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Article

Correspondence Analysis for Assessing Departures from Perfect Symmetry Using the Cressie-Read Family of Divergence Statistics

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Abstract: Recently, there appeared in this journal (Beh and Lombardo 2022, *Symmetry*, 14, 1103) a paper that showed how to perform a correspondence analysis on a two-way contingency table where Bowker's statistic lies at the numerical heart of this analysis. Thus, we showed how this statistic can be used to visually identify departures from perfect symmetry. Interestingly, Bowker's statistic is a special case of the symmetry-version of the Cressie-Read family of divergence statistics. Therefore, this paper presents a new framework for visually assessing departures from perfect symmetry using a second-order Taylor series approximation of the Cressie-Read family of divergence statistics.

Keywords: Bowker's chi-squared statistic; correspondence analysis; Cressie-Read family of divergence statistics; singular value decomposition

1. Introduction

The correspondence analysis of a symmetric $S \times S$ contingency table, \mathbf{N} has been a topic of research undertaken by, for example, Greenacre [22] and Beh and Lombardo [10]. Both approaches involve the partition of \mathbf{N} such that

$$\mathbf{N} = \mathbf{Y} + \mathbf{K} = \frac{1}{2}(\mathbf{N} + \mathbf{N}^T) - \frac{1}{2}(\mathbf{N} - \mathbf{N}^T) \quad (1)$$

where \mathbf{Y} is the matrix that reflects the symmetric part of the table and \mathbf{K} reflects the skew-symmetric part. This partition was considered in various context by many including, but certainly not limited to, Bove [14], Constantine and Gower [16, Section 3] and Gower [25].

The methods of Greenacre [22] and Beh and Lombardo [11] approach the visualisation of the departure from perfect symmetry using correspondence analysis by partitioning the transformed contingency table into a skew matrix and a skew-symmetric matrix, as (1) does. While both correspondence analysis approaches have (1) as a common thread, they are quite different. Greenacre [22] uses Pearson's chi-squared statistic, X^2 , that centres the elements of the contingency table with respect to the mean of the row and column marginal totals, yielding two low-dimensional displays; one depicting departures from perfect symmetry and the other depicting departures from skew-symmetry. On the other-hand, Beh and Lombardo [11] use Bowker's chi-squared statistic, X_B^2 [15], producing a single low-dimensional display depicting departures from perfect symmetry.

While the difference between X^2 and X_B^2 is that the former assesses departures from complete independence while the latter assesses departures from perfect symmetry, both can be expressed as a special case of the Cressie-Read family of divergence statistics (Cressie and Read [17]) when viewed as a goodness-of-fit measure. For more on how correspondence analysis can be performed on a two-way contingency table using the Cressie-Read family of divergence statistics refer to Beh and Lombardo [11]. Their method includes, as special cases, the classical approach to correspondence analysis [6,8,21,28],

log-ratio analysis (LRA) [23,24] and the Hellinger Distance Decomposition (HDD) method [18,19]. Rather than using Pearson's statistic as the numerical foundation, LRA and HDD use the modified log-likelihood ratio statistic [29] and the Freeman-Tukey statistic [20], respectively.

This paper presents a family of symmetry divergent statistics based on the two recent works of Beh and Lombardo [10,11] by demonstrating how a correspondence analysis can be performed for assessing departures from perfect symmetry of \mathbf{N} . To do so, this paper is divided into five further sections. Section 2 gives an overview of the classic test of the departure from perfect symmetry for a two-way contingency table. It also describes how the Cressie-Read family of divergence statistics can be used for performing such a test. This family is dependent on the power parameter δ where changes in δ lead to special cases of the family. In this paper we focus on the second order approximation of this family which yields *exactly* Pearson's chi-squared statistic ($\delta = 1$), the Freeman-Tukey statistic ($\delta = 1/2$) and the modified likelihood ratio statistic ($\delta = 0$). Section 3 describes the core interest of this paper; the development of a correspondence analysis framework that can be applied to a two-way contingency table to visualise sources of departure from perfect symmetry when using the Cressie-Read family of divergence statistics. As part of this discussion, we also show that when $\delta = 1$, the method described by Beh and Lombardo [10] is a special case of this new framework. Two examples are given that demonstrate the various features of this new framework. Section 4 studies a 4×4 artificial contingency table which exhibits perfect symmetry when a constant $C = 0$ is added to a cell frequency. As C increases, the artificial table exhibits features consistent with increasing departures from perfect symmetry and so this example examines the features of this correspondence analysis framework as C and δ change. Our second example (Section 5) examines the data of Wiepkema [33] that is concerned with 12 pre- and post-courtship behaviours of a small European fish called a bitterling (*Rhodeus amarus* Bloch). Some final comments on the framework outlined here are made in Section 6.

2. Test of Perfect Symmetry and the Cressie-Read Family of Divergence Statistics

2.1. Notation

Suppose we have an $S \times S$ contingency table, \mathbf{N} , where the (i, j) th cell entry has a frequency of n_{ij} for $i = 1, 2, \dots, S$ and $j = 1, 2, \dots, S$. Let the grand total of \mathbf{N} be n and let the matrix of relative frequencies be \mathbf{P} so that its (i, j) th cell entry is $p_{ij} = n_{ij}/n$ where $\sum_{i=1}^S \sum_{j=1}^S p_{ij} = 1$. Define the i th row marginal proportion by $p_{i\bullet} = \sum_{j=1}^S p_{ij}$. Similarly, define the j th column marginal proportion as $p_{\bullet j} = \sum_{i=1}^S p_{ij}$.

2.2. Testing Departures from a Hypothesised p_{ij}

Testing whether there is evidence of a statistically significant association between the row and column variables of \mathbf{N} can be made by considering any member of the Cressie-Read family of divergence statistics

$$\text{CR}(\delta) = \frac{2n}{\delta(\delta+1)} \sum_{i=1}^S \sum_{j=1}^S p_{ij} \left[\left(\frac{p_{ij}}{\hat{p}_{ij}} \right)^\delta - 1 \right], \quad (2)$$

for any $\delta \in (-\infty, \infty)$, where \hat{p}_{ij} is some value of p_{ij} under a well defined null hypothesis. When assessing departures from complete independence $\hat{p}_{ij} = p_{i\bullet}p_{\bullet j}$ for $i, j = 1, 2, \dots, S$ so that (2) is a chi-squared random variable with $(S-1)^2$ degrees of freedom. However, Cressie and Read [17] also presented a second order approximation of (2) around $(p_{ij}/\hat{p}_{ij})^\delta = 1$ that is very useful for the purposes of applying correspondence analysis to \mathbf{N} . This approximation is

$$\text{CR}(\delta) \approx \text{CR}^*(\delta) = n \sum_{i=1}^S \sum_{j=1}^S \hat{p}_{ij} \left\{ \frac{1}{\delta} \left[\left(\frac{p_{ij}}{\hat{p}_{ij}} \right)^\delta - 1 \right] \right\}^2. \quad (3)$$

2.3. Testing Departures from Complete Independence

Of course, any reasonable choice of \hat{p}_{ij} may be defined but we will confine ourselves to briefly discussing its definition under complete independence and perfect symmetry. For the case where one is interested in assessing departures from complete independence of the variables of \mathbf{N} , (2) is expressed as

$$CR_A(\delta) = \frac{2n}{\delta(\delta+1)} \sum_{i=1}^S \sum_{j=1}^S p_{ij} \left[\left(\frac{p_{ij}}{p_{i\bullet}p_{\bullet j}} \right)^\delta - 1 \right]. \quad (4)$$

The subscript “A” has been added to the left-hand side to show that this family of statistics assesses departures from complete association. The general nature of (2), and (4), ensures that specific values of δ lead to well defined and well understood measures of association, all of which are chi-squared random variables. These include Pearson’s chi-squared statistic, the log-likelihood ratio statistic, and the Freeman-Tukey statistic which are $X^2 = CR(\delta = 1)$, $G^2 = CR(\delta = 0)$ and $T^2 = CR(\delta = -1/2)$, respectively. The modified chi-squared statistic, the modified log-likelihood ratio statistic and the Cressie-Read statistic are also special cases such that $N^2 = CR(\delta = -2)$ and $M^2 = CR(\delta = -1)$, and $C^2 = CR(\delta = 2/3)$, respectively.

Beh and Lombardo [11] showed that a correspondence analysis of \mathbf{N} when assessing departures from independence can be undertaken by making use of the second order Taylor series approximation of (2) around

$$\left(\frac{p_{ij}}{\hat{p}_{ij}} \right)^\delta = \left(\frac{p_{ij}}{p_{i\bullet}p_{\bullet j}} \right)^\delta = 1$$

resulting in

$$CR_A(\delta) \approx CR_A^*(\delta) = n \sum_{i=1}^S \sum_{j=1}^S p_{i\bullet}p_{\bullet j} \left\{ \frac{1}{\delta} \left[\left(\frac{p_{ij}}{p_{i\bullet}p_{\bullet j}} \right)^\delta - 1 \right] \right\}^2.$$

This approximation may be obtained by substituting $\hat{p}_{ij} = p_{i\bullet}p_{\bullet j}$ into (3). See also Cressie and Read [17, pp. 94 – 95] for a derivation of this approximation of (4). This family of statistics gives *exactly* the following commonly used chi-squared statistics: Pearson’s statistic $X^2 = CR_A(1) = CR_A^*(1)$, the Freeman-Tukey statistic $T^2 = CR_A(-1/2) = CR_A^*(1/2)$ and the modified log-likelihood ratio statistic $M^2 = CR_A(-1) = CR_A^*(0)$. In the context of correspondence analysis, X^2 serves as numerical foundations of the traditional approach, T^2 is the foundations of the method described in Beh, Lombardo and Alberti [12] and Cuadras and Cuadras [18], while M^2 serves as the foundations of LRA, a variant of correspondence analysis described by Greenacre [23]. A fourth chi-squared statistic that is commonly used in the context of contingency table analysis, and was discussed from a correspondence analysis perspective by Beh and Lombardo [11] is when $\delta = 2/3$ yielding $CR_A^*(2/3)$, a second order approximation of the Cressie-Read statistic so that $CR_A(2/3) \approx CR_A^*(2/3)$.

2.4. Testing for Departures from Perfect Symmetry

Sometimes it is the case that the two variables of \mathbf{N} are, say, identical but measured over two different time periods. It may be that these same variables are collected between two different cohorts. In cases such as these, it is more typical to analyse the departures from perfect symmetry between the rows and columns of \mathbf{N} . Therefore, when testing for departures from perfect symmetry, the null hypothesis is

$$H_0 : p_{ij} = p_{ji} \quad (5)$$

for $i = 1, 2, \dots, I$ and for $j = 1, 2, \dots, J$.

Assessing whether there exists any evidence of symmetry between the variables of \mathbf{N} in the population, Agresti [1, p. 427] and Anderson [4, p. 321] showed that the most appropriate choice of \hat{p}_{ij} is

$$\hat{p}_{ij} = \frac{p_{ij} + p_{ji}}{2}. \quad (6)$$

Therefore, the Cressie-Read family of divergence statistics can be defined for testing departures from perfect symmetry so that (2) can be expressed as

$$CR_S(\delta) = \frac{2n}{\delta(\delta+1)} \sum_{i=1}^S \sum_{j=1}^S p_{ij} \left[\left(\frac{2p_{ij}}{p_{ij} + p_{ji}} \right)^\delta - 1 \right], \quad (7)$$

and is a chi-squared random variable with $S(S-1)/2$ degrees of freedom. The subscript “S” has been added to the left-hand side of (7) to show that this family of statistics assesses departures from perfect symmetry. This statistic has been the topic of interest by Tomizawa, Seo and Yamamoto [30], Ando, Hoshi, Ishii and 98izawa [5] and Altun and Saraçbaşı [2]. Our focus will be to examine the role of a second order approximation of $CR_S(\delta)$ for performing a correspondence analysis to visually detect departures from perfect symmetry. Therefore, it presents a more general framework to the correspondence analysis discussed by Greenacre [22] and Beh and Lombardo [11].

2.5. A Second Order Approximation

A second-order Taylor series approximation of (7) around

$$\left(\frac{p_{ij}}{\hat{p}_{ij}} \right)^\delta = \left(\frac{2p_{ij}}{p_{ij} + p_{ji}} \right)^\delta = 1$$

can be obtained by substituting (6) into (3). Doing so yields the family of asymptotically chi-squared random variables with $S(S-1)/2$ degrees of freedom under the null hypothesis of perfect symmetry, (6),

$$CR_S^*(\delta) = \frac{n}{2} \sum_{i=1}^S \sum_{j=1}^S (p_{ij} + p_{ji}) \left\{ \frac{1}{\delta} \left[\left(\frac{2p_{ij}}{p_{ij} + p_{ji}} \right)^\delta - 1 \right] \right\}^2 \quad (8)$$

or, alternatively but equivalently,

$$CR_S^*(\delta) = n \sum_{i>j} (p_{ij} + p_{ji}) \left\{ \frac{1}{\delta} \left[\left(\frac{2p_{ij}}{p_{ij} + p_{ji}} \right)^\delta - 1 \right] \right\}^2.$$

There are three special cases of this family of divergence statistics that we shall consider in our analysis of symmetry in a two-way contingency table. The first is when $\delta = 1$:

$$X_S^2 = CR_S^*(1) = \frac{n}{2} \sum_{i=1}^S \sum_{j=1}^S \frac{(p_{ij} - p_{ji})^2}{p_{ij} + p_{ji}} = n \sum_{i>j} \frac{(p_{ij} - p_{ji})^2}{p_{ij} + p_{ji}} \quad (9)$$

which is just Bowker’s chi-squared statistic [15]. Beh and Lombardo [10] used this statistic as the basis for performing correspondence analysis to assess departures from perfect symmetry in \mathbf{N} .

Secondly, suppose that we consider the case where (8) is evaluated when $\delta = 1/2$. Then, we can show that

$$T_S^2 = CR_S^*\left(\frac{1}{2}\right) = 4n \sum_{i=1}^S \sum_{j=1}^S \left(\sqrt{p_{ij}} - \sqrt{\frac{p_{ij} + p_{ji}}{2}} \right)^2 \quad (10)$$

is the Freeman-Tukey statistic when assessing departures from perfect symmetry.

The third special case of (8) is when $\delta = 0$. For this value of δ , (8) does not exist. However, we can obtain the limiting value of (8) as $\delta \rightarrow 0$. Doing so means that we can use the Box-Cox transformation so that

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \left[\left(\frac{2p_{ij}}{p_{ij} + p_{ji}} \right)^\delta - 1 \right] = \ln \left(\frac{2p_{ij}}{p_{ij} + p_{ji}} \right).$$

Therefore,

$$\begin{aligned} \text{CR}_S^*(0) &= \lim_{\delta \rightarrow 0} \text{CR}_S^*(\delta) \\ &= n \sum_{i=1}^S \sum_{j=1}^S \frac{(p_{ij} + p_{ji})}{2} \lim_{\delta \rightarrow 0} \left\{ \frac{1}{\delta} \left[\left(\frac{2p_{ij}}{p_{ij} + p_{ji}} \right)^\delta - 1 \right] \right\}^2 \end{aligned}$$

simplifies to

$$M_S^2 = \text{CR}_S^*(0) = n \sum_{i=1}^S \sum_{j=1}^S \frac{(p_{ij} + p_{ji})}{2} \ln \left(\frac{2p_{ij}}{p_{ij} + p_{ji}} \right) \quad (11)$$

and is the modified version of the log-likelihood ratio statistic when testing for perfect symmetry in \mathbf{N} . Note that Bishop, Fienberg and Holland [13, eq. (8.2-11)], Haberman [26, p. 489] and Ireland, Ku and Kullback [27, eq. (1.3)] gave the (unmodified) log-likelihood ratio statistic:

$$\tilde{M}_S^2 = n \sum_{i=1}^S \sum_{j=1}^S p_{ij} \ln \left(\frac{2p_{ij}}{p_{ij} + p_{ji}} \right)$$

for assessing departures from perfect symmetry in a $S \times S$ contingency table.

When there is perfect symmetry between the variables of \mathbf{N} so that (5) holds, (9), (10) and (11) will be zero. When there exists a statistically significant departure from perfect symmetry, we can visually assess the statistical significance of this departure then using correspondence analysis. We shall now show how (8) can be used to perform a correspondence analysis on \mathbf{N} when assessing these departures.

3. Correspondence Analysis & Perfect Symmetry

3.1. The Divergence Residual

To perform a correspondence analysis on \mathbf{N} under the null hypothesis of perfect symmetry we first define the $S \times S$ matrix of *divergence residuals*, \mathbf{S}_δ , where its (i, j) th element is

$$s_{ij}(\delta) = \frac{1}{\delta} \sqrt{\frac{p_{ij} + p_{ji}}{2}} \left[\left(\frac{2p_{ij}}{p_{ij} + p_{ji}} \right)^\delta - 1 \right]. \quad (12)$$

Therefore, the sum-of-squares of these residuals gives (8) so that

$$\begin{aligned} \text{CR}_S^*(\delta) &= n \sum_{i=1}^S \sum_{j=1}^S s_{ij}^2(\delta) \\ &= n \text{trace}(\mathbf{S}_\delta^T \mathbf{S}_\delta) \end{aligned} \quad (13)$$

$$= n \text{trace}(\mathbf{S}_\delta \mathbf{S}_\delta^T). \quad (14)$$

Note that when $i = j$ (so that we are concerned with the diagonal elements of \mathbf{S}_δ) these residuals are zero for all δ . Three examples of the form that (12) takes is when $\delta = 1$, $1/2$ and (approaching) 0. Respectively, these values of δ give the residuals

$$\begin{aligned}
s_{ij}(1) &= \frac{1}{\sqrt{2}} \frac{p_{ij} - p_{ji}}{\sqrt{p_{ij} + p_{ji}}} \\
s_{ij}\left(\frac{1}{2}\right) &= \sqrt{2} \left(\sqrt{2p_{ij}} - \sqrt{p_{ij} + p_{ji}} \right) = 2 \left(\sqrt{p_{ij}} - \sqrt{\frac{p_{ij} + p_{ji}}{2}} \right) \\
s_{ij}(0) &= \sqrt{\frac{p_{ij} + p_{ji}}{2}} \ln \left(\frac{2p_{ij}}{p_{ij} + p_{ji}} \right).
\end{aligned}$$

The first of these, $s_{ij}(1)$, is the (i, j) th *Bowker residual* described by Beh and Lombardo [10, eq. (7)] so that n times its sum-of-squares produces Bowker's statistic, (9).

The second and third residuals are akin to the *Freeman-Tukey residual* and *modified log-likelihood ratio residual*, respectively, described by Beh and Lombardo [11], but are used when assessing departures from perfect symmetry. Note that n times the sum of squares of these two residuals gives (10) and (11), respectively.

For $s_{ij}(0)$, it is assumed that all cells of the contingency table have non-zero frequencies so that $0 < p_{ij} < 1$ for $i = 1, 2, \dots, S$ and $j = 1, 2, \dots, S$. This is to avoid any problems with calculating the natural logarithm of zero. In the event that a zero cell frequency is observed, a simple remedy is to replace it with a small value, say 0.01. Alternatively, one may use more objective methods to accommodate for a zero cell frequency. Other residuals can also be obtained using alternative values of δ .

3.2. Is the Matrix of Divergence Residuals Skew-Symmetric?

One of the benefits of using Bowker's statistic as the numerical basis on which to perform correspondence analysis is that the resulting matrix of divergence residuals is skew-symmetric. That is, when $\delta = 1$, \mathbf{S}_1 has the property that $\mathbf{S}_1^T = -\mathbf{S}_1$. Therefore, $s_{ii}(1) = 0$ and $s_{ij}(1) = -s_{ji}(1)$ for $i \neq j, \forall i, j$. It also means that the singular values, and the left and right singular vectors, of \mathbf{S}_1 can be calculated by applying an eigen-decomposition to $\mathbf{S}_1^T \mathbf{S}_1$ or, equivalently, $-\mathbf{S}_1^2$. Ward and Gray [32] and Gower [25, p. 113] discuss that for a $S \times S$ skew-symmetric matrix, like \mathbf{S}_1 , that if S is odd then there will always be a zero eigen-value and $S - 1$ positive eigen-values. If S is even there will always be S eigen-values that exist in pairs [16].

When $\delta \neq 1$, \mathbf{S}_δ is not a skew-symmetric matrix, since there will be at least one cell where $s_{ij}(\delta) \neq -s_{ji}(\delta), \forall i \neq j$, unless there is perfect symmetry between the variables of \mathbf{N} .

3.3. Singular Value Decomposition and the Divergence Residual

When assessing departures from perfect symmetry in \mathbf{N} , the correspondence analysis approach of Beh and Lombardo [10] involves applying a singular value decomposition (SVD) to the matrix \mathbf{S}_1 . Since Bowker's statistic is a special case of (8), this suggests that a more general family of correspondence analysis techniques can be developed for visualising departures from perfect symmetry. Such a general family can be developed using the family of statistics generated from (8). Therefore, a new general family of correspondence analysis techniques can be obtained by applying a SVD to \mathbf{S}_δ such that, for the (i, j) cell,

$$\frac{1}{\delta} \sqrt{\frac{p_{ij} + p_{ji}}{2}} \left[\left(\frac{2p_{ij}}{p_{ij} + p_{ji}} \right)^\delta - 1 \right] = \sum_{m=1}^M a_{im}(\delta) \lambda_m(\delta) b_{jm}(\delta) \quad (15)$$

where

$$\sum_{i=1}^I a_{im}(\delta) a_{im'}(\delta) = \begin{cases} 1 & m = m' \\ 0 & m \neq m' \end{cases}, \quad \sum_{j=1}^J b_{jm}(\delta) b_{jm'}(\delta) = \begin{cases} 1 & m = m' \\ 0 & m \neq m' \end{cases}, \quad (16)$$

and M is the maximum number of dimensions required to depict all of the association that exists between the variables of the contingency table. When $\delta = 1$ then $M = S$ if S is even and $M = S - 1$ if S is odd. For other values of δ , $M = S$. The quantities $a_{im}(\delta)$ and $b_{jm}(\delta)$ are the i th and j th element,

respectively, of the m th left and right singular vectors of the matrix of divergence residuals for a fixed δ . The m th largest singular value is $\lambda_m(\delta)$ so that $1 > \lambda_1(\delta) > \lambda_2(\delta) > \dots, \lambda_{M^*}(\delta) > 0$.

The matrix form of (15) and (16) is

$$\mathbf{S}_\delta = \mathbf{A}_\delta \Delta_\delta \mathbf{B}^T \quad (17)$$

with

$$\mathbf{A}_\delta^T \mathbf{A}_\delta = \mathbf{I}_M \quad \text{and} \quad \mathbf{B}_\delta^T \mathbf{B}_\delta = \mathbf{I}_M$$

being the matrix form of (16). Here, \mathbf{I}_M is a $M \times M$ identity matrix, \mathbf{A}_δ is the $S \times M$ matrix where the (i, m) th element is $a_{im}(\delta)$, \mathbf{B}_δ is the $S \times M$ matrix where the (j, m) th element is $b_{jm}(\delta)$, and Δ_δ is the $M \times M$ diagonal matrix of singular values with λ_m as its (m, m) th element.

While the Cressie-Read family of divergence statistics can be expressed in terms of \mathbf{S}_δ – see (13) and (14) – it can also be expressed in terms of its singular values. To show this, substituting (17) into (13) leads to

$$\begin{aligned} \text{CR}^*(\delta) &= n \text{trace} \left(\left(\mathbf{A}_\delta \Delta_\delta \mathbf{B}_\delta^T \right)^T \left(\mathbf{A}_\delta \Delta_\delta \mathbf{B}_\delta^T \right) \right) \\ &= n \text{trace} \left(\mathbf{B}_\delta \Delta_\delta^2 \mathbf{B}_\delta^T \right) \\ &= n \text{trace} \left(\Delta_\delta^2 \right) \end{aligned}$$

when \mathbf{B}_δ is of full rank so that $\mathbf{B}_\delta \mathbf{B}_\delta^T = \mathbf{B}_\delta^T \mathbf{B}_\delta = \mathbf{I}_M$. Therefore, the total inertia of \mathbf{N} can be expressed as the sum-of-squares of the squared singular values so that

$$\frac{\text{CR}^*(\delta)}{n} = \sum_{m=1}^M \lambda_m^2(\delta). \quad (18)$$

Expressing the total inertia in this manner is analogous to the total inertia of Beh and Lombardo [11, p. 22] when the Cressie-Read family of divergence statistics is used as the numerical basis of the correspondence analysis of a two-way contingency table.

3.4. The Principal Inertia Values

Beh and Lombardo [10] showed that when assessing departures from perfect symmetry when \mathbf{N} is a 2×2 contingency table, \mathbf{S}_1 , has two equal singular-values whose squared values are

$$\lambda_1^2 = \lambda_2^2 = s_{21}^2(1) = \frac{1}{\sqrt{2}} \frac{p_{21} - p_{12}}{\sqrt{p_{21} + p_{12}}} \quad (19)$$

and are the principal inertia values of the first two dimensions of the correspondence plot when analysing a two-way contingency table. Similarly, when symmetry is of concern for 3×3 contingency table, the three principal inertia values are

$$\lambda_1^2 = \lambda_2^2 = \frac{\text{CR}^*(1)}{2n} \quad \text{and} \quad \lambda_3^2 = 0. \quad (20)$$

For both sized \mathbf{N} , the sum of their squared singular values gives Bowker's statistic.

When analysing the symmetry of a two-way table using the Cressie-Read family of divergence statistics we can consider values of $\delta \neq 1$. For example, when \mathbf{S}_δ is of rank 2 then amending Appendix A of [10] for $\delta \neq 1$ shows that there will be two unequal singular values whose squares are

$$\lambda_1^2(\delta) = \max(s_{12}^2(\delta), s_{21}^2(\delta)) \quad (21)$$

$$\lambda_2^2(\delta) = \min(s_{12}^2(\delta), s_{21}^2(\delta)). \quad (22)$$

Note that when $\delta = 1$ these squared singular values simplify to (19).

When $\delta \neq 1$, then (19) and (20) are satisfied only when there exists perfect symmetry between the variables of \mathbf{N} (in which case all squared singular values will be zero). This is because when $\delta \neq 1$, \mathbf{S}_δ is not a skew-symmetric matrix.

We now turn our attention to the construction of the M -dimensional correspondence plot by defining and describing the principal coordinates for each row and column of \mathbf{N} .

3.5. Principal Coordinates

When visually portraying the categories of \mathbf{N} , define the metric matrix by

$$\tilde{\mathbf{D}} = \frac{\mathbf{D}_I + \mathbf{D}_J}{2}.$$

Then the matrix of row and column principal coordinates is

$$\mathbf{F}_\delta = \tilde{\mathbf{D}}^{-1/2} \mathbf{A}_\delta \mathbf{\Delta}_\delta \quad (23)$$

$$\mathbf{G}_\delta = \tilde{\mathbf{D}}^{-1/2} \mathbf{B}_\delta \mathbf{\Delta}_\delta, \quad (24)$$

respectively. These provide a more general set of principal coordinates than those of Beh and Lombardo [10, eqs. (14) & (15)] who were concerned only with the case when $\delta = 1$. Although, the principal coordinates of Beh and Lombardo [10, eqs. (14) & (15)] can be obtained by simply substituting $\delta = 1$ into (23) and (24).

Defining the row and column principal coordinates by (23) and (24), respectively, means that the row and column spaces have the same metric that is based on the aggregation of p_{ij} across the two variables, irrespective of the value of δ . Such an aggregation is done since (7) relies only on the cell proportions p_{ij} and p_{ji} .

Post-multiplying both sides of (23) by \mathbf{B}_δ^T and simplifying gives us an alternative expression for the row principal coordinates

$$\mathbf{F}_\delta = \tilde{\mathbf{D}}^{-1/2} \mathbf{S}_\delta \mathbf{B}_\delta.$$

Similarly, it can be shown that the column principal coordinates can be expressed in terms of \mathbf{S}_δ such that

$$\mathbf{G}_\delta = \tilde{\mathbf{D}}^{-1/2} \mathbf{S}_\delta^T \mathbf{A}_\delta.$$

As we have already shown, \mathbf{S}_δ is not a skew-symmetric matrix unless $\delta = 1$. In the event that $\delta = 1$ then, as shown by Beh and Lombardo [10],

$$\mathbf{G}_1 = \tilde{\mathbf{D}}^{-1/2} \mathbf{A}_1 \mathbf{\Delta}_1 \mathbf{J}_M^T = \mathbf{F}_1 \mathbf{J}_M^T$$

where \mathbf{J}_M is an $M \times M$ block-diagonal and orthogonal skew-symmetric matrix so that

$$\mathbf{J}_M^T \mathbf{J}_M = \mathbf{J}_M \mathbf{J}_M^T = \mathbf{M}.$$

See Beh and Lombardo [10, Section 5.1] for examples of \mathbf{J}_M when $M = 2, 3$ and 4 .

3.6. On the Total Inertia and the Origin

The total inertia of the two-way contingency table can be expressed in terms of the matrices of row and column principal coordinates given by (23) and (24). To show this, suppose we consider the total inertia in terms of the row principal coordinates. Then,

$$\begin{aligned}
\text{trace}(\mathbf{F}_\delta^T \tilde{\mathbf{D}} \mathbf{F}_\delta) &= \text{trace}\left(\left(\tilde{\mathbf{D}}^{-1/2} \mathbf{A}_\delta \Delta_\delta\right)^T \tilde{\mathbf{D}} \left(\tilde{\mathbf{D}}^{-1/2} \mathbf{A}_\delta \Delta_\delta\right)\right) \\
&= \text{trace}\left(\Delta_\delta \mathbf{A}_\delta^T \mathbf{A}_\delta \Delta_\delta\right) \\
&= \text{trace}\left(\Delta_\delta^2\right) \\
&= \frac{\text{CR}^*(\delta)}{n}.
\end{aligned}$$

Similarly, we can also show that the total inertia can be expressed in terms of the column principal coordinates so that

$$\frac{\text{CR}^*(\delta)}{n} = \text{trace}\left(\mathbf{G}_\delta^T \tilde{\mathbf{D}} \mathbf{G}_\delta\right).$$

Therefore, if there is perfect symmetry between the rows and columns of our two-way contingency table then the total inertia will be zero. When this happens, the position of the row and column principal coordinates will be located at the origin. Therefore, the origin is interpreted as the point in the low-dimensional space where there is perfect symmetry between the row and column variables. The further a point is away from this origin then the more deviation it has from the null hypothesis of perfect symmetry. When assessing departures from complete independence assessing the contribution of a row and column point to the association structure can be undertaken using the closed-form equations that yield confidence regions for each point; see Beh [7] and Beh and Lombardo [9] when $\delta = 1$ and Alzahrani, Beh and Stojanovski [3] for other values of δ . Such regions have not yet been developed for studying departures from perfect symmetry and so we shall leave this for future study.

4. Example 1: Artificial Data

4.1. The Data

To examine how the Cressie-Read family of divergence statistics can be used for the purposes of applying correspondence analysis to visually assess departures from perfect symmetry we consider the artificial data set given in Table 1. Beh and Lombardo [10] used this contingency table to highlight the features obtained when using Bowker's statistic and so we shall focus on showing the features of the correspondence analysis using (8) for $\delta = 1, 1/2$ and 0. Thus the numerical foundations of this variant of correspondence analysis uses the modified log-likelihood ratio statistic, $M_{\mathcal{S}}^2$, the Freeman-Tukey statistic $T_{\mathcal{S}}^2$ and Pearson's chi-squared statistic $X_{\mathcal{S}}^2$. In Table 1 the (2, 1)th cell frequency is $20 + C$ where $C \geq -20$ is a constant; note that Beh and Lombardo [10] considered the case where $C \geq 0$ for their analysis of Table 1. When $C = 0$ then the variables of Table 1 exhibit perfect symmetry and as $C \rightarrow \infty$ the departure from perfect symmetry between the variables becomes more apparent. The sample size of Table 1 is $n = 680 + C$.

Table 1. A near-symmetric artificial contingency table where C is a non-negative integer.

Rows	Columns				Total
	C1	C2	C3	C4	
R1	10	20	30	40	100
R2	$20 + C$	50	60	70	$200 + C$
R3	30	60	20	40	150
R4	40	70	40	80	230
Total	$100 + C$	200	150	230	$680 + C$

4.2. The Family of Divergence Statistics

Since there exists perfect symmetry in all but two cells of Table 1 we only need to confine ourselves to examining the difference between the (2, 1)th and (1, 2)th cells. Of course, when there is perfect symmetry then $s_{12}(\delta) = s_{21}(\delta)$ and this happens only when $C = 0$. We shall examine the changes

in these two values as C and δ change. Therefore, to assess the departure from perfect symmetry in Table 1 we shall do so by comparing

$$\sqrt{n} \cdot s_{12}(\delta) = \frac{1}{\delta} \sqrt{\frac{40+C}{2}} \left[\left(\frac{40}{40+C} \right)^{\delta} - 1 \right] \quad (25)$$

and

$$\sqrt{n} \cdot s_{21}(\delta) = \frac{1}{\delta} \sqrt{\frac{40+C}{2}} \left[\left(\frac{40+2C}{40+C} \right)^{\delta} - 1 \right] \quad (26)$$

for $\delta \neq 0$, otherwise

$$\sqrt{n} \cdot s_{12}(0) = \sqrt{\frac{40+C}{2}} \ln \left(\frac{40}{40+C} \right)$$

and

$$\sqrt{n} \cdot s_{21}(0) = \sqrt{\frac{40+C}{2}} \ln \left(\frac{40+2C}{40+C} \right)$$

using the Box-Cox transformation.

Therefore, when assessing departures from perfect symmetry for the data in Table 1 the Cressie-Read family of divergence statistics can be expressed in terms of $C \geq -20$ and $\delta \in (-\infty, \infty)$ so that

$$\begin{aligned} \text{CR}^*(\delta) &= n \left(s_{12}^2(\delta) + s_{21}^2(\delta) \right) \\ &= \frac{1}{\delta^2} \left(\frac{40+C}{2} \right) \left[\left(\left(\frac{40}{40+C} \right)^{\delta} - 1 \right)^2 + \left(\left(\frac{40+2C}{40+C} \right)^{\delta} - 1 \right)^2 \right]. \end{aligned} \quad (27)$$

We can immediately see that this family of statistics can also be derived by substituting (21) and (22) into (18) (since $M = 2$) yielding the equivalent expression

$$\text{CR}^*(\delta) = n \left(\lambda_1^2(\delta) + \lambda_2^2(\delta) \right).$$

Substituting $\delta = 1$ into (27) for Table 1 of

$$X_S^2 = \text{CR}^*(1) = \frac{C^2}{40+C}$$

which is Bowker's statistic derived by Beh and Lombardo [10, eq. (21)]. Similarly, the Freeman-Tukey statistic, (10), and modified log-likelihood ratio statistic, (11), can be written in terms of C so that

$$\begin{aligned} T_S^2 &= \text{CR}^*\left(\frac{1}{2}\right) = 8 \left[40+C - \frac{1}{2} \sqrt{40+C} \left(\sqrt{40} + \sqrt{40+2C} \right) \right] \\ M_S^2 &= \text{CR}^*(0) = (40+C) \left| \ln \left(\frac{40(40+2C)}{(40+C)^2} \right) \right|. \end{aligned}$$

A visual representation of X_S^2 , T_S^2 , and M_S^2 versus $C \in [-20, 100]$ at unitary increments is given in Figure 1; the horizontal line is the quantile of the chi-squared distribution with 6 degrees of freedom for $\alpha = 0.05$ so that $\chi_{0.95}^2(6) = 12.5916$. Figure 1 shows that all three statistics are quite similar, especially for values of $C \in [-15, 50]$. Note that when $C = 0$ these three statistics are all zero showing there is perfect symmetry in Table 1. When $C \in [-20, 0)$ all three statistics decrease to zero and then increase for $C > 0$. Therefore, there is a minimum value of C that will lead to the rejection of the null hypothesis of perfect symmetry. We now investigate what this value of C is for X_S^2 , T_S^2 and M_S^2 .

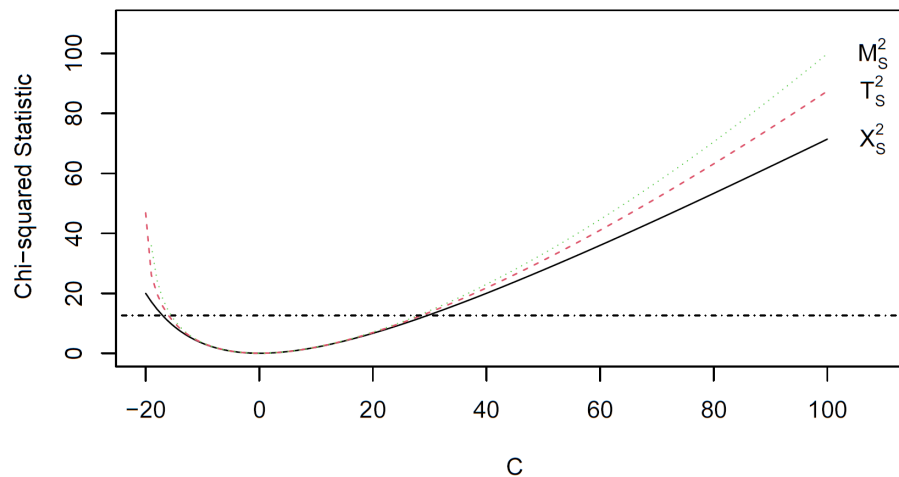


Figure 1. X_S^2 , T_S^2 and M_S^2 versus $C \in [-20, 100]$ at unitary increments for Table 1.

4.3. On the Departure from Perfect Symmetry

Beh and Lombardo [10] show that for Bowker's statistic, $CR^*(1)$, there is a statistically significant departure from perfect symmetry at the α level of significance, when

$$C > \frac{\chi_\alpha^2(6) + \sqrt{(\chi_\alpha^2(6))^2 + 160\chi_\alpha^2(6)}}{2}, \quad (28)$$

where $\chi_\alpha^2(6)$ is $1 - \alpha$ quantile of the chi-squared distribution with $S(S - 1)/2 = 4(4 - 1)/2 = 6$ degrees of freedom. For example, the minimum value of C when $\alpha = 0.05$ is 29.59, or 30 when rounded up to an integer value. Hence, the (2, 1)th cell frequency must be at least 50 to detect any departure from perfect symmetry when performing the test at the 0.05 level of significance using Bowker's statistic.

There is also a second solution to C that leads to a rejection of the null hypothesis of perfect symmetry. This is when

$$-20 \leq C < \frac{\chi_\alpha^2(6) - \sqrt{(\chi_\alpha^2(6))^2 + 160\chi_\alpha^2(6)}}{2} \quad (29)$$

yielding an upper bound of this interval of $C = -17.01$. Thus, there is a rejection of the null hypothesis of perfect symmetry when the (2, 1)th cell frequency is less than 2.99, or 2 when rounding down to an integer value.

Figure 1 shows fairly similar values of C (and n_{11}) are required when assessing the test of perfect symmetry using T_S^2 and M_S^2 . Although obtaining a simple expression to determine these cell counts, like (28) and (29) do for Bowker's statistic, is not straightforward. However, numerical methods show that the values of C that produce a statistically significant T_S^2 are when $C < -15.99$ and $C > 28.50$, for $\alpha = 0.05$ and for 6 degrees of freedom. So, when using T_S^2 , the values of the (2, 1)th cell frequency that ensure that the null hypothesis of perfect symmetry is rejected at the 5% level of significance is $n_{11} \geq 49$ and $0 \leq n_{11} \leq 4$.

Similarly, numerical methods show that when using M_S^2 the value of C that rejects the null hypothesis of perfect symmetry is $C < -15.52$ and $C > 27.94$. Therefore, the values of the (2, 1)th cell frequency that ensure that the null hypothesis of perfect symmetry is rejected at the 5% level of significance is $n_{11} \geq 48$ and $0 \leq n_{11} \leq 4$.

To keep any further analysis of Table 1 simple we shall now confine our attention to values of $C \geq 0$ like Beh and Lombardo [10] did for their analysis of the table.

4.4. Features of Correspondence Analysis & Symmetry

4.4.1. The Matrix of Divergence Residuals

We can derive the matrix of divergence residuals, \mathbf{S}_δ for Table 1. Based on (25) and (26) this matrix is

$$\mathbf{S}_\delta = \frac{1}{\delta} \sqrt{\frac{40+C}{2(680+C)}} \begin{pmatrix} 0 & \left(\frac{40}{40+C}\right)^\delta - 1 & 0 & 0 \\ \left(\frac{40+2C}{40+C}\right)^\delta - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (30)$$

when $\delta \neq 0$. For example, when $\delta = 1$ then

$$\mathbf{S}_1 = \frac{C}{\sqrt{2(680+C)(40+C)}} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

which is a skew-symmetric matrix since $s_{ij}(1) \neq -s_{ji}(1)$, $i, j = 1, \dots, 4$. When $\delta = 0$ and $\delta = 1/2$, then the matrix of divergence residuals is

$$\mathbf{S}_0 = \sqrt{\frac{40+C}{2(680+C)}} \begin{pmatrix} 0 & \ln\left(\frac{40}{40+C}\right) & 0 & 0 \\ \ln\left(\frac{40+2C}{40+C}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathbf{S}_{1/2} = \sqrt{\frac{2}{680+C}} \begin{pmatrix} 0 & \sqrt{40} - \sqrt{40+C} & 0 & 0 \\ \sqrt{40+2C} - \sqrt{40+C} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

respectively. These two matrices are not skew-symmetric matrix since $s_{12}(\delta) \neq -s_{21}(\delta)$ for $\delta = 0$ and $1/2$ unless $C = 0$. In this case there is perfect symmetry in Table 1 so that $s_{12}(\delta) = s_{21}(\delta) = 0$.

4.4.2. The Singular Values

The structure of the 4×4 matrix \mathbf{S}_δ given by (30) is identical to the 2×2 matrix obtained by removing the zero rows and columns of the matrix. Beh and Lombardo [10, Appendix A] derived the two singular values of \mathbf{S}_1 for Table 1 and showed them to be

$$\lambda_1(1) = \lambda_2(1) = \frac{C}{\sqrt{2(680+C)(40+C)}} \quad (31)$$

for $C > 0$ and are both zero when $C = 0$. When $\delta \neq 1$ the two singular values are not equivalent since, for these δ values, \mathbf{S}_δ is not a skew-symmetric matrix. Although, the two singular values will be approximately equivalent when $\delta \approx 1$. Adjusting the derivation of Beh and Lombardo [10, Appendix A] for $\delta \neq 0$ and $C \geq 0$, the two singular values of Table 1 are

$$\lambda_1(\delta) = \frac{|s_{12}(\delta)|}{\sqrt{n}} = \frac{1}{\delta} \sqrt{\frac{40+C}{2(680+C)}} \left| \left(\frac{40}{40+C}\right)^\delta - 1 \right| \quad (32)$$

$$\lambda_2(\delta) = \frac{|s_{21}(\delta)|}{\sqrt{n}} = \frac{1}{\delta} \sqrt{\frac{40+C}{2(680+C)}} \left| \left(\frac{40+2C}{40+C}\right)^\delta - 1 \right| \quad (33)$$

so that $\lambda_1(\delta) \geq \lambda_2(\delta)$ when $C \geq 0$. Therefore, while the 4×4 diagonal matrix of eigen-values consists of zero values for the (3, 3)th and (4, 4)th elements, the 2×2 matrix of non-zero eigen-values is

$$\Delta_\delta = \frac{(40 + C)^{1/2-\delta}}{\delta\sqrt{2(680 + C)}} \begin{pmatrix} (40 + C)^\delta - 40^\delta & 0 \\ 0 & (40 + 2C)^\delta - (40 + C)^\delta \end{pmatrix} \quad (34)$$

so that both singular values remain positive when $C > 0$. For example, when $\delta = 1$ these two singular values simplify to (31). Similarly, when $\delta = 1/2$,

$$\begin{aligned} \lambda_1\left(\frac{1}{2}\right) &= \frac{|s_{21}(1/2)|}{\sqrt{n}} = \sqrt{\frac{2}{680 + C}} \left(\sqrt{40 + C} - \sqrt{40} \right) \\ \lambda_2\left(\frac{1}{2}\right) &= \frac{|s_{21}(1/2)|}{\sqrt{n}} = \sqrt{\frac{2}{680 + C}} \left(\sqrt{40 + 2C} - \sqrt{40 + C} \right). \end{aligned}$$

When $\delta = 0$, then applying the Box-Cox transformation to (32) and (33) yields, for $C \geq 0$, the two singular values

$$\begin{aligned} \lambda_1(0) &= \frac{s_{12}(0)}{\sqrt{n}} = \sqrt{\frac{40 + C}{2(680 + C)}} \left| \ln\left(\frac{40}{40 + C}\right) \right| \\ \lambda_2(0) &= \frac{s_{21}(0)}{\sqrt{n}} = \sqrt{\frac{40 + C}{2(680 + C)}} \left| \ln\left(\frac{40 + 2C}{40 + C}\right) \right|. \end{aligned}$$

Figure 2 displays $\lambda_1(\delta)$ versus $C \in [0, 100]$ for $\delta = 0, 1/2$ and 1, while Figure 3 shows $\lambda_2(\delta)$ versus $C \in [0, 100]$; the vertical axis of both figures are identically scaled to enable an easy comparison of the two singular values. These two figures show that, for all values of C , $\lambda_1(1) = \lambda_2(1)$ as expected since \mathbf{S}_1 is a skew-symmetric matrix. It also shows that $\lambda_1(\delta) = \lambda_2(\delta) = 0$ when $C = 0$. A comparison of Figures 2 and 3 shows that $\lambda_1(\delta) > \lambda_2(\delta)$ for $C > 0$.

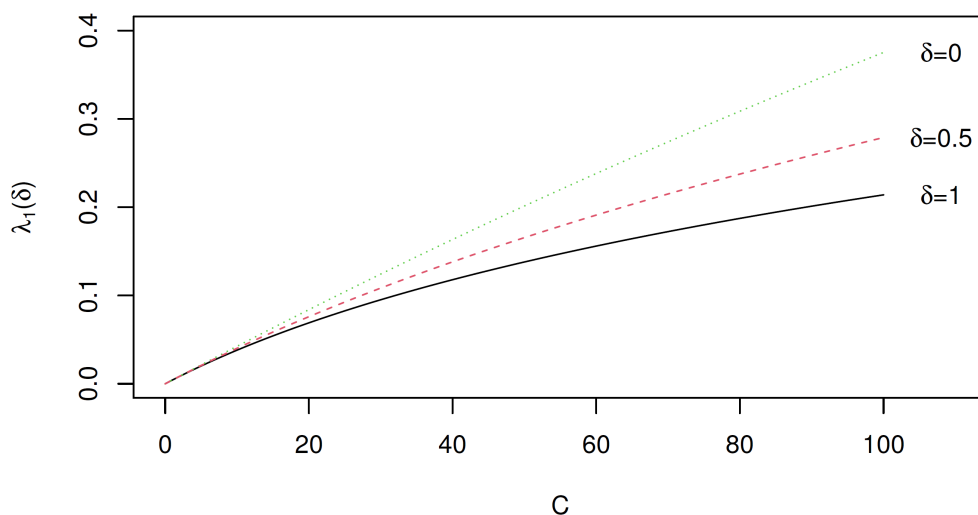


Figure 2. $\lambda_1(\delta)$ versus $C \in [0, 100]$ for Table 1; $\delta = 1, 1/2$ and 0

Figure 2 shows that, for $\lambda_1(\delta)$, as δ moves from 0 to 1 the singular value decreases in magnitude for all C . However, the values of $\lambda_2(\delta)$ increases as δ goes from 0 to 1, although any difference between values of $\lambda_2(\delta)$ for a given $C > 0$ is not as large as the differences observed between the $\lambda_1(\delta)$ values.

Suppose we define $\lambda_{\text{Diff}}(\delta) = \lambda_1(\delta) - \lambda_2(\delta)$ so that, from (34),

$$\lambda_{\text{Diff}}(\delta) = \frac{(40 + C)^{1/2-\delta}}{\delta\sqrt{2(680 + C)}} \left\{ 2(40 + C)^\delta - (40 + 2C)^\delta - 40^\delta \right\} \geq 0$$

for all values of δ ; note that this difference is zero when $\delta = 1$. This difference is also zero when $C = 0$ irrespective of the choice of δ . A plot of this difference versus $C \in [0, 100]$ is given in Figure 4. It confirms that $\lambda_1(1) = \lambda_2(1)$ while the difference between the two singular values is at its largest when $\delta = 0$. Thus, LRA will produce a more heavily dominant first dimension than its second dimension when compared with the correspondence analysis approach of Beh and Lombardo [10]. In fact, Figures 2 and 3 show that when $\delta = 0$ the first singular value will be larger than the first singular value when performing HDD and correspondence analysis to assess departures from perfect symmetry. Therefore, the first dimension of an LRA will always account for a larger proportion of any departure from perfect symmetry than HDD and correspondence analysis.

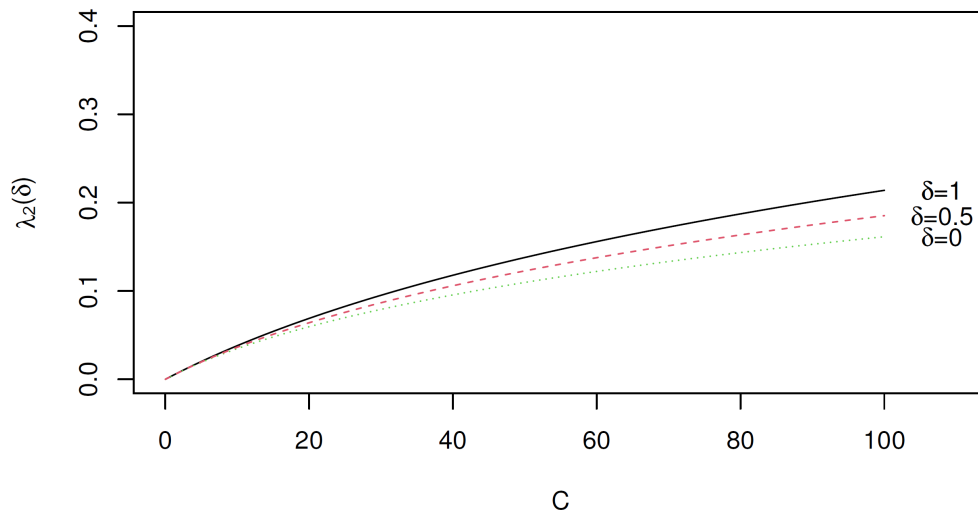


Figure 3. $\lambda_2(\delta)$ versus $C \in [0, 100]$ for Table 1; $\delta = 1, 1/2$ and 0

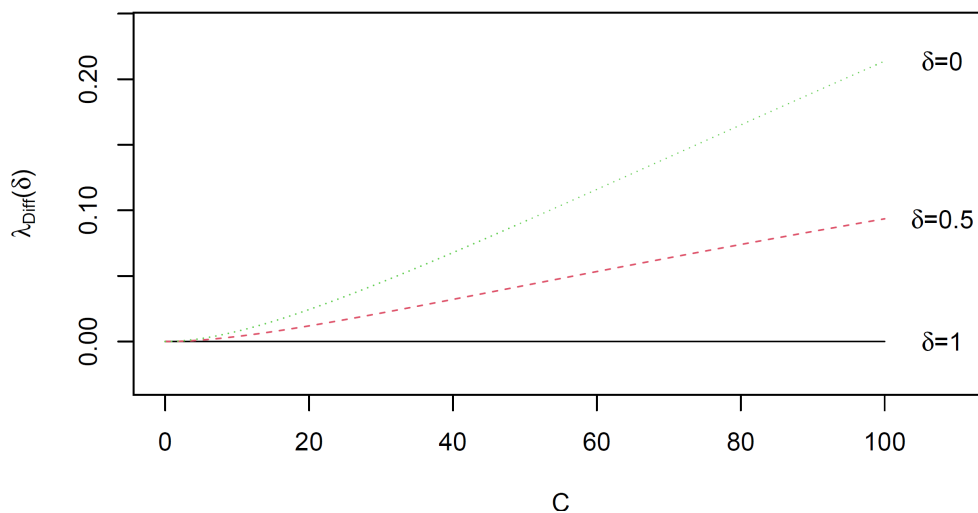


Figure 4. $\lambda_{\text{Diff}}(\delta)$ versus $C \geq 0$ for Table 1; $\delta = 1, 1/2$ and 0

4.4.3. Principal Coordinates

To derive the row and principal coordinates, (23) and (24), we first need to determine $\tilde{\mathbf{D}}$. Beh and Lombardo [10, p. 11] showed that for Table 1

$$\tilde{\mathbf{D}} = \begin{pmatrix} \frac{200+C}{2(680+C)} & 0 & 0 & 0 \\ 0 & \frac{400+C}{2(680+C)} & 0 & 0 \\ 0 & 0 & \frac{150}{680+C} & 0 \\ 0 & 0 & 0 & \frac{230}{680+C} \end{pmatrix}$$

so that

$$\tilde{\mathbf{D}}^{-1/2} = \sqrt{680+C} \begin{pmatrix} \sqrt{\frac{2}{200+C}} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{2}{400+C}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{150}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{230}} \end{pmatrix}. \quad (35)$$

We also have the matrix of left and right singular vectors which are

$$\mathbf{A}_\delta = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B}_\delta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad (36)$$

when $\delta \in [0, 1)$ and, when $\delta \geq 1$,

$$\mathbf{A}_\delta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{B}_\delta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \quad (37)$$

When $\delta \in [0, 1)$ then, using (34), (35), and \mathbf{A}_δ from (36), the elements of the matrix of row principal coordinates, (23), can be expressed in terms of δ and C so that

$$\mathbf{F}_\delta = -\frac{(40+C)^{1/2-\delta}}{\delta} \begin{pmatrix} \frac{(40+C)^\delta - 40^\delta}{\sqrt{200+C}} & 0 \\ 0 & \frac{(40+2C)^\delta - (40+C)^\delta}{\sqrt{400+C}} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, changing δ and C does not influence the position of the principal coordinates of the third and fourth rows of Table 1. This make sense since there is perfect symmetry for these two rows and so that their position in the correspondence plot is at the origin. Note that for $\delta \in (0, 1)$, the (1, 1)th and (2, 2)th elements of \mathbf{F}_δ , denoted by $f_{11}(C, \delta)$ and $f_{22}(C, \delta)$, respectively, are both negative for $C > 0$. The link between them is

$$\frac{f_{11}(C, \delta)}{f_{22}(C, \delta)} = K(C, \delta) \sqrt{\frac{400+C}{200+C}} > \sqrt{\frac{400+C}{200+C}}$$

where

$$K(C, \delta) = \frac{1 - \left(\frac{40}{40+C}\right)^\delta}{\left(\frac{40+2C}{40+C}\right)^\delta - 1} > 1$$

for $\delta \in [0, 1)$ and $C > 0$. Thus, the magnitude of $f_{11}(C, \delta)$ will always be at least $\sqrt{(400+C)/(200+C)}$ times larger than the magnitude of $f_{22}(C, \delta)$ for all $\delta \in (0, 1)$. For example, when $C = 50$ in Table 1, the lower bound of this ratio is $\sqrt{450/250} = 3/\sqrt{5} = 1.3416$ and this will occur as $\delta \rightarrow 1$. Therefore, when $C > 0$, $f_{11}(C, \delta)$ will always lie at least 1.3416 times further from the origin than $f_{22}(C, \delta)$. Thus row 1 of Table 1 will contribute more to any departure from perfect symmetry than row 2, irrespective of the choice of δ . When $\delta = 1$ then $K(C, 1) = 1$ so that the row principal coordinates of Beh and Lombardo [10, eq. (23)] are derived. Also, when $\delta = 1$, the link between $f_{12}(C, 1)$ and $f_{21}(C, 1)$ can be established using (37) instead of (36) and is

$$\frac{f_{12}(C, 1)}{f_{21}(C, 1)} = \sqrt{\frac{400 + C}{200 + C}}.$$

This is identical to the ratio derived by Beh and Lombardo [10, Section 6.3] when using Bowker's statistic to assess departures from perfect symmetry.

We can also obtain similar expressions for the column principal coordinates. Substituting (34), and (35), and \mathbf{B}_δ from (36), into (24) leaves us with

$$\mathbf{G}_\delta = \frac{(40 + C)^{1/2-\delta}}{\delta} \begin{pmatrix} 0 & -\frac{(40+2C)^\delta - (40+C)^\delta}{\sqrt{200+C}} \\ \frac{(40+C)^\delta - 40^\delta}{\sqrt{400+C}} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Note that the choice of δ and C does not influence the position of the third and fourth columns in the two-dimensional correspondence plot, where they lie at the origin. This makes sense since these columns of Table 1 are perfectly symmetrical with the third and fourth rows of the contingency table. Something else to note is that the (1, 2)th element of \mathbf{G}_δ , denoted by $g_{12}(\delta)$ is negative for $\delta \in (0, 1)$. Also, the (2, 1)th element of \mathbf{G}_δ , denoted by $g_{21}(\delta)$ is always positive for these values of δ . Therefore, the ratio of these two coordinates is always negative and is

$$\frac{g_{12}(C, \delta)}{g_{21}(C, \delta)} = -\frac{1}{K(C, \delta)} \sqrt{\frac{400 + C}{200 + C}}.$$

Therefore,

$$K(C, \delta) = \sqrt{-\frac{f_{11}(C, \delta)}{f_{22}(C, \delta)} \cdot \frac{g_{21}(C, \delta)}{g_{12}(C, \delta)}}$$

and shows that the relationship between the first and second row and column principal coordinates remains constant for some given value of C and δ .

4.5. The Correspondence Plots

Figure 5 gives the correspondence plot of Table 1 for $\delta = 1, 1/2$ and 0; these are constructed with X_S^2 , T_S^2 and M_S^2 , respectively, as their numerical foundation with $C = 50$.

Suppose we consider first the correspondence plot (Figure 5a) which can also be obtained using the technique outlined in Beh and Lombardo [10]. It shows that $R3$, $R4$, $C3$ and $C4$ are located at the origin. This should not be surprising for two related reasons: (1) there is perfect symmetry between $R3$ and $C3$, and between $R4$ and $C4$, and (2) these rows and columns are not influenced by the magnitude of C . Thus, these four categories of Table 1 play no part in determining the magnitude of Bowker's statistic. Instead, X_B^2 is influenced solely by the row categories $R1$ and $R2$, and the column categories $C1$ and $C2$, since C impacts on the symmetry (or lack thereof) of the (1, 2)th and (2, 1)th cell frequencies of Table 1. However, there is a noticeable difference in the position of $R1$ and $C1$ showing that there is a large departure from perfect symmetry between these categories; a feature present because $C = 50$. Similarly, $R2$ and $C2$ are situated at quite a distance from each other showing the influence of C on their position in the correspondence plot. However, since this distance appears shorter than between $R1$ and $C1$ this shows the influence of C impacts more on the symmetry between $R1$ and $C1$ than it does on the symmetry between $R2$ and $C2$.

The configuration of points in Figure 5b,c are quite similar, although appear quite different when compared with the configuration of points in Figure 5a. However, since $\lambda_1(1) = \lambda_2(1)$ then the configuration of points in Figure 5a remains unchanged if it is rotated clockwise 90 degrees and reflected along the first dimension. Doing so produces a configuration of points that is comparable to Figure 5b,c and, since $M = 2$ for our three values of δ , the three correspondence plots in Figure 5

depict all of the departures that exist from perfect symmetry. The only noticeable difference between the three plots is the percentage of the total inertia accounted for by the two dimensions. While all three plots display 100% of the departures from perfect symmetry (and are therefore excellent visual depictions) the first dimension is very much the most dominant when $\delta = 0$, accounting for 77.1% of M_S^2 , while 64.5% of T_S^2 is accounted for along this dimension when $\delta = 1/2$. This confirms the findings in our discussion of Figure 4.

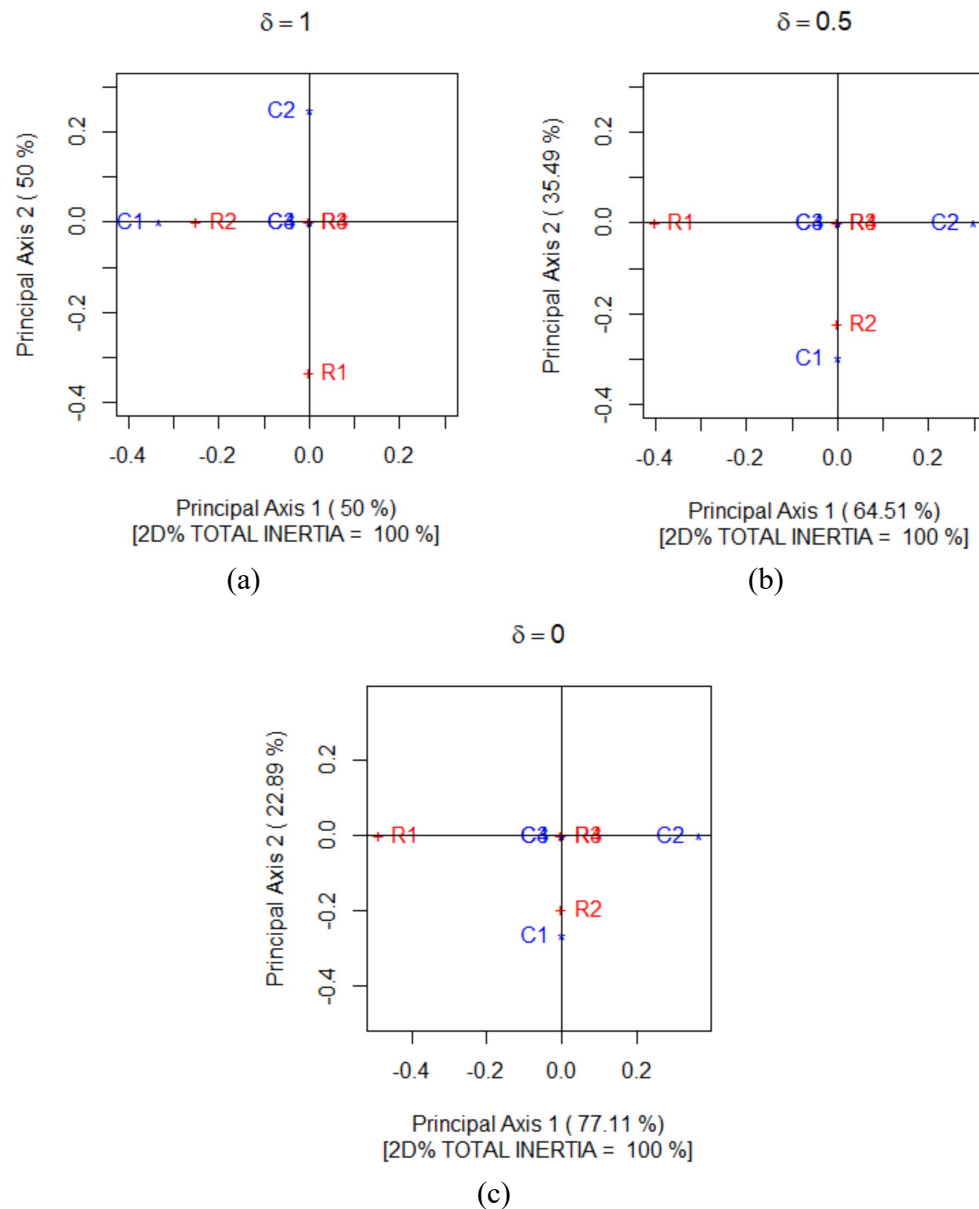


Figure 5. Correspondence plot for Table 1 with $C = 50$ where (a) $\delta = 1$, (b) $\delta = 1/2$ and (c) $\delta = 0$

5. Example 2: Pre- and Post-Courtship Behaviour of Bitterlings

5.1. The Data

We now move away from the analysis in Section 4 of the artificial contingency table and turn our attention to a more practical application. Consider Table 2 where $S = 12$ that originally comes from the extensive study of Wiepkema [33, Table II]. The data concerns the pre- and post-courtship behaviour of male bitterlings (*Rhodeus amarus* Bloch), a small European fish where the behaviour is classified according to 12 traits. Here we use the (*pre*/*POST*)-courtship labelling convention that is an adaptation

of the one used by Wiepkema [33] and van der Heijden [31]: jerking (*jk*/*JK*), turning beats (*tu*/*TU*), head butting (*hb*/*HB*), chasing (*cs*/*CS*), fleeing (*fl*/*FL*), quivering (*qu*/*QU*), leading (*le*/*LE*), head-down posture (*hd*/*HD*), skimming (*sk*/*SK*), snapping (*sn*/*SN*), chafing (*cf*/*CF*) and fin-flickering (*ff*/*FF*).

Table 2. The pre- and post-courtship behaviour of bitterlings (*Rhodeus amarus* Bloch, Source: [33]).

Post-	Pre-Courtship Behaviour												Total
	<i>jk</i>	<i>tu</i>	<i>hb</i>	<i>cs</i>	<i>fl</i>	<i>qu</i>	<i>le</i>	<i>hd</i>	<i>sk</i>	<i>sn</i>	<i>cf</i>	<i>ff</i>	
<i>JK</i>	654	128	172	56	27	25	1	28	0	46	14	18	1169
<i>TU</i>	101	132	62	27	5	1	1	11	0	8	5	9	362
<i>HB</i>	171	62	197	130	0	25	0	50	14	18	14	12	693
<i>CS</i>	60	22	152	135	0	8	0	43	16	15	12	4	467
<i>FL</i>	19	2	0	0	419	19	0	2	0	17	5	11	494
<i>QU</i>	36	1	18	5	12	789	119	295	26	70	1	14	1386
<i>LE</i>	4	0	0	0	0	57	167	73	0	8	0	0	309
<i>HD</i>	22	9	40	37	5	245	7	171	287	53	8	13	897
<i>SK</i>	3	2	7	38	0	120	8	134	19	28	4	9	363
<i>SN</i>	42	2	17	16	20	70	11	67	9	225	12	12	503
<i>CF</i>	18	3	10	13	6	5	0	8	0	24	97	9	193
<i>FF</i>	27	3	6	5	10	13	0	18	0	10	8	29	129
Total	1157	366	681	462	504	1377	314	900	371	522	180	131	6965

Table 2 was the subject of a classical correspondence analysis performed by van der Heijden [31] where departures from complete independence were assessed. Given the symmetric nature of the variables, we shall now perform a correspondence analysis using (8) to assess any departures from perfect symmetry that may exist in the data.

5.2. Test of the Departure from Perfect Symmetry

Of the 144 cells in Table 2 there are 22 zero cell frequencies (or 15.3% of the cells). The affect of this is that there are 16 values of $p_{ij} + p_{ji}$ that are zero which means that (8) involves 16 instances where a division by zero occurs. To overcome this problem 0.01 has been added to each cell of the contingency table. Doing this leads to Bowker’s statistic, (9), of 277.801 while (10) and (11) are 333.9 and 671.0, respectively. With $12(12 - 1)/2 = 66$ degrees of freedom, these three statistics have a p-value that is less than 0.0001. Therefore, there is enough evidence in Table 2 to conclude that there is a statistically significant departure from perfect symmetry. That is, there is at least one of the 12 pairings of the pre- and post-courtship behaviour that is statistically different.

5.3. On the Divergence Residuals

One may evaluate where these departures from perfect symmetry lie by observing the elements of S_δ . Table 3 gives these residuals for $\delta = 1, 1/2$ and 0. Note that since all diagonal elements of S_δ are zero they have been omitted from Table 3. Those residuals designated “< 0.001” are residuals lying within the interval $\pm(0.0000001, 0.0001)$.

The largest (negative and positive) divergence residuals for our three values of δ appear in bolded text in Table 3. We can see that the largest positive residuals are for the pre- and post-courtship pairs (*hd*, *LE*), (*sk*, *HD*) and (*qu*, *SK*). These combinations reflect that there are more observations in these cells than what would be expected if there were perfect symmetry between the variables. For example, for (*LE*, *hd*) the observed cell count is 167 while the expected number of observations under perfect symmetry is $(167 + 7)/2 = 87$. On the other hand, the largest negative residuals are for the pairs (*QU*, *sk*), (*HD*, *le*) and (*SK*, *hd*) and reflect those cells where the observed cell count is smaller than what is expected under perfect symmetry. This can be seen with the (*HD*, *le*) pairing where the observed cell count is 7 and the expected number of observations under perfect symmetry is 87 (as we showed

above). Therefore, these three pairs of pre- and post-courtship behaviour are the reverse of those pairings with a large positive divergence residual.

Table 3. The matrix of divergence residuals, S_{δ} for $\delta = 1, 1/2$ and 0.

		Pre-Courtship Behaviour											
Post-	δ	<i>jk</i>	<i>tu</i>	<i>hb</i>	<i>cs</i>	<i>fl</i>	<i>qu</i>	<i>le</i>	<i>hd</i>	<i>sk</i>	<i>sn</i>	<i>cf</i>	<i>ff</i>
<i>JK</i>	1		0.015	<0.001	-0.003	0.010	-0.012	-0.011	0.007	-0.015	0.004	-0.006	-0.011
	1/2		0.015	<0.001	-0.003	0.010	-0.013	-0.014	0.007	-0.027	0.004	-0.006	-0.012
	0		0.014	<0.001	-0.003	0.009	-0.013	-0.017	0.007	-0.074	0.004	-0.006	-0.013
<i>TU</i>	1	-0.015		0	0.006	0.010	0	0.008	0.004	-0.012	0.016	0.006	0.015
	1/2	-0.016		0	0.006	0.009	0	0.007	0.004	-0.022	0.014	0.006	0.013
	0	-0.016		0	0.006	0.008	0	0.006	0.004	-0.056	0.013	0.005	0.012
<i>HB</i>	1	<0.001	0		-0.011	0	0.009	0	0.009	0.013	0.001	0.007	0.012
	1/2	<0.001	0		-0.011	0	0.009	0	0.009	0.012	0.001	0.007	0.011
	0	<0.001	0		-0.012	0	0.008	0	0.008	0.011	0.001	0.006	0.010
<i>CS</i>	1	0.003	-0.006	0.011		0	0.007	0	0.006	-0.025	-0.002	-0.002	-0.003
	1/2	0.003	-0.006	0.011		0	0.007	0	0.006	-0.029	-0.002	-0.002	-0.003
	0	0.003	-0.006	0.011		0	0.006	0	0.005	-0.033	-0.002	-0.002	-0.003
<i>FL</i>	1	-0.010	-0.010	0	0		0.011	0	-0.010		0	-0.004	-0.003
	1/2	-0.010	-0.011	0	0		0.010	0	-0.011		0	-0.004	-0.003
	0	-0.011	-0.013	0	0		0.010	0	-0.013		0	-0.004	-0.003
<i>QU</i>	1	0.012	0	-0.009	-0.007	-0.011		0.040	0.018	-0.066	0	-0.014	0.002
	1/2	0.011	0	-0.009	-0.008	-0.011		0.037	0.018	-0.083	0	-0.017	0.002
	0	0.011	0	-0.010	-0.008	-0.012		0.034	0.017	-0.106	0	-0.023	0.002
<i>LE</i>	1	0.011	-0.008	0	0	0	-0.040		0.063	-0.024	-0.006	0	0
	1/2	0.010	-0.015	0	0	0	-0.044		0.053	-0.046	-0.006	0	0
	0	0.009	-0.034	0	0	0	-0.049		0.046	-0.144	-0.006	0	0
<i>HD</i>	1	-0.007	-0.004	-0.009	-0.006	0.010	-0.018	-0.063		0.063	-0.011	0	-0.008
	1/2	-0.007	-0.004	-0.009	-0.006	0.009	-0.019	-0.088		0.058	-0.011	0	-0.008
	0	-0.008	-0.004	-0.009	-0.006	0.008	-0.019	-0.132		0.054	-0.012	0	-0.008
<i>SK</i>	1	0.015	0.012	-0.013	0.025	0	0.066	0.024	-0.063		0.026	0.017	0
	1/2	0.012	0.010	-0.014	0.023	0	0.058	0.020	-0.070		0.024	0.014	0
	0	0.010	0.008	-0.016	0.021	0	0.051	0.017	-0.079		0.021	0.012	0
<i>SN</i>	1	-0.004	-0.016	-0.001	0.002	0.004	0	0.006	0.011	-0.026		-0.017	0.004
	1/2	-0.004	-0.020	-0.001	0.002	0.004	0	0.006	0.011	-0.031		-0.019	0.004
	0	-0.004	-0.024	-0.001	0.001	0.004	0	0.005	0.010	-0.037		-0.021	0.003
<i>CF</i>	1	0.006	-0.006	-0.007	0.002	0.003	0.014	0	0	-0.017	0.017		0.002
	1/2	0.006	-0.006	-0.007	0.002	0.002	0.012	0	0	-0.032	0.016		0.002
	0	0.006	-0.007	-0.008	0.002	0.002	0.011	0	0	-0.090	0.015		0.002
<i>FF</i>	1	0.011	-0.015	-0.012	0.003	-0.002	-0.002	0	0.008	0	-0.004	-0.002	
	1/2	0.011	-0.017	-0.013	0.003	-0.002	-0.002	0	0.007	0	-0.004	-0.002	
	0	0.010	-0.020	-0.015	0.003	-0.002	-0.002	0	0.007	0	-0.004	-0.002	

Table 3 also shows that $s_{ij}(1) = -s_{ji}(1)$ for all $i, j = 1, 2, \dots, 12$. For our other two values of δ there is either perfect or near perfect symmetry since the (i, j) th and (j, i) th divergence residuals are of the same or similar magnitude (differing only in their sign). Although, there are clear differences in magnitude of some of these residuals. For example $s_{87}(0) = -0.132$ (corresponding to the (HD/le) pair) while $s_{78}(0) = 0.046$ (corresponding to the (LE/hd) pair). Comparing these divergence residuals shows that the negative interaction between HD and le is about three times greater than the positive interaction between hd and LE .

The similarities, and differences, in these divergence residuals can be visualised using the correspondence plot. We now turn our attention to the correspondence plot of Table 2 when $\delta = 1, 1/2$ and 0.

5.4. Visualising the Departures from Perfect Symmetry

To visualise where the departures from perfect symmetry exist we construct the correspondence plot using the principal coordinates of (23) and (24) for $\delta = 1, 1/2$ and 0. These plots are given in

Figure 6 where departures from perfect symmetry are assessed using the statistics, (9) (11) and (10) for $\delta = 1, 1/2$ and 0, respectively. These three correspondence plots provide an excellent visual depiction of departures from perfect symmetry in Table 2 since they all account for about 84% of the total inertia calculated using X_{δ}^2 , T_{δ}^2 and M_{δ}^2 .

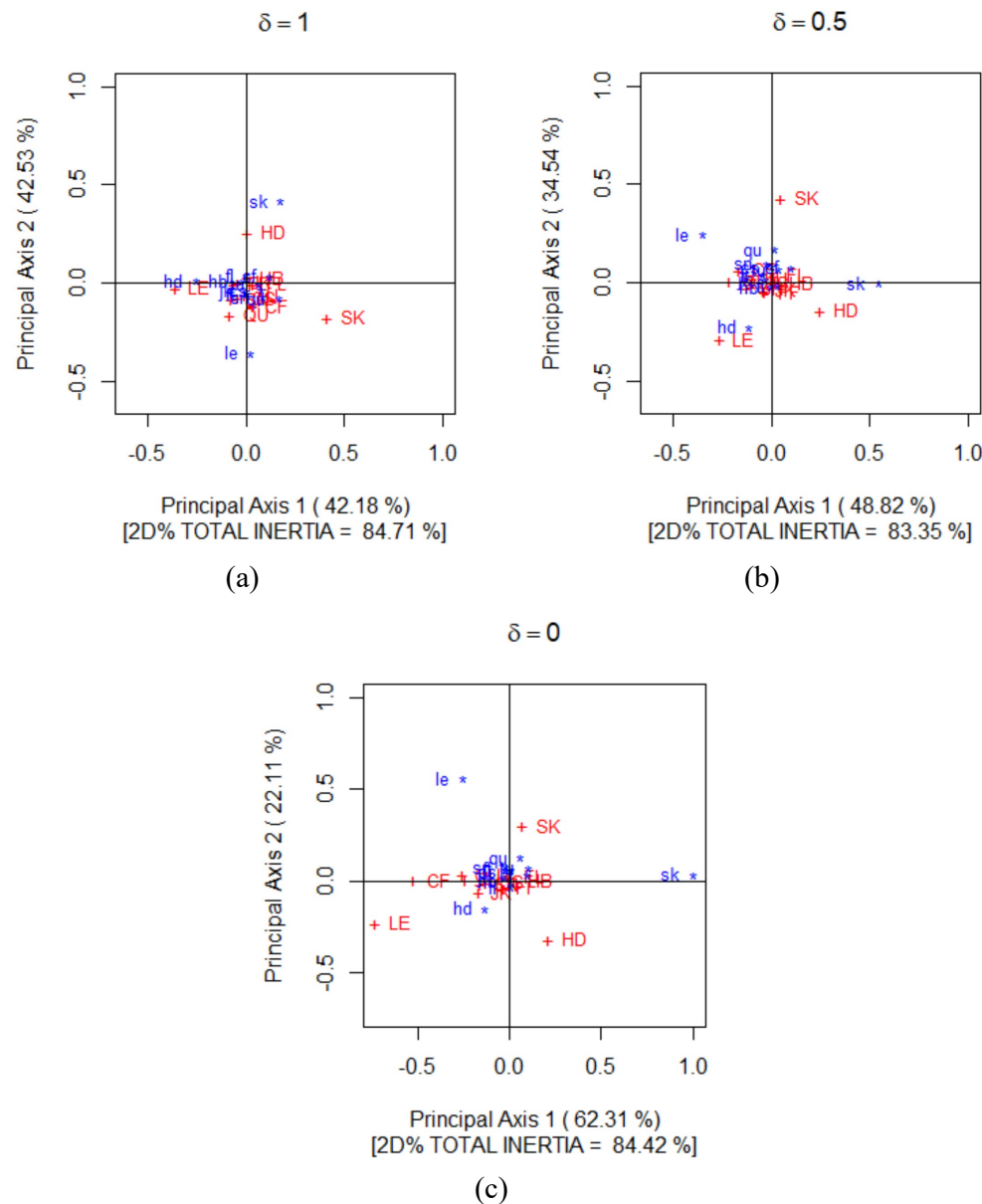


Figure 6. Correspondence plot for Table 2 where (a) $\delta = 0$, (b) $\delta = 1/2$ and (c) $\delta = 1$

The first thing to note about the configuration of points in the three correspondence plots of Figure 6 is that there is a large cluster of points that lie close to the origin. In fact, most of the categories of Table 2 lie at, or near, the origin with only a few categories that lie at a distance from the origin. Therefore, the three plots of Figure 6 show that most of the categories of Table 2 are fairly consistent with what is expected under perfect symmetry. Note that we are not saying here that all of the categories located in close proximity to the origin are perfectly symmetric. This can be achieved by determining the $100(1 - \alpha)\%$ confidence region for each category (for some level of significance, α) which is beyond the scope of this paper. Although, when assessing departures from complete independence, such regions were recently developed by Alzahrani, Beh and Stojanovski [3] and are based on those described in Beh [7] and Beh and Lombardo [9].

We now turn our attention to those categories that are located relatively far from the origin. The three plots of Figure 6 show these to be *le/LE* (pre- and post-courtship leading), *sk/SK* (pre- and post-courtship skimming), and *hd/HD* (pre- and post-courtship head-down posture). Therefore it is these three behaviours that deviate the most from what would be expected if there were perfect symmetry in Table 2, and are the dominant source for why the p-value of (9), (10) and (11) is very small. Interestingly, these are three of the four behaviours that Wiepkema [33, p. 131] and van der Heijden [31, p. 56] note as being the sexual factors that underlay bitterling courtship behaviour. The fourth trait they identified was quivering (*qu/QU*). Note that there is a relatively large negative divergence residual between *sk* and *QU* in Table 3 which suggests that a pre-courtship skimming behaviour is unlikely to lead to a quivering post-courtship behaviour. While *QU* lies relatively close to the origin for all four values of δ , it does lie at a distance from *sk*. However, there are many other post-courtship behaviours that lie close to the origin of their correspondence plot and hence at a distance from *sk* and have a relatively small divergence residual. So is there really an under-count of pre-courtship skimming behaviour and post-courtship quivering? While adding a third dimension does not add a great deal to our visual display of the departures from perfect symmetry, they do show, for our three δ values, that the proximity of *sk* from the origin is matched by the proximity that *le* and/or *hd* (depending on the choice of δ). Therefore, the third dimension does add additional context to the differences highlighted in Table 3 between *sk* and *QU*.

Suppose we now discuss other courtship behaviours and pairs that are located relatively far from each other. The first thing to point out here is that for our three values of δ , *LE*, *HD* and *SK* are all located in different parts of their correspondence plot. This suggests that the post-courtship behaviours of leading, head-down posture and skimming all contribute differently to the lack of perfect symmetry in Table 2. So too are their pre-courtship behaviours *le*, *hd* and *sk*. Interestingly, each of these three pre-courtship behaviours is not followed by their post-courtship behaviour. That is, for example, a pre-courtship display of leading is not followed by a post-courtship display of leading. In fact, Figure 6 shows that the differences between these three courtship behaviours is quite consistent.

While there are differences in pre- and post-courtship behaviours there are also some clearly defined pairings that can be identified by observing where departures from perfect symmetry exist. These are for the pairings of (*hd*, *LE*), (*sk*, *HD*) when $\delta = 1$ and $1/2$; recall that the divergence residuals for these pairs in Table 3 is relatively large and positive. This suggests that when assessing the departures from independence using the statistics (9) and (10), a pre-courtship display of head-down posture is followed by a leading post-courtship display, while a pre-courtship display of skimming is followed by a post-courtship display of head-down posture. Only when $\delta = 0$ does there appear to be quite a difference between *sk* and *HD*; in fact, Figure 6a shows that a pre-courtship display of skimming is equally likely to lead to a post-courtship behaviour of head-down posture and skimming, although the link between the (*sk*, *SK*) and (*sk*, *HD*) pairs is not strong when $\delta = 0$.

6. Discussion

When numerically assessing departures from perfect symmetry one need not be confined to Bowker's statistic [15], defined here by (9). There are a range of alternative statistics that can be considered and have been available for many decades; here we have focused our attention on the Freeman-Tukey statistic, T^2_S , and the modified log-likelihood ratio statistic M^2_S . These statistics are special cases of the Cressie-Read family of divergence statistics, defined by (2), as well as the second order Taylor series approximation of this family; see (3).

This paper has demonstrated how (3) can be used as the numerical foundations for performing a correspondence analysis to visualise departures from perfect symmetry. A special case of this family is when $\delta = 1$ leading to the correspondence analysis technique recently described by Beh and Lombardo [10]. While we have discussed that any value of δ can be considered when performing this analysis, there are advantages in considering $\delta = 1$, $\delta = 1/2$ and $\delta = 0$. With such flexibility in the choice of δ , one may well ask *what is the most appropriate choice of δ to use?* We discussed this issue when

showing the links between (4) and correspondence analysis when assessing departures from complete independence; see Beh and Lombardo [11]. We described in that paper that the choice of δ may depend on many factors, including “the structure of the data, the output that is generated from the analysis or the ease and interpretability that a value of δ provides” (p. 38). However, there are other factors that may impact on the choice of δ . As the applications have shown, one may wish to choose the value of δ that yields the greatest percentage of the total inertia in a two-dimensional, say, correspondence plot; this depends greatly on the data structure that is being assessed for departures from perfect symmetry. One may consider $\delta = 1$ to be an ideal choice for numerous reasons including (1) it leads to the more traditional correspondence analysis (2) the total inertia is measured using the well known and well understood Bowker’s statistic, and (3) the first two dimensions will account for the same percentage of the total inertia. This third reason also means that the analyst is provided with flexibilities to rotate and/or reflect the configuration of points around either dimension without affecting the general interpretability of the configuration. As the application to Table 2 also shows, of the three values we considered, $\delta = 1$ also leads to the greatest percentage of the total inertia being visualised.

The next step in the evolution of this method of correspondence analysis is to derive the confidence regions alluded to Section 3.6 for visualising those categories that are statistically significant contributors to the global measure of the departure from perfect symmetry. Such regions expand upon those describe by Alzahrani, Beh and Stojanovski [3] and complement the correspondence analysis framework developed by Beh and Lombardo [11]. We shall leave this, and other further developments of the method of correspondence analysis outlined in this paper, for future work.

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