_0,T];E) \to \mathbb{R}.$$

To this aim, in this section we state and prove the compactness of the set of all mild solutions of problem (P). We preface the following lemma, which can be immediately deduced by [24, Lemma 1].

Lemma 5.1. *Assume that F and k respectively satisfy (F1)-(F5) and (k).*

Then, for every $j = 1,\ldots,m$ and every set of functions $\{y_i \in C([t_i,t_{i+1}],E) : i = 0,\ldots,j-1\}$, the multimap $F_j : [t_j,t_{j+1}] \times E \times E \multimap E$ defined by

$$F_j(t,v,w) := F\left(t,v,w + \sum_{i=0}^{j-1} \int_{t_i}^{t_{i+1}} k(t,s)y_i(s)ds\right),\ t \in [t_j,t_{j+1}], v,w \in E \tag{32}$$

satisfies (F1)-(F5).

Theorem 5.1 (Compactness of the mild solutions set under the impulses' effect)**.** *Suppose that $\{A(t)\}_{t\in[0,T]}$, F, k, and $I_1,\ldots,I_m$ respectively satisfy (A), (F1)-(F5), (k), and (I).*

Then, the set of all mild solutions of problem (P) is a nonempty compact subset of $\mathcal{PC}([t_0,T];E)$.

Proof. Let us denote the set of all mild solutions of (P) as $\mathcal{S}$. It is a nonempty set (cf. [24, Theorem 1]).

In order to prove the compactness of the set of solutions $\mathcal{S}$, we suppose that the number of impulse times is $m = 1$. Clearly, if $m > 1$ we will only have to iterate the procedure a finite number of times to achieve the same conclusions.

We proceed by steps.

Step 1. Let us consider the non-impulsive Cauchy problem

$$(P_1)\begin{cases} y'(t) \in A(t)y(t) + F\left(t,y(t),\int_{t_0}^t k(t,s)y(s)ds\right), & t \in [t_0,t_1],\\ y(t_0) = y_0. \end{cases}$$

We can apply Theorem 4.1 to (P_1) with $[a,b] = [t_0,t_1]$ and $v = y_0$, so that the set

$$\mathcal{S}_1 := \{y \in C([t_0,t_1];E) : y \text{ mild solution to } (P_1)\}$$

is a nonempty compact subset of $C([t_0,t_1];E)$.

Step 2. Let us fix any $y_1 \in \mathcal{S}_1$ and consider the problem

$$(P_2)_{y_1}\begin{cases} y'(t) \in A(t)y(t) + F_1\left(t,y(t),\int_{t_1}^t k(t,s)y(s)ds\right), & t \in [t_1,t_2],\\ y(t_1) = y_1(t_1) + I_1(y_1(t_1)), \end{cases}$$

 doi:10.20944/preprints202406.0710.v1

where $F_1 : [t_1, t_2] \times E \times E \multimap E$ is defined by

$$F_1(t, v, w) := F\Big(t, v, w + \textstyle\int_{t_0}^{t_1} k(t, s) y_1(s) ds\Big), \; t \in [t_1, t_2], \; v, w \in E. \tag{33}$$

Clearly, problem $(P_2)_{y_1}$ is of the type (P_v), just taking $[a, b] = [t_1, t_2]$, $v = y_1(t_1) + I_1(y_1(t_1))$, and $F = F_1$. Since by Lemma 5.1 the multimap F_1 satisfies properties (F1)-(F5), we can apply Theorem 4.1 and claim that the set

$$\mathcal{S}_{2,y_1} := \{y \in C([t_1, t_2]; E) : y \text{ mild solution to } (P_2)_{y_1}\}$$

is a nonempty compact subset of $C([t_1, t_2]; E)$.

Step 3. Put

$$\mathcal{S}_2 := \cup_{y_1 \in \mathcal{S}_1} \mathcal{S}_{2,y_1},$$

we are going to show that it is a compact subset of $C([t_1, t_2]; E)$ as well.

To this aim, let us consider the multimap $\Phi_1 : \mathcal{S}_1 \multimap C([t_1, t_2]; E)$ given by

$$\Phi_1(y) = \mathcal{S}_{2,y}, \; y \in \mathcal{S}_1. \tag{34}$$

From what shown before, we know that this multimap takes nonempty compact values.

Let us prove that it is an upper semicontinuous multifunction. Suppose, on the contrary, that there exist $\bar{y} \in \mathcal{S}_1$, $\varepsilon_0 > 0$, and sequences $\{y_n\}_n \subset \mathcal{S}_1$ with $y_n \to \bar{y}$, $\{w_n\}_n \subset C([t_1, t_2], E)$ with $w_n \in \Phi_1(y_n)$ for every $n \in \mathbb{N}$, such that

$$w_n \notin V_{\varepsilon_0}(\Phi_1(\bar{y})), \; n \in \mathbb{N}, \tag{35}$$

where V_{ε_0} is a ε_0-neighbourhood of $\Phi_1(\bar{y})$.

Clearly, since w_n belongs to $\Phi_1(y_n) = \mathcal{S}_{2,y_n}$ for every $n \in \mathbb{N}$, then there exists a sequence $\{f_n\}_n \subset L^1([t_1, t_2], E)$ with (cf. (33))

$$f_n(s) \in F_1(s, w_n(s), \textstyle\int_{t_1}^{s} k(s, r) w_n(r)\, dr), \text{ a.e. } s \in [t_1, t_2],$$

and such that

$$w_n(t) = U(t, t_1) T_1(y_n) + \int_{t_1}^{t} U(t, s) f_n(s)\, ds, \; t \in [t_1, t_2], \; n \in \mathbb{N}, \tag{36}$$

where the function $T_1 : \mathcal{S}_1 \to E$ is defined by

$$T_1(y) = y(t_1) + I_1(y(t_1)), \; y \in \mathcal{S}_1.$$

It is easy to see that, according to the continuity of I_1, the mapping T_1 is continuous.

The set $\{w_n\}_n$ is bounded in $C([t_1, t_2]; E)$. In fact, fixed $n \in \mathbb{N}$ and put

$$m_n(t) := \sup_{t_1 \le s \le t} \|w_n(s)\|, \; t \in [t_1, t_2],$$

by using the same arguments as in the proof of Theorem 4.1 we have

$$m_n(t) \le D(\|T_1(y_n)\| + \|\alpha\|_{L^1}) + \int_{t_1}^{t} D\alpha(s)[1 + M(t_2 - t_1)] m_n(s)\, ds.$$

By the continuity of T_1 and the convergence of $\{y_n\}_n$, the set $\{T_1(y_n)\}_n$ is bounded. Thus, by the Gronwall inequality the boundedness of $\{w_n\}_n$ follows.

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 June 2024 doi:10.20944/preprints202406.0710.v1

Moreover, by (F4) of F_1 and by (6) we have

$$\|f_n(t)\| \leq \|F_1(t, w_n(t), \int_{t_1}^t k(t,s)w_n(s)\,ds)\|$$
$$\leq \alpha(t)\Big(1 + \|w_n(t)\| + \int_{t_1}^{t_2} M\|w_n(s)\|\,ds\Big),\ a.e.\ t \in [t_1, t_2].$$

So, recalling the boundedness of $\{w_n\}_n$, we obtain that the integrably boundedness of the sequence $\{f_n\}_n$.

Further, the set $\{f_n(t)\}_n$ is relatively compact for a.e. $t \in [t_1, t_2]$. Indeed, by the properties of the Hausdorff measure of noncompactness, (F5) of F_1 and recalling the definition (2), we have the estimate

$$\begin{aligned}
\chi(\{f_n(t)\}_n) &\leq \chi\Big(\{F_1(t, w_n(t), \int_{t_1}^t k(t,s)w_n(s)\,ds)\}_n\Big)\\
&\leq h(t)\Big[\chi(\{w_n(t)\}_n) + \chi\Big(\{\int_{t_1}^t k(t,s)w_n(s)\,ds\}_n\Big)\Big]\\
&\leq e^{Lt}h(t)\gamma(\{w_n\}_n),\ \text{a.e. } t \in [t_1, t_2],
\end{aligned}$$

which is analogous to (28). So, with the same reasonings as in the proof of Theorem 4.1, we can claim that $\gamma(\{w_n\}_n) = 0$ (cf. (29)). Thus $\chi(\{f_n(t)\}_n) = 0$ for a.e. $t \in [t_1, t_2]$.

We therefore have that the set $\{f_n\}_n$ is semicompact.

Now, considered the generalized Cauchy operator on $[t_1, t_2]$, i.e. $G_1 : L^1([t_1, t_2]; E) \to C([t_1, t_2]; E)$,

$$G_1 f(t) = \int_{t_1}^t U(t,s)f(s)\,ds,\ t \in [t_1, t_2], \tag{37}$$

by Proposition 2.4 we have that the set $\{G_1 f_n\}_n$ is relatively compact. Further, since T_1 is continuous, we have both

$$y_n \to \bar{y}\ \text{ and }\ T_1(y_n) \to T_1(\bar{y}).$$

We can hence say that the set $\{w_n\}_n$ is relatively compact in $C([t_1, t_2]; E)$. Therefore there exists $\bar{w} \in C([t_1, t_2]; E)$ such that, eventually passing to a subsequence,

$$w_n \to \bar{w}. \tag{38}$$

Now, we prove that $\bar{w} \in \Phi_1(\bar{y})$. In fact, we can use Proposition 2.3 which yields that there exists $\bar{f} \in L^1([t_1, t_2], E)$ such that $f_n \rightharpoonup \bar{f}$. Then, we apply Lemma 4.1 to the operator

$$\mathcal{N}_{F_1}(y) := \{f \in L^1([t_1, t_2], E) : f(t) \in F_1(t, w(t), \int_{t_1}^t k(t,s)w(s)\,ds),\ a.e.\ t \in [t_1, t_2]\},$$

so that

$$\bar{f}(t) \in F_1(t, \bar{w}(t), \int_{t_1}^t k(t,s)\bar{w}(s)\,ds),\ a.e.\ t \in [t_1, t_2]. \tag{39}$$

Moreover, by Proposition 2.4 we get

$$G_1 f_n \to G_1 \bar{f}.$$

Therefore, for every $t \in [t_1, t_2]$, on the one hand it is (see (36))

$$w_n(t) \to U(t, t_1)T_1(\bar{y}) + \int_{t_1}^t U(t,s)\bar{f}(s)\,ds,$$

on the other we had $w_n(t) \to \bar{w}(t)$ (see (38)). Therefore, by the uniqueness of the limit algorithm, we obtain

$$\bar{w}(t) = U(t, t_1)T_1(\bar{y}) + \int_{t_1}^t U(t,s)\bar{f}(s)\,ds$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 June 2024 doi:10.20944/preprints202406.0710.v1

with (see (39) and (33))

$$\begin{aligned}\bar{f}(s) &\in F_1(s, w_n(s), \textstyle\int_{t_1}^{s} k(s,r)w_n(r)\,dr) \\ &= F(s, w_n(s), \textstyle\int_{t_1}^{s} k(s,r)w_n(r)\,dr + \int_{t_0}^{t_1} k(s,r)y_n(r)\,dr), \text{ a.e. } s \in [t_1, t_2].\end{aligned}$$

Hence, $\bar{w}$ is a mild solution to $(P_2)_{\bar{y}}$, i.e. (see (34))

$$\bar{w} \in \mathcal{S}_{2,\bar{y}} = \Phi_1(\bar{y}).$$

Thus $w_n \in V_{\varepsilon_0}(\Phi_1(\bar{y}))$ definitively, leading to a contradiction to (35).

So far we have proven that Φ_1 takes compact values and is upper semicontinuous. Therefore it maps compact sets into compact sets (see, e.g. [13, Theorem 1.1.7]), and hence we can conclude that the set $\Psi_1(\mathcal{S}_1)$ is compact. From the equality

$$\mathcal{S}_2 = \Psi_1(\mathcal{S}_1),$$

the compactness of $\mathcal{S}_2$.

Step 4. Let us observe that the Banach space $\mathcal{PC}([t_0, T]; E)$ is isomorphic to the space $C([t_0, t_1]; E) \times C([t_1, t_2]; E)$ endowed with the Chebyshev norm

$$\|(y_1, y_2)\|_C = \max\left\{ \sup_{t\in[t_0,t_1]} \|y_1(t)\|, \ \sup_{t\in[t_1,t_2]} \|y_2(t)\| \right\}.$$

Indeed, first of all recall that $m = 1$, so $T = t_2$.

Then, let us define the mapping $J : \mathcal{PC}([t_0, T]; E) \to C([t_0, t_1]; E) \times C([t_1, t_2]; E)$ as

$$J(y) = (y_1, y_2), \ y \in \mathcal{PC}([t_0, T]; E),$$

where

$$y_1(t) := y(t), \ t \in [t_0, t_1], \ \text{ and } \ y_2(t) := \begin{cases} y(t), & t \in]t_1, t_2], \\ y(t_1^+), & t = t_1. \end{cases} \tag{40}$$

It is easy to check that it is injective and continuous.

Moreover, also its inverse function $J^{-1} : C([t_0, t_1]; E) \times C([t_1, t_2]; E) \to \mathcal{PC}([t_0, T]; E)$,

$$J^{-1}(y_1, y_2) = y(t) := \begin{cases} y_1(t), & t \in [t_0, t_1], \\ y_2(t), & t \in]t_1, t_2], \end{cases}$$

is continuous.

Step 5. We show that $J(\mathcal{S}) \subset \mathcal{S}_1 \times \mathcal{S}_2$. To this aim, let $y \in \mathcal{S}$ be arbitrarily fixed. By (9), there exists $f \in L^1([t_0, T]; E)$ with $f(s) \in F\Big(s, y(s), \int_{t_0}^{s} k(s,r)y(r)dr\Big)$ for a.e. $s \in [t_0, T]$, and such that

$$y(t) = U(t, t_0)y_0 + \sum_{t_0 < t_j < t} U(t, t_j)I_j(y(t_j)) + \int_{t_0}^{t} U(t,s)f(s)\,ds, \ t \in [t_0, T].$$

We put

$$f_1(s) = f(s), \ s \in [t_0, t_1], \ \text{ and } \ f_2(s) = f(s), \ s \in [t_1, t_2].$$

Consider now $J(y) = (y_1, y_2)$ where y_1 and y_2 are defind by (40). It is immediate to verify that

$$y_1(t) = U(t, t_0)y_0 + \int_{t_0}^{t} U(t,s)f_1(s)ds, \ t \in [t_0, t_1],$$

Preprints.org (www.preprints.org) | NOT PEER-REVIEWED | Posted: 11 June 2024 doi:10.20944/preprints202406.0710.v1

and $f_1(s) \in F\Big(s, y_1(s), \int_{t_0}^{s} k(s,r) y_1(r) dr\Big)$ a.e. $s \in [t_0, t_1]$. Therefore y_1 a mild solution of (P_1), i.e. $y_1 \in \mathcal{S}_1$.

About y_2, we have

$$\begin{aligned} y_2(t) &= U(t,t_0)y_0 + U(t,t_1)I_1(y(t_1)) + \int_{t_0}^{t} U(t,s)f(s)ds \\ &= U(t,t_1)\Big[U(t_1,t_0)y_0 + I_1(y_1(t_1)) + \int_{t_0}^{t_1} U(t_1,s)f_1(s)ds\Big] + \int_{t_1}^{t} U(t,s)f_2(s)ds \\ &= U(t,t_1)[y_1(t_1) + I_1(y_1(t_1))] + \int_{t_1}^{t} U(t,s)f_2(s)ds,\ t \in [t_0,t_1]. \end{aligned}$$

Further, recalling (32), we have

$$\begin{aligned} f_2(s) &\in F\Big(s, y(s), \textstyle\int_{t_0}^{s} k(s,r)y(r)dr\Big) \\ &= F\Big(s, y_2(s), \textstyle\int_{t_0}^{t_1} k(s,r)y_1(r)dr + \int_{t_1}^{s} k(s,r)y_2(r)dr\Big) \\ &= F_1\Big(s, y_2(s), \textstyle\int_{t_1}^{s} k(s,r)y_2(r)dr\Big), \text{ a.e. } s \in [t_1,t_2]. \end{aligned}$$

Hence y_2 is a mild solution of $(P_2)_{y_1}$ and then belongs to $\mathcal{S}_{2,y_1}$.

So we achieve

$$J(y) \in \{y_1\} \times \mathcal{S}_{2,y_1} \subset \mathcal{S}_1 \times \mathcal{S}_2.$$

Step 6. Now, we prove that the nonempty set $\mathcal{S}$ is closed.

To this aim, let us fix any sequence $\{y_n\}_n$ in $\mathcal{S}$ converging to a function $\bar{y} \in \mathcal{PC}([t_0,T];E)$. Since each $y_n \in \mathcal{S}$, then there exists a sequence $\{f_n\}_n$ in $L^1([t_0,T];E)$ such that for y_n the representation in (9) holds, i.e. there exists a sequence $\{f_n\}_n$ in $L^1([t_0,T];E)$ with $f_n(s) \in F\Big(s, y_n(s), \int_{t_0}^{s} k(s,r)y_n(r)dr\Big)$ a.e. $s \in [t_0,T]$, such that

$$y_n(t) = U(t,t_0)y_0 + U(t,t_1)I_1(y_n(t_1)) + \int_{t_0}^{t} U(t,s)f_n(s)\,ds,\ t \in [t_0,T].$$

Notice that $\mathcal{S}$ isomorphic to the set $J(\mathcal{S})$, which is a subset of the compact set $\mathcal{S}_1 \times \mathcal{S}_2$. Therefore $\mathcal{S}$ is bounded in $\mathcal{PC}([t_0,T];E)$. We can therefore use the same arguments as before to say that $\{f_n\}_n$ is semicompact. Then, by Proposition 2.3 on $[t_0,T]$, there exists $\bar{f} \in L^1([t_0,T];E)$ such that w.l.o.g. $f_n \rightharpoonup \bar{f}$. Hence $\bar{f} \in \mathcal{N}_F(\bar{y})$.

Moreover, by the continuity of I_1 and by Proposition 2.4 applied to $G : L^1([t_0,T];E) \to C([t_0,T];E)$,

$$Gf(t) = \int_{t_0}^{t} U(t,s)f(s)ds,\ t \in [t_0,T],$$

we obtain

$$y_n(t) \to U(t,t_0)y_0 + U(t,t_1)I_1(\bar{y}(t_1)) + \int_{t_0}^{t} U(t,s)\bar{f}(s)\,ds,\ t \in [t_0,T].$$

Thus, by the uniqueness of the limit we have

$$\bar{y}(t) = U(t,t_0)y_0 + U(t,t_1)I_1(\bar{y}(t_1)) + \int_{t_0}^{t} U(t,s)\bar{f}(s)\,ds,\ t \in [t_0,T],$$

and then $\bar{y} \in \mathcal{S}$.

Step 7. In conclusion we have

$$\mathcal{S} \cong J(\mathcal{S}) \subset \mathcal{S}_1 \times \mathcal{S}_2$$

 doi:10.20944/preprints202406.0710.v1

with $\mathcal{S}$ closed and $\mathcal{S}_1 \times \mathcal{S}_2$ compact, from which the compactness of $\mathcal{S}$. □

Remark 5.1. *Let us note that, unlike the case without delay, in this case we do not have $J(\mathcal{S}) = \mathcal{S}_1 \times \mathcal{S}_2$, but only $J(\mathcal{S}) \subset \mathcal{S}_1 \times \mathcal{S}_2$. In fact, considering a pair $(y_1, y_2) \in \mathcal{S}_1 \times \mathcal{S}_2$, there are no reasons why y_2 should belong exactly to $\mathcal{S}_{2,y_1}$. In other words, an element of $\mathcal{S}_2$ has a past that may not be y_1.*

Remark 5.2. *A different approach to the problem of the compactness of the solutions for the problem (P) could be to avoid the extensio-with-memory process by addressing the solutions globally over the entire interval at a single time. In this case the proof of the Theorem 4.1 should be retraced, but using the Gronwall-Bellman inequality of [2, Lemma 1] established in the impulsive case. To be able to apply it, it would be necessary to strengthen the hypothesis of continuity on the impulse functions, assuming $\|I_j(v)\| \le \beta_j \|v\|$ for every $v \in E$ and some $\beta_j \in E$, $j = 1, \cdot, m$, thus making the result to all intents and purposes a mere corollary of the previous Theorem 5.1.*

We can finally provide the existence of optimal solutions to our problem (P), whose proof is immediate according to the compactness of the solutions set.

Theorem 5.2 (Existence of optimal solutions)**.** *Assume the same hypotheses as Theorem 5.1 and let $\mathcal{J} : \mathcal{PC}([t_0, T]; E) \to \mathbb{R}$ be a cost functional for (P).*

If $\mathcal{J}$ is lower semicontinuous, then there exists a mild solution y_* of (P) such that

$$\mathcal{J}(y_*) = \min_{y \in \mathcal{S}} \mathcal{J}(y);$$

if $\mathcal{J}$ is upper semicontinuous, then there exists a mild solution y^* of (P) such that

$$\mathcal{J}(y^*) = \max_{y \in \mathcal{S}} \mathcal{J}(y),$$

where $\mathcal{S}$ is the set of all the mild solutions of (P).

6. Existence of Optimal Solutions for Feedback Control Systems under Impulses' Effects

We now deal with applying the theory up to now developed to the following feedback control system

$$y'(t) = A(t)y(t) + f\Big(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds, \eta(t)\Big),\ t \in [t_0, T],\ t \neq t_j,\ j = 1, \dots, m, \tag{41}$$

$$\eta(t) \in H(t, y(t), \int_{t_0}^{t} k(t,s)y(s)ds)\ ,\ t \in [t_0, T], \tag{42}$$

satisfying the initial condition and subject to the impulses' action

$$y(t_0) = y_0; \quad y(t_j^+) = y(t_j) + I_j(y(t_j))\ ,\ j = 1, \dots, m. \tag{43}$$

Here $f : [t_0, T] \times E \times E \times E_1 \to E$, E_1 separable Banach space, is a function with the properties:

(f1) $f(\cdot, v, w, \eta) : [t_0, T] \to E$ is measurable for every $(v, w, \eta) \in E \times E \times E_1$;
(f2) $f(t, \cdot, \cdot, \cdot) : E \times E \times E_1 \to E$ is continuous for a.e. $t \in [t_0, T]$;
(f3) $\|f(t, v_1, w_1, \eta) - f(t, v_2, w_2, \eta)\| \le q(t)(\|v_1 - v_2\| + \|w_1 - w_2\|)$, for every $v_1, v_2, w_1, w_2 \in E$, $\eta \in E_1$, where $q \in L^1_+([t_0, T])$.

Moreover, $H : [t_0, T] \times E \times E \multimap E_1$ is a multifunction such that the next conditions hold:

(H1) H takes compact values;
(H2) $H(\cdot, v, w) : [t_0, T] \multimap E_1$ is measurable for every $v, w \in E$;
(H3) $H(t, \cdot, \cdot) : E \times E \multimap E_1$ is upper semicontinuous for a.e. $t \in [t_0, T]$;

 doi:10.20944/preprints202406.0710.v1

(H4) H is superpositionally measurable, i.e. for every measurable multifunction $Q : [t_0, T] \multimap E \times E$ with compact values, the multifunction $\mathcal{Q} : [t_0, T] \multimap E$, $\mathcal{Q}(t) = H(t, Q(t))$, is measurable;
(H5) the set

$$F(t, v, w) := f(t, v, w, H(t, v, w)) \tag{44}$$

is convex for all $(t, v, w) \in [t_0, T] \times E \times E$;
(H6) the multimap F satisfies the sublinear growth (F4);
(H7) the set $f(t, v, w, H(t, D_1, D_2))$ is relativly compact for every $(t, v, w) \in [t_0, T] \times E \times E$ and D_1, D_2 bounded subsets of E.

A pair (y, η), where $y \in \mathcal{PC}([t_0, T]; E)$ and $\eta : [t_0, T] \to E_1$ is measurable, is said to be a *mild solution of the control system* (41)-(43) if

$$\begin{aligned} y(t) = U(t, t_0)y_0 \ &+ \sum_{t_0 < t_j < t} U(t, t_j)I_j(y(t_j)) \\ &+ \int_{t_0}^{t} U(t, s) f\left(s, y(s), \textstyle\int_{t_0}^{s} k(s, r)y(r)dr, \eta(s)\right) ds, \ t \in [t_0, T], \end{aligned}$$

with $\eta(s) \in H(s, y(s), \int_{t_0}^{s} k(s, r)y(r)dr)$, $s \in [t_0, T]$.
The piecewise continuous function y is the *mild trajectory* and the measurable function η is the *control*.

Theorem 6.1. *Let $\mathcal{J} : \mathcal{PC}([t_0, T]; E) \to \mathbb{R}$ be a cost functional for the control system (41)-(42). Suppose that $\{A(t)\}_{t\in[0,T]}$, f, k, H, and $I_1, \ldots, I_m$ respectively satisfy (A), (f1)-(f3), (k), (H1)-(H7), and (I).*

If $\mathcal{J}$ is lower semicontinuous, then there exists a mild solution (y_*, η_*) of the control system (41)-(43) such that

$$\mathcal{J}(y_*) = \min_{y \in \mathcal{S}} \mathcal{J}(y);$$

if $\mathcal{J}$ is upper semicontinuous, then there exists a mild solution (y^*, η^*) of the control system (41)-(43) such that

$$\mathcal{J}(y^*) = \max_{y \in \mathcal{S}} \mathcal{J}(y);$$

where $\mathcal{S}$ is the set of all mild trajectories of the control system with initial datum $y(t_0) = y_0$ and subject to the impulses' action $y(t_j^+) = y(t_j) + I_j(y(t_j))$, $j = 1, \ldots, m$.

Proof. Assume that $\mathcal{J}$ is lower semicontinuous. In the case of upper semicontinuity the proof will be analogous.

From the system (41)-(42), by using the function F defined in (44) we obtain the associated integro-differential inclusion

$$y'(t) \in A(t)y(t) + F(t, y(t), \textstyle\int_{t_0}^{t} k(t, s)y(s)ds). \tag{45}$$

Notice that the multifunction F defined in (44) satisfies properties (F1)-(F5). This is a consequence of the hypotheses (f1)-(f3), (k), and (H1)-(H7), and of the basic properties of multifunctions. The detailed proof can be immediately deduced by the one of [13, Theorem 5.2.3], so we refer the interested reader to that.

We are hence in position to apply our Theorem 5.2 to the problem (45), (43), so that there exists a function $y_* \in \mathcal{PC}([t_0, T]; E)$ minimizing the cost functional $\mathcal{J}$ over the nonempty compact set of all the mild solution problem (45), (43).

Thus, the function y_* has the representation (9), i.e.

$$y_*(t) = U(t, t_0)y_0 + \sum_{t_0 < t_j < t} U(t, t_j)I_j(y_*(t_j)) + \int_{t_0}^{t} U(t, s)g(s)\, ds, \ t \in [t_0, T],$$

 doi:10.20944/preprints202406.0710.v1

where $g : [t_0, T] \to E$ is a L^1-function on $[t_0, T]$ such that (see (44))

$$g(s) \in F\Big(s, y_*(s), \int_{t_0}^{s} k(s,r) y_*(r) dr\Big)$$
$$= f\Big(s, y_*(s), \int_{t_0}^{s} k(s,r) y_*(r) dr, H(s, y_*(s), \int_{t_0}^{s} k(s,r) y_*(r) dr)\Big),$$

for a.e. $s \in [t_0, T]$.

By (H4) and the Filippov Implicit Function Lemma (see, e.g. [13, Theorem 1.3.3]), there exists a measurable selection η_* of $H(\cdot, y_*(\cdot), \int_{t_0}^{(\cdot)} k(\cdot, r) y_*(r) dr)$ such that

$$g(s) = f\Big(s, y_*(s), \int_{t_0}^{s} k(s,r) y_*(r) dr, \eta_*(s)\Big), \ a.e.\ t \in [t_0, T].$$

The function η_* is the control which realizes the mild solution y_* to be a mild trajectory of the control system (41)-(43).

The pair (y_*, η_*) is therefore an optimal solution of the control system (41)-(43). □

7. Optimal Solutions for a Feedback Control Population Dynamics Model with Impulses and Fading Memory

In this section we apply our optimality result to a feedback control population dynamics model subject to the action of instantaneous external forces and with fading memory.

The differential equation describing the population dynamics we deal with is

$$u_t(t,x) = -b(t,x)u(t,x) + g\Big(t, u(t,x), \int_{t_0}^{t} \frac{e^{-(t-s)/\tau}}{\tau}\, u(s,x)\, ds\Big) + \omega(t,x),$$
$$t \in [t_0, T]\,,\ t \neq t_j\,,\ j = 1, \dots, m,\ a.e.\ x \in [0,1]. \tag{46}$$

Here $u(t,x)$ represents the local and instantaneous population density (in the normalized spatial interval $[0,1]$); $b(t,x)$ is the removal rate coefficient, due to death and displacement; g is the nonlinear law of population development; the Volterra integral $\int_{t_0}^{t} \frac{e^{-(t-s)/\tau}}{\tau}\, u(s,x)\, ds$ describes the distributed delay which affects the evolution of the population, where the positive number τ establishes the width of the action of a fading memory kernel; the control $\omega(t,x)$ belongs to a set of feedback controls

$$\omega(t,\cdot) \in \Omega(u(t,\cdot)),\ t \in [t_0, T]. \tag{47}$$

The system must satisfy the initial datum $u_0 \in L^2([0,1])$ and the effects induced by the impulses $\mathcal{I}_j : \mathbb{R} \to \mathbb{R}$, $j = 1, \dots, m$, i.e.

$$u(t_0, x) = u_0(x) \ \text{ and } u\big(t_j^+, x\big) = u(t_j, x) + \mathcal{I}_j(u(t_j, x))\,,\ j = 1, \dots, m,\ a.e.\ x \in [0,1]. \tag{48}$$

The controllability of the system (46)-(48) has already been demonstrated in our earlier work [24], even on the half-line $[t_0, +\infty[$. By reducing the therein assumptions to $[t_0, T]$, we can follow that paper and state the next propositions.

Proposition 7.1. *(cf. [24, Proposition 4.1]) Assume that the function $b : [0,T] \times [0,1] \to \mathbb{R}^+$ satisfies properties*

(b1) b is measurable;
(b2) there exists $s \in L^1([0,T])$ such that

$$0 < b(t,x) \leq s(t)\,,\ \textit{for every } t \in [0,T],\ a.e.\ x \in [0,1];$$

(b3) for every $x \in [0,1]$, the function $b(\cdot, x) : [0,T] \to \mathbb{R}^+$ is continuous.

Then, the family $\{A(t)\}_{t\in[0,T]}$, $A(t) : L^2([0,1]) \to L^2([0,1])$, $t \in [0,T]$, *defined by*

$$A(t)v(x) = -b(t,x)v(x),\ v \in L^2([0,1]),\ x \in [0,1],$$

satisfies property (A).

Moreover, the (noncompact) evolution system generated by $\{A(t)\}_{t\in[0,T]}$ *is given by*

$$[U(t,s)v](x) = e^{\int_s^t -b(\sigma,x)d\sigma}v(x),$$

for every $0 \le s \le t \le T$, $v \in L^2([0,1])$, $x \in [0,1]$.

Proposition 7.2. *(cf. [24, Theorem 4.1]) Assume that the function* $b : [0,T] \times [0,1] \to \mathbb{R}^+$ *satisfies properties (b1)-(b3). Suppose also that the function* $g : [t_0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies*

(g1) for every $t \in [t_0,T]$, *the map* $x \mapsto g(t,v(x),w(x))$ *belongs to* $L^2([0,1])$, *for every* $v,w \in L^2([0,1])$;
(g2) for every $p,q \in \mathbb{R}$, *the function* $g(\cdot,p,q)$ *is (strongly) measurable;*
(g3) for a.e. $t \in [t_0,T]$, *the function* $g(t,\cdot,\cdot)$ *is continuous;*
(g4) there exists $\varphi \in L^1_+([t_0,T])$ *such that* $|g(t,p,q)| \le \varphi(t)$, *for a.e.* $t \in [t_0,T]$ *and every* $p,q \in \mathbb{R}$;
(g5) there exists $m \in L^1_+([t_0,T])$ *such that*

$$\chi_{L^2}(g(t,D_1(\cdot),D_2(\cdot))) \le m(t)[\chi_{L^2}(D_1) + \chi_{L^2}(D_2)]\,,$$

for a.e. $t \in [t_0,T]$ *and every bounded* $D_1, D_2 \subset L^2([0,1])$.

Moreover, assume that for the multifunction $\Omega : L^2([0,1]) \multimap L^2([0,1])$ *the next properties hold:*

(Ω1) Ω *takes compact convex values;*
(Ω2) Ω *is upper semicontinuous;*
(Ω3) there exists $Q > 0$ *such that* $\chi_{L^2}(\Omega(D)) \le Q\chi_{L^2}(D)$, *for every bounded* $D \subset L^2([0,1])$;
(Ω4) there exists $R > 0$ *such that* $\|\Omega(v)\|_{L^2} \le R(1 + \|v\|_{L^2})$, *for every* $v \in L^2([0,1])$.

Then, the problem (46)-(48) is controllable, i.e., there exists a pair mild trajectory-control (u,ω), *where* $u,\omega : [t_0,T] \times [0,1] \to \mathbb{R}$ *with* $u(t,\cdot) \in L^2([0,1])$ *for every* $t \in [t_0,T]$, $u(\cdot,x) \in \mathcal{PC}([t_0,T],\mathbb{R})$ *for all* $x \in [0,1]$, *and* ω *measurable, such that*

$$\begin{aligned} u(t,x) &= e^{\int_{t_0}^t -b(\sigma,x)d\sigma}u_0(x) + \sum_{t_0<t_j<t} e^{\int_{t_j}^t -b(\sigma,x)d\sigma}\mathcal{I}_j(u(t_j,x)) + \\ &+ \int_{t_0}^t e^{\int_s^t -b(\sigma,x)d\sigma}\Big[g\Big(s,u(s,x),\int_{t_0}^s \tfrac{e^{-(s-\tau)/\tau}}{\tau}\,u(\tau,x)\,d\tau\Big) + \omega(s,x)\Big]\,ds, \end{aligned}$$

for every $t \in [t_0,T]$, $x \in [0,1]$, *where* $\omega(s,\cdot) \in \Omega(u(s,\cdot))$, *a.e.* $s \in [t_0,T]$.

We show now that the control system (46)-(48) admits optimal solutions. To this aim, we take the following positions:

- $E = E_1 = L^2([0,1])$;
- $y : [t_0,T] \to L^2([0,1])$,
$$y(t)(x) = u(t,x),\ t \in [t_0,T],\ x \in [0,1];$$
- $\eta : [t_0,T] \to L^2([0,1])$,
$$\eta(t)(x) = \omega(t,x),\ t \in [t_0,T],\ x \in [0,1];$$
- $H : [t_0,T] \times L^2([0,1]) \times L^2([0,1]) \multimap L^2([0,1])$,
$$H(t,v,w) = \Omega(v)\,,\ t \in [t_0,T],\ v,w \in L^2([0,1]); \tag{49}$$

- $f : [t_0, T] \times L^2([0,1]) \times L^2([0,1]) \times L^2([0,1]) \to L^2([0,1])$,

$$f(t, v, w, \eta)(x) = g(t, v(x), w(x)) + \eta(x), t \in [t_0, T],\ v, w, \eta \in L^2([0,1]),\ x \in [0,1]; \quad (50)$$

- $k : \Delta = \{(t,s) : t_0, \leq s \leq t \leq T\} \to \mathbb{R}^+$,

$$k(t,s) = \frac{e^{-(t-s)/\tau}}{\tau},\ (t,s) \in \Delta; \quad (51)$$

- $y_0 = u_0$;
- $I_j : L^2([0,1]) \to L^2([0,1]), j = 1\dots, m$,

$$I_j(v)(x) = \mathcal{I}_j(v(x)), v \in L^2([0,1]),\ x \in [0,1]. \quad (52)$$

In this way, the feedback control population dynamics system (46)-(48) assumes the form of a control system of type (41)-(43).

To prove our goal it is therefore sufficient to show that the maps f, H, and $I_1, \dots, I_m$ above defined satisfy the hypotheses of Theorem 6.1.

Proposition 7.3. *Suppose that g, Ω, and $\mathcal{I}_1, \dots, \mathcal{I}_m$ satisfy properties (g1)-(g4), (Ω1), (Ω2), (Ω4), and*

(g6) there exists $q \in L^1_+([t_0, T])$ such that, for a.e. $t \geq 0$

$$\|g(t, v_1(\cdot), w_1(\cdot)) - g(t, v_2(\cdot), w_2(\cdot))\|_{L^2} \leq q(t)(\|v_1 - v_2\|_{L^2} + \|w_1 - w_2\|_{L^2}),$$

for all $v_1, v_2, w_1, w_2 \in L^2([0,1])$ and $\eta \in L^2([0,1])$;
(g7) the map $t \mapsto \|g(t, 0_{L^2}, 0_{L^2})\|_{L^2}$ belongs to $L^1_+([t_0, T])$;
(Ω5) Ω is compact, i.e. maps bounded sets into relatively compact sets;
($\mathcal{I}$) the functions $\mathcal{I}_1, \dots, \mathcal{I}_m$ are bounded and continuous.

Then, the functions f, H, and $I_1, \dots, I_m$ defined respectively in (50), (49), and (52) satisfy (f1)-(f3), (H1)-(H7), and (I).

Proof. First of all, f is well-defined by (g1). Now, we show that it satisfies (f1)-(f3).

Property (f1) easily follows from (g2), while (f2) comes from (g3) and the Lebesgue dominated converge theorem (see (g4)); indeed, for any $t \in [t_0, T]$, $(v_0, w_0, \eta_0) \in (L^2([0,1]))^3$, and $(v_n, w_n, \eta_n)_n \to (v_0, w_0, \eta_0)$ in $(L^2([0,1]))^3$, we get

$$\begin{aligned} &\|f(t, v_n, w_n, \eta_n) - f(t, v_0, w_0, \eta_0)\|_{L^2} \\ &\quad \leq \left[\int_0^1 |g(t, v_n(x), w_n(x)) - g(t, v_0(x), w_0(x))|^2 dx\right]^{1/2} + \|\eta_n - \eta_0\|_{L^2} \to 0. \end{aligned}$$

About (f3), fixed $v_1, v_2, w_1, w_2 \in L^2([0,1])$ and $\eta \in L^2([0,1])$, for a.e. $t \in [t_0, T]$ by (g6) we have

$$\begin{aligned} \|f(t, v_1, w_1, \eta) - f(t, v_2, w_2, \eta)\|_{L^2} &= \|g(t, v_1(\cdot), w_1(\cdot)) - g(t, v_2(\cdot), w_2(\cdot))\|_{L^2} \\ &\leq q(t)(\|v_1 - v_2\|_{L^2} + \|w_1 - w_2\|_{L^2}). \end{aligned}$$

On the other hand, H satisfies (H1)-(H7), as we are going to prove.

First of all, condition (Ω1) and (Ω3) imply respectively (H1) and (H3). Moreover, (H2) is immediate since H with respect to t is constant.

Let us prove that (H4) holds. To this aim, let us consider any measurable multifunction $Q = (Q_1, Q_2) : [t_0, T] \multimap L^2([0,1]) \times L^2([0,1])$ with compact values and consider the multifunction $\mathcal{Q} : [t_0, T] \multimap L^2([0,1])$ defined by $\mathcal{Q}(t) = H(t, Q_1(t), Q_2(t))$. By the definition of H, we have

$$\mathcal{Q}(t) = \Omega(Q_1(t)).$$

 doi:10.20944/preprints202406.0710.v1

Since Ω is a upper semicontinuous multifunction with compact values (cf. (Ω2) and (Ω1)), then we can use [13, Proposition 1.3.1] and claim that it is superpositionally measurable, so $\mathcal{Q}$ is measurable.

Property (H5) is satisfied, since (cf. (44), (49), (50))

$$F(t,v,w) = f(t,v,w,H(t,v,w)) = f(t,v,w,\Omega(v)) = g(t,v(\cdot),w(\cdot)) + \Omega(v)$$

for every $t \in [t_0, T]$, $v, w \in L^2([0,1])$, and (Ω1) holds.

Now, let us check property (H6). For every $t \in [t_0, T]$, $v, w \in L^2([0,1])$ we have

$$\|F(t,v,w)\|_{L^2} \leq \|g(t,v(\cdot),w(\cdot))\|_{L^2} + \|\Omega(v)\|_{L^2}.$$

By using (Ω4) and (g6) we obtain

$$\begin{aligned}\|F(t,v,w)\|_{L^2} &\leq \|g(t,v(\cdot),w(\cdot))\|_{L^2} + R(1+\|v\|_{L^2})\\ &\leq q(t)(\|v\|_{L^2} + \|w\|_{L^2}) + \|g(t,0(\cdot),0(\cdot))\|_{L^2} + R(1+\|v\|_{L^2})\\ &\leq (q(t) + \|g(t,0(\cdot),0(\cdot))\|_{L^2} + R)(1+\|v\|_{L^2} + \|w\|_{L^2}).\end{aligned}$$

Recalling the hypothesis (g7), property (H6) is true taking $h(t) = q(t) + \|g(t,0(\cdot),0(\cdot))\|_{L^2} + R$.

To check that (H7) holds, it is enaugh to use (Ω5); in fact for every $t \in [t_0, T]$, $v, w \in L^2([0,1])$, and D_1, D_2 bounded subsets of $L^2([0,1])$ we achieve

$$\chi_{L^2}(f(t,v,w,H(t,D_1,D_2))) = \chi_{L^2}(g(t,v(\cdot),w(\cdot)) + \Omega(D_1)) \leq \chi_{L^2}(\Omega(D_1)) = 0,$$

thus the relative compactness of $f(t,v,w,H(t,D_1,D_2))$.

Finally, by ($\mathcal{I}$) and ($\mathcal{I}$2) we can apply the Lebesgue dominated convergence theorem, hence for every $j = 1,\ldots,m$ and every $v_0 \in L^2([0,1])$, $v_n \to v_0$ in $L^2([0,1])$, we get

$$\|\mathcal{I}_j(v_n) - \mathcal{I}_j(v_0)\|^2_{L^2} \to 0.$$

Therefore, the functions $I_1,\ldots,I_m$ satisfy (I). $\square$

Remark 7.1. *Notice that (g6) implies (g5) and (Ω5) implies (Ω3), so Proposition 7.2 still holds.*

Conclusion. In the end, by Propositions 7.1 and 7.3 and the continuity of k (cf. 51), we can apply Theorem 6.1. Theorefore, the feedback control population dynamics system (46)-(48) admits optimal solutions, that is a pair mild trajectory-control (u,ω),

$$\begin{aligned}u(t,x) &= e^{\int_{t_0}^t -b(\sigma,x)d\sigma} u_0(x) + \sum_{t_0<t_j<t} e^{\int_{t_j}^t -b(\sigma,x)d\sigma} \mathcal{I}_j(u(t_j,x)) +\\ &\quad + \int_{t_0}^t e^{\int_s^t -b(\sigma,x)d\sigma}\left[g\left(s,u(s,x),\int_{t_0}^s \frac{e^{-(s-\tau)/\tau}}{\tau} u(\tau,x)\,d\tau\right) + \omega(s,x)\right] ds,\\ &\qquad t \in [t_0,T],\ x \in [0,1],\\ \omega(s,\cdot) &\in \Omega(u(s,\cdot)),\ a.e.\ s \in [t_0,T].\end{aligned}$$

minimizing or maximizing a cost functional to the system, depending if it is lower or upper semicontinuous.

Acknowledgments: The author is member of the national group GNAMPA of INdAM (National Institute of Higher Mathematics, Italy) and of the group TAA of UMI (Italian Mathematical Union). This study was partly funded by: Research project of MUR (Ministry of University and Research, Italy) PRIN 2022 "Nonlinear differential problems with applications to real phenomena" - Grant Number: 2022ZXZTN2.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Araújo, R. O.; Marinho, S. S.; Prates Filho, J. S.; Uniform stability of a non-autonomous semilinear Bresse system with memory. Appl. Math. Comput. 387 (2020), 124418, 11 pp.
2. Baĭnov, D. ; Covachev, V.; Stamova, I.; Estimates of the solutions of impulsive quasilinear functional-differential equations, Ann. Fac. Sci. Toulouse Math. (5) 12 (2) (1991) 149–161.
3. Benchohra, M.; Henderson, J.; Ntouyas, S.; Impulsive differential equations and inclusions. Contemporary Mathematics and Its Applications, 2. Hindawi Publishing Corporation, New York, 2006.
4. Bungardi, S.; Cardinali, T.; Rubbioni, P.; Nonlocal semilinear integro-differential inclusions via vectorial measures of noncompactness. Appl. Anal. 96 (2017), no. 15, 2526-2544.
5. Cardinali, T.; Matucci, S.; Rubbioni, P.; Uniform asymptotic stability of a PDE's system arising from a flexible robotics model, submitted.
6. Cardinali, T.; O'Regan, D.; Rubbioni, P.; Mönch sets and fixed point theorems for multimaps in locally convex topological vector spaces. Fixed Point Theory 18 (2017), no. 1, 147-153.
7. Cardinali, T.; Rubbioni, P.; On the existence of mild solutions of semilinear evolution differential inclusions. J. Math. Anal. Appl. 308 (2005), no. 2, 620-635.
8. Cardinali, T.; Rubbioni, P.; Impulsive mild solutions for semilinear differential inclusions with nonlocal conditions in Banach spaces. Nonlinear Anal. 75 (2012), no. 2, 871–879.
9. Cardinali, T.; Rubbioni, P.; Hereditary evolution processes under impulsive effect, Mediterr. J. Math. 18 (2021), no. 3, Paper No. 91, 26 pp.
10. Chen, B.; Timoshin, S. A.; Optimal control of a population dynamics model with hysteresis. Acta Math. Sci. Ser. B (Engl. Ed.) 42 (2022), no. 1, 283–298.
11. Djebali, S.; Górniewicz, L.; Ouahab, A.; Existence and structure of solution sets for impulsive differential inclusions: a survey. Lecture Notes in Nonlinear Analysis, 13. Juliusz Schauder Center for Nonlinear Studies, Toruń, 2012. 148 pp.
12. Graef, J. R.; Henderson, J.; Ouahab, A,; Impulsive differential inclusions. A fixed point approach. De Gruyter Series in Nonlinear Analysis and Applications, 20. De Gruyter, Berlin, 2013.
13. Kamenskii, M.; Obukhovskii, V.; Zecca, P.; Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces, De Gruyter Ser. Nonlinear Anal. Appl. 7, Walter de Gruyter, Berlin - New York, 2001.
14. Kreĭn, S.G.; Linear differential equations in Banach space. Translated from the Russian by J. M. Danskin. Translations of Mathematical Monographs, Vol. 29. American Mathematical Society, Providence, R.I., 1971.
15. Lakshmikantham, V.; Rama Mohana Rao, M.; Theory of integro-differential equations. Stability and Control: Theory, Methods and Applications, 1. Gordon and Breach Science Publishers, Lausanne, 1995.
16. Leiva, H.; Garcia, K.; Lucena, E.; Existence of solutions for semilinear retarded equations with non-instantaneous impulses, non-local conditions, and infinite delay. Open Math. 21 (2023), no. 1, Paper No. 20230106, 17 pp.
17. Li, Y.; Zhao, H.; Cheng, Y.; Modeling mosquito control by an impulsive reaction-diffusion mosquito model with periodic evolution domain. Commun. Nonlinear Sci. Numer. Simul. 130 (2024), Paper No. 107738, 29 pp.
18. Ma, W.; Yang, B.; Zheng, Y.; Practical exponential stability of impulsive stochastic functional differential systems with distributed-delay dependent impulses. Nonlinear Anal. Hybrid Syst. 51 (2024), Paper No. 101424, 15 pp.
19. Metwali, M. M. A.; Cichoń, K.; On solutions of some delay Volterra integral problems on a half-line. Nonlinear Anal. Model. Control 26 (2021), no. 4, 661–677.
20. Pazy, A.; Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, 44. Springer-Verlag, New York, 1983.

 doi:10.20944/preprints202406.0710.v1

21. Perestyuk, N. A.; Plotnikov, V. A.; Samoilenko, A. M.; Skripnik, N. V.; Differential equations with impulse effects. Multivalued right-hand sides with discontinuities. De Gruyter Studies in Mathematics, 40. Walter de Gruyter & Co., Berlin, 2011.
22. Refaai, D. A.; El-Sheikh, M. M. A.; Ismail, G. A. F.; Bahaaeldin, A.; Thabet, A.; Hyers-Ulam stability of impulsive Volterra delay integro-differential equations. Adv. Difference Equ. 2021, Paper No. 477, 13 pp.
23. Rubbioni, P.; Asymptotic stability of solutions for some classes of impulsive differential equations with distributed delay. Nonlinear Anal. Real World Appl. 61 (2021), Paper No. 103324, 17 pp.
24. Rubbioni, P.; Solvability for a Class of Integro-Differential Inclusions Subject to Impulses on the Half-Line, Mathematics 2022, 10(2), 224.
25. Smith, H.; An introduction to delay differential equations with applications to the life sciences. Texts in Applied Mathematics, 57. Springer, New York, 2011.